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A MEASURE EXTENSION WITH RESPECT TO A MEASURE PRESERVING MAPPING

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Abstract. Abstract The present paper shows that a measure defined on a σ-algebra and invariant with respect to a measurable mapping may be extended onto a greater σ-algebra such that the mapping is strict measurable with respect to this greater σ-algebra.

1. Preliminaries.

It is known that the continuous image of a Borel set need not be Borel. This result belongs to M. Souslin, which in [5] corrected an error of H. Lebesgue, which in [4] stated that the continuous image of a measurable set is measurable. Let J be the set of all irrational numbers with the standard topology. Then J is homeomorphic to the product of countable many copies of the countable discrete topological space [3, p. 32]. Using results of Exercise 6 of [1, pp. 152 – 153] it may be proved the existence of a continuous mapping $T : J \to J$ such that the set $T(J)$ is not Borel.

In the whole paper we consider a quadruple $(X, \mathcal{A}, m, T)$, where X is a set, $\mathcal{A}$ is a σ-algebra on X, m is a σ-finite measure on $\mathcal{A}$ and $T : X \to X$ is an $\mathcal{A}$-measurable m-preserving mapping, i.e. $T^{-1}(A) \in \mathcal{A}$ and $m(T^{-1}(A)) = m(A)$ for any $A \in \mathcal{A}$, see [6, p. 19]. In this case the measure m is said to be T-invariant.

The mapping T preserving the measure m is almost surjective in the following sense. For $A \in \mathcal{A}$ with $A \cap T(X) = \emptyset$ we have $m(A) = m(T^{-1}(A)) = m(\emptyset) = 0$. Particularly, $m(X \setminus T(X)) = 0$ whenever $T(X) \in \mathcal{A}$. As it was said, in a general case $\mathcal{A}$-measurability of a mapping T does not imply $\mathcal{A}$-measurability of the set $T(X)$, i.e. $T(X) \in \mathcal{A}$. However, for a strict $\mathcal{A}$-measurable mapping $T : X \to X$ we can guarantee $T(X) \in \mathcal{A}$ and more generally $T^n(X) \in \mathcal{A}$ for all natural n. The definition of a strict measurable mapping follows.

Definition 1.1. Let X be a set, $\mathcal{A}$ be a σ-algebra on X and $T : X \to X$ be a mapping. Then T is said to be strict $\mathcal{A}$-measurable if $A \in \mathcal{A}$ if and only if $T^{-1}(A) \in \mathcal{A}$ for any $A \subset X$.

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The present paper constructs a natural extension $\tilde{m}$ of the measure m onto a greater σ-algebra $\tilde{\mathcal{A}}$, such that T is strict $\tilde{\mathcal{A}}$-measurable and $\tilde{m}$-preserving.

2. One step extension and extension by induction.

Let and $T : X \to X$ be a mapping and $\mathcal{A}$ be a σ-algebra on X. Put $\mathcal{A}^T = \{A : A \subset X \text{ and } T^{-1}(A) \in \mathcal{A}\}$.

Proposition 2.1.

(i) *$\mathcal{A}^T$ is a σ-algebra.*
(ii) *If T is $\mathcal{A}$-measurable, then $\mathcal{A} \subset \mathcal{A}^T$ and T is $\mathcal{A}^T$-measurable.*
(iii) *$T(X) \in \mathcal{A}^T$.*
(iv) *T is strict $\mathcal{A}$-measurable, if and only if $\mathcal{A} = \mathcal{A}^T$.*

Example 2.1. Let $T : X \to X$ be an $\mathcal{A}$-measurable mapping such that $T(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$. Then $\mathcal{A}^T$ consists of the sets of the form $C = A \cup B$, where $A \in \mathcal{A}$ and $B \cap T(X) = \emptyset$. If moreover $T(X) = X$ then T is strict $\mathcal{A}$-measurable.

Proposition 2.2. *Let m be a measure on $\mathcal{A}$. Put $m^T(A) = m(T^{-1}(A))$ for $A \in \mathcal{A}^T$.*

(i) *m^T is a measure on $\mathcal{A}^T$.*
(ii) *If the measure space $(X, \mathcal{A}, m)$ is complete, then so is $(X, \mathcal{A}^T, m^T)$.*
(iii) *If T is $\mathcal{A}$-measurable and m-preserving, then m^T is a unique T-invariant extension of m onto $\mathcal{A}^T$; if T is not strict $\mathcal{A}$-measurable then m^T is a nontrivial extension of m.*

Proof. Part (i) is obvious. We shall prove (ii).
Let $B \subset A \in \mathcal{A}^T$ and $m^T(A) = 0$. Then $T^{-1}(A) \in \mathcal{A}$ and $m(T^{-1}(A))$. Since $(X, \mathcal{A}, m)$ is complete, we have $T^{-1}(B) \in \mathcal{A}$ and $B \in \mathcal{A}^T$. It proves (ii). Now, let T be $\mathcal{A}$-measurable and m-preserving. Take $A \in \mathcal{A}$. Then $m^T(A) = m(T^{-1}(A)) = m(A)$, because T is m-preserving. In means that the measure m^T is an extension of m. Take $A \in \mathcal{A}^T$, then $T^{-1}(A) \in \mathcal{A}$ and as we have shown $m^T(T^{-1}(A)) = m(T^{-1}(A))$. The definition of m^T yields $m^T(A) = m(T^{-1}(A))$. Therefore $m^T(T^{-1}(A)) = m^T(A)$ and T is m^T-preserving. Let μ be another T-invariant extension of m. For $A \in \mathcal{A}^T$ we have $\mu(A) = \mu(T^{-1}(A)) = m(T^{-1}(A)) = m^T(A)$, because $T^{-1}(A) \in \mathcal{A}$ and μ is an extension of m. If T is not strict $\mathcal{A}$-measurable, then $\mathcal{A} \subset \mathcal{A}^T$ but $\mathcal{A} \neq \mathcal{A}^T$ by Proposition 2.1.

Obviously, we can continue extension procedure by induction. Put $\mathcal{A}_0 = \mathcal{A}$, $m_0 = m$ and $\mathcal{A}_{m+1} = \mathcal{A}_n^T$, $m_{n+1} = m_n^T$. Then the measure m_{n+1} is an extension of m_n onto A_{n+1}. The union $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra. For $A \in \mathcal{A}_n$ put $\mu(A) = m_n(A)$. Then we obtain a measure μ defined on the algebra $\bigcup_{n=1}^{\infty} \mathcal{A}_n$. Denote by $\mathcal{A}_\omega$ the σ-algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{A}_n$. The measure μ may by uniquely extended onto $\mathcal{A}_\omega$ [2, p. 40]. Denote this extension by m_ω. It is clear that the mapping $T : X \to X$ is $\mathcal{A}_n$-measurable for all natural n and $T^{-1}(A) \in \mathcal{A}_n$ implies $A \in \mathcal{A}_{n+1}$ for any $A \subset X$. However, we are not able to prove that the mapping

$T : X \to X$ is strict $\mathcal{A}_\omega$-measurable. The problem of a strict measurability of the mapping T will be solved in the following section. We shall show that the σ-algebra $\mathcal{A}_\omega$ has some interesting properties.

Proposition 2.3. *The σ-algebra $\mathcal{A}_\omega$ contains $T^n(X)$ for all natural n and their intersection $\bigcap_{n=1}^{\infty} T^n(X)$ as well.*

Proof. Note that for any $A \subset X$ the iterated preimage $(T^{-1})^n(A)$ and the preimage under iterated mapping $(T^n)^{-1}(A)$ coincide. This set will be denoted by $T^{-n}(A)$. For any natural n we have $T^{-n}(T^n(X)) = X \in \mathcal{A} = \mathcal{A}_0$. By induction $T^{-(n-k)}(T^n(X)) \in \mathcal{A}_k$ for all natural k, $0 \leq k \leq n$. (The equality $T^{-(n-k)}(T^n(X)) = T^k(X)$ does not hold generally, but it is true for $k = n$.) It means $T^n(X) \in \mathcal{A}_n$ and $T^n(X) \in \mathcal{A}_\omega$. Therefore $\mathcal{A}_\omega$ contains the set $\bigcap_{n=1}^{\infty} T^n(X)$.

Corollary 2.1. *If T is strict $\mathcal{A}$-measurable then $T^n(X) \in \mathcal{A}$ for all natural n.*

Corollary 2.2. *Let $\bigcap_{n=1}^{\infty} T^n(X) \in \mathcal{A}$. Then $m(X \setminus \bigcap_{n=1}^{\infty} T^n(X)) = 0$.*

Proof. (Note, that we suppose nothing about the sets $T^n(X)$.) Since all degrees T^n preserve the measure m, they preserve also the measure m_ω. Therefore $m_\omega(X \setminus T^n(X)) = m_\omega(T^{-n}(X \setminus T^{-n}(X))) = m_\omega(\emptyset) = 0$ and $m(X \setminus \bigcap_{n=1}^{\infty} T^n(X)) = m_\omega(X \setminus \bigcap_{n=1}^{\infty} T^n(X)) = 0$.

3. Extension of $\mathcal{A}$ by transfinite induction.

Now, consider only a σ-algebra $\mathcal{A}$ on X and an $\mathcal{A}$-measurable mapping $T : X \to X$. Denote by $Ext(\mathcal{A})$ the class of all σ-algebras $\mathcal{B}$ on X such that:

(i) $\mathcal{A} \subset \mathcal{B}$.
(ii) T is strict $\mathcal{B}$-measurable.

We shall show, that the class $Ext(\mathcal{A})$ contains the smallest σ-algebra $\tilde{\mathcal{A}}$, which may be described by transfinite induction. Let ω_1 be the first uncountable ordinal. Put

$$\begin{aligned}
&\mathcal{A}_0 = \mathcal{A}, \\
&\mathcal{A}_\alpha = \mathcal{A}_{\alpha-1}^T \ , \ \text{when } \alpha < \omega_1 \text{ is an unlimit ordinal} \\
&\mathcal{A}_\alpha = \sigma\left(\bigcup_{\beta<\alpha} \mathcal{A}_\beta\right) \ , \ \text{when } \alpha \text{ is a limit ordinal and} \\
&\tilde{\mathcal{A}} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha \ .
\end{aligned}$$

Proposition 3.1. *The σ-algebra $\tilde{\mathcal{A}}$ is the smallest element of the class $Ext(\mathcal{A})$. The mapping $T : X \to X$ is strict $\tilde{\mathcal{A}}$-measurable.*

Proof. Let $\mathcal{B}$ be the element of the class $Ext(\mathcal{A})$. All inclusions $\mathcal{A}_\alpha \subset \mathcal{B}$ for $\alpha < \omega_1$ follows immediately by transfinite induction from the strict $\mathcal{B}$-measurability of the mapping T and the inclusion $\mathcal{A} \subset \mathcal{B}$. We shall show that T is $\tilde{\mathcal{A}}$-measurable. It suffices to prove that T is $\mathcal{A}_\alpha$-measurable for all $\alpha < \omega_1$. The case $\alpha = 0$ is obvious. If $\alpha < \omega_1$ is an unlimit ordinal, then $\mathcal{A}_\alpha$-measurability follows from $\mathcal{A}_{\alpha-1}$-measurability and Proposition 2.1. If $\alpha < \omega_1$ is a limit ordinal then $\mathcal{A}_\alpha$ contains $T^{-1}(A)$ for all $A \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta$ by the inductive assumption. Since $\mathcal{A}_\alpha$ is a σ-algebra, it contains $T^{-1}(A)$ for all $A \in \sigma\left(\bigcup_{\beta<\alpha} \mathcal{A}_\beta\right) = \mathcal{A}_\alpha$. It shows $\tilde{\mathcal{A}}$-measurability of T. Finally, if $T^{-1}(A) \in \tilde{\mathcal{A}}$ then $T^{-1}(A) \in \mathcal{A}_\alpha$ and $A \in \mathcal{A}_{\alpha+1}$. It completes the proof.

4. Extension of a measure onto $\tilde{\mathcal{A}}$.

Put $m_0 = m$, $m_\alpha = m^T_{\alpha-1}$ for any unlimit countable ordinal $\alpha > 0$. For a limit countable ordinal α the measure m_α will be defined in the following way. Note that $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$ is an algebra on X. Take $A \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. Then $A \in \mathcal{A}_\beta$ for some $\beta < \alpha$ and put $\mu_\alpha(A) = m_\beta(A)$. Then we obtain a σ-finite measure defined on the algebra $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$. The measure μ_α may be uniquely extended onto σ-algebra $\sigma\left(\bigcup_{\beta<\alpha} \mathcal{A}_\beta\right)$, [2, p. 40]. This extension will be denoted by m_α. Finally put $\tilde{m}(A) = m_\alpha(A)$ whenever $A \in \mathcal{A}_\alpha$ for some $\alpha < \omega_1$.

Theorem 4.1. *The measure $\tilde{m}$ is a unique T-invariant extension of the measure m onto $\tilde{\mathcal{A}}$.*

Proof. Since $\tilde{\mathcal{A}} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, it suffices to prove that m_α is a unique T-invariant extension of m onto $\mathcal{A}_\alpha$.
The case $\alpha = 0$ is trivial.
Let $\alpha > 0$ be an unlimit countable ordinal. Suppose that $m_{\alpha-1}$ is a unique T-invariant extension of m onto $\mathcal{A}_{\alpha-1}$. By Proposition 2.2. m_α is a unique T-invariant extension of $m_{\alpha-1}$ onto $\mathcal{A}_\alpha$. Therefore m_α is a unique T-invariant extension of m onto $\mathcal{A}_\alpha$. Now, let α be a limit countable ordinal. Suppose that for all $\beta < \alpha$ the measure m_β is a unique T-invariant extension of m onto $\mathcal{A}_\beta$. Then μ_α is a unique T-invariant extension of m onto the algebra $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$. The measure m_α is a unique extension of μ_α onto $\mathcal{A}_\alpha$ in the realm of measures. It suffices to prove that m_α is T-invariant. To see this for $A \in \mathcal{A}_\alpha$ put $v_\alpha(A) = m_\alpha(T^{-1}(A))$. Then v_α is a measure, which coincide with μ_α on the algebra $\bigcup_{\beta<\alpha} \mathcal{A}$. It means that v_α coincide with μ_α on $\mathcal{A}_\alpha$ and the measure m_α is T-invariant.

5. Complete extension.

In this section we describe an extension $\hat{m}$ of the measure m onto a σ-algebra $\hat{\mathcal{A}}$, such that the measure space $(X, \hat{\mathcal{A}}, \hat{m})$ will be also complete.

Let us recall the notion of the completion of a measure space. Let $(X, \mathcal{A}, m)$ be a measure space. Denote be $\overline{\mathcal{A}}$ the system of all sets A of the form $A = A_1 \cup A_0$, where $A_1 \in \mathcal{A}$ and $A_0 \subset B_0$ for some $B_0 \in \mathcal{A}$ with $m(B_0) = 0$, and for such a set A define $\overline{m}(A) = m(A_1)$. Then $(X, \overline{\mathcal{A}}, \overline{m})$ is a complete measure space and $\overline{m}$ is an extension of m. It is easy to see, that any $\mathcal{A}$-measurable m-preserving mapping $T : X \to X$ is also $\overline{\mathcal{A}}$-measurable and $\overline{m}$-preserving. It means that T is also $\overline{\tilde{\mathcal{A}}}$-measurable and $\overline{\tilde{m}}$-preserving ($\tilde{\mathcal{A}}$ and $\tilde{m}$ has been constructed in the preceding sections.) Unfortunately, there are no arguments that T is strict $\overline{\tilde{\mathcal{A}}}$-measurable.

So we modify the construction of $\tilde{\mathcal{A}}$ and $\tilde{m}$ in the following way.
Put $\mathcal{A}_0 = \overline{\mathcal{A}}$ and $= \overline{m}$. When $\alpha > 0$ is unlimit countable ordinal put $\mathcal{A}_\alpha = \mathcal{A}^T_{\alpha-1}$ and $m_\alpha = m^T_{\alpha-1}$. (If $m_{\alpha-1}$ is complete then m_α is complete by Proposition 2.2.) For a limit countable ordinal α the σ-algebra $\mathcal{A}_\alpha$ and the measure m_α defined in preceding sections must be replace by their completions $\overline{\mathcal{A}}_\alpha$ and $\overline{m}_\alpha$. Put $\hat{\mathcal{A}} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ and $\hat{m}(A) = m_\alpha(A)$ for $A \in \mathcal{A}_\alpha$.

Theorem 5.1. *The measure space $(X, \hat{\mathcal{A}}, \hat{m})$ is complete and the mapping $T : X \to X$ is strict $\hat{\mathcal{A}}$-measurable and $\hat{m}$-preserving.*

The proof of the last theorem is simiral to the proof of Theorem 4.1.

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