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GAUGE-NATURAL TRANSFORMATIONS OF SOME COTANGENT BUNDLES

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Dedicated to Anton Dekrét on the occasion of his 65-th birthday

ABSTRACT. For an arbitrary vector bundle $E \to M$, we determine all gauge-natural transformations $T^*E \to T^*E^*$. In order to describe the result geometrically, we characterize some properties of T^*E in terms of an original approach to the concept of double vector bundle.

In [4] and [7], some important relations between the cotangent bundle of the tangent bundle T^*TM and the cotangent bundle of the cotangent bundle T^*T^*M of a manifold M were studied and applied. In the present paper we discuss the "pure" case of the cotangent bundle T^*E of any vector bundle $E \to M$ and the cotangent bundle T^*E^* of its dual E^*. First of all we show that T^*E with the canonical projections to E and E^* has the structure of double vector bundle. The origin of such a concept goes back to [3], [6], [9], but our approach in Section 2 is new and we find it very simple. In Section 3 we construct a canonical isomorphism $\varepsilon : T^*E \to T^*E^*$. In the next section we use the viewpoint of the theory of gauge-natural bundles, [1], [2], and we determine all gauge-natural (in other words: geometrical) transformations $T^*E \to T^*E^*$. In Section 5 we clarify how the canonical isomorphism ε relates a linear vector field on E and the dual vector field on E^*. All manifolds and maps are assumed to be infinitely differentiable.

1. TWO VECTOR BUNDLE STRUCTURES ON T^*E

Consider a vector bundle $p : E \to M$ and its dual bundle $p^* : E^* \to M$. Write $q_1 : T^*E \to E$ for the cotangent bundle of E. According to [2], p.227, there exists another projection $q_2 : T^*E \to E^*$. For every linear map $w : T_yE \to \mathbb{R}$, $y \in E$, we define $q_2(w)$ to be the restriction of w to the vertical tangent space V_yE, which is canonicaly identified with the fiber E_x, $x = py$. Let x^i, y^p be some linear fiber coordinates on E and x^i, z_p be the the dual coordinates on E^*. The additional coordinates on T^*E are given by $w = v_i dx^i + u_p dy^p$. Then the coordinate form of q_1 or q_2 is

$$q_1(x^i, y^p, u_p, v_i) = (x^i, y^p), \qquad q_2(x^i, y^p, u_p, v_i) = (x^i, u_p) \tag{1}$$

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We are going to show that $q_2 : T^*E \to E^*$ is a vector bundle too. We shall use the well-known fact there are two vector bundle structures $\pi_E : TE \to E$ and $Tp : TE \to TM$ on TE, [2]. If $X^i = dx^i$, $Y^p = dy^p$ are the additional coordinates on TE, then $Tp(x^i, y^p, X^i, Y^p) = (x^i, X^i)$. The vector addition and the multiplication by reals on Tp have the following form

$$(x^i, y_1^p, X^i, Y_1^p) + (x^i, y_2^p, X^i, Y_2^p) = (x^i, y_1^p + y_2^p, X^i, Y_1^p + Y_2^p),$$
$$k(x^i, y^p, X^i, Y^p) = (x^i, ky^p, X^i, kY^p). \tag{2}$$

Consider $w_1 \in T^*_{y_1}E$ and $w_2 \in T^*_{y_2}E$ satisfying $q_2(w_1) = q_2(w_2)$. Let us decompose, with respect to Tp, a vector $Y \in T_{y_1+y_2}E$ as $Y_1 + Y_2$, $Y_1 \in T_{y_1}E$, $Y_2 \in T_{y_2}E$, $Tp(Y) = Tp(Y_1) = Tp(Y_2)$. Then we define $w_1 + w_2 \in T^*_{y_1+y_2}E$ by

$$(w_1 + w_2)(Y) = w_1(Y_1) + w_2(Y_2) \tag{3}$$

The correctness follows from the coordinate form of (3). We have $Y = (x^i, y_1^p + y_2^p, X^i, Y^p)$, $Y_1 = (x^i, y_1^p, X^i, Y_1^p)$, $Y_2 = (x^i, y_2^p, X^i, Y_2^p)$ with $Y^p = Y_1^p + Y_2^p$ and $w_1 = (x^i, y_1^p, u_p, v_{1i})$, $w_2 = (x^i, y_2^p, u_p, v_{2i})$. Then $w_1(Y_1) + w_2(Y_2) = u_p Y_1^p + v_{1i}X^i + u_p Y_2^p + v_{2i}X^i = u_p Y^p + (v_{1i} + v_{2i})X^i$. Moreover, the latter expression implies

$$(x^i, y_1^p, u_p, v_{1i}) + (x^i, y_2^p, u_p, v_{2i}) = (x^i, y_1^p + y_2^p, u_p, v_{1i} + v_{2i}) \tag{4}$$

Further, if $w \in T^*_yE$, $0 \neq k \in \mathbb{R}$ and $Y \in T_{ky}E$, we construct $\frac{1}{k}Y$ with respect to the vector bundle structure Tp. Hence $\frac{1}{k}Y \in T_yE$ and we set

$$(kw)(Y) = kw(\frac{1}{k}Y) \tag{5}$$

For $k = 0$, we consider the restriction T_0E of E to the zero section $0 : M \to E$ and the tangent map $T0 : TM \to TE$. We have the following decomposition $T_0E = TM \times_M E$. We set $pr_1Y = Tp(Y)$. Then $Y - T0(pr_1Y)$ is a vertical vector, which is identified with $pr_2Y \in E$. Now we define $0w \in T^*_{0(x)}E$, $x = py$, by

$$(0w)(Y) = (q_2w)(pr_2Y), \qquad Y \in T_{0(x)}E. \tag{6}$$

In coordinates, one finds easily

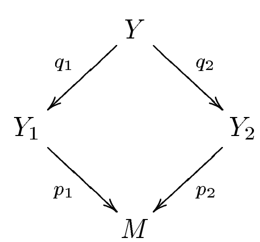
$$k(x^i, y^p, u_p, v_i) = (x^i, ky^p, u_p, kv_i), \qquad k \in \mathbb{R} \tag{7}$$

Clearly, (4) and (7) imply, that $q_2 : T^*E \to E^*$ is a vector bundle.

2. Double vector bundles

We define a fibered square to be a commutative diagram

(8)



in which all arrows are fibered manifolds. If there is no danger of confusion, we shall write Y for (8). The diagonal map in (8) will be denoted by $q : Y \to M$. If $(\bar{Y}, \bar{Y}_1, \bar{Y}_2, \bar{M}, \bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2)$ is another fibered square, then a morphism $Y \to \bar{Y}$ means a quadruple of maps $f : Y \to \bar{Y}$, $f_1 : Y_1 \to \bar{Y}_1$, $f_2 : Y_2 \to \bar{Y}_2$, $f_0 : M \to \bar{M}$ such that all pairs (f, f_1), (f, f_2), (f_1, f_0), (f_2, f_0) are fibered manifold morphisms. Hence we have a category $\mathcal{FS}$ of fibered squares.

A fibered square will be called a vector bundle square, if all arrows in (8) are vector bundles. For example, both $(T^*E, E, E^*, M, q_1, q_2, p, p^*)$ and $(TE, E, TM, M, \pi_E, Tp, p, \pi_M)$ are vector bundle squares. An $\mathcal{FS}$-morphism (f, f_1, f_2, f_0) of two vector bundle squares is said to be linear, if all pairs (f, f_1), (f, f_2), (f_1, f_0), (f_2, f_0) are vector bundle morphisms. Hence we obtain a category $\mathcal{VBS}$ of vector bundle squares.

Consider a manifold M and three vector spaces V_1, V_2, V_3. Put

$$Y = M \times V_1 \times V_2 \times V_3, \qquad Y_1 = M \times V_1, \qquad Y_2 = M \times V_2. \tag{9}$$

Then we have canonical vector bundle structures on $Y \to Y_1$, $Y \to Y_2$, $Y_1 \to M$ and $Y_2 \to M$. We shall say that $Y = M \times V_1 \times V_2 \times V_3$ is a trivial double vector bundle.

Definition 1. *A vector bundle square* (8) *will be called a double vector bundle, if for every $x \in M$ there exists its neighbourhood $U \subset M$ such that $q^{-1}(U)$ is $\mathcal{VBS}$-isomorphic to a trivial double vector bundle.*

A morphism between two double vector bundles is an $\mathcal{VBS}$-morphism. Thus, we obtain a category $\mathcal{DVB}$ of double vector bundles.

Denote by $HY \subset Y$ the set of elements which are projected by q_1 into a zero vector in Y_1 and by q_2 into a zero vector in Y_2. By J. Pradines, [5], [6], HY is called the heart of the double vector bundle Y. In the trivial case we have $H(M \times V_1 \times V_2 \times V_3) = M \times V_3$. This implies that even in the general case both vector bundle structures q_1 and q_2 coincide on HY and $HY \to M$ is a vector bundle.

If $D \subset \mathbb{R}^m$ is an open subset and V is a vector space, then Section 1 implies

$$T^*(D \times V) = D \times V \times V^* \times \mathbb{R}^{m*}.$$

Quite similarly,

$$T(D \times V) = D \times V \times \mathbb{R}^m \times V.$$

Then one verifies easily

Proposition 1. *For every vector bundle E, both TE and T^*E are double vector bundles.*

We remark that a direct characterization of double vector bundles in terms of the underlying vector bundle structures can be deduced directly from the results of [8].

3. The canonical isomorphism $T^*E \to T^*E^*$

We are going to construct a canonical map $\varepsilon : T^*E \to T^*E^*$. Consider the evaluation map $e : E \times_M E^* \to \mathbb{R}$. Its differential

$$de : TE \times_{TM} TE^* \to \mathbb{R}$$

is the second component of the tangent map $Te : TE \times_{TM} TE^* \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$.

Proposition 2. *For every covector $C \in T^*_y E$, there exists a unique element $z \in E^*$ satisfying $py = p^*z$ and a unique covector $\varepsilon C \in T^*_z E^*$ such that every vectors $A \in T_y E$ and $B \in T_z E^*$ over the same vector $Tp(A) = Tp^*(B) \in TM$ satisfy*

$$de(A, B) = C(A) - (\varepsilon C)(B) \tag{10}$$

Proof. The coordinate form of the evaluation map e is $y^p z_p$, so that de is of the form

$$z_p dy^p + y^p dz_p.$$

Consider $C = (x^i, y^p, u_p, v_i)$, $A = (x^i, y^p, a^i, a^p)$, $B = (x^i, g_p, a^i, b_p)$ and write $D = (x^i, g_p, h^p, k_i)$. Then the condition $de(A, B) = C(A) - D(B)$ reads

$$g_p a^p + y^p b_p = v_i a^i + u_p a^p - h^p b_p - k_i a^i.$$

Since a^i, a^p and b_p are arbitrary, the unique $D = \varepsilon C$ is of the form

$$g_p = u_p, \qquad h^p = -y^p, \qquad k_i = v_i. \tag{11}$$

Definition 2. *The map* (11) *from Proposition 2 will be called the canonical isomorphism $\varepsilon : T^*E \to T^*E^*$.*

Clearly, ε is an isomorphism of double vector bundles.

4. All gauge-natural transformations $T^*E \to T^*E^*$

Using the viewpoint of the theory of gauge-natural bundles, [1], [2], we can say that $\varepsilon : T^*E \to T^*E^*$ is a gauge-natural transformation. We are going to determine all gauge-natural transformations $T^*E \to T^*E^*$. For every $w \in T^*E$, we have $q_1(w) \in E$, $q_2(w) \in E^*$ and $p_1(q_1(w)) = p_2(q_2(w))$, so that we can evaluate $\langle q_1(w), q_2(w) \rangle \in \mathbb{R}$. Write $\eta \mapsto (k)_1\eta$ or $\eta \mapsto (k)_2\eta$, $k \in \mathbb{R}$, $\eta \in T^*E^*$, for the scalar multiplication with respect to the first or second vector bundle structure on T^*E^*, respectively.

Proposition 3. *All gauge-natural transformations $T^*E \to T^*E^*$ are of the form*

$$w \mapsto A(\langle q_1(w), q_2(w) \rangle)_1 B(\langle q_1(w), q_2(w) \rangle)_2 \varepsilon(w), \tag{12}$$

where $A(t)$ and $B(t)$ are two arbitrary smooth functions of one variable.

Proof. As remarked in [2], p.409, the category $\mathcal{VB}_{m,n}$ of vector bundles with m-dimensional bases and n-dimensional fibers and their local isomorphisms is naturally equivalent to the category $\mathcal{PB}_m(GL(n))$ of $GL(n)$-principal bundles with m-dimensional bases and their local isomorphisms. Consider the trivial vector bundle

$\mathbb{R}^m \times \mathbb{R}^n$ and write $S = (T^*(\mathbb{R}^m \times \mathbb{R}^n))_0$ and $Z = (T^*(\mathbb{R}^m \times \mathbb{R}^{n*}))_0$, $0 \in \mathbb{R}^m$. Then both S and Z are $W^1_m GL(n)$-spaces and all gauge-natural transformations $T^*E \to T^*E^*$ are in the bijection with the $W^1_m GL(n)$-equivariant maps $S \to Z$, [2], Chapter XII.

On S, we have the coordinates y^p, u_p, v_i from Section 1. Every element of $T^*(\mathbb{R}^m \times \mathbb{R}^{n*})$ can be written in the form $\xi_i dx^i + \mu^p dz_p$, so that z_p, μ^p, ξ_i are the corresponding coordinates on Z. The elements of $W^1_m GL(n)$ are of the form (a^i_j, a^p_q, a^p_{qi}), $\det a^i_j \neq 0$, $\det a^p_q \neq 0$, [2], p.153. Using standard evaluations, one finds easily the following actions of $W^1_m GL(n)$ on S and Z

$$\bar{y}^p = a^p_q y^q, \qquad \bar{u}_p = \tilde{a}^q_p u_q, \qquad \bar{v}_i = \tilde{a}^j_i v_j - \tilde{a}^j_i \tilde{a}^p_r a^r_{qj} u_p y^q \tag{13}$$

$$\bar{z}_p = \tilde{a}^q_p z_q, \qquad \bar{\mu}^p = a^p_q \mu^g \qquad \bar{\xi}_i = \tilde{a}^j_i \xi_j + \tilde{a}^j_i \tilde{a}^p_r a^r_{qj} z_p \mu^q \tag{14}$$

where $\tilde{a}^i_j$ or $\tilde{a}^p_q$ is the inverse matrix to a^i_j or a^p_q, respectively. To determine all $W^1_m GL(n)$-equivariant maps $f : S \to Z$, we shall use the methods from [2]. Thus, let us start from an arbitrary map

$$z_p = f_p(y, u, v), \qquad \mu^p = f^p(y, u, v), \qquad \xi_i = f_i(y, u, v). \tag{15}$$

Consider first the equivariancy with respect to the canonical injection $GL(m) \times GL(n) \to W^1_m GL(n)$, which is characterized by $a^p_{qi} = 0$. Using the homotheties in $GL(m)$, we find that f_p and f^p are independent of v and f_i is linear in v_i. The equivariancy with respect to the whole group $GL(m)$ yields

$$f_i = g(y, u) v_i. \tag{16}$$

Then we consider equivariancy with respect to $GL(n)$. The tensor evaluation theorem, [2], Section 26, yields

$$f^p = -A(y^q u_q) y^p, \qquad f_p = B(y^q u_q) v_p. \tag{17}$$

Consider now the equivariancy with respect to the kernel K of the canonical projection $W^1_m GL(n) \to GL(m) \times GL(n)$, which is characterized by $a^i_j = \delta^i_j$, $a^p_q = \delta^p_q$. This yields

$$a^p_{qi} A u_p B y^q = a^p_{qi} g u_p y^q.$$

Thus,

$$g(y, u) = A(y^p u_p) B(y^q u_q) \tag{18}$$

Clearly, (16)-(18) is the coordinate form of (12). $\square$

5. Linear vector fields

In general, every vector field $X : N \to TN$ on a manifold N defines a function $\widetilde{X} : T^*N \to \mathbb{R}$, $\widetilde{X}(z) = \langle X(x), z\rangle$, $z \in T^*_x N$. Conversely, every function $f : T^*N \to \mathbb{R}$ linear on each fiber is of the form $f = \widetilde{X}$ for a vector field $X : N \to TN$.

Consider a linear vector field $X : E \to TE$, [2], p.379. Its coordinate form is

$$X^i(x)\frac{\partial}{\partial x^i} + X^p_q(x)y^q\frac{\partial}{\partial y^p}.$$

Using flows, one constructs the dual vector field $X^* : E^* \to TE^*$, [2], p.380, whose coordinate expression is

$$X^i(x)\frac{\partial}{\partial x^i} - X^q_p(x)z_q\frac{\partial}{\partial z_p}.$$

Proposition 4. *For every linear vector field $X : E \to TE$, we have*

$$\widetilde{X} = \widetilde{X^*} \circ \varepsilon.$$

Proof. In the coordinates of the proof of Proposition 2, $\widetilde{X} = X^i(x)v_i + X^p_q(x)y^q u_p$, $\widetilde{X^*} = X^i(x)k_i - X^p_q(x)g_p h^q$. Then (11) yields our claim.

6. Remark

It can be expected from the trivialization $T(D \times V^*) = D \times V^* \times \mathbb{R}^m \times V^*$, that there is no natural isomorphism $TE \to TE^*$. To confirm it rigorously, one can determine all gauge-natural transformations $TE \to TE^*$. Using the basic methods from [2], one obtains easily the following result. Consider the projection $Tp : TE \to TM$, the homothetic transformation $k_M : TM \to TM$, $v \mapsto kv$, $k \in \mathbb{R}$, and the tangent map $T0 : TM \to TE^*$ of the zero section $0 : M \to E^*$. All gauge-natural transformations $TE \to TE^*$ are of the form

$$T0 \circ k_M \circ Tp : TE \to TE^*, \qquad k \in \mathbb{R}.$$

Clearly, none of them is an isomorphism.

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EXISTENCE OF INVARIANT TORI OF CRITICAL DIFFERENTIAL EQUATION SYSTEMS DEPENDING ON MORE-DIMENSIONAL PARAMETER. PART I.

Rudolf Zimka

Dedicated to Anton Dekrét on the occasion of his 65-th birthday

Abstract. In the paper a system of differential equations depending on more-dimensional parameter with the matrix of the first linear approximation P having pure imaginary eigenvalues while the others do not lie on the imaginary axis is studied. Conditions under which such a system has invariant tori are presented (section 1). In sections 2, 3 the cases when P has one and two pairs of pure imaginary eigenvalues are investigated. In Part II the cases with three and four pairs of pure imaginary eigenvalues will be analysed.

Introduction

In the monograph [1] Yu. N. Bibikov studies the system of differential equations depending on a small non-negative parameter μ:

$$\dot{x} = X(x,\mu) + X^*(x,\mu) \ , \tag{1}$$

where $x = (x_1, \ldots, x_n)$, $X(x,\mu)$ - a vector polynomial with respect to x, μ, $X(0,0) = 0$, $X^*(x,\mu) : \mathbb{M} \to \mathbb{R}^n$, $M = \{(x,\mu) : ||x|| < K, \ 0 \le \mu < L\}$ - a continuous vector function with the property:

$$X^*(\sqrt{\mu}x, \mu) = (\sqrt{\mu})^{3p+2}\tilde{X}(x,\mu) \ ,$$

p - a natural number, $\tilde{X}(x,\mu)$ - a function of the class $C^{10}_{x\mu}(\mathbb{M})$. It is supposed that the spectrum of the linear approximation matrix P of the polynomial $X(x,\mu)$ consists of m pairs of pure imaginary eigenvalues while the others have non-zero real parts. Yu. N. Bibikov found conditions under which to every small parameter μ there exists an invariant manifold of the system (1) that is homeomorphic with a torus. He also presents in [1] an idea how these results can be utilized in the case when the parameter μ is m-dimensional one, where m is the number of the pairs of pure imaginary eigenvalues of the matrix P.

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In applications the dimension of the parameter μ is not a function of the number of pure imaginary eigenvalues of P but it follows from the character of a process which is described by the considered system. Therefore it is worth studying the system (1) which depend on the more-dimensional parameter μ with an arbitrary dimension.

In this article the system (1) is investigated on the domain:

$$(2)\quad \mathbb{M} = \{(x, y) : x = (x_1, \ldots, x_n),\ \mu = (\mu_1, \ldots, \mu_d),\ d \geq 1,\ ||x|| < K,\ ||\mu|| < L\}$$

(in the whole article Euclidean norm is used).

Let us take an arbitrary parameter $\mu \in \mathbb{M}$. Consider the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 \leq \varepsilon < L,\ \mu_0 = \frac{\mu}{||\mu||}\}$ (index "o" at parameters μ will always have this meaning). The system (1) depending on parameters $\mu \in \delta(\mu_0)$ has the form:

$$(3)\qquad \dot{x} = X(x, \varepsilon\mu_0) + X^*(x, \varepsilon\mu_0),\ 0 \leq \varepsilon < L\ .$$

The system (3) is the system of differential equations depending on one-dimensional non-negative parameter ε. It means the system (3) is the system of the kind (1) which was studied in [1]. Such an access enables to investigate the system (1) on the domain (2) and utilize the results achieved in [1]. Doing it the problem of determining subsets of the set $\mathbb{M}$ with respect to μ on which invariant manifolds of the system (1) exist arises.

In section 1 preliminary transformations of the system (1) depending on parameters μ from the domain (2) are performed enabling to utilize the results from [1]. In sections 2, 3 the cases when the matrix P has one and two pairs of pure imaginary eigenvalues are studied.

1. The existence of invariant tori

Consider the system of differential equations

$$(1.1)\qquad \dot{x} = X(x, \mu) + X^*(x, \mu)\ ,$$

where $x = (x_1, \ldots, x_n)$, $\mu = (\mu_1, \ldots, \mu_d)$, $\dot{x} = \frac{dx}{dt}$, $X(x, \mu)$ - a vector polynomial with respect to x, μ, $X(0, 0) = 0$, $X^*(x, \mu) : \mathbb{M} \to \mathbb{R}^n$, $\mathbb{M} = \{x, \mu) : ||x|| < K,\ ||\mu|| < L\}$ - a continuous function with the property:

$$(1.2)\qquad X^*(\sqrt{\varepsilon}x, \varepsilon\mu_0) = (\sqrt{\varepsilon})^{3p+2}\tilde{X}(x, \varepsilon, \mu_0),$$

$0 \leq \varepsilon < L$, $\mu \in \mathbb{M}$, p - a natural number, $\tilde{X}(x, \varepsilon, \mu_0)$ - a continuous function with respect to x, ε, μ_0 of the class $C^1_x(\mathbb{M})$.

We suppose that the matrix $P = \frac{\partial X(0,0)}{\partial x}$ has m pairs of pure imaginary eigenvalues $\pm i\lambda_1, \ldots, \pm i\lambda_m$ and the others $\lambda_{2m+1}, \ldots, \lambda_n$ have non-zero real parts. Further we suppose that $\det P \neq 0$.

Note 1.1. The requirements on the functions $X(x,\mu)$, $X^*(x,\mu)$ in (1.1) are not very limiting as every system $\dot{x} = f(x,\mu)$, $f(x,\mu) \in C^{3p+3}(\mathbb{M})$, $f(0,0) = 0$, can be expressed in the form (1.1). For that it is sufficient to introduce the function $f(x,\mu)$ in the form of the Taylor polynomial with the Lagrange form of the remainder. In this case $X(x,\mu) = \sum_{k=0}^{N} X_k(x_1,\dots,x_n)\, \mu_1^{k_1} \dots \mu_d^{k_d}$, $k = k_1 + \dots + k_d$, N - the whole part of the number $\frac{3p+1}{2}$, $X_k(x_1,\dots,x_n)$ - polynomials of the degree not higher then $3p+1-2k$.

Let us denote $F(x,\mu) = X(x,\mu) + X^*(x,\mu)$. In the power of (1.2) $F(0,0) = 0$. This means that the origin $(x,\mu) = (0,0)$ is the state of equilibrium of the system (1.1). Since

$$\frac{\partial X^*(x,\mu)}{\partial x} = \frac{\partial X^*(\sqrt{\varepsilon}y, \varepsilon\mu_0)}{\partial x} = \frac{\partial}{\partial x}\left[(\sqrt{\varepsilon})^{3p+2}\tilde{X}(y,\varepsilon,\mu_0)\right] =$$
$$= \frac{\partial}{\partial y}\left[(\sqrt{\varepsilon})^{3p+2}\tilde{X}(y,\varepsilon,\mu_0)\right]\cdot\frac{\partial y}{\partial x} = (\sqrt{\varepsilon})^{3p+1}\frac{\partial}{\partial y}\tilde{X}(y,\varepsilon,\mu_0)\ ,$$

we have:

$$\left|\frac{\partial F(0,0)}{\partial x}\right| = \left|\frac{\partial X(0,0)}{\partial x} + \frac{\partial X(0,0)}{\partial x}\right| = |P| \neq 0\ .$$

Using Implicit Function Theorem on the function $F(x,\mu)$ we get that in a small neighbourhood $O(0)$ of the origin $\mu = 0$ there exists a function $x = \psi(\mu)$ with the following properties:

1. $\psi(0) = 0$
2. $F[\psi(\mu),\mu] = 0$ for $\mu \in O(0)$.

We see that to every small enough parameter $\mu^* \in \mathbb{M}$ there exists the state of equilibrium of the system (1.1) $x^* = \psi(\mu^*)$. It will be shown that to such a μ^* there exists also under certain conditions an invariant manifold of the system (1.1) which is homeomorphic with a torus. When such a situation realizes we say that at $\mu = 0$ the bifurcation of an invariant torus arises from the state of equilibrium $x = 0$.

Lemma 1.1. *System (1.1) can be reduced by the transformation*

$$x = S\xi + T\mu\ , \tag{1.3}$$

where $\xi = col(y,\bar{y},z)$, $y = col(y_1,\dots,y_m)$, y *- the complex conjugate vector to* y *(in the article the symbol "*$\bar{a}$*" always means the complex conjugate expression to* a*,* $z = col(z_1,\dots,z_{n-2m})$, S *- a regular* $n \times n$*-matrix,* T *-* $n \times d$*-matrix, to the system*

$$\begin{aligned} \dot{y} &= i\lambda y + Y(y,\bar{y},z,\mu) + Y^*(y,\bar{y},z,\mu) \\ \dot{\bar{y}} &= i\lambda\bar{y} + \bar{Y}(y,\bar{y},z,\mu) + \bar{Y}^*(y,\bar{y},z,\mu) \\ \dot{z} &= Jz + Z(y,\bar{y},z,\mu) + Z^*(y,\bar{y},z,\mu)\ , \end{aligned} \tag{1.4}$$

where $\lambda = diag(\lambda_1,\dots,\lambda_m), J$ *- a Jordan canonical lower matrix,* $Y,\bar{Y},Z$ *- vector polynomials without scalar and linear terms,* $Y^*,\bar{Y}^*,Z^*$ *- continuous functions having the property (1.2), i.e. for example*

$$Y(\sqrt{\varepsilon}y,\sqrt{\varepsilon}\bar{y},\sqrt{\varepsilon}z,\ \varepsilon\mu_0) = (\sqrt{\varepsilon})^{3p+2}\tilde{Y}(y,\bar{y},z,\varepsilon,\mu_0)\ ,$$

$\tilde{Y}$ - *a continuous function of the class* $C^1_{y,\bar{y},z}$ *in a neighbourhood of the point* $y = 0, z = 0, 0 \le \varepsilon < L,\ \mu \in \mathbb{M}$. *The second equation in (1.4) is conjugated to the first one in (1.4) and can be gained from this by the change* y *for* $\bar{y}$, $\bar{y}$ *for* y *and* i *for* $-i$. *Further equations which will be conjugated to another ones will not be written.*

Proof. Expressing (1.1) in the form

$$\dot{x} = Px + Qx + X^1(x,\mu) + X^*(x,\mu) \tag{1.5}$$

and putting (1.3) into (1.5) we get:

$$S\dot{\xi} = P(S\xi + T\mu) + Q\mu + X^1(S\xi + T\mu, \mu) + X^*(S\xi + T\mu, \mu)\ .$$

From this we have:

$$\dot{\xi} = S^{-1}PS\xi + (S^{-1}PT + S^{-1}Q)\mu + S^{-1}X^1 + S^{-1}X^*\ . \tag{1.6}$$

If the matrices S, T are taken in the way to get: $S^{-1}PS = diag(i\lambda, -i\lambda, J),\ T = -P^{-1}Q$, then (1.6) gives the system (1.4). The proof is over.

Consider now the system

$$\begin{aligned} \dot{y} &= i\lambda y + Y(y,\bar{y},z,\mu) \\ \dot{z} &= Jz + Z(y,\bar{y},z,\mu)\ , \end{aligned} \tag{1.7}$$

which is gained from the system (1.4) by taking away the functions Y^*, Z^*.

Lemma 1.2. *Let the eigenvalues* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *of the matrix* P *satisfy the condition:*

$$q_1\lambda_1 + \cdots + q_m\lambda_m \ne 0 \quad \text{for} \quad 0 < |q| \le 3p + 2\ , \tag{1.8}$$

$|q| = |q_1| + \cdots + |q_m|,\ q_i$ *- integer numbers,* $i = 1, \ldots, m$.

There exists a polynomial transformation

$$\begin{aligned} y &= u + h(u,\bar{u},\mu) \\ z &= v + g(u,\bar{u},\mu)\ , \end{aligned} \tag{1.9}$$

where $u = (u_1, \ldots, u_m),\ v = (v_1, \ldots, v_{n-2m}),\ h, g$ are polynomials without scalar and linear terms, that reduces the system (1.7) to the system

$$\begin{aligned} \dot{u} &= i\lambda u + uU(u\cdot\bar{u},\mu) + U^0(u,\bar{u},v,\mu) + U^*(u,\bar{u},v,\mu) \\ \dot{v} &= Jv + V^0(u,\bar{u},v,\mu) + V^*(u,\bar{u},v,\mu)\ , \end{aligned} \tag{1.10}$$

where $U(u\cdot\bar{u},\mu)$ - a vector polynomial with respect to $u\cdot\bar{u},\mu$ without scalar terms, $U^0(u,\bar{u},0,\mu)=0$, $V^0(u,\bar{u},0,\mu)=0$, U^*,V^* have the property (1.2).

Proof. Differentiating (1.9) with respect to t and taking into account (1.7) and (1.10) we obtain:

$$i\lambda(u+h)+Y(u+h,\bar{u}+\bar{h},v+g,\mu)=i\lambda u+uU+U^0+U^*+$$
$$+\frac{\partial h}{\partial u}(i\lambda u+uU+U^0+U^*)+\frac{\partial h}{\partial \bar{u}}(-i\lambda\bar{u}+\bar{u}\bar{U}+\bar{U}^0+\bar{U}^*)$$
$$J(v+g)+Z(u+h,\bar{u}+\bar{h},v+g,\mu)=Jv+V^0+V^*+\frac{\partial g}{\partial u}(i\lambda u+uU+U^0+U^*)+$$
$$+\frac{\partial g}{\partial \bar{u}}(-i\lambda\bar{u}+\bar{u}\bar{U}+\bar{U}^0+\bar{U}^*)\ .$$

Giving away expressions with the property (1.2) and putting $v=0$ we get from these equations:

(1.11)
$$i\lambda u\frac{\partial h}{\partial u}-i\lambda\bar{u}\frac{\partial h}{\partial \bar{u}}-i\lambda h=Y(u+h,\bar{u}+\bar{h},g,\mu)-uU\frac{\partial h}{\partial u}-\bar{u}\bar{U}\frac{\partial h}{\partial \bar{u}}-uU$$
$$i\lambda u\frac{\partial g}{\partial u}-i\lambda\bar{u}\frac{\partial g}{\partial \bar{u}}-Jg=Z(u+h,\bar{u}+\bar{h},g,\mu)-uU\frac{\partial g}{\partial u}-\bar{u}\bar{U}\frac{\partial g}{\partial \bar{u}}\ .$$

Expressing the polynomials h,g in the form of the sum of vector homogenous polynomials $h^{(s)},g^{(s)},s$ - the degree, we get from (1.11) that $h^{(s)},g^{(s)}$ are determined by the equations:

(1.12)
$$i\lambda u\frac{\partial h^{(s)}}{\partial u}-i\lambda\bar{u}\frac{\partial h^{(s)}}{\partial \bar{u}}-i\lambda h^{(s)}=P^{(s)}(h^{(i)},g^{(j)})-(uU)^{(s)}$$
$$i\lambda u\frac{\partial g^{(s)}}{\partial u}-i\lambda\bar{u}\frac{\partial g^{(s)}}{\partial \bar{u}}=R^{(s)}(h^{(i)},g^{(j)}),\ i<s,j<s\ .$$

We see that if we calculate $h^{(s)},g^{(s)}$ in the direction of arising s then the functions $P^{(s)},R^{(s)}$ will be known for every s. For the coefficients $h_k^{(q,\tilde{q},r)}$, $g_k^{(q,\tilde{q},r)}$, $q=(q_1,\ldots,q_m)$, $\tilde{q}=(\tilde{q}_1,\ldots,\tilde{q}_m)$, $r=(r_1,\ldots,r_d)$ of the polynomials $h^{(s)}=col(h_1^{(s)},\ldots,h_m^{(s)})$, $g^{(s)}=col(g_1^{(s)},\ldots,g_{n-2m}^{(s)})$ we get from (1.12) the equations:

(1.13)
$$i\left[\sum_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_k\right]h_k^{(q,\tilde{q},r)}=P_k^{(q,\tilde{q},r)}-U_k^{(q,\tilde{q},r)},\ k=1,\ldots,m$$

(1.14)
$$i\left[\sum_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_{2m+l}\right]g_l^{(q,\tilde{q},r)}=R_l^{(q,\tilde{q},r)},\ l=1,\ldots,n-2m.$$

When $(q,\tilde{q},r)$ is such a set that $q_j=\tilde{q}_j,\ q_k=\tilde{q}_k+1,\ j=1,\dots,m,\ j\neq k$, then $\sum\limits_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_k=0$ in (1.13). In this case we put $U_k^{(q,\tilde{q},r)}=P_k^{(q,\tilde{q},r)}$ and $h_k^{(q,\tilde{q},r)}=0$. For other sets $(q,\tilde{q},r)$ in the power of (1.8) $\sum\limits_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_k\neq 0$. In these cases we put $U_k^{(q,\tilde{q},r)}=0$. Then the corresponding coefficient $h_k^{(q,\tilde{q},r)}$ is determined by equation (1.13) uniquely. The coefficients $g_l^{(q,\tilde{q},r)}$ in (1.14) are determined uniquely for every set of $(q,\tilde{q},r)$ as $\sum\limits_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_{2m+l}\neq 0$ since $Re\lambda_{2m+l}\neq 0,\ l=1,\dots,n-2m$. The proof is over.

Let us perform the transformation (1.9) on the system (1.4). We again get system (1.10) but this time with another functions U^*,V^* having again the property (1.2). Introducing into this system polar coordinates

$$u=\rho e^{i\varphi},\quad \bar{u}=\rho e^{-i\varphi}\ , \tag{1.15}$$

$\rho=col(\rho_1,\dots,\rho_m)$, $\varphi=col(\varphi_1,\dots,\varphi_m)$, $e^{i\varphi}=col(e^{i\varphi_1},\dots,e^{i\varphi_m})$, we get:

$$\begin{aligned}
&\dot{\rho}=\rho F(\rho^2,\mu)+F^0(\rho,\varphi,v,\mu)+F^*(\rho,\varphi,v,\mu)\\
&\dot{\varphi}=\lambda+\Phi(\rho^2,\mu)+\rho^{-1}[\Phi^0(\rho,\varphi,v,\mu)+\Phi^*(\rho,\varphi,v,\mu)]\\
&\dot{v}=Jv+V^0(\rho e^{i\varphi},\rho e^{-i\varphi},v,\mu)+V^*(\rho e^{i\varphi},\rho e^{-i\varphi},v,\mu)\ ,
\end{aligned} \tag{1.16}$$

where $\rho^2=(\rho_1^2,\dots,\rho_m^2)$, $\rho^{-1}=(\rho_1^{-1},\dots,\rho_m^{-1})$, $F=ReU(\rho^2,\mu)$, $\Phi=ImU(\rho^2,\mu)$, $F^0+F^*=Ree^{-i\varphi}[U^0(\rho e^{i\varphi},\ \rho e^{-i\varphi},v,\mu)+U^*(\rho e^{i\varphi},\ \rho e^{-i\varphi},v,\mu)]$, $\Phi^0+\Phi^*=Ime^{-i\varphi}[U^0(\rho e^{i\varphi},\ \rho e^{-i\varphi},v,\mu)+U^*(\rho e^{i\varphi},\ \rho e^{-i\varphi},v,\mu)]$, $F^0(\rho,\varphi,0,\mu)=0$, $\Phi^0(\rho,v,0,\mu)=0$, $F^*(\sqrt{\varepsilon}\rho,\varphi,\sqrt{\varepsilon}v,\varepsilon\mu_0)=(\sqrt{\varepsilon})^{3p+2}\tilde{F}(\rho,\varphi,v,\varepsilon,\mu_0)$, $\Phi^*(\sqrt{\varepsilon}\rho,\varphi,\sqrt{\varepsilon}v,\varepsilon\mu_0)=\sqrt{\varepsilon}^{3p+2}\tilde{\Phi}(\rho,\varphi,v,\varepsilon,\mu_0)$, $\tilde{F}$, $\tilde{\Phi}$ - continuous functions with respect to all variables of the class $C^1_{\rho,\varphi,v}$. All functions in (1.16) depending on φ are 2π-periodic with respect to all components of the vector φ.

Denote the linear parts of the function $F(\rho^2,\mu)$ by the expression $B\rho^2+C\mu$, where

$$B=\begin{pmatrix}B_{11} & \dots & B_{1m}\\ \dots & \dots & \dots\\ B_{m1} & \dots & B_{mm}\end{pmatrix}\ ,\qquad C=\begin{pmatrix}C_{11} & \dots & C_{1o}\\ \dots & \dots & \dots\\ C_{m1} & \dots & C_{mo}\end{pmatrix}\ .$$

The equation

$$B\rho^2+C\mu=0 \tag{1.17}$$

is called the bifurcation equation of system (1.16).

Let us suppose that $\det B\neq 0$ and that at least one element of the matrix C is different from zero.

Take an arbitrary $\mu\in\mathbb{M}$. The bifurcation equation (1.17) on the beam $\delta(\mu_0)=\{\varepsilon\mu_0:\ 0\leq\varepsilon<L\}$ has the form:

$$B\rho^2+\varepsilon C\mu_0=0\ .$$

Solving this equation with respect to ρ^2 we have:

$$\rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0) \ ,$$

where $\alpha^2(\mu_0) = col[\alpha^2(\mu_0), \dots, \alpha_m^2(\mu_0)] = \Lambda\mu_0$,

$$\Lambda = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1o} \\ \dots & \dots & \dots \\ \alpha_{m1} & \dots & \alpha_{mo} \end{pmatrix} .$$

We say that the bifurcation equation (1.17) satisfies the condition of positiveness at $\mu \in \mathbb{M}$ if $\alpha^2(\mu_0)$ is positive at every component $\alpha_k^2(\mu_0)$, $k = 1, \dots, m$. Let $\mathcal{DP}$ denote the subset of all parameters $\mu \in \mathbb{M}$ at which the bifurcation equation satisfies the condition of positiveness. We shall call this subset $\mathcal{DP}$ the domain of positiveness of the bifurcation equation (1.17).

Lemma 1.3. *The domain of positiveness $\mathcal{DP}$ of the bifurcation equation (1.17) is an open cone with the apex at the origin $\mu = 0$ consisting of beams $\delta(\mu_0) = \{\varepsilon\mu_0 : \mu \in \mathbb{M},\ 0 < \varepsilon < L,\ \alpha_k^2(\mu_0) > 0,\ k = 1, \dots, m\}$.*

Proof. Consider an arbitrary $\mu^* \in \mathcal{DP}$ and take an arbitrary $\mu \in \delta(\mu_0^*)$, $\mu = \varepsilon\mu_0^*$, $\varepsilon = ||\mu||$. As $\alpha^2(\mu_0) = col[\alpha_1^2(\mu_0), \dots, \alpha_m^2(\mu_0)]$ and $\alpha_k^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{k1}\mu_1 + \dots + \alpha_{kd}\mu_d) = \frac{1}{||\mu||}(\alpha_{k1}\varepsilon\frac{\mu_1^*}{||\mu^*||} + \dots + \alpha_{kd}\varepsilon\frac{\mu_d^*}{||\mu^*||}) = \frac{1}{||\mu^*||}(\alpha_{k1}\mu_1^* + \dots + \alpha_{kd}\mu_d^*) = \alpha_k^2(\mu_0^*) > 0$, $k = 1, \dots, m$, we get that $\delta(\mu_0^*) \subset \mathcal{DP}$. This means that $\mathcal{DP}$ is a cone. We need to show yet that to this μ^* there exists such $\sigma > 0$ that the sphere $O_\sigma(\mu^*) \subset \mathcal{DP}$. As $\mu^* \in \mathcal{DP}$ so $\alpha_k^2(\mu_0^*) = \nu_k > 0$, $k = 1, \dots, m$. Take an arbitrary μ from a sphere $O_\sigma(\mu^*)$, $\mu \neq \mu^*$. Then $\alpha_k^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{k1}\mu_1 + \dots + \alpha_{kd}\mu_d) = \frac{1}{||\mu||}[\alpha_{k1}(\mu_1^* + \sigma_1) + \dots + \alpha_{kd}(\mu_d^* + \sigma_d)]$, $-\sigma < \sigma_j < \sigma$, $j = 1, \dots, d$, $k = 1, \dots, m$. From this equation we have:

$$\alpha_k^2(\mu_0) > \frac{1}{||\mu^*|| + \sigma}(\alpha_{k1}\mu_1^* + \dots + \alpha_{kd}\mu_d^*) - \frac{1}{||\mu^*|| - \sigma}d\alpha\sigma \ ,$$
$$\alpha = \max\{|\alpha_{kl}|\},\ k = 1, \dots, m;\ l = 1, \dots, d \ .$$

If we take $\sigma = \frac{||\mu^*||}{s}$ then we get from the last inequality:

$$\alpha_k^2(\mu_0) > \frac{s}{s+1}\alpha_k^2(\mu_0^*) - \frac{d\alpha}{s-1} > \frac{s}{s+1}\nu - \frac{d\alpha}{s-1} > 0$$

for big enough natural number s, $\nu = \min\{\nu_1, \dots, \nu_m\}$, $k = 1, \dots, m$. The proof is over.

Let us take an arbitrary $\mu \in \mathcal{DP}$. On the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 \le \varepsilon < L\}$ the system (1.16) has the form:

$$\begin{aligned} \dot\rho &= \rho F(\rho^2, \varepsilon\mu_0) + F^0(\rho, \varphi, v, \varepsilon\mu_0) + F^*(\rho, \varphi, v, \varepsilon\mu_0) \\ \dot\varphi &= \lambda + \Phi(\rho^2, \varepsilon\mu_0) + \rho^{-1}[\Phi^0(\rho, \varphi, v, \varepsilon\mu_0) + \Phi^*(\rho, \varphi, v, \varepsilon\mu_0)] \\ \dot v &= Jv + V^0(\rho e^{i\varphi}, \rho e^{-i\varphi}, v, \varepsilon\mu_0) + V^*(\rho e^{i\varphi}, \rho e^{-i\varphi}, v, \varepsilon\mu_0) \ . \end{aligned} \tag{1.18}$$

The system (1.18) is the system of differential equations depending on one-dimensional non-negative parameter ε with the bifurcation equation satisfying the condition of positiveness. As it was shown in [1] the system (1.18) can be reduced introducing new variables x_1, φ_1, v_1 by the relations $\rho = \sqrt{\varepsilon}[\alpha(\mu_0) + x_1], \varphi = \varphi_1, \ v = \sqrt{\varepsilon} v_1$ to the system

$$
\begin{aligned}
\dot{x}_1 &= \varepsilon X_1(x_1, \varepsilon, \mu_0) + \sqrt{\varepsilon} X_1^0(x_1, \varphi_1, v_1, \varepsilon, \mu_0) + \\
&\quad + (\sqrt{\varepsilon})^{3p+1} \tilde{X}_1(x_1, \varphi_1, v_1, \varepsilon, \mu_0) \\
\dot{\varphi}_1 &= \lambda_1(\varepsilon) + \varepsilon \Phi_1(x_1, \varepsilon, \mu_0) + \sqrt{\varepsilon} \Phi_1^0(x_1, \varphi_1, v_1, \varepsilon, \mu_0) + \\
&\quad + (\sqrt{\varepsilon})^{3p+1} \tilde{\phi}_1(x_1, \varphi_1, v_1, \varepsilon, \mu_0) \\
\dot{v}_1 &= J v_1 + \sqrt{\varepsilon} V_1^0(x_1, \varphi_1, v_1, \varepsilon, \mu_0) + (\sqrt{\varepsilon})^{3p+1} \tilde{V}_1(x_1, \varphi_1, v_1, \varepsilon, \mu_0) \ ,
\end{aligned} \tag{1.19}
$$

where X_1, Φ_1 - vector polynomials, $X_1(0,0,\mu_0) = 0, \ \Phi_1(0,\varepsilon,\mu_0) = 0, \ \lambda(0) = \lambda, \ X_1^0, \Phi_1^0, V_1^0, \ \tilde{X}_1, \tilde{\Phi}_1, \tilde{V}_1$ - continuous functions in all variables of the class $C^1_{x_1,\varphi_1,v_1}$ on the domain $\mathbb{M}_1 = \{(x_1, \varphi_1, v_1, \varepsilon, \mu) : ||x_1|| < K_1, \ ||v_1|| < K_1, \ \varphi_1 \in \mathbb{R}^m, \ 0 \le \varepsilon < L, \ \mu \in \mathcal{DP}\}, \ X_1^0, \ \Phi_1^0, \ V_1^0$ vanishing at $v_1 = 0$ and

$$
P_1(\mu) = \frac{\partial X_1(0,0,\mu_0)}{\partial x_1} = 2[\text{diag } \alpha(\mu_0)] B [\text{diag } \alpha(\mu_0)] \ . \tag{1.20}
$$

We say that $P_1(\mu)$ is non-critical at $\mu \in \mathcal{DP}$ if its eigenvalues do not lie on the imaginary axis and is critical at $\mu \in \mathcal{DP}$ if it has at least one pair of pure imaginary eigenvalues while the others have non-zero real parts. Let $\mathcal{DC}$ denote the subset of all parameters $\mu \in \mathcal{DP}$ at which the matrix $P_1(\mu)$ is critical. We shall call this subset $\mathcal{DC}$ the domain of criticalness of the bifurcation equation (1.17).

Theorem 1.1. *To every $\mu \in \mathcal{DP} \setminus \mathcal{DC}$ of the bifurcation equation (1.17) there exists the invariant manifold of the system (1.19) which is defined by the equations*

$$
\begin{aligned}
x_1 &= ||\mu|| \eta_1(\varphi_1, ||\mu||, \mu_0) \\
v_1 &= ||\mu||^2 \Theta_1(\varphi_1, ||\mu||, \mu_0) \ ,
\end{aligned} \tag{1.21}
$$

where $\eta_1(\varphi_1, ||\mu||, \mu_0), \ \Theta_1(\varphi_1, ||\mu||, \mu_0)$ are continuous functions 2π-periodic in all components of $\varphi_1, \ \varphi_1 \in \mathbb{R}^m, \ 0 \le ||\mu|| < L, \ \mu \in \mathcal{DP} \setminus \mathcal{DC}$. The natural number p in (1.2) can be taken $p = 1$.

Proof. Consider an arbitrary $\mu \in \mathcal{DP} \setminus \mathcal{DC}$. The parameter μ lies on the beam $\delta(\mu_0) = \{\varepsilon \mu_0 : \ 0 \le \varepsilon < L\}$. On this beam the system (1.16) can be reduced to the system (1.19). According to Theorem from section 3 of Chapter 1 in [1] there exists to every $\varepsilon, \ 0 < \varepsilon < L$ (in the case of necessity L is taken smaller) the invariant manifold

$$
\begin{aligned}
x_1 &= \varepsilon \eta_1(\varphi_1, \varepsilon, \mu_0) \\
v_1 &= \varepsilon^2 \Theta_1(\varphi_1, \varepsilon, \mu_0) \ ,
\end{aligned}
$$

where η_1, Θ_1 are continuous functions 2π-periodic in all components $\varphi_1, \ \varphi_1 \in \mathbb{R}^m, \ 0 \le \varepsilon < L, \ p$ can be $p = 1$. In our case $\varepsilon = ||\mu||$. The proof is over.

2. One pair of pure imaginary eigenvalues

Suppose that the eigenvalues of the matrix P of the system (1.1) are: $\pm i\lambda,\ \lambda_3, \dots, \lambda_n$, $Re\lambda_k \neq 0,\ k = 3, \dots, n$.

The bifurcation equation (1.17) of system (1.16) is:

$$B\rho^2 + C\mu = 0\ , \tag{2.1}$$

where $B \in \mathbb{R}, C = (C_1, \dots, C_d),\ C_k \in \mathbb{R},\ k = 1, \dots, d$.

We suppose that $B \neq 0$ and the vector C has at least one element different from zero.

Theorem 2.1. *If the matrix P of the system (1.1) has one pair of pure imaginary eigenvalues and the others have non-zero real parts then:*

1. *$\mathcal{DP}$ of the bifurcation equation (2.1) is the whole half-sphere of the sphere $O = \{\mu = (\mu_1, \dots, \mu_d)\ :\ 0 < ||\mu|| < L\}$ which is determined by the hyperplane $C_1\mu_1 + \dots + C_d\mu_d = 0$ and by a point $\mu^* \in O$ at which $-\frac{1}{B}(C_1\mu_1^* + \dots + C_d\mu_d^*) > 0$.*
2. *$\mathcal{DC}$ of the bifurcation equation (2.1) is empty set.*

Proof. Let us take an arbitrary $\mu \in \mathbb{M}$. The bifurcation equation (2.1) has on the beam $\delta(\mu_0) = \{\varepsilon\mu_0 :\ 0 \le \varepsilon < L\}$ the form: $B\rho^2 + \varepsilon C\mu_0 = 0$. Solving this equation with respect to ρ^2 we get: $\rho^2 = \varepsilon\alpha^2(\mu_0)$, where $\alpha^2(\mu_0) = -\frac{1}{B||\mu||}(C_1\mu_1 + \dots + C_d\mu_d)$.

$\mathcal{DP}$ is the set of all $\mu \in \mathbb{M}$ at which

$$\alpha^2(\mu_0) = -\frac{1}{B||\mu||}(C_1\mu_1 + \dots + C_d\mu_d) > 0\ .$$

From this inequality the first assertion of Theorem 2.1 follows.

The matrix $P_1(\mu)$ of the system (1.19) has on $\mathcal{DP}$ according to (1.20) this form:

$$P_1(\mu) = 2[\text{diag}\ \alpha(\mu_0)]\ B[\text{diag}\ \alpha(\mu_0)] = 2\sqrt{-\frac{1}{B||\mu||}(C_1\mu_1 + \dots + C_d\mu_d)}\cdot$$

$$B\sqrt{-\frac{1}{B||\mu||}(C_1\mu_1 + \dots + C_d\mu_d)} = -\frac{2}{||\mu||}(C_1\mu_1 + \dots + C_d\mu_d) \neq 0$$

for all $\mu \in \mathcal{DP}$. The proof is over.

Consequence of Theorem 1.1 and Theorem 2.1. To every $\mu \in \mathcal{DP}$ of the bifurcation equation (2.1) there exists the invariant manifold of the system (1.19) of the kind (1.21).

3. Two pairs of pure imaginary eigenvalues

Suppose that the matrix P of the system (1.1) has two pairs of pure imaginary eigenvalues $\pm i\lambda_1,\ \pm i\lambda_2$ and the others $\lambda_5,\dots,\lambda_n$ have non-zero real parts.

The bifurcation equation (1.17) of the system (1.16) is:

$$B\rho^2 + C\mu = 0 \ , \tag{3.1}$$

where

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} , \qquad C = \begin{pmatrix} C_{11} & \dots & C_{1d} \\ C_{21} & \dots & C_{2d} \end{pmatrix} .$$

We suppose that $\det B \neq 0$ and the matrix C has at least one element different from zero.

Let us take an arbitrary $\mu \in \mathbb{M}$. The equation (3.1) has on the beam $\delta(\mu_0) = \{\varepsilon\mu_0 :\ 0 \le \varepsilon < L\}$ the form: $B\rho^2 + \varepsilon C\mu_0 = 0$. Solving this equation with respect to ρ^2 we get

$$\rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0) \ , \tag{3.2}$$

where

$$\alpha^2(\mu_0) = \begin{pmatrix} \alpha_1^2(\mu_0) \\ \alpha_2^2(\mu_0) \end{pmatrix} = \Lambda\mu_0, \quad \Lambda = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1d} \\ \alpha_{21} & \dots & \alpha_{2d} \end{pmatrix} .$$

The matrix $P_1(\mu)$ which is defined by (1.20) has the form:

$$P_1(\mu) = 2\begin{pmatrix} \alpha_1^2(\mu_0)B_{11} & \alpha_1(\mu_0)\alpha_2(\mu_0)B_{12} \\ \alpha_1(\mu_0)\alpha(\mu_0)B_{21} & \alpha_2^2(\mu_0)B_{22} \end{pmatrix} ,$$

where

$$\alpha_1(\mu_0) = \sqrt{\frac{1}{||\mu||}(\alpha_{11}\mu_1 + \dots + \alpha_{1d}\mu_d)}, \quad \alpha_2(\mu_0) = \sqrt{\frac{1}{||\mu||}(\alpha_{21}\mu_1 + \dots + \alpha_{2d}\mu_d)} \ .$$

Lemma 3.1. *The matrix $P_1(\mu)$ is critical at $\mu \in \mathcal{DP}$ only if the following two conditions are satisfied:*

$$\begin{aligned} &1. \quad \det B > 0 \\ &2. \quad a_1(\mu_0) = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22} = 0 \ . \end{aligned} \tag{3.3}$$

Proof. The characteristic equation of the matrix $\frac{P_1(\mu)}{2}$ which is similar to $P_1(\mu)$ is:

$$\lambda^2 - a_1(\mu_0)\lambda + a_2(\mu_0) = 0 \ , \tag{3.4}$$

where $a_1(\mu_0) = Tr\frac{P_1(\mu)}{2} = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22}$, $a_2(\mu_0) = \det\frac{P_1(\mu)}{2} = \alpha_1^2(\mu_0)\alpha_2^2(\mu_0)\cdot \det B$.

Comparing (3.4) with its expression by means of its pure imaginary roots we gain the conditions for $P_1(\mu)$ to have a pair of pure imaginary eigenvalues:

$$a_1(\mu_0) = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22} = 0, \quad a_2(\mu_0) = \alpha_1^2(\mu_0)\alpha_2^2(\mu_0)\det B > 0 \ .$$

Taking into account that $\alpha_1^2(\mu_0) > 0,\ \alpha_2^2(\mu_0) > 0$ at every $\mu \in \mathcal{DP}$ we get the assertion of Lemma 3.1.

Theorem 3.1. *Let the rank $h(\Lambda)$ of the matrix Λ in (3.2) be 1. Then the following holds for $\mathcal{DP}$ and $\mathcal{DC}$ of the bifurcation equation (3.1):*

1. $\mathcal{DP} \neq \emptyset \Leftrightarrow \alpha_2 = k\alpha_1,\ k > 0.$
2. $\mathcal{DC} \neq \emptyset \Leftrightarrow \{(\det B > 0) \wedge [(B_{11} = B_{22} = 0) \vee (B_{11} = -kB_{22})]\}.$
3. *If* $\mathcal{DC} \neq \emptyset \Rightarrow \mathcal{DC} \equiv \mathcal{DP}.$

Proof. The domain of positiveness of the bifurcation equation (3.1) is determined by the inequalities:

$$\alpha_1^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d) > 0$$

(3.5)

$$\alpha_2^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d) > 0\ .$$

The first inequality in (3.5) is satisfied at all parameters $\mu \in \mathbb{M}$ which belong to that half-sphere of the sphere $O = \{\mu = (\mu_1, \ldots, \mu_d) :\ 0 < ||\mu|| < L\}$ which is determined by the hyperplane $\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d = 0$ and by a point $\mu^* \in O$ at which $\alpha_1^2(\mu_0^*) > 0$. As $h(\Lambda) = 1$ so there exists $k \in \mathbb{R}$ such that $\alpha_2 = k\alpha_1$. Using this we can express the second inequality in (3.5) in the form: $\frac{k}{||\mu||}(\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d) > 0$. From this inequality it follows that the parameters μ which satisfy the first inequality in (3.5) will also satisfy the second inequality in (3.5) only when $k > 0$. This gives the first assertion of Theorem 3.1.

Let $\mathcal{DC} \neq \emptyset$. Take an arbitrary $\mu \in \mathcal{DP}$. As $\alpha_2 = k\alpha_1,\ k > 0$, so $\alpha_2^2(\mu_0) = k\alpha_1^2(\mu_0)$. Therefore the conditions of criticalness (3.3) of the matrix $P_1(\mu)$ can be written in the form:

$$\begin{aligned} &1. \qquad \det B > 0 \\ &2. \qquad a_1(\mu_0) = \alpha_1^2(\mu_0)(B_{11} + kB_{22}) = 0\ . \end{aligned} \tag{3.6}$$

The equation (3.6) is satisfied only when $B_{11} = B_{22} = 0$ or $B_{11} = -kB_{22}$. From this equation also follows that when $B_{11} = B_{22} = 0$ or $B_{11} = -kB_{22}$ then (3.6) is satisfied at every $\mu \in \mathcal{DP}$. This gives the second and the third assertion of Theorem 3.1. The proof is over.

Theorem 3.2. *Let the rank $h(\Lambda)$ of the matrix Λ in (3.2) be 2. Then the following holds:*

1. $\mathcal{DP} \neq \emptyset$
2. $\mathcal{DC} \neq \emptyset \Leftrightarrow \{(\det B > 0) \wedge [(B_{11} = B_{22} = 0) \vee (B_{11}B_{22} < 0)]\}$
3. $\mathcal{DC} \equiv \mathcal{DP} \Leftrightarrow [(\det B > 0) \wedge (B_{11} = B_{22} = 0)].$

Proof. As $h(\Lambda) = 2$ then from the definition of the rank of a matrix follows that the dimension o of the parameter μ is at least 2, i.e. $o \geq 2$. The domain of positivenes $\mathcal{DP}$ of the equation (3.1) is determined by the inequalities

$$\frac{1}{||\mu||}(\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d) > 0$$

(3.7)

$$\frac{1}{||\mu||}(\alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d) > 0\ .$$

Expressing (3.7) in the form of equations we get:

$$
(3.8) \qquad \begin{aligned} &\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d - t_1 &&= 0 \\ &\alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d &&- t_2 = 0\ ,\ t_1 > 0, t_2 > 0\ . \end{aligned}
$$

As the rank of the matrix of the system (3.8) is 2 this system has infinite number of solutions with $t_1 > 0, t_2 > 0$. Therefore the inequalities (3.7) have solutions $\mu^* = (\mu_1^*, \dots, \mu_d^*)$. As parameters $\mu = \varepsilon\mu^*$ for $0 < \varepsilon < L$ also satisfy (3.7) so $\mathcal{DP} \neq \emptyset$. This gives the first assertion of Theorem 3.2.

Let $\mathcal{DC} \neq \emptyset$. The conditions of the criticalness of the matrix $P_1(\mu)$ are:

$$
(3.9) \qquad a_1(\mu_0) = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22}, \quad \det B > 0\ .
$$

Let $\mu^* \in \mathcal{DC}$. It means that $a_1(\mu_0^*) = 0$, $\det B > 0$. But as at the same time $\mu^* \in \mathcal{DP}$ so $\alpha_1^2(\mu_0^*) > 0$, $\alpha_2^2(\mu_0^*) > 0$. From (3.9) it follows that $B_{11} = B_{22} = 0$ or $B_{11}B_{22} < 0$.

Let

$$
(3.10) \qquad (\det B > 0) \wedge [(B_{11} = B_{22} = 0) \vee (B_{11}B_{22} < 0)]\ .
$$

$\mathcal{DC}$ is the set of parameters $\mu \in \mathcal{DP}$ satisfying the relations:

$$
(3.11) \qquad \begin{aligned} \alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d &> 0 \\ \alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d &> 0 \\ (\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d)B_{11} + (\alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d)B_{22} &= 0\ . \end{aligned}
$$

We shall show that under the assumptions (3.10) these relations have solutions. Expressing (3.11) in the form of equations we get:

$$
(3.12) \qquad \begin{aligned} \alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d - t_1 \qquad &= 0 \\ \alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d \qquad - t_2 &= 0 \\ (\alpha_{11}B_{11} + \alpha_{21})B_{22}\mu_1 + \cdots + (\alpha_{1d}B_{11} + \alpha_{2d}B_{22})\mu_d o &= 0\ . \end{aligned}
$$

If $B_{11} = B_{22} = 0$ then the third equation in (3.12) is satisfied for every $\mu \in \mathcal{DP}$ and $\mathcal{DC} \equiv \mathcal{DP}$.

If $B_{11}B_{22} < 0$ then the system (3.12) can be reduced to the form:

$$
(3.13) \qquad \begin{aligned} \alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d - t_1 \qquad &= 0 \\ \alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d \qquad - t_2 &= 0 \\ B_{11}t_1 + B_{22}t_2 &= 0\ . \end{aligned}
$$

As $h(\Lambda) = 2$ so the rank of the system (3.13) is 3. One of $d-1$ parameters of this system is t_2. For t_1 we get: $t_1 = -\frac{B_{22}}{B_{11}}t_2 > 0$ as $t_2 > 0$. So the system (3.13) has infinite number of solutions $(\mu_1^*, \dots, \mu_d^*, t_1^*, t_2^*)$ with $t_1^* > 0, t_2^* > 0$. This means that at these solutions $\alpha_1^2(\mu_0^*) > 0$, $\alpha_2^2(\mu_0^*) > 0$ and $a_1(\mu_0^*) = 0$. Thus parameters

$\mu = \varepsilon\mu^*,\ 0 < \varepsilon < L$, belong to $\mathcal{DC}$ and therefore $\mathcal{DC} \neq \emptyset$. This gives the second assertion of Theorem 3.2 and the relation: $[(\det B > 0) \wedge (B_{11} = B_{22} = 0)] \Rightarrow \mathcal{DC} \equiv \mathcal{DP}$.

Suppose that $\mathcal{DC} \equiv \mathcal{DP}$. Consider the parameters μ^*, μ^+ which are the solutions of (3.8) corresponding to the pairs $(t_1^* = 1, t_2^* = 1),\ t_1^+ = 1, t_2^+ = 2)$ respectively and an arbitrary choice of the other parameters of (3.8). Then we have from (3.8):

$$\alpha_{11}\mu_1^* + \cdots + \alpha_{1d}\mu_d^* = 1$$
$$\alpha_{21}\mu_1^* + \cdots + \alpha_{2d}\mu_d^* = 1$$

and also

$$\alpha_{11}\mu_1^+ + \cdots + \alpha_{1d}\mu_d^+ = 1$$
$$\alpha_{21}\mu_1^+ + \cdots + \alpha_{2d}\mu_d^+ = 2\ .$$

Take an arbitrary $\varepsilon_0, 0 < \varepsilon_0 < L$ and consider the parameters $\mu^*(\varepsilon_0) = \varepsilon_0\mu^* \in \mathcal{DP},\ \mu^+(\varepsilon_0) = \varepsilon_0\mu^+ \in \mathcal{DP}$. According to the assumption $\mathcal{DC} \equiv \mathcal{DP}$ we have: $\mu^*(\varepsilon_0) \in \mathcal{DC},\ \mu^+(\varepsilon_0) \in \mathcal{DC}$. Thus the conditions of criticalness at $\mu^*(\varepsilon_0),\ \mu^+(\varepsilon_0)$ are satisfied what means:

$$\alpha_1^2[\mu_0^*(\varepsilon_0)]B_{11} + \alpha_2^2[\mu_0^*(\varepsilon_0)]B_{22} = 0$$

(3.14)

$$\alpha_1^2[\mu_0^+(\varepsilon_0)]B_{11} + \alpha_2^2[\mu_0^+(\varepsilon_0)]B_{22} = 0\ .$$

As $\alpha_1^2[\mu_0^*(\varepsilon_0)] = \alpha_2^2[\mu_0^*(\varepsilon_0)] = \frac{1}{||\mu^*||}$ and $\alpha_1^2[\mu_0^+(\varepsilon_0)] = \frac{1}{||\mu^+||},\ \alpha_2^2[\mu_0^+(\varepsilon_0)] = \frac{2}{||\mu^+||}$, the equations (3.14) have the form:

$$\frac{1}{||\mu^*||}B_{11} + \frac{1}{||\mu^*||}B_{22} = 0$$
$$\frac{1}{||\mu^+||}B_{11} + \frac{2}{||\mu^+||}B_{22} = 0\ .$$

But this system is satisfied only when $B_{11} = B_{22} = 0$. This gives the relation: $\mathcal{DC} \equiv \mathcal{DP} \Rightarrow [(\det B > 0) \wedge (B_{11} = B_{22} = 0)]$. The proof of Theorem 3.2 is over.

According to Theorem 1.1 to every $\mu \in \mathcal{DP} \setminus \mathcal{DC}$ there exists an invariant manifold (1.21) which is homeomorphic with an invariant torus. Suppose now that $\mu \in \mathcal{DC}$ of the bifurcation equation (3.1). This means that $P_1(\mu)$ is critical on the beam of parameters $\delta(\mu_0) = \{\varepsilon\mu_0 :\ 0 \leq \varepsilon < L\}$. On this beam the system

(3.15) $$\dot{x}_1 = \varepsilon X_1(x_1, \varepsilon, \mu_0)$$

which is gained from the first equation of (1.19) is two-dimensional system with the critical matrix $P_1(\mu) = \frac{\partial X_1(0,0,\mu_0)}{\partial x_1}$. Denote its eigenvalues $\pm i\lambda^1$. The system (3.15) is the system of the same character as the system $\dot{x} = X(x, \varepsilon, \mu_0)$ which is gained from the system (1.1) being expressed on the beam $\delta(\mu_0)$. As it was shown in [1] we can do on (1.19) the analogical sequence of transformations as it was done on the system (1.1). During this process we get the bifurcation equation

$$B_1\rho_1^2+\varepsilon C_1(\mu_0)=0, \quad B_1\in\mathbb{R},\ C_1(\mu_0)\in\mathbb{R}\ . \tag{3.16}$$

If (3.16) satisfies the condition of positiveness, i.e. $\rho_1^2=\varepsilon[-\frac{1}{B_1}C_1(\mu_0)]=\varepsilon\alpha^2(\mu_0)$, $\alpha^2(\mu_0)\ >0$, the system (1.19) can be reduced to the system

$$\begin{aligned}
\dot{x}_2 &= \varepsilon^2X_2(x_2,\varepsilon,\mu_0)+X_2^0(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)+\\
&\quad +(\sqrt{\varepsilon})^{3p}\tilde{X}_2(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)\\
\dot{\varphi}_{12} &= \lambda_1(\varepsilon)+\varepsilon^2\Phi_{12}(x_2,\varepsilon,\mu_0)+\Phi_{12}^0(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)+\\
&\quad +(\sqrt{\varepsilon})^{3p}\tilde{\Phi}_{12}(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)\\
\dot{\varphi}_{22} &= \varepsilon\lambda_2(\varepsilon)+\varepsilon^2\Phi_{22}(x_2,\varepsilon,\mu_0)+\Phi_{22}^0(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)+\\
&\quad +(\sqrt{\varepsilon})^{3p}\tilde{\Phi}_{22}(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)\\
\dot{v}_{12} &= Jv_{12}+V_{12}^0(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)+\\
&\quad +(\sqrt{\varepsilon})^{3p+1}\tilde{V}_{12}(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)\ ,
\end{aligned} \tag{3.17}$$

where $\dim x_2=\dim\varphi_{22}=1,\ \dim\varphi_{12}=2,\ \dim v_{12}=n-4,\ \lambda_1(0)=\lambda,\ \lambda_2(0)=\lambda^1$ and the functions $X_2,\Phi_{12},\Phi_{22},X_2^0,\Phi_{12}^0,\Phi_{22}^0,V_{12}^0,\tilde{X}_2,\tilde{\Phi}_{12},\tilde{\Phi}_{22},\tilde{V}_{12}$ have the same character as the analogical functions in (1.19).

Denote $P_2(\mu)=\frac{\partial X_2(0,0,\mu_0)}{\partial x_2}$. It was shown in [1] that $P_2(\mu)=2\alpha^2(\mu_0)B_1=-2C_1(\mu_0)$. As the bifurcation equation (3.16) satisfies the condition of positiveness we have $P_2(\mu)\neq 0$ what means that $P_2(\mu)$ is non-critical. Therefore according to Theorem of section 3 Chapter 1 in [1] the following assertion is valid.

Theorem 3.3. *Let $\mu\in\mathcal{DC}$ of the bifurcation equation (3.1). If the bifurcation equation (3.16) satisfies at μ the condition of positiveness then p in (1.2) can be taken $p=2$ and to this μ there exists the invariant manifold of the system (3.17) which is defined by the equations*

$$\begin{aligned}
x_2 &= ||\mu||\eta_2(\varphi_{12},\varphi_{22},||\mu||,\mu_0)\\
v_{12} &= ||\mu||^2\Theta_2(\varphi_{12},\varphi_{22},||\mu||,\mu_0)\ ,
\end{aligned}$$

where η_2,Θ_2 are continuous functions in all variables 2π-periodic at $\varphi_{12},\varphi_{22},\varphi_{12}\in\mathbb{R}^2,\ \varphi_{22}\in\mathbb{R}^1,\ 0\le\varepsilon<L$.

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MINIMAL ECCENTRIC SEQUENCES WITH LEAST ECCENTRICITY THREE

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Dedicated to Anton Dekrét on the occasion of his 65-th birthday

Abstract. An eccentric sequence is called minimal if it has no proper eccentric subsequence with the same number of distinct eccentricities. All minimal eccentric sequences with least value 2 were found by R. Nandakumar (see [1]). In the paper it is shown that there are exactly 13 minimal eccentric sequences with least eccentricity three (see Theorem 5.1).

1. Introduction

In this paper we consider undirected connected graphs without loops and multiple edges. If G is a graph we denote by $V(G)$ the set of all its vertices and by $E(G)$ the set of all its edges. We write $|V(G)|$ for the cardinality of $V(G)$. The subgraph of G induced by a set of vertices $\{v_1, \dots, v_n\}$ will be denoted by $< v_1, \dots, v_n >$. Let $\deg v$ denote the degree of a vertex v and $d(u, v)$ denote the distance between vertices u, v. If u, v are vertices of a graph then uv is the edge which is incident with each of two vertices u and v. Let us denote by $\operatorname{diam} G$ the diameter of the graph G.

The eccentricity of a vertex $u \in V(G)$ is the integer

$$e_G(u) = \max\{d(u, v); v \in V(G)\}.$$

We write simply $e(u)$ when no confusion can arise. The eccentric sequence of a graph G is a list of the eccentricities of its vertices in nondecreasing order. Since often there are many vertices having the same eccentricity we will simplify the sequence by listing it as

$$e_1^{m_1}, e_2^{m_2}, \dots, e_s^{m_s}$$

where the e_i are the eccentricities for which $e_i < e_{i+1}$ and m_i is the multiplicity of e_i. For example 3^3, 4^3, 5^2 is the eccentricity sequence of the graph in Fig. 1.1 (at each vertex its eccentricity can be found).

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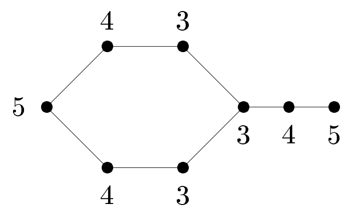


Fig. 1.1

Fig. 1.2

Isomorphic graphs have the same eccentric sequence (we will identify them). Obviously, the converse is not true.

A sequence of positive integers is called eccentric if there is a graph which realizes the considered sequence as the sequence of its eccentricities. An eccentric sequence is called minimal (by R. Nandakumar) if it has no proper eccentric subsequence with the same number of distinct eccentricities. For example, the eccentric sequence $3^3, 4^3, 5^2$ is not minimal since the graph in Fig. 1.2 has the eccentric sequence $3^2, 4^2, 5^2$.

Now we remind some well known properties of eccentric sequences (see [1]).

For every eccentric sequence $e_1^{m_1}, e_2^{m_2}, \ldots, e_s^{m_s}$ holds

1. $e_{i+1} = e_i + 1$ for $i = 1, 2, \ldots, s-1$, i.e. the $e_i's$ are consecutice positive itegers,
2. $e_s \le 2e_1$, i.e. the diameter is at most twice the radius.

The next assertions can be found in [2].

Theorem 1.1. *If $e_1^{m_1}, e_2^{m_2}, \ldots, e_s^{m_s}$ is an ecentric sequence then $m_i \ge 2$ for every $i \ge 2$.*

Theorem 1.2. *A sequence of positive integers is eccentric if and only if some its subsequence with the same number of distinct integers is eccentric.*

All minimal eccentric sequences with least value 2 were found by R. Nandakumar (see [1]). In this paper it is shown that there are exactly 13 minimal eccentric sequences with least eccentricity three, namely 3^6; $3^5, 4^2$; $3^4, 4^4$; $3^3, 4^6$; $3^2, 4^8$; $3, 4^{10}$; $3, 4^2, 5^{12}$; $3, 4^3, 5^9$; $3, 4^4, 5^7$; $3, 4^5, 5^4$; $3, 4^7, 5^2$; $3^2, 4^2, 5^2$; $3, 4^2, 5^2, 6^2$, (Theorem 5.1).

Now we will give some statements which we will use in the paper.

Lemma 1.3. *Let G be a graph and let $uv \in E(G)$. If $\deg v = 1$ and $e(u) \ge 2$ then $e(v) = e(u) + 1$.*

Proof. Any nontrivial path contained the vertex v also contains the vertex u.

Lemma 1.4. *If $uv \in E(G)$, $vw \in E(G)$, $\deg v = 2$, $\deg w = 1$, $e(u) \ge 3$ then $e(w) = e(v) + 1 = e(u) + 2$.*

Proof. It is obvious.

2. Eccentric sequences of type $3, 4^l$ and $3, 4^l, 5^r$, $l + r = 9$.

In this section we will show that there are only two 10-vertices graphs with eccentric sequences of type $3, 4^l$ or $3, 4^l, 5^r$. From this (by Theorem 1.2) it is possible to obtain some properties of minimal eccentric sequences (Corollary 2.5 - 2.7).

Lemma 2.1. *If the eccentric sequence of a graph G is $3, 4^l$ or $3, 4^l, 5^r$ then G contains the subgraph in Fig. 2.1 satisfying $e(v_4) = 3$ and*

$$d(v_3, v_7) = d(v_1, v_5) = 4, \tag{2.1}$$

or the subgraph in Fig. 2.2 satisfying $e(v_4) = 3$ and

$$d(v_4, v_1) = d(v_4, v_7) = d(v_4, w_3) = 3 \tag{2.2}$$

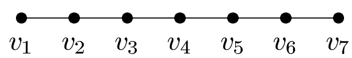


Fig. 2.1

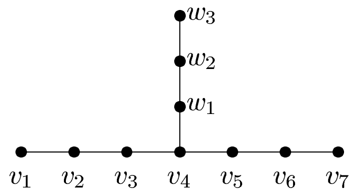


Fig. 2.2

Proof. Let v_4 be a vertex of G for which $e(v_4) = 3$. Since $e(v_4) = 3$ there is a path v_4, v_3, v_2, v_1 with the property $d(v_4, v_1) = 3$. Let $V_1 = V(G) - \{v_1, v_2, v_3, v_4\}$. If $d(v, v_4) \leq 2$ for any vertex $v \in V_1$ then $e(v_3) \leq 3$, a contradiction. If for any vertex $v \in V_1$ for which $d(v, v_4) = 3$ we would have that a shortest path between v, v_4 contains one of the vertices v_2, v_3 then again $e(v_3) \leq 3$, which is impossible. Hence the graph G contains a path v_4, v_5, v_6, v_7 satisfying $d(v_4, v_7) = 3$ that is disjoint with the path v_1, v_2, v_3. Thus the graph G contains the subgraph in Fig. 2.1.

If the distance between vertices from the set $V_2 = V(G) - \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and the vertex v_4 is at most two then $d(v_3, v_7) = d(v_1, v_5) = 4$ (because $e(v_3) = e(v_5) = 4$). Let there exist a vertex $w_3 \in V_2$ for which $d(v_4, w_3) = 3$. If for every vertex $x \in V_2$ satisfying $d(v_4, x) = 3$ every shortest path between vertices v_4, x contains at least one of the vertices v_2, v_3, v_5, v_6 then without loss of generality we can assume that (2.1) is satisfied. In the opposite case G contains the subgraph in Fig. 2.2 satisfying $e(v_4) = 3$ and (2.2). $\square$

Lemma 2.2. *Let the eccentric sequence of a graph G be $3, 4^l$ or $3, 4^l, 5^r$ and $e(v_4) = 3$. Let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be a path for which $d(v_1, v_4) = d(v_7, v_4) = 3$ and let G_1, G_2 be subgraphs of the graph G satisfying the following conditions*

- (j) $V(G_1) \cap V(G_2) \subset \{v_4\}$,
- (jj) $V(G) - \{v_4\} \subset V(G_1) \cup V(G_2)$,
- (jjj) $v_1 \in V(G_1)$, $v_7 \in V(G_2)$.

Then none of the following assertions holds

- (a) *there is no edge uv with $u \in V(G_1) - \{v_4\}$ and $v \in V(G_2) - \{v_4\}$,*
- (b) *$uv \in E(G)$, $u \in V(G_1) - \{v_4\}$, $v \in V(G_2) - \{v_4\}$ and $e_{G_1}(u) \leq 2$, $e_{G_2}(v) \leq 3$ (or $e_{G_1}(u) \leq 3$, $e_{G_2}(v) \leq 2$).*

Proof. In the case (a) a shortest path between vertices v_1, v_7 contains the vertex v_4 and it yields that $d(v_1, v_7) = d(v_1, v_4) + d(v_4, v_7) = 3 + 3 = 6$. Then $e(v_1) \geq 6$, a contradiction.

In the case (b) we have $e_G(v) \leq 3$, $v \neq v_4$ but v_4 is the only vertex with eccentricity less than four, a contradiction. □

Corollary 2.3. *There exists no graph satisfying all assumptions of Lemma 2.2 and simultaneously* $\operatorname{diam} G_1 \leq 2$, $\operatorname{diam} G_2 \leq 3$ *(or* $\operatorname{diam} G_1 \leq 3$, $\operatorname{diam} G_2 \leq 2$*).*

Theorem 2.4. *If the eccetric sequence of a graph G is* $3, 4^l$ *or* $3, 4^l, 5^k$ *and* $|V(G)| = 10$ *then G is the graph in Fig 2.3 or in Fig. 2.4.*

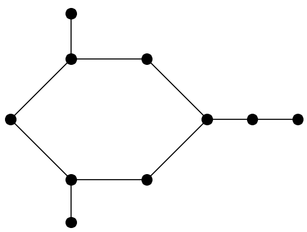


Fig. 2.3

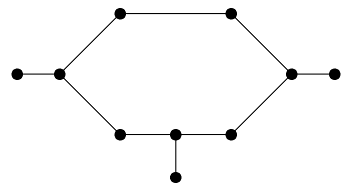


Fig. 2.4

Proof. Let $V(G) = \{v_1, \ldots, v_{10}\}$. We distinguish two cases (with respect to Lemma 2.1).

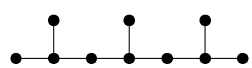
I. Let G contain a path $v_1, v_2, \ldots, v_7$ and let $e(v_4) = 3$ and (2.1) hold.

From (2.1) follows that the subgraph $\langle v_1, v_2, \ldots, v_7 \rangle$ has six edges. It is easy to see that neither $\deg v_1 = 1$ and $\deg v_2 = 2$ nor $\deg v_7 = 1$ and $\deg v_6 = 2$ is possible (see Lemma 1.4).

Since $e(v_4) = 3$ it is sufficient to consider the following cases with respect to symmetry, suitable denotation and to the number of vertices from the set $\{v_8, v_9, v_{10}\}$ which are adjacent to at least one of the vertices from the set $\{v_2, v_3, v_4, v_5, v_6\}$.

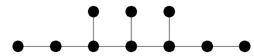
a) Each of the vertices v_8, v_9, v_{10} is adjacent to at least one vertex from the set $V' = \{v_2, v_3, v_4, v_5, v_6\}$.

$a_1)$ $\quad v_4v_8, v_2v_9, v_6v_{10} \in E(G)$



By Lemma 2.2 we have $\deg v_8 = 1$ (in the opposite case the graph G would not exist). Since any shortest path between v_1, v_7 can not contain the vertex v_4 we have that an edge of type v_iv_j, $i \in \{1, 2, 3, 9\}$, $j \in \{5, 6, 7, 10\}$ belongs to G. With respect to (2.1), $e(v_2) \geq 4$ and $e(v_6) \geq 4$ this is possible only for $i = 9$, $j = 10$, i.e. $v_9v_{10} \in E(G)$. Therefore, the graph G contains the subgraph in Fig. 2.4 with the eccentric sequence $3, 4^7, 5^2$. It is easy to check that if we add an edge to the graph in Fig. 2.4 we obtain a graph in which eccentricities at least two vertices are at most three.

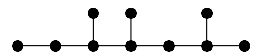
a_2) $v_4v_8, v_3v_9, v_5v_{10} \in E(G)$



Let the subgraphs $H_1 = \langle v_1, v_2, v_3, v_9\rangle$ and $H_2 = \langle v_5, v_6, v_7, v_{10}\rangle$ have at least four edges (i.e. their diameters are at most two). The inequality $\deg v_8 > 1$ is impossible by Lemma 2.2. So, let $\deg v_8 = 1$. Since any shortest path between vertices v_1, v_7 can not contain v_4 and (1.2) holds we get $v_iv_j \in E(G)$, $i \in \{3, 9\}$, $j \in \{5, 10\}$. Hence $e(v_i) \leq 3$ (and $e(v_j) \leq 3$, too), a contradiction.

Thus (with respect to symmetry) we can assume that the subgraph H_1 has three edges. Since $\deg v_1 > 1$ or $\deg v_2 > 2$ we get (with respect to (2.1)) $v_2v_8 \in E(G)$. Since $\deg v_7 > 1$ or $\deg v_6 > 2$ we get a contradiction ($e(v_8) \leq 3$ or by Lemma 2.2).

a_3) $v_4v_8, v_3v_9, v_6v_{10} \in E(G)$



Let the graph $H_1 = \langle v_1, v_2, v_3, v_9\rangle$ have at least four edges. By Lemma 2.2 we have again $\deg v_8 = 1$. With respect to (2.1) any shortest path between v_1, v_7 can not contain v_4 and thus an edge of type v_iv_j, $i \in \{2, 9\}$, $j \in \{5, 10\}$ belongs to $E(G)$. Hence $e(v_i) \leq 3$, a contradiction. Now, we can assume that $|E(H_1)| = 3$. Then $\deg v_1 = 1$ (by (2.1)). By Lemma 2.2 we get $\deg v_2 = 2$, a contradiction (see Lemma 1.4).

a_4) $v_4v_8, v_4v_9 \in E(G)$ and the vertex v_{10} is adjecent to at least one of the vertices v_2, v_3.

Deg $v_7 = 1$ (by (2.1)) and so $\deg v_6 > 2$ (by Lemma 1.4). Hence we have (with respect to (2.1) and Lemma 2.2) one of the cases a_1), a_3).

a_5) Let $v_6v_8 \in E(G)$ and each of the vertices v_9, v_{10} be adjacent to at least one of the vertices v_2, v_3.

This is impossible by Lemma 2.2.

a_6) Let $v_5v_8 \in E(G)$ and each of the vertices v_9, v_{10} be adjacent to at least one of the vertices v_2, v_3.

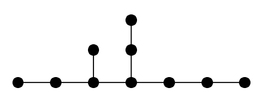
Since $\deg v_7 > 1$ or $\deg v_6 > 2$ we obtain (with respect to (2.1)) that the graph G does not exist by Lemma 2.2.

a_7) Each of the vertices v_8, v_9, v_{10} is adjacent to at least one vertex from the set $\{v_2, v_3, v_4\}$.

From (2.1) it follows $\deg v_7 = 1$. Since $\deg v_6 > 2$ we get one of the previous cases.

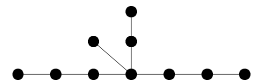
b) Let $v_8v_9 \in E(G)$ and each of the vertices v_8, v_{10} be adjacent to at least one vertex from the set $V' = \{v_2, \ldots, v_6\}$. We distinguish six subcases.

b_1) $v_4v_8, v_3v_{10} \in E(G)$



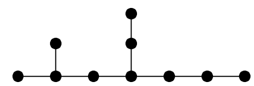
Since neither $\deg v_1 = 1$ and $\deg v_2 = 2$ nor $\deg v_7 = 1$ and $\deg v_6 = 2$ can holds, we have (by (2.1)) that $e(v_8) \leq 3$ or the graph G does not exist by Lemma 2.2, a contradiction.

$b_2)$ $v_4v_8, v_4v_{10} \in E(G)$



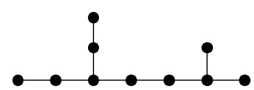
Similarly, since neither $\deg v_1 = 1$ and $\deg v_2 = 2$ nor $\deg v_7 = 1$ and $\deg v_6 = 2$ holds we have (by (2.1)) that either $e(v_8) \le 3$ or $e(v_{10}) \le 3$ holds or the graph G does not exist by Lemma 2.2, a contradiction.

$b_3)$ $v_4v_8, v_2v_{10} \in E(G)$



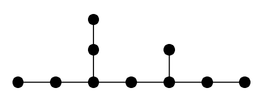
If $\deg v_7 > 1$ then $v_7v_9 \in E(G)$ by (2.1) and the graph G does not exist by Lemma 2.2. If $\deg v_6 > 2$ then $v_6v_{10} \in E(G)$ or the graph G does not exist by Lemma 2.2. If $v_6v_{10} \in E(G)$ we get that the graph G contains the subgraph in Fig. 2.3 with the eccentric sequence $3, 4^5, 5^4$. It is easy to check that if we add an edge to the graph in Fig. 2.5 we obtain a graph in which eccentricities at least two vertices are at most most three, a contradiction.

$b_4)$ $v_3v_8, v_6v_{10} \in E(G)$



Since $\deg v_1 > 1$ or $\deg v_2 > 2$ we obtain (with respect to (2.1)) that the graph G does not exist by Lemma 2.2.

$b_5)$ $v_3v_8, v_5v_{10} \in E(G)$



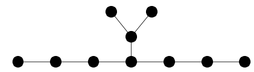
Since $\deg v_1 > 1$ or $\deg v_2 > 2$ the subgraph $\langle v_1, v_2, v_3, v_8, v_9\rangle$ has at least five edges (with respect to (2.1)). Since $\deg v_7 > 1$ or $\deg v_6 > 2$ the graph G does not exist by Lemma 2.2.

$b_6)$ $v_3v_8 \in E(G)$ and the vertex v_{10} is adjacent to at least one vertex from the set $\{v_2, v_3, v_4\}$.

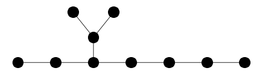
By (2.1) we have $\deg v_7 = 1$. If $\deg v_2 > 2$ then $e(v_8) \le 3$ or it takes place the case $b_4)$.

c) It remains the next three cases:

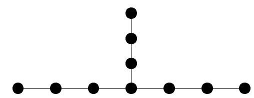
(*c*) $v_4v_8, v_8v_9, v_8v_{10} \in E(G)$



(*d*) $v_3v_8, v_8v_9, v_8v_{10} \in E(G)$



(*e*) $v_4v_8, v_8v_9, v_9v_{10} \in E(G)$



Since neither $\deg v_1 = 1$ and $\deg v_2 = 2$ nor $\deg v_7 = 1$ and $\deg v_6 = 2$ is possible we have $e(v_8) \leq 3$ or $e(v_9) \leq 3$ (with respect to (2.1)), a contradiction.

II. If it does not take place the case I then by Lemma 2.1 the graph G has the subgraph in Fig. 2.2 and without loss of generality we can asuume

$$d(v_3, w_3) = d(w_1, v_7) = d(v_5, v_1) = 4 \tag{2.3}$$

We first show that at least two of the vertices v_1, v_7, w_3 have degree one. On the contrary we may suppose (without loss of generality) that $\deg v_1 > 1$ and $\deg v_7 > 1$. This is possible only (with regard to (2.3)) if $v_1w_2 \in E(G)$ and $v_7v_2 \in E(G)$. Then we obtain $e(v_2) \leq 3$, a contradiction. So let $\deg v_1 = \deg w_3 = 1$. The inequality $d(v_1, w_3) \leq 5$ yields $d(v_2, w_2) \leq 3$. With respect to (2.3) we get that $v_2w_2 \notin E(G)$ and any shortest path between the vertices v_2, w_2 can contain neither vertex v_3 nor v_7. It can contain only the vertex w_1 and this is possible only if $v_2w_1 \in E(G)$. Since $e(v_2) = 4$ we get $\deg v_7 = 1$ and so $d(v_6, w_2) \leq 3$. As in the previous case we obtain $v_6w_2 \notin E(G)$ and moreover, any shortest path between vertices w_2, v_6 can not contain the vertex w_1. This path can contain only the vertex v_5 and this is possible only if $w_2v_5 \in E(G)$. Then $e(w_2) \leq 3$, a contradiction. □

Corollary 2.5. *The eccentric sequence* $3, 4^{10}$ *is minimal.*

Proof. The sequence $3, 4^{10}$ is the eccentric sequence of the graph in Fig. 2.5. According to Theorem 2.4 and Theorem 1.2 there is no graph with eccentric sequence $3, 4^l$, $l < 10$. □

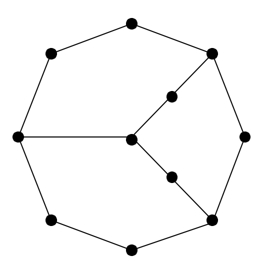


Fig. 2.5

Corollary 2.6. *The eccentric sequences* $3, 4^5, 5^4$ *and* $3, 4^7, 5^2$ *are minimal.*

Proof. The sequence $3, 4^5, 5^4$ is the eccentric sequence of the graph in Fig 2.3. The sequence $3, 4^7, 5^2$ is the eccentric sequence of the graph in Fig 2.4. Now we will show that there is no graph with eccentric sequence $3, 4^l, 5^r$, $l + r < 9$. On the contrary, let such graph G exist. According to Theorem 1.2 it is sufficient to consider the case $l + r = 8$. Let $e(v_4) = 3$ and let we obtain a graph G' from the graph G by adding a vertex $v \notin V(G)$ and an edge v_4v. Then, evidently, $e_{G'}(v) = 4$ and for any vertex $u \in V(G)$ it holds $e_G(u) = e_{G'}(u)$ (because $d_{G'}(u, v) \leq 4$). Then the graph G' has to be one of the graph in Fig. 2.3 or Fig. 2.4 but it is easily to see that G' is not any of them, a contradiction. $\square$

Corollary 2.7. *A sequence* $3, 4^6, 5^r$ *is not a minimal eccentric sequence for any* r.

Proof. There is no graph with the eccentric sequence $3, 4^6, 5^3$ according to Theorem 2.4. Then it is sufficient to use Corollary 2.6. $\square$

Minimal eccentric sequences with least eccentricity 2 can be found in [1], but with a mistake in Fig. 9.4. Therefore we will touch this case in the last part of this section.

Theorem 2.8. *The eccentric sequence* $2, 3^6$ *is minimal and it is realized by only the graph in Fig. 2.6.*

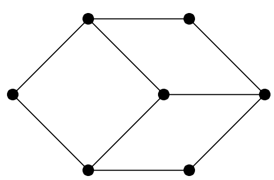


Fig. 2.6

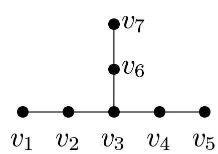


Fig. 2.7

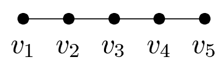


Fig. 2.8

Proof. In the same way as in the proof of Lemma 2.1 we can show that the graph G with the eccentric sequence $2, 3^k$ contains the subgraph in Fig. 3.4 where $e(v_3) = 2$ and $d(v_1, v_3) = d(v_5, v_3) = d(v_7, v_3) = 2$ or it contains the subgraph in Fig. 2.8 satisfying $e(v_3) = 2$ and $d(v_1, v_4) = d(v_2, v_5) = 3$ (if the vertices of G are suitable denoted).

I. In the first part of the proof we will show that the eccentric sequence $2, 3^6$ is minimal.

Suppose on the contrary that there exists a graph G with the eccentric sequence $2, 3^5$. It is obvious that the graph in Fig. 2.8 is a subgraph of G, where $e(v_3) = 2$ and $d(v_1, v_4) = d(v_2, v_5) = 3$.

Since $e(v_3) = 2$ it is sufficient to consider two possibilities for the vertex v_6 .

a)Let $v_2v_6 \in E(G)$. The equality $d(v_2, v_5) = 3$ implies that $\deg v_5 = 1$ and so $e(v_5) = e(v_4) + 1$ (by Lemma 1.3), a contradiction.

b) Let $v_3v_6 \in E(G)$. Since $e(v_2) = e(v_4) = 3$ we get $\deg v_1 > 1$ and $\deg v_5 > 1$ (by Lemma 1.3) and so $v_1v_6 \in E(G)$ and $v_5v_6 \in E(G)$, which gives $e(v_6) \leq 2$, a contradiction.

II. We will show that the eccentric sequence $2, 3^6$ is realized by only the graph in Fig. 2.6.

a) Let G contain the subgraph in Fig. 2.7 where $e(v_3) = 2$ and $d(v_1, v_3) = d(v_5, v_3) = d(v_7, v_3) = 2$.

Firstly, let $v_5v_7 \in E(G)$. Since $\deg v_1 > 1$ (by Lemma 1.3) we have $e(v_4) \le 2$ or $e(v_6) \le 2$, a contradiction.

Secondly, let $v_5v_7 \notin E(G)$. With respect to symmetry we can also assume that $v_1v_7 \notin E(G)$ and $v_1v_5 \notin E(G)$. Since $d(v_3, v_5) = 2$ and $\deg v_5 > 1$ (by Lemma 1.3) we can assume (without loss of generality) that $v_5v_6 \in E(G)$. From $\deg v_1 > 1$ and $e(v_6) = 3$ we have that $v_1v_4 \in E(G)$. From $\deg v_7 > 1$ and $e(v_4) = 3$ we have $v_2v_7 \in E(G)$. Hence the graph in Fig. 2.6 is a subgraph of G. It is easy to check that if we add an edge to the graph in Fig. 2.6 we obtain a graph in which eccentricities of at least two vertices are at most 2.

b) Let G contain a subgraph in Fig. 2.8 where $e(v_3) = 2$ and $d(v_1, v_4) = d(v_2, v_5) = 3$.

In this case the subgraph $\langle v_1, v_2, v_3, v_4, v_5 \rangle$ has 4 edges. Since $e(v_3) = 2$, it is sufficient to consider (with respect to symmetry) the next four possibilities.

1) $v_2v_6, v_2v_7 \in E(G)$

Since $d(v_2, v_5) = 3$ we get $\deg v_5 = 1$ which is impossible (by Lemma 1.3).

2) $v_3v_6, v_3v_7 \in E(G)$

In this case a shortest path between v_1 and v_5 contains a vertex v_i, $i \in \{6, 7\}$ and so $e(v_i) \le 2$, a contradiction.

3) $v_2v_6, v_4v_7 \in E(G)$

If $v_1v_6 \notin E(G)$ we obtain that $e(v_4) \le 2$ (because $\deg v_1 > 1$ and $\deg v_6 > 1$ by Lemma 1.3), a contradiction. Now, let $v_1v_6 \in E(G)$. Any shortest path between v_1 and v_5 does not contain the vertex v_3 (because $d(v_1, v_5) \le 3$) and so there exists an edge $v_iv_j \in E(G)$, where $i \in \{1, 2, 6\}$, $j \in \{4, 5, 7\}$. This yields $e(v_j) \le 2$, a contradiction.

4) $v_2v_6, v_3v_7 \in E(G)$

Since $\deg v_5 > 1$ and $e(v_2) = 3$ we obtain $v_5v_7 \in E(G)$. Analogously as in the previous case it can be shown that $v_1v_6 \notin E(G)$. Since $\deg v_1 > 1$ and $d(v_1, v_4) = 3$ we have $v_1v_7 \in E(G)$. Similarly, $\deg v_6 > 1$ and $e(v_7) = 3$ imply $v_6v_4 \in E(G)$. Thus we obtain the graph in Fig. 2.6. □

Remark. Theorem 2.8 implies that the graph corresponding to the eccentric sequence $2, 3^6$ in [1, Fig. 9.4] is wrong. Its eccentric sequence is $2, 3^4, 4^2$ and it is easy to see that it is not even a subgraph of the graph in Fig. 2.6.

3. Minimal eccentric sequences of type $3^k, 4^l$, $k \ge 2$.

Lemma 3.1. *Let G be a graph with an eccentric sequence $3^k, 4^l$, $k \ge 2$. Then G contains the subgraph in Fig. 3.1 satisfying $e(v_4) = 3$ and*

$$d(v_1, v_4) = d(v_4, v_7) = 3, \tag{3.1}$$

or the subgraph in Fig. 3.2 satisfying $e(v_3) = e(v_4) = 3$ and

$$d(v_1, v_4) = d(v_3, v_6) = 3 \tag{3.2}$$

(provided that the vertices of the graph G are suitable denoted).

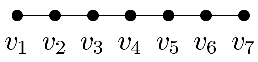


Fig. 3.1

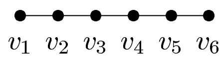


Fig. 3.2

Proof. Let $v_4 \in V(G)$, $e(v_4) = 3$ and let the graph G do not contain the graph in Fig. 3.1 satisfying (3.1). There exists a vertice $v_1 \in V(G)$ with $d(v_1, v_4) = 3$. Let v_1, v_2, v_3, v_4 be a path from v_1 to v_4 of the length 3. We denote

$$V' = V(G) - \{v_1, v_2, v_3, v_4\}.$$

Evidently, $V' \neq \emptyset$. If $v \in V'$ with $d(v, v_4) = 3$ then every shortest path from v to v_4 contains the vertex v_2 or v_3. Therefore, $d(v, v_3) \leq 2$. If $v \in V'$ with $d(v, v_4) = 1$ then again $d(v, v_3) \leq 2$. If also for each vertex $v \in V'$ with $d(v, v_4) = 2$ we have $d(v, v_3) \leq 2$ then $e(v_3) \leq 2$, which is impossible. Hence there exists a vertex v which satisfies $d(v, v_4) = 2$ and $d(v, v_3) = 3$. Consequently, the graph G contains the subgraph in Fig. 2 satisfying (3.2) (we put $v_6 = v$). $\square$

Lemma 3.2. *Let G be a graph with an eccentric sequence $3^k, 4^l$ and let v_1, v_2, v_3, v_4, v_5, v_6 be a path of G such that $e(v_3) = e(v_4) = 3$ and (3.2) are satisfied. Let G_1 and G_2 be subgraphs of G satisfying the following three conditions:*

(k) $\quad V(G_1) \cap V(G_2) = \emptyset,$

(kk) $\quad V(G_1) \cup V(G_2) = V(G),$

(kkk) $\quad v_1, v_2, v_3 \in V(G_1), \quad v_4, v_5, v_6 \in V(G_2).$

We claim that

a) *there exists an edge* $uv \in E(G)$ *for which*
$u \in V(G_1) - \{v_3\}, \quad v \in V(G_2) - \{v_4\},$

b) $uv \in E(G)$, $u \in V(G_1) - \{v_3\}$, $v \in V(G_2) - \{v_4\}$ *and*
$\operatorname{diam}(G_1) \leq 3$, $\operatorname{diam}(G_2) \leq 2$ *give* $k \geq 3$,

c) $uv \in E(G)$, $u \in V(G_1) - \{v_3\}, v \in V(G_2) - \{v_4\}$ *and*
$\operatorname{diam}(G_1) \leq 2$, $\operatorname{diam}(G_2) \leq 2$ *give* $k \geq 4$.

Proof. a) Suppose, contrary to our claim, that there is no edge uv with $u \in V(G_1) - \{v_3\}$, $v \in V(G_2) - \{v_4\}$. Then every shortest path between the vertices v_1 and v_6 contains the vertex v_3 or v_4, which implies $d(v_1, v_6) \geq 5$. This contradicts to our assumption ($e(v_1) \leq 4$).
b) It is clear that $e(u) \leq 3$.
c) It follows easily that $e(u) \leq 3$ and $e(v) \leq 3$. $\square$

Lemma 3.3. *Let G be a graph with an eccentric sequence of type $3^k, 4^l$ and let $V(G) = \{v_1, v_2, \ldots, v_8\}$. If a circle $v_1, v_2, \ldots, v_m, v_1$ with $m \in \{4, 5, 6, 7\}$ is a subgraph of G and each vertex v_j, $j > m$, is adjacent to at least one vertex from the set $\{v_1, v_2, \ldots, v_m\}$, then $k \geq 4$.*

Proof. It is clear that $k \geq 5$ if $m = 7$. Let $m = 6$. At most one vertex of the circle $v_1, v_2, \ldots, v_m$ is at distance 4 from v_7 and at distance 4 from v_8. It implies that $l \leq 4$ and so $k \geq 4$.

Let $m \in \{4, 5\}$. It is easily seen that $d(v_i, v_j) \leq 3$ for each $i \in \{1, 2, \ldots, m\}$, $j \in \{m+1, \ldots, 8\}$. It implies $e(v_i) \leq 3$ for each $i = 1, 2, \ldots, m$. □

Lemma 3.4. *Let G be a connected graph such that $|V(G)| = 7$ and let a circle of the length at least 6 be its subgraph. Then the eccentricities of at least 5 vertices of G do not exceed 3.*

Proof. It is immediate. □

Lemma 3.5. *A graph G with the eccentric sequence $3^2, 4^7$ does not exist.*

Proof. Suppose that there is a graph G with the eccentric sequence 3^2, 4^7. Let $V(G) = \{v_1, v_2, \ldots, v_9\}$. Suppose that the assertion is false. We will use Lemma 3.1 and hence we distinguish two cases.

I. Let the graph G contains a path $v_1, v_2, \ldots, v_7$ such that $e(v_4) = 3$ and the equalities (3.1) hold.

If $\deg v_1 = \deg v_7 = 1$ then (by Lemma 1.1) $e(v_2) = 3$ and $e(v_6) = 3$, a contradiction. Therefore we will suppose that at least one of the vertices v_1, v_7 has the degree at least 2. If we take symmetry and the equality $e(v_4) = 3$ into account, we see that it is sufficient to distinguish the following subcases.

a) Each of the vertices v_8, v_9 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4, v_5, v_6\}$.

a_1) The vertex v_8 is adjacent to v_2 or v_3 and the vertex v_9 to v_5 or v_6.
Without loss of generality we may assume that both vertices with the eccentricity 3 belong to the set $\{v_1, v_2, v_3, v_4, v_8\}$.

Firstly, suppose that $v_2 v_8 \in E(G)$ or the subgraph $\langle v_1, v_2, v_3, v_8 \rangle$ of G has at least 4 edges. Any shortest path between v_1 and v_7 does not contain v_4 (with respect to (3.1)) hence there is an edge $v_i v_j$, $i \in \{1, 2, 3, 8\}$, $j \in \{5, 6, 7, 9\}$. It gives $e(v_j) \leq 3$, a contradiction.

Secondly, let $v_3 v_8 \in E(G)$ and the subgraph $\langle v_1, v_2, v_3, v_8 \rangle$ has exactly 3 edges. The assumption $\deg v_1 > 1$ implies $e(v_5) \leq 3$, a contradiction. If $\deg v_1 = 1$ then $\deg v_2 \geq 3$ (by Lemma 1.4) and so $E(G)$ contains an edge $v_2 v_j$, $j \in \{5, 6, 7, 9\}$. Thus $e(v_j) \leq 3$, a contradiction.

a_2) Each of the vertices v_8, v_9 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4\}$ and $v_4 v_9 \notin E(G)$.

Firstly, let $\deg v_7 > 1$. In this case we have that eccentricities of at least two vertices from v_2, v_3, v_8, v_9 are at most 3, a contradiction. (Note that $v_4 v_8 \in E(G)$ gives $v_7 v_8 \notin E(G)$ because $d(v_4, v_7) = 3$.)

Secondly, $\deg v_7 = 1$ gives $e(v_6) = 3$ and $\deg v_6 > 2$ (by Lemma 1.3 and 1.2). Hence we have either the case a_1) or at least one of the inequalities $e(v_2) \leq 3$, $e(v_3) \leq 3$ holds, which is impossible.

a_3) Let $v_4v_8, v_4v_9 \in E(G)$.
The inequality $\deg v_1 > 1$ gives (with respect to (3.1)) $e(v_5) \leq 3$ and $e(v_6) \leq 3$, which is impossible. Similarly, $\deg v_7 > 1$ yields $e(v_2) \leq 3$ and $e(v_3) \leq 3$.

b) The vertex v_9 is not adjacent to any vertex from the set $\{v_2, v_3, v_4, v_5, v_6\}$.

In this case we have $v_8v_9 \in E(G)$ and it is sufficient to consider two possibilities.
b_1) Let $v_4v_8 \in E(G)$.
Without loss of generality we may assume that both vertices with the eccentricity 3 belong to the set $\{v_1, v_2, v_3, v_4, v_8, v_9\}$.

Firstly, if $\deg v_1 > 1$ we obtain $v_1v_9 \in E(G)$ (because $e(v_5) = 4$). By our assumption $e(v_7) = e(v_6) = 4$ and so $\deg v_7 > 1$. Therefore eccetricities of at least two vertices from the set $\{v_2, v_3, v_8, v_9\}$ are at most 3, a contradiction.

Secondly, if $\deg v_1 = 1$ then $e(v_2) = 3$ and $\deg v_2 > 2$ (by Lemmas 1.3 and 1.4). Since $e(v_8) = 4$ we obtain that $\deg v_9 > 1$ (by Lemma 1.3). Therefore, $v_7v_9 \in E(G)$ holds. From $\deg v_2 > 2$ we get $v_2v_j \in E(G)$, $j > 4$. This yields $e(v_j) \leq 3$, a contradiction.
b_2) Let $v_3v_8 \in E(G)$.
If $\deg v_7 > 1$ then eccentricities of at least two vertices from the set $\{v_2, v_3, v_8\}$ are at most 3. The equality $\deg v_7 = 1$ guarantees $e(v_6) = 3$ and $\deg v_6 > 2$. This forces $e(v_2) \leq 3$ or $e(v_8) \leq 3$. It again contradicts our assumption.

II. Let the graph G contain a path $v_1, v_2, \ldots, v_6$ (see Fig. 3.2) such that $e(v_3) = e(v_4) = 3$ and (3.2) hold.

From $e(v_2) = e(v_5) = 4$ we get $\deg v_1 > 1$ and $\deg v_6 > 1$ (by Lemma 1.3). Since we may assume that the case I does not take place and $e(v_3) = e(v_4) = 3$, it is sufficient to consider the next possibilities.

a) Each of the vertices v_7, v_8, v_9 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4, v_5\}$.
a_1) If at least one of the vertices v_3, v_4 is adjacent to no vertex from the set $\{v_7, v_8, v_9\}$ then we obtain a contradiction (by Lemma 3.2).
a_2) Let $v_3v_7, v_4v_8 \in E(G)$.
We may suppose that the vertex v_9 is adjacent to v_2 or v_3. Since $\deg v_6 > 1$ we have either $e(v_2) \leq 3$ or $v_8v_6 \in E(G)$. In the second case we again obtain a contradiction by Lemma 3.2.

b) Let $v_3v_7, v_7v_8 \in E(G)$ and let the vertex v_8 be adjacent to no vertex from the set $\{v_2, v_3, v_4, v_5\}$.
b_1) In each of the subcases $v_2v_9 \in E(G)$, $v_3v_9 \in E(G)$, $v_7v_9 \in E(G)$ we can obtain $e(v_2) \leq 3$ or $e(v_7) \leq 3$ (because $\deg v_6 > 1$).
b_2) Let $v_4v_9 \in E(G)$.
If $v_6v_9 \notin E(G)$ then $e(v_2) \leq 3$ or $e(v_7) \leq 3$ (because $\deg v_6 > 1$ by Lemma 1.3). If $v_6v_9 \in E(G)$ then $e(v_2) \leq 3$ (because $\deg v_1 > 1$ and $d(v_1, v_4) = 3$) or the graph G does not exist by Lemma 3.2.
b_3) Let $v_5v_9 \in E(G)$.
If the subgraph $\langle v_1, v_2, v_3, v_7, v_8 \rangle$ contains at least 5 edges using Lemma 3.2 we obtain a contradiction. Otherwise, we have $e(v_5) \leq 3$ (because $\deg v_1 > 1$ and $\deg v_8 > 1$ by Lemma 1.3). □

Lemma 3.6. *A graph with the eccentric sequence* $3^3, 4^5$ *does not exist.*

Proof. Suppose that there is a graph G with the eccentric sequence 3^3, 4^5. We again use Lemma 3.1 and hence the proof splits into two parts.

I. Let G contain a path $v_1, v_2, \dots, v_7$ such that $e(v_4) = 3$ and the equalities (3.1) hold.

Since $e(v_4) = 3$ the vertex v_8 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4, v_5, v_6\}$. We consider two cases.

a) The subgraph $\langle v_1, v_2, \dots, v_7 \rangle$ of G contains at least 7 edges.

Evidently $d(v_1, v_7) \leq 4$ and in accordance with Lemma 3.3 it is sufficient to consider two subcases.

a_1) $v_2v_5 \in E(G)$

Since $e(v_5) \geq 3$ we obtain $v_3v_8 \in E(G)$. In the case $\deg v_7 > 1$ we may use (3.1) and Lemma 3.3 and we get a contradiction. The equality $\deg v_7 = 1$ yields $\deg v_6 > 2$ (by Lemma 1.4) and we have again (by Lemma 3.3) a contradiction.

a_2) $v_3v_5 \in E(G)$

In this case $e(v_3) \leq 3$ and $e(v_5) \leq 3$. The case a_1) implies that $v_2v_5 \notin E(G)$ and with respect to symmetry also $v_3v_6 \notin E(G)$. Then using the Lemma 3.3 we get that every shortest path between v_1 and v_7 must contain the vertex v_8 and so $e(v_8) \leq 3$.

b) The subgraph $\langle v_1, v_2, \dots, v_7 \rangle$ contains only 6 edges.

By Lemma 1.4 we have $v_1v_8 \in E(G)$ or $v_2v_8 \in E(G)$. Analogously, $v_6v_8 \in E(G)$ or $v_7v_8 \in E(G)$. Hence we get a contradiction (with Lemma 3.3 or $e(v_8) \leq 2$).

II. The graph G contains a path $v_1, v_2, \dots, v_6$ where $e(v_3) = e(v_4) = 3$ and (3.2) holds.

Since we can assume that the case I does not take place it is sufficient to consider the next possibilities.

a) Each of the vertices v_7, v_8 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4, v_5\}$.

We distinguish two cases.

a_1) The subgraph $\langle v_1, v_2, \dots, v_6 \rangle$ contains at least 6 edges.

In this case we get $v_1v_6 \in E(G)$ or $v_2v_5 \in E(G)$ (in accordance with (3.2)), which is impossible by Lemma 3.3.

a_2) The subgraph $\langle v_1, v_2, \dots, v_6 \rangle$ contains only 5 edges.

The case $\deg v_6 = 1$ and $\deg v_5 = 2$ is impossible (by Lemma 1.4). If we take symmetry into account it is sufficient to distinguish the next subcases.

1) $v_2v_7, v_5v_8 \in E(G)$.

In this subcase we have that eccentricities of at least 4 vertices are at most 3 (by Lemma 3.2).

2) $v_2v_7, v_4v_8 \in E(G)$

Since $\deg v_6 > 1$ or $\deg v_5 > 2$ we can again get that eccentricities of at least 4 vertices are at most 3 (by Lemmas 3.2 and 3.3).

3) $v_2v_7, v_3v_8 \in E(G)$

With respect to (3.2) and Lemma 3.3 we have $\deg v_7 = 1$. Then $\deg v_6 > 2$ and we obtain a contradiction (by Lemmas 3.2 and 3.3).

4) $v_2v_7, v_2v_8 \in E(G)$

By Lemma 3.3 we obtain $\deg v_6 = 1$ and $\deg v_5 = 2$, a contradiction (by Lemma 1.4).

5) $v_3v_7, v_4v_8 \in E(G)$

If $\deg v_5 > 2$ or $\deg v_2 > 2$ we obtain one of previous cases. In the opposite case we have (by Lemma 1.4) $\deg v_1 > 1$, $\deg v_6 > 1$ and this is impossible by Lemma 3.2.

6) $v_3v_7, v_3v_8 \in E(G)$

With respect to the equalities (3.2) we get $\deg v_6 = 1$. If $\deg v_5 > 2$ we obtain the case 2). Otherwise we have a contradiction by Lemma 1.4.

b) The vertice v_8 is adjacent to no vertex from the set $\{v_2, v_3, v_4, v_5\}$.
With respect to symmetry we may assume that $v_3v_7, v_7v_8 \in E(G)$. We distinguish two possibilities.

b_1) A subgraph $\langle v_1, v_2, \ldots, v_6\rangle$ has at least 6 edges.
In this case we get $v_1v_6 \in E(G)$ or $v_2v_5 \in E(G)$. By Lemma 3.3 we get $\deg v_8 = 1$ and $\deg v_7 = 2$. Accordingly to Lemma 1.4 it is impossible.

b_2) A subgraph $\langle v_1, v_2, \ldots, v_6\rangle$ has 5 edges.

Firstly, let $v_4v_7 \notin E(G)$ and so $d(v_4, v_8) = 3$. With respect to the previous case we may assume that the subgraph $\langle v_3, v_4, v_5, v_6, v_7, v_8\rangle$ has 5 edges which implies $\deg v_6 = 1$ and $\deg v_5 = 2$, a contradiction (by Lemma 1.4).

Secondly, let $v_4v_7 \in E(G)$. By Lemma 3.2 we obtain that $\deg v_8 = 1$. Hence, a shortest path between v_1 and v_6 contains v_7 and so $e(v_7) \le 2$ (because $v_1v_7 \notin E(G)$ and $v_6v_7 \notin E(G)$ by (3.2)). □

Lemma 3.7. *A graph G with the eccentric sequence $3^4, 4^3$ does not exist.*

Proof. On the contrary, let there exist such a graph G.

Similarly as at the previous proofs we distinguish two cases.

a) Let G contain a path $v_1, v_2, \ldots, v_7$ where $e(v_4) = 3$ and (3.1) holds.

By (3.1) and Lemma 3.4 we deduce that $\deg v_7 = 1$. From this it follows that $\deg v_6 > 2$ (by Lemma 1.4). By Lemma 3.4 and the equalities (3.1) we obtain $v_iv_6 \in E(G)$ for some $i \in \{2, 3\}$. It implies $e(v_i) \le 2$, a contradiction.

b) Let G contain a path $v_1, v_2, \ldots, v_6$ where $e(v_3) = e(v_4) = 3$ and (3.2) holds.

It suffices to consider the next two cases.

b_1) $v_2v_7 \in E(G)$
Applying (3.2) and Lemma 3.4 gives $\deg v_6 = 1$. Therefore $e(v_5) > 2$ by Lemma 1.4. This clearly forces $v_5v_i \in E(G)$, $i \in \{2, 7\}$ (with respect to (3.2)) an so $e(v_i) \le 2$, a contradiction.

b_2) $v_3v_7 \in E(G)$
We have $\deg v_6 = 1$ using Lemma 3.4 and the equalities (3.2). This gives $e(v_5) > 2$ by Lemma 1.4. Consequently, $v_5v_i \in E(G)$, $i \in \{2, 7\}$.

Firstly, if $v_5v_2 \in E(G)$ then $e(v_2) \le 2$, a contradiction.

Secondly, if $v_5v_7 \in E(G)$ (and $v_5v_2 \notin E(G)$) then a shortest path between v_1 and v_6 contains v_7 and in consequence, $e(v_7) \le 2$ (because $d(v_6, v_7) = 2$ and $d(v_1, v_6) \le 4$). □

4. Minimal eccentric sequences of type $3, 4^l, 5^r$, $l + r > 9$.

In this section we will use the following basic idea.

Let G be a graph. Fix the vertex with the eccentricity 3 and denote it by s (a source). In this case we denote

$A := \{v \in V(G); d(s, v) = 1\}$,

$B := \{v \in V(G); d(s,v) = 2\}$,
$C := \{v \in V(G); d(s,v) = 3\}$.
If $v \in C$ ($v \in B$), there exists a vertice $u \in B$ ($u \in A$) such that $uv \in E(G)$. If $v \in A$ then $vs \in E(G)$. On the other hand, $vs \notin E(G)$ if $v \in B \cup C$ and $vu \notin E(G)$ if $v \in C$, $u \in A$. The distance of vertices u, v is at most 2 if $u, v \in A$, is at most 3 if $v \in A$, $u \in B$, respectively, etc. In the case $\deg s = 2$ we divide the vertices of G more carefully. In this case we put $A = \{a, a'\}$. Further, we will denote

by $b_1, b_2, \dots, b_n$ the vertices from the set $\{v \in B; d(v,a) = 1\}$,

by $b'_1, b'_2, \dots, b'_m$ the vertices from the set $\{v \in B; d(v,a') = 1\}$,

by $c_1, c_2, \dots c_p$ the vertices from the set C which are adjacent to at least one vertex from the set $\{b_1, b_2, \dots, b_n\}$,

by $c'_1, c'_2, \dots, c'_q$ the vertices from the set C which are adjacent to at least one vertex from the set $\{b'_1, b'_2, \dots, b'_m\}$.

It is possible that, $c_i = c'_j$ for some i, j. On the other hand, if $aa' \in E(G)$ then $e(a) \leq 3$, which is impossible in the considered case. Similarly, if $b_i = b'_j$ for some i, j then $e(b_i) \leq 3$.

Lemma 4.1. *Let G be a graph with the eccentric sequence $3, 4^l, 5^r$, where $l \leq 4$. Let $e(s) = 3$ and $\deg s \geq 3$ hold. Then*

a) $|C| > 2$,

b) $d(c_i, c_j) > 2$ for every $i \neq j$ if $|C| = 3$.

Proof. By our assumption $|A| \geq 3$ and $e(v) = 4$ if $v \in A$. Therefore the eccentricity of at most one vertex from B is 4 (because $l \leq 4$).

a) The inequality $|C| \geq 2$ follows by Lemma 2.1. Let $C = \{c_1, c_2\}$. If the eccentricity of a vertex $b \in B$ is 5 then $d(b, c_1) = 5$ or $d(b, c_2) = 5$. If a shortest path between c_1 and c_2 contains at least two vertices $b_i, b_j \in B$ then $e(b_i) \leq 4$ and $e(b_j) \leq 4$, a contradiction. Otherwise, $d(c_1, c_2) \leq 2$ and there exist edges $c_1b_i, c_2b_j \in E(G)$, $i \neq j$ (by Lemma 2.1) and so $e(b_i) \leq 4$ and $e(b_j) \leq 4$, a contradiction.

b) Let $C = \{c_1, c_2, c_3\}$ and let (on the contrary) $d(c_1, c_2) \leq 2$. It is sufficient to distinguish the following two cases.

b_1) $c_1c_2 \in E(G)$

Firstly, let $d(c_1, c_3) \leq 2$ or $d(c_2, c_3) \leq 2$. By Lemma 2.1 there are edges c_ib_j, c_sb_t, $j \neq t$. Then we conclude that $e(b_j) \leq 4$ and $e(b_t) \leq 4$, which contradicts our assumption $l \leq 4$.

Secondly, let $d(c_1, c_3) > 2$ and $d(c_2, c_3) > 2$. In this case we consider a shortest path between c_1 and c_3

$c_1 = u_1, u_2, \dots, u_i, u_{i+1} = c_3$

and a shortest path between c_2 and c_3

$c_2 = v_1, v_2, \dots, v_j, v_{j+1} = c_3$.

Both of the paths contain at least two vertices from B. If one of the paths contains all vertices from C then the eccetricities of all vertices of this path from B are at most 4, a contradiction. For otherwise we have either $e(u_2) \leq 4$ and $e(v_2) \leq 4$ provided that $u_2 \neq v_2$ or $e(u_2) \leq 4$ and $e(u_i) \leq 4$ provided that $u_2 = v_2$.

b_2) $b_1c_1, b_1c_2 \in E(G)$

We distinguish three cases.

If $d(c_1, c_3) \leq 3$ and $d(c_2, c_3) \leq 3$ then there exists an edge $c_i b_j \in E(G)$, $j > 1$ (by Lemma 2.1) which implies $e(b_1) \leq 4$ and $e(b_j) \leq 4$, a contradiction.

If $d(c_1, c_3) > 3$ and $d(c_2, c_3) > 3$ we obtain a contradiction in the same way as in the second case of the part b_1).

Let $d(c_1, c_3) \leq 3$ and $d(c_2, c_3) > 3$. Obviously, $e(b_1) \leq 4$. Let

$c_2 = v_1, v_2, \ldots, v_i, v_{i+1} = c_3$

be a shortest path between the vertices c_2 and c_3. If this path contains b_1 or c_1 we have $e(v_i) \leq 4$. Otherwise we get $e(v_2) \leq 4$. These conclusions contradict $l \leq 4$. □

Lemma 4.2. *If $b_i c_j \in E(G)$ and b_i is a cut vertex of G then $e(c_j) = e(b_i) + 1$.*

Proof. It is obvious that $e(c_j) \geq 3$. If $d(u, c_j) \geq 3$ then a shortest path between u and c_j contains the vertex b_i. □

Lemma 4.3. *A graph G with the eccentric sequence $3, 4^2, 5^r$ exists if and only if $r \geq 12$.*

Proof. In this case $A = \{a, a'\}$. Vertices from B adjacent to the vertex a will be denoted by $b_1, b_2, \ldots b_n$ and vertices of B adjacent to a' will be denoted by $b'_1, b'_2, \ldots, b'_m$, etc.

It is easily seen that $aa' \notin E(G)$ (in the opposite case we get $e(a) \leq 3$ which is impossible). Further, $b_i \neq b'_j$ and $b_i b'_j \notin E(G)$ for each $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$ (otherwise, we have $e(b_i) \leq 4$ which is impossible). Analogously $c_i \neq c'_j$ for each $i \in \{1, 2, \ldots p\}$, $j \in \{1, 2, \ldots, q\}$. Since $d(c_1, c'_1) \leq 5$ there exists an edge $c_i c'_j \in E(G)$, for some i, j.

We denote the subgraph $\langle a, b_1, \ldots, b_n, c_1, \ldots c_p \rangle$ by H. If $diam H \leq 3$ we obtain $e(c'_j) \leq 4$ (because $c_i c'_j \in E(G)$), a contradiction. Thus, we can suppose that $diam H = 4$. Now we are going to show that $|V(H)| \leq 6$ is impossible.

On the contrary, suppose that $diam H = 4$ and $|V(H)| \leq 6$. Without loss of generality we can assume that the following conditions are satisfied: $d_H(c_1, c_2) = 4$, a shortest path between c_1 and c_2 is c_1, b_1, a, b_2, c_2, $\deg_H c_1 = 1$ and if $p = 3$ (i.e. there is a vertex c_3) then $c_3 b_2 \in E(H)$. If there is no edge of type $c_1 c'_i$ then we have $e(b_1) = 4$ (because $e(c_1) = 5$), a contradiction. Analogously, if $\deg_H c_2 = 1$ then $c_2 c'_k \in E(G)$ for some k. If $\deg_H c_2 > 1$ then either $c_2 b_3 \in E(H)$ or $c_2 c_3 \in E(G)$. If there is no edge of type $c_j c'_k$, $j \in \{2, 3\}$ then we have $e(a) = 3$ (because $e(c_2) = 5$), a contradiction. Thus we may assume that $c_1 c'_i, c_j c'_k \in E(G)$ for some i, j, k, $j > 1$. Let b'_t be a vertex adjacent to c'_k. Then, obviously, we get $e(b'_t) \leq 4$, a contradiction. Thus, we conclude that $|V(H)| \geq 7$.

If $H' = \langle a', b'_1, \ldots, b'_m, c'_1, \ldots, c'_q \rangle$ then with respect to symmetry we get $|V(H')| \geq 7$, too. Therefore, the graph G has at least 15 vertices.

The graph with the eccentric sequence $3, 4^2, 5^{12}$ is in Fig.4.1. □

It is possible to show that the eccentric sequence $3, 4^2, 5^{12}$ is realized by only the graph in Fig. 4.1, but we will not deal with this point here.

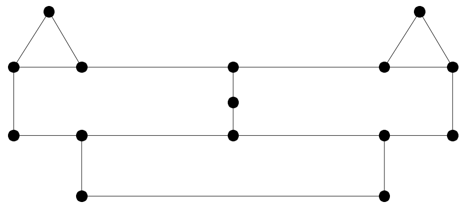


Fig. 4.1

Lemma 4.4. *A graph G with the eccentric sequence $3, 4^3, 5^8$ does not exist.*

Proof. Suppose on the contrary, that there is such a graph G. We distinguish two cases.

I. Let $\deg s = 2$.

Suppose that the vertices of the graph G are denoted as in the proof of Lemma 4.3. If there exists an edge of type $b_i b'_j$ then $e(b_i) \le 4$ and $e(b'_j) \le 4$, i.e. there exist at least 4 vertices with eccentricity at most 4, a contradiction. Since $d(c_1, c'_1) \le 5$, either $c_i = c'_j$ or G contains an edge of type $c_i c'_j$ for some i, j.

a) Let $c_i = c'_j$.

Without loss of generality we can assume that $c_1 = c'_1$ and the graph G contains the subgraph in Fig. 4.2.

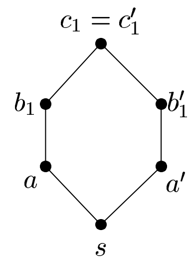


Fig. 4.2

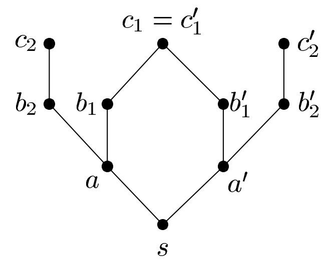


Fig. 4.3

Then $e(c_1) \le 4$ and so each vertex from the set $V(G) - \{s, a, a', c_1\}$ has the eccentricity 5. It implies that G contains the subgraph in Fig. 4.3 satisfying

$$(f) \qquad d(b_1, c'_2) = 5 = d(b'_1, c_2).$$

Since $d(c_2, c'_2) \le 5$ there exists a path

$c_2 = v_1, v_2, \ldots, v_s, v_{s+1} = c'_2$

of the length at most 5 (i.e. $s \le 5$). The path contains no vertex from the set $\{s, a, a', b_1, b'_1, c_1\}$ because (f) holds. For the same reason this path contains at least one vertex $v \notin \{b_2, b'_2, c_2, c'_2\}$. It follows easily that $e(b_2) \le 4$ or $e(b'_2) \le 4$, a contradiction.

b) Let $c_i c'_j \in E(G)$.

Without loss of generality we may assume that the graph in Fig. 4.4 is a subgraph of G.

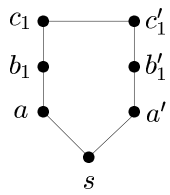


Fig. 4.4

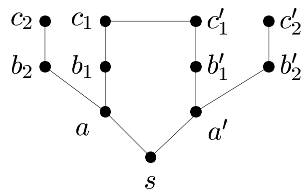


Fig. 4.5

Since at most one vertex from the set $\{b_1, c_1, b_1', c_1'\}$ has the eccentricity 4 the graph in Fig. 4.5 is a subgraph of G, too.

Besides the vertices of the graph in Fig. 4.5 the graph G has one another vertex w. Without loss of generality we may assume that $d(a, w) \le 2$. The inequality $\deg c_2' > 1$ yields either $e(a) \le 3$ or $e(b_1) \le 4$ and $e(c_1) \le 4$, a contradiction.

Let $\deg c_2' = 1$, i.e. $e(b_2') = 4$ (by Lemma 1.3). Then $\deg b_2' > 2$ by Lemma 1.4 and we conclude $e(c_1) \le 4$ or $e(b_2) \le 4$, a contradiction.

II. Let $\deg s = 3$.

Since $|A| = 3$ we have $|B| + |C| = 8$ and every vertex from the set $B \cup C$ has the eccentricity 5. Then, by Lemma 4.2, $\deg c_i \ge 2$ for every vertex $c_i \in C$. Lemma 4.1 implies that $|C| > 2$ and if $|C| = 3$ then $|B| \ge 6$ (a contradiction). Therefore, we can suppose that $|C| \ge 4$. We will distinguish three cases.

1) Let $|C| = |B| = 4$.

Denote $H = \langle c_1, c_2, c_3, c_4, b_1, b_2, b_3, b_4 \rangle$. We consider two possibilities.

a) The vertices c_1, c_2, c_3, c_4 belong to the same component H_1 of H.

If $V|H_1| = 8$ then there exists a vertex v for which $e_{H_1}(v) \le 4$ (for example, we can take a vertex from the centre of a spanning tree of H_1). This clearly forces $e_G(v) \le 4$, a contradiction.

Let $V|(H_1)| \le 7$ and suppose that b_4 does not belong to $V(H_1)$. Since every vertex $c \in C$ is adjacent to a vertex $b \in B$ we may assume that $b_1c_1, b_1c_2 \in E(G)$. Clearly $e(b_1) = 5$ and so we may assume (without loss of generality) that $d(b_1, c_4) = 5$. This yields that a shortest path between b_1 and c_4 in the subgraph H_1 contains the vertices b_2, b_3, c_3. It implies either $e(b_2) \le 4$ or $e(b_3) \le 4$, a contradiction.

b) The vertices c_1, c_2, c_3, c_4 do not belong to the same component of H.

With respect to Lemma 4.2 they belong to only two components. It is sufficient to consider two subcases.

b_1) Let $H_1 = \langle b_1, b_2, c_1, c_2, c_3 \rangle$, $H_2 = \langle b_3, b_4, c_4 \rangle$.

A shortest path between c_1 and c_4 contains a vertex b_i, $i \in \{1, 2\}$, which implies $e(b_i) \le 4$, a contradiction.

b_2) Let $H_1 = \langle b_1, b_2, c_1, c_2 \rangle$, $H_2 = \langle b_3, b_4, c_3, c_4 \rangle$. Consider two paths P_1 and P_2 from four shortest paths (in G) between vertices c_i and c_j, $i \in \{1, 2\}$, $j \in \{3, 4\}$, which contain a vertex b_k, $k \in \{1, 2\}$. If P_1 and P_2 have the same initial vertex then $e(b_k) \le 4$. In the opposite case we have $e(b_m) \le 4$, $m \in \{3, 4\}$, where b_m is a vertex of P_1 or P_2.

2) Let $|B| = 3$, $|C| = 5$.

We put $H = \langle c_1, c_2, c_3, c_4, c_5, b_1, b_2, b_3 \rangle$. The vertices c_1, c_2, c_3, c_4, c_5 belong to the same component H_1 of H (by Lemma 4.2). If $|V(H_1)| = 8$ then there exists a vertex v with $e_{H_1}(v) \le 4$ and so $e_G(v) \le 4$. If $|V(H)| < 8$ and the vertex b_3

does not belong to H_1 then $e(b_1) \leq 4$, $e(b_2) \leq 4$. In the both cases we get a contradiction.

3) Let $|B| = 2$, $|C| = 6$.

It is easy to check that we again get $e(b_1) \leq 4$ and $e(b_2) \leq 4$, which is impossible.

Since the set B contains at least two elements (by Lemma 2.1), the proof is complete. □

Lemma 4.5. *There is no graph G with the eccentric sequence* $3, 4^4, 5^6$.

Proof. Suppose, contrary to our claim, that there is such a graph G. Let $s \in V(G)$ be the vertex with the eccentricity 3. We again distinguish two cases.

I. Let $\deg s = 2$.

We will suppose that the vertices of G are denoted as in the proof of Lemma 4.3.

a) Suppose there exists an edge $b_i b'_j \in E(G)$.

We may assume that the graph G contains the subgraph in Fig. 4.6.

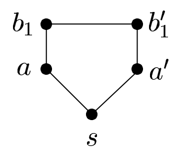


Fig. 4.6

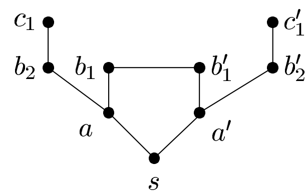


Fig. 4.7

Since $e(a) = e(a') = 4$, also the graph in Fig. 4.7 is a subgraph of G, where

$$d(a, c'_1) = d(a', c_1) = 4. \qquad \text{(g)}$$

Therefore, the eccentricity of every vertex from $V(G) - \{s, a, a', b_1, b'_1\}$ is 5. We may also assume that $d(b_2, c'_1) = 5$.

Besides the vertices of the graph in Fig. 4.7 the graph G has another vertices x, y. With respect to symmetry it is sufficient to consider the next cases.

a_1) Let $d(x, a') \leq 2$ and $d(y, a') \leq 2$. Then $\deg c_1 = 1$ and so $e(c_1) \neq e(b_2)$ (by Lemma 1.3) or $e(a') \leq 3$, a contradiction.

a_2) Suppose that $d(a, x) \leq 2$ and $d(a', y) \leq 2$.

Let

$c_1 = v_1, v_2, \ldots, v_k = c'_1$

be a shortest path between vertices c_1 and c'_1. Since $d(b_2, c'_1) = 5$ and $d(c_1, c'_1) \leq 5$ we obtain $v_2 \neq b_2$ and $k \in \{5, 6\}$. The equalities (g) and $e(b'_1) = 4$ yield that $v_2 = x$ and $v_3 = y$ and so $e(y) \leq 4$, a contradiction.

b) Let $c_i = c'_j$ for some i, j.

We may assume that the graph G contains the subgraph in Fig. 4.2 and, moreover, only one vertex from $V(G) - \{s, a, a', c_1\}$ has the eccentricity 4.

Let $e(b'_1) = 5$. Then G contains the subgraph in Fig. 4.8, where $d(c_2, b'_1) = 5$.

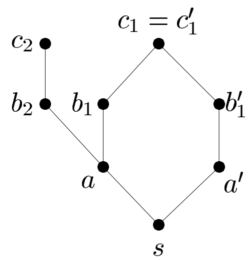


Fig. 4.8

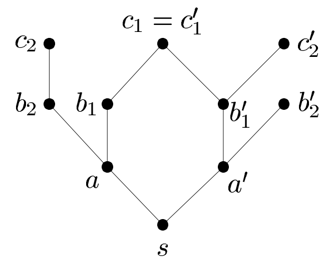


Fig. 4.9

It is sufficient to consider two possibilities.

b_1) Let G contain a subgraph in Fig. 4.3 satisfying $d(c'_2, b_1) = 5$ and let $d(a', x) \leq 2$ for a remaining vertex x.

If $\deg c_2 = 1$ then $e(b_2) = 4$ and $\deg b_2 > 2$ (see Lemma 1.4). Hence, we obtain $e(b'_1) \leq 4$ or $e(b'_2) \leq 4$. If $\deg c_2 > 1$ then $d(b'_1, c_2) < 5$. In both cases we have a contradiction.

b_2) In the opposite case to b1) the graph G contains the subgraph in Fig. 4.9 (because $e(b_1) = e(a) = 4$) and the eccentricity of every vertex from $V(G) - \{s, a, a', b_1, c_1\}$ is 5. Then clearly $\deg c_2 > 1$ (by Lemma 1.1). If the remaining vertex x satisfies $d(a', x) \leq 2$ we obtain $d(b'_1, c_2) < 5$, a contradiction. Let $d(a, x) \leq 2$. If a shortest path P between c_2 and c'_2 contains b_2 then we get $e(b_2) \leq 4$, a contradiction. In the opposite case we get $d(c_2, b'_1) < 5$ or $e(a) \leq 3$, a contradiction.

c) Let $c_i c'_j \in E(G)$ for some i, j.

We may assume that the graph G contains the subgraph in Fig. 4.10 (otherwise the eccentricity each of the vertices b_1, b'_1, c_1, c'_1 is at most 4) and moreover $d(b'_1, c_2) = 5$. We distinguish two subcases.

c_1) The graph G contains the subgraph in Fig. 4.5 satisfying $d(b_1, c'_2) = 5$.

Let $\deg c_2 = \deg c'_2 = 1$. If $b_2 b'_2 \in E(G)$, then $e(a) \leq 3$. In the opposite case (i.e. $b_2 b'_2 \notin E(G)$) we denote by v a vertex of a shortest path between b_2 and b'_2 satisfying $v \neq b_2$, $v \neq b'_2$. Since $d(b_2, b'_2) \leq 3$ (it follows from $d(c_2, c'_2) \leq 5$) we have $v \neq s$ and $e(v) \leq 3$, a contradiction.

The inequality $\deg c_2 > 1$ implies $d(b'_1, c_2) \leq 4$ (and similarly for c'_2), a contradiction.

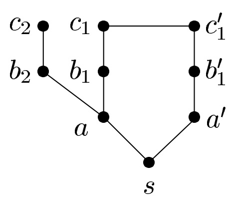


Fig. 4.10

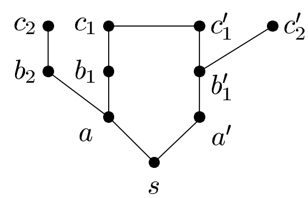


Fig. 4.11

c_2) In the opposite case to c_1) the graph G contains the subgraph in Fig. 4.11 (note that $b'_1 c'_2 \notin E(G)$ gives $e(a) \leq 3$) and the eccentricity of every vertex from $V(G) - \{s, a, a', b_1, c_1\}$ is 5.

Firstly, let $d(a', x) \leq 2$ for the remaining vertex x. The equality $\deg c_2 = 1$

yields $e(b_2) = 4$ (by Lemma 1.3), a contradiction. If $\deg c_2 > 1$ then $d(b_1', c_2) \le 4$, a contradiction.

Secondly, let $d(a, x) \le 2$. If $\deg c_2' = 1$ then $e(b_1') = 4$ (by Lemma 1.3), a contradiction. On the other hand, if $\deg c_2' > 1$ then $e(a) \le 3$ or $e(b_1) \le 3$, a contradiction.

II. Let $\deg s \ge 3$.

a) Suppose that $\deg s = 4$.

In this case we have $|B| + |C| = 6$ and the eccentricity of every vertex from $B \cup C$ is 5. Denote by H the subgraph of G induced by the set $B \cup C$. All vertices from C belong to the same component H_1 of H by Lemmas 4.1 and 4.2. We can assume (with respect to Lemma 4.2) that $b_1, b_2 \in V(H_1)$. Since $e(b_1) = 5$ we may suppose (without loss of generality) that $d(b_1, c_1) = 5$. Therefore, $|V(H_1)| = 6$ and a shortest path between b_1 and c_1 (in H_1) contains every vertex from $B \cup C$. This yields $e(b_2) \le 4$, a contradiction.

b) Suppose that $\deg s = 3$.

Then $|B| + |C| = 7$ and the eccentricity of at most one vertex from B is 4. By Lemma 4.2 the degree of at most one vertex from C is 1. We get (by Lemma 4.1) $|C| > 2$ and moreover if $|C| = 3$ then $|B| \ge 5$ (a contradiction).

Therefore, we may assume that $|C| \ge 4$. We distinguish two subcases.

1) Let $|C| = 4$, $|B| = 3$.

Consider the subgraph $H = \langle c_1, c_2, c_3, c_4, b_1, b_2, b_3 \rangle$ of G. There are two possibilities.

A) All vertices from C belong to the same component H_1 of H.

Firstly, if $|V(H_1)| = 7$ then the eccentricities of at least 3 vertices from $V(H_1)$ are at most 4 (in H_1 and so in G, too), a contradiction.

Secondly, if (for example) $b_3 \notin V(H_1)$ then $e(b_1) \le 4$ and $e(b_2) \le 4$, a contradiction.

B) In the opposite case to A) the vertices c_1, c_2, c_3, c_4 belong to two components H_1, H_2 of the graph H (by Lemma 4.2). We may assume, without loss of generality, that $b_1, c_1 \in V(H_1)$ and $b_2, b_3, c_4 \in V(H_2)$. This yields $e(b_1) = 4$ by Lemma 4.2. A shortest path between c_1 and c_4 (in G) contains the vertex b_2 or b_3 and so $e(b_2) \le 4$ or $e(b_3) \le 4$, a contradiction.

2) Let $|C| = 5$, $|B| = 2$.

By Lemmas 2.1 and 4.2 the graph $H = \langle c_1, \ldots, c_5, b_1, b_2 \rangle$ is connected and so $e(b_1) \le 4$ and $e(b_2) \le 4$.

The set B has at least two elements by Lemma 2.1 and so the proof is complete. $\square$

Theorem 5.1. *There are exactly 13 minimal eccentric sequences with least eccentricity three, namely*

$S_1:$	3^6	$S_7:$	$3, 4^2, 5^{12}$
$S_2:$	$3^5, 4^2$	$S_8:$	$3, 4^3, 5^9$
$S_3:$	$3^4, 4^4$	$S_9:$	$3, 4^4, 5^7$
$S_4:$	$3^3, 4^6$	$S_{10}:$	$3, 4^5, 5^4$
$S_5:$	$3^2, 4^8$	$S_{11}:$	$3, 4^7, 5^2$
$S_6:$	$3, 4^{10}$	$S_{12}:$	$3^2, 4^2, 5^2$

$$S_{13}: \quad 3, 4^2, 5^2, 6^2$$

Proof. The sequence S_1 is the eccentric sequence of the graph in Fig 5.1. We will show that it is minimal. Let G be a connected graph with $|V(G)| \leq 5$ and let H be its spanning tree. Let a vertex v belong to the centre of H. It is obvious that $e_H(v) \leq 2$ and so $e_G(v) \leq 2$.

The sequences S_2, S_3, S_4, S_5 and S_6 are the eccentric sequences of the graphs in Figs. 5.2, 5.3, 5.4, 5.5 and 2.5, respectively.

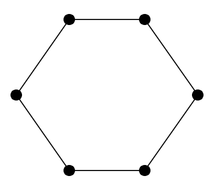


Fig. 5.1

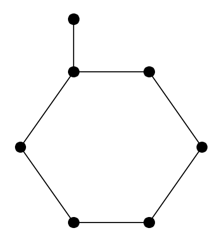


Fig. 5.2

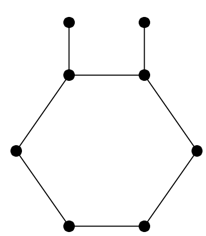


Fig. 5.3

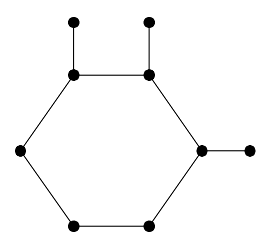


Fig. 5.4

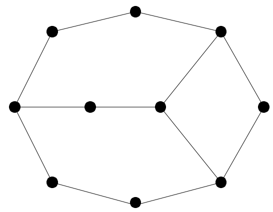


Fig. 5.5

By Theorems 1.1, 1.2, Lemmas 3.7, 3.6, 3.5 and Corollary 2.5 the eccentic sequences S_2, S_3, S_4, S_5 and S_6 are minimal and there are no other minimal eccentric sequences of type $3^k, 4^l$.

S_7, S_8, S_9, S_{10}, S_{11} and S_{12} are the eccentric sequences of the graphs in Figs. 4.1, 5.6, 5.7, 2.3, 2.4 and 5.8.

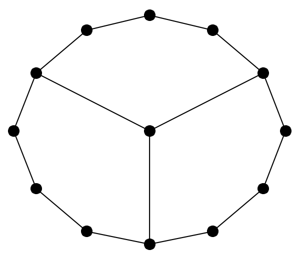


Fig. 5.6

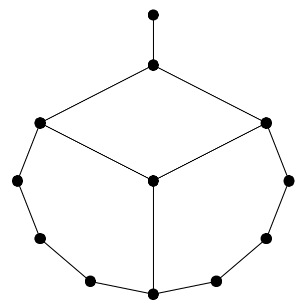


Fig. 5.7

Fig. 5.8

Fig. 5.9

By Theorems 1.1, 1.2, Lemmas 4.3, 4.4, 4.5 and Corollaries 2.6, 2.7 the eccentric sequences S_7, S_8, S_9, S_{10}, S_{11} and S_{12} are minimal and there are no other minimal eccentric sequences of type $3^k, 4^l, 5^r$.

S_{13} is the eccentric sequence of the graph in Fig 5.9. By Theorem 1.1 it is minimal $\square$

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AFFINE COMPLETENESS OF KLEENE ALGEBRAS II

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Abstract. A characterization of affine complete algebras in the variety of all Kleene algebras was given in [8]. Also local polynomial functions of Kleene algebras and locally affine complete algebras were characterized there. In this paper alternative proofs to these three main results of [8] are presented. Also examples illustrated the results are given.

1. Introduction

A *polynomial function* of an algebra A is a function that can be obtained by composition of the basic operations of A, the projections and the constant functions. *A local polynomial function* of A is a function which can be represented by a polynomial function on any finite subset of its domain. A well-known fact about polynomial and local polynomial functions of any algebra A is that they are *compatible* functions in the following sense: a function $f:\ A^n \to A$ is compatible if, for any congruence θ of A, $(a_i, b_i) \in \theta$, $i = 1, \ldots, n$, implies that

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta.$$

An algebra in which (local) polynomial functions are the only compatible functions is called *(locally) affine complete.* (The concept 'locally affine complete' has sometimes also another meaning in the literature - see e.g. [11].) The problem of characterizing algebras which are affine complete was originally formulated in [6]. Since every algebra is a reduct of an affine complete algebra (for example, of that which contains all its compatible functions among the basic operations) and hence affine complete algebras are in general very diverse, in [3] the problem was reducted into the following formulation: *characterize affine complete algebras in your favourite variety.* Many varieties for which the problem has already been solved are mentioned in [3] or [9].

In [8] we characterized (locally) affine complete algebras in the variety of all Kleene algebras. Previously, only a finite case was entirely solved: a finite Kleene algebra is affine complete if and only if it is a Boolean algebra (see [7]). Moreover, in [8] we characterized locally polynomial functions of Kleene algebras as those which

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preserve the congruences and one important binary relation called 'uncertainty order'.

The aim of this paper is to give alternative proofs to the main three results of the preceding paper [8] which are presented here in Theorems 3.3, 3.4 and 3.9. Close to our considerations are some ideas of the papers [5], [7]-[10] and [12]-[13]. We use several preliminary results of [8] which are summarized in section 2. Our alternative approach to the main results of [8] starts in section 3 with crucial Lemmas 3.1 and 3.2. In addition to [8] we present examples at the end which illustrate the results.

2. Preliminaries

First we recall a few basic facts about Kleene algebras. For more information we refer the reader, for example, to [1] or [2].

A Kleene algebra is an algebra $(K, \vee, \wedge, ', 0, 1)$ where $(K, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, $'$ is a unary operation of complementation and the identities

$$0' = 1\,,\ x'' = x\,,\ (x \vee y)' = x' \wedge y'\,,\ (x \wedge x') \vee (y \vee y') = y \vee y'$$

and their duals are satisfied. Every Boolean algebra is clearly a Kleene algebra, a smallest Kleene algebra which is not Boolean is $\mathbf{3} = \{0, a, 1\}$ with $0 < a < 1$ and $a' = a$. The algebra $\mathbf{3}$ is subdirectly irreducible and generates the variety of Kleene algebras.

Two subsets of a Kleene algebra K often play an important role: a subset $K^{\vee} = \{x \vee x' | x \in K\}$, which is a filter of the distributive lattice K, and a dually defined ideal $K^{\wedge}$. The complementation operation clearly induces an antiisomorphism between $K^{\vee}$ and $K^{\wedge}$. Further, the union $K^{\vee} \cup K^{\wedge}$ is a subalgebra of the Kleene algebra K. The variety of Kleene algebras has the congruence extension property and we have the following lemma.

1.1 Lemma ([8; 1.1])**.** *For every Kleene algebra K, any congruence of the lattice $K^{\vee}$ is a restriction of some congruence of the Kleene algebra K.*

In [7] it was proved that a Kleene algebra K with a finite filter $K^{\vee}$ is (locally) affine complete if and only if it is a Boolean algebra. To characterize affine complete Kleene algebras in general, we will need the following generalization of affine completeness: if A is a subalgebra of an algebra B then A is *affine complete in* B if every compatible function on A can be interpolated by a polynomial of B. (Hence we allow elements of B to be used as constants to represent compatible functions of A.)

We can establish a canonical way of defining any n-ary polynomial function of a Kleene algebra in the following way: to every pair of subsets $\alpha_0, \alpha_1 \subseteq \underline{n} = \{1, \ldots, n\}$ we assign the n-ary Kleene term

$$C_{\alpha}(x_1, \ldots, x_n) = (\bigvee_{i \in \alpha_0} x_i) \vee (\bigvee_{i \in \alpha_1} x_i')\,.$$

From the axioms of Kleene algebras it follows that that every n-ary Kleene polynomial can be represented as a meet of so-called elementary polynomials $k_{\alpha} \vee C_{\alpha}$ where k_{α} are constants from K.

Let K be an affine complete Kleene algebra. Let $g : (K^\vee)^n \to K^\vee$ be a compatible function of the lattice $K^\vee$. We can extend the function g to a compatible function
$f : K^n \to K$ by

$$f(x_1, \ldots, x_n) = g(x_1 \vee x_1', \ldots, x_n \vee x_n') \quad \text{for all } x_1, \ldots, x_n \in K.$$

This function must be polynomial, hence representable as a meet of elementary polynomials of K. One can show that its restriction to $K^\vee$, which is the function g, is therefore a lattice polynomial (obtained by omitting all x_i' in the representation of f). Hence we get:

2.1 Lemma. *Let K be a Kleene algebra. If K is affine complete, then $K^\vee$ and $K^\wedge$ are (as lattices) affine complete in K.*

The following lemma, which is a special case of [7; Theorem 1], can be proved similarly.

2.2 Lemma ([8; 2.2])**.** *If K is a locally affine complete Kleene algebra then the lattices $K^\vee$ and $K^\wedge$ are locally affine complete.* □

To describe situations in which the lattices $K^\vee$ and $K^\wedge$ are affine complete in the lattice K we will use the following two concepts introduced in [12] (see also [9]): a filter F of a distributive lattice L is *almost principal* if for every $x \in L$ the filter $F \cap \uparrow x = \{y \in F \mid y \geq x\}$ is principal, i.e. has a smallest element. An *almost principal ideal* of L is defined dually. Further, a filter or an ideal of L is proper if it is not equal to L while an interval of L is proper if it contains at least two elements.

2.3 Lemma ([8; 2.3])**.** *Let D be a sublattice of a distributive lattice L. Suppose that D is affine complete in L. Then*

($\mathcal{B}$) *D does not contain a proper Boolean interval;*

($\mathcal{F}$) *for every proper almost principal filter F in D there exists $b \in L$ such that $F = D \cap \uparrow b$;*

($\mathcal{I}$) *for every proper almost principal ideal I in D there exists $c \in L$ such that $I = D \cap \downarrow c$.* □

Let us summarize the known results for distributive lattices.

2.4 Theorem.

(1) *A bounded distributive lattice is affine complete if and only if it does not contain a proper Boolean interval* ([5])*.*

(2) *A distributive lattice is locally affine complete if and only if it does not contain a proper Boolean interval* ([4; p. 102])*.*

(3) *A distributive lattice is affine complete if and only if the following conditions are satisfied:*

(i) it does not contain a proper Boolean interval;

(ii) it does not contain a proper almost principal ideal without a largest element;

(iii) it does not contain a proper almost principal filter without a smallest element ([12; 2.7])*.* □

Throughout the paper we assume that the Kleene algebra K is embedded in $\mathbf{3}^I$, a power of the Kleene algebra $\mathbf{3} = \{0, a, 1\}$. Accordingly, we will write the elements of K in the form $x = (x_i)_{i \in I}$. Clearly, $s \in K^\vee$ iff $s_i \in \{a, 1\}$ for every $i \in I$.

As the proofs of the following two lemmas from [8] are very short, we include them here.

2.5 Lemma. *Let $f : K^n \to K$ be compatible, $x_j, y_j \in K$, $j = 1, \ldots, n$ and $i \in I$. Then $x_{1i} = y_{1i}, \ldots, x_{ni} = y_{ni}$ implies $f(x_1, \ldots, x_n)_i = f(y_1, \ldots, y_n)_i$.*

Proof. Consider the compatibility relative to the kernel congruence of the i-th projection. □

For any $s \in K^\vee$ we define the subalgebra K^s of K:

$$K^s = \{x \in K | x \vee x' \geq s\} .$$

2.6 Lemma. *Let $s \in K^\vee$. If two n-ary compatible functions of K coincide on $\{0, s, 1\}^n$ then they coincide on $(K^s)^n$.*

Proof. Let f and g coincide on $\{0, s, 1\}^n$. We prove that

$$f(x_1, \ldots, x_n)_i = g(x_1, \ldots, x_n)_i$$

for every $x_1, \ldots, x_n \in K^s$ and $i \in I$.

First we define for every x_j the element $y_j \in \{0, s, 1\}$ having the same i-th component as x_j:

$$y_j = \begin{cases} 0 & \text{if} \quad x_{ji} = 0; \\ 1 & \text{if} \quad x_{ji} = 1; \\ s & \text{if} \quad x_{ji} = a . \end{cases}$$

Now by Lemma 2.5,

$$f(x_1, \ldots, x_n)_i = f(y_1, \ldots, y_n)_i = g(y_1, \ldots, y_n)_i = g(x_1, \ldots, x_n)_i. \quad \square$$

The *uncertainty order* of a Kleene algebra K is the binary relation $\sqsubseteq$ defined by

$$x \sqsubseteq y \quad \iff \quad x \wedge s \leq y \leq x \vee s' \text{ for some } s \in K^\vee .$$

Hence the uncertainty order on $K = \mathbf{3}$ is the relation

$$\{(0,0), (a,a), (1,1), (0,a), (1,a)\}.$$

This relation on $\mathbf{3}$ can really be found under the name 'uncertainty order' in the literature.

2.7 Lemma ([8; 3.4])**.** *The uncertainty order on K is inherited from the uncertainty order on $\mathbf{3}$, i.e. $x \sqsubseteq y$ iff $x_i \sqsubseteq y_i$ for every $i \in I$.* □

It can easily be seen that $\sqsubseteq$ is indeed a partial order relation on K which is a subalgebra of $K \times K$. Hence every local polynomial function preserves $\sqsubseteq$.

2.8 Lemma ([8; 3.6])**.** *If all compatible functions on the lattice $K^\vee$ are order preserving then all compatible functions on the Kleene algebra K preserve $\sqsubseteq$.* □

3. The results - an alternative approach

In this section we give alternative proofs to the three main results presented in [8]. Our approach is based on calculations presented in the following two lemmas.

3.1 Lemma. *If a compatible function $f: K \to K$ preserves $\sqsubseteq$ then, for every $s, t \in K^\vee$,*

(1) $f(1) \wedge s \leq f(s) \leq f(1) \vee s'$;
(2) $f(0) \wedge s' \leq f(s) \leq f(0) \vee s$;
(3) $f(s') \wedge s' \leq f(s) \leq f(s') \vee f(1)$;
(4) $f(s) \leq s \vee f(t)$;
(5) *if* $s \geq f(0)$ *or* $s \geq f(1)$ *then* $s \geq f(s)$.

Proof. We prove that $f(1)_i \wedge s_i \leq f(s)_i \leq f(1)_i \vee s_i'$ for every $i \in I$. If $s_i = 1$ then $f(s)_i = f(1)_i$ by 2.5. Let $s_i = a$. Now the case $f(s)_i = a$ is trivial, let $f(s)_i \in \{0, 1\}$. Since $1 \sqsubseteq s$, we have $f(1)_i \sqsubseteq f(s)_i$, which is only possible if $f(s)_i = f(1)_i$. Thus, (1) is proved.

(2) is trivial on those components i where $s_i = 1$ or $f(s)_i = a$. The remaining case is $s_i = a$ and $f(s)_i \in \{0, 1\}$. Then, by 2.5, $f(s)_i = f(s')_i$ and from $0 \sqsubseteq s'$ we deduce that $f(0)_i = f(s)_i$.

To see (3), notice that $s_i = a$ implies $f(s')_i = f(s)_i$, while $s_i = 1$ implies $f(s)_i = f(1)_i$.

It is clear that $f(s)_i \leq s_i \vee f(t)_i$ if $s_i = 1$ or $s_i = t_i$ or $f(s)_i \leq a$. The remaining case is $s_i = a$, $f(s_i) = 1$ and $t_i = 1$. Then $1 \sqsubseteq s$ implies that $f(1)_i \sqsubseteq f(s)_i = 1$, hence $f(t)_i = f(1)_i = 1 = f(s)_i$. This proves (4).

(5) follows from (2) and (4). $\square$

3.2. Lemma. *If a compatible function $f: K \to K$ preserves $\sqsubseteq$ then, for every $s \in K^\vee$,*

$$\begin{aligned} f(s) =& (f(s) \wedge f(0) \wedge f(1)) \vee (f(1) \wedge s) \vee ((f(s') \vee f(0) \vee f(1)) \wedge s') = \\ & (f(1) \vee s') \wedge (f(s') \vee f(0) \vee f(1)) \wedge ((f(s) \wedge f(0) \wedge f(1)) \vee s). \end{aligned}$$

Proof. The equality of the last two expressions follows from the distributivity, since $s' \leq s$ and $f(s) \wedge f(0) \wedge f(1) \leq f(1) \leq f(s') \vee f(0) \vee f(1)$.

Obviously, $f(s) \geq f(s) \wedge f(0) \wedge f(1)$. By 3.1 we have $f(s) \geq f(1) \wedge s \geq f(1) \wedge s'$, $f(s) \geq f(0) \wedge s'$ and $f(s) \geq f(s') \wedge s'$, hence $f(s) \geq (f(s) \wedge f(0) \wedge f(1)) \vee (f(1) \wedge s) \vee ((f(s') \vee f(0) \vee f(1)) \wedge s')$.

It remains to prove the inverse inequality. By 3.1, $f(s) \leq f(1) \vee s'$, $f(s) \leq f(s') \vee f(1)$, $f(s) \leq f(0) \vee s$ and $f(s) \leq f(1) \vee s$ and obviously $f(s) \leq f(s) \vee s$, which completes the proof. $\square$

The previous lemma will be used to characterize local polynomial functions of Kleene algebras and consequently also locally affine complete Kleene algebras.

3.3 Theorem ([8; 4.1])**.** *Let f be an n-ary compatible function on a Kleene algebra K. Then the following conditions are equivalent:*

(1) *f is a local polynomial function of K;*
(2) *f preserves the uncertainity order of K;*
(3) *f can be interpolated by a polynomial on K^s for every $s \in K^\vee$.*

Proof. Since $\sqsubseteq$ is a subalgebra of $K \times K$, we have (1) $\Longrightarrow$ (2). Clearly, every finite subset of K is contained in some K^s, $s \in K^\vee$, which yields (3) $\Longrightarrow$ (1). Hence the key implication is (2) $\Longrightarrow$ (3).

By 2.6 it suffices to interpolate f on the set $\{0, s, 1\}$. We proceed by induction on arity n of f. The claim is obviously true for $n = 0$. Suppose now that $n > 0$ and that the implication (2) $\Longrightarrow$ (3) is true for all functions of arity less than n. Hence, the $(n-1)$-ary functions $f(0, x_2, \ldots, x_n)$, $f(s, x_2, \ldots, x_n)$, $f(s', x_2, \ldots, x_n)$, $f(1, x_2, \ldots, x_n)$ (of variables $x_2, \ldots, x_n$) are representable by polynomials $p_0, p_s, p_{s'}, p_1$, respectively. Let us set

$$p(x_1, \ldots, x_n) = (p_s \wedge p_0 \wedge p_1) \vee (p_1 \wedge x_1) \vee (p_0 \wedge x_1') \vee ((p_{s'} \vee p_0 \vee p_1) \wedge x_1 \wedge x_1').$$

We claim that p represents f on $\{0, s, 1\}^n$. Let $x_1, \ldots, x_n \in \{0, s, 1\}$. It is easy to see that
$p(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ whenever $x_1 \in \{0, 1\}$. Finally, for $x_1 = s$ we have $s' \leq s$ and therefore $p(s, x_2, \ldots, x_n) = (p_s \wedge p_0 \wedge p_1) \vee (p_1 \wedge s) \vee ((p_{s'} \vee p_0 \vee p_1) \wedge s')$, which is equal to $f(s, x_2, \ldots, x_n)$ by 3.2. (Apply 3.2 to the unary function g defined by $g(y) = f(y, x_2, \ldots, x_n)$.) $\square$

3.4 Theorem ([8; 4.2]). *Let K be a Kleene algebra. The following are equivalent:*

(1) *K is locally affine complete;*
(2) *$K^\vee$ is a locally affine complete lattice;*
(3) *$K^\vee$ does not contain a proper Boolean interval.*

Proof. The equivalence of (2) and (3) was given by 2.4(2). We stated (1) $\Longrightarrow$ (2) in 2.2. By (2), every compatible function of the lattice $K^\vee$ is order preserving and by 2.8 and the previous theorem, every compatible function on the Kleene algebra K is a local polynomial function. $\square$

Before characterizing affine complete Kleene algebras in general we can already state the following special result.

3.5 Proposition ([8; 4.3]). *Let K be a Kleene algebra such that $K^\vee$ has a smallest element. The following are equivalent:*

(1) *K is affine complete;*
(2) *$K^\vee$ is an affine complete lattice;*
(3) *$K^\vee$ does not contain a proper Boolean interval.*

Proof. The equivalence of (2) and (3) was given by 2.4(1). The implication (1) $\Longrightarrow$ (3) follows the fact that every affine complete algebra is locally affine complete and from Theorem 3.4. If (3) holds, then the algebra K is locally affine complete by 3.4 and hence by 2.6 every compatible function of K can be interpolated by a polynomial function on any K^s. But clearly $K = K^s$ where s is the smallest element of $K^\vee$, which completes the proof. $\square$

For a subset Y of an ordered set X we denote $\uparrow Y = \{x \in X \mid x \geq y \text{ for some } y \in Y\}$ and $\downarrow Y = \{x \in X \mid x \leq y \text{ for some } y \in Y\}$.

3.6 Lemma **([8; 5.1]).** *Let $f : K \longrightarrow K$ be a local polynomial function of a Kleene algebra K. Then $K^\vee \cap \uparrow f(K^\vee)$ is an almost principal filter in $K^\vee$ and $K^\wedge \cap \downarrow f(K^\wedge)$ is an almost principal ideal in $K^\wedge$.*

Proof. Denote $F = K^\vee \cap \uparrow f(K^\vee) = \{x \in K^\vee \mid f(z) \le x \text{ for some } z \in K^\vee\}$. We show that, for $x \in K^\vee$,

$$x \vee f(x) = \min\{y \in F \mid x \le y\}.$$

Clearly, $x \le x \vee f(x) \in F$. Conversely, let $x \le y \in F$. Then $y \ge f(z)$ for some $z \in K^\vee$. By Lemma 3.1, $f(x) \le x \vee f(z)$, hence $x \vee f(x) \le x \vee f(z) \le y$.

It remains to show that F is closed under meets. Let $x, y \in F$, $z = x \wedge y$, $t = \min\{u \in F \mid z \le u\}$. Then $z \le t \le x$, $t \le y$, thus $z = t \in F$. We showed that F is an almost principal filter in $K^\vee$.

The other statement can be proved dually. □

Let P denote the set of all pairs $\alpha = (\alpha_0, \alpha_1)$ with $\alpha_0, \alpha_1 \subseteq \underline{n}$, $\alpha_0 \cap \alpha_1 = \emptyset$. We introduce an order relation on P by $\alpha \le \beta$ iff $\alpha_0 \subseteq \beta_0$ and $\alpha_1 \subseteq \beta_1$.

Suppose now that a Kleene algebra K satisfies the following conditions:

($\mathcal{B}$) $K^\vee$ does not contain a proper Boolean interval;

($\mathcal{F}$) for every proper almost principal filter F in $K^\vee$ there exists $b \in K$ such that $F = K^\vee \cap \uparrow b$.

Since $'$ is a dual automorphism of the lattice K, ($\mathcal{F}$) is equivalent to the dual condition

($\mathcal{I}$) for every proper almost principal ideal I in $K^\wedge$ there exists $c \in K$ such that $I = K^\wedge \cap \downarrow c$.

Let $f : K^n \to K$ be a compatible function. By ($\mathcal{B}$) and 3.4, f is a local polynomial function. For every $\alpha \in P$ we define a unary function $f_\alpha : K \to K$ by the rule

$$f_\alpha(y) = f(x_1, \ldots, x_n), \text{ where } x_i = \begin{cases} 0 & \text{if } i \in \alpha_0; \\ 1 & \text{if } i \in \alpha_1; \\ y & \text{otherwise.} \end{cases}$$

It is clear that the functions f_α are compatible. Therefore by ($\mathcal{F}$) and ($\mathcal{I}$) we have constants b_α, c_α such that

$$(*) \qquad \begin{aligned} K^\vee \cap \uparrow f_\alpha(K^\vee) &= K^\vee \cap \uparrow b_\alpha; \\ K^\wedge \cap \downarrow f_\alpha(K^\wedge) &= K^\wedge \cap \downarrow c_\alpha. \end{aligned}$$

From the proof of Lemma 3.6 we see that

$$\begin{aligned} x \vee f_\alpha(x) &= \min\{y \in K^\vee \cap \uparrow f_\alpha(K^\vee) \mid x \le y\} = b_\alpha \vee x; \\ z \wedge f_\alpha(z) &= \max\{y \in K^\wedge \cap \downarrow f_\alpha(K^\wedge) \mid z \ge y\} = c_\alpha \wedge z \end{aligned}$$

for every $x \in K^\vee$, $z \in K^\wedge$.

3.7 Lemma. *If $\alpha \leq \beta$ then $K^{\vee} \cap \uparrow f_{\alpha}(K^{\vee}) \supseteq K^{\vee} \cap \uparrow f_{\beta}(K^{\vee})$.*

Proof. It suffices to deal with the case when $(\beta_0 \cup \beta_1) \setminus (\alpha_0 \cup \alpha_1)$ is a one-element set, say, $\{j\}$. Let $x \in K^{\vee} \cap \uparrow f_{\beta}(K^{\vee})$. Then $x \geq f_{\beta}(y)$ for some $y \in K^{\vee}$ and by 3.1 we have $x \geq f_{\beta}(y) \vee x \geq f_{\beta}(x)$. Let us define a unary compatible function g as follows:

$$g(y) = f(x_1, \dots, x_n), \text{ where } x_i = \begin{cases} 0 & \text{if} \quad i \in \alpha_0; \\ 1 & \text{if} \quad i \in \alpha_1; \\ y & \text{if} \quad i = j; \\ x & \text{otherwise.} \end{cases}$$

If $j \in \beta_0$ then $g(0) = f_{\beta}(x)$. If $j \in \beta_1$ then $g(1) = f_{\beta}(x)$. Hence, either $x \geq g(0)$ or $x \geq g(1)$. By 3.1(5) then $x \geq g(x) = f_{\alpha}(x)$, hence $x \in K^{\vee} \cap \uparrow f_{\alpha}(K^{\vee})$. □

3.8 Lemma. *The constants b_{α}, c_{α} in (*) can be chosen in such a way that*

(i) *if $\alpha_0 \cup \alpha_1 = \underline{n}$ then both b_{α} and c_{α} are equal to the value of the constant function f_{α};*
(ii) *if $\alpha \leq \beta$ then $b_{\alpha} \leq b_{\beta} \leq c_{\beta} \leq c_{\alpha}$.*

Proof. If $\alpha_0 \cup \alpha_1 = \underline{n}$ then f_{α} is a constant function equal to some $k \in K$. We set $b_{\alpha} = c_{\alpha} = k$. Clearly, (*) is satisfied.

Let b_{α}, c_{α} be arbitrary elements satisfying (*). We set $b_{\alpha}^1 = \bigwedge_{\beta \geq \alpha} b_{\beta}$, $c_{\alpha}^1 = \bigvee_{\beta \geq \alpha} c_{\beta}$. Now the constants b_{α}^1, c_{α}^1 fulfil (ii) (notice that for $\beta_0 \cup \beta_1 = \underline{n}$ we have $b_{\beta} = c_{\beta}$) and it remains to show that (*) is valid when we replace b_{α}, c_{α} by b_{α}^1, c_{α}^1.

For any $x, y \in K$ we have $K^{\vee} \cap \uparrow(x \wedge y) = (K^{\vee} \cap \uparrow x) \vee (K^{\vee} \cap \uparrow y)$, i.e. $K^{\vee} \uparrow(x \wedge y)$ is the least filter containing both $K^{\vee} \cap \uparrow x$ and $K^{\vee} \cap \uparrow y$. By induction we obtain that, for any $\alpha \in P$,

$$K^{\vee} \cap \uparrow b_{\alpha}^1 = \bigvee_{\alpha \leq \beta} K^{\vee} \cap \uparrow b_{\beta} = \bigvee_{\alpha \leq \beta} K^{\vee} \cap \uparrow f_{\beta}(K^{\vee}) = K^{\vee} \cap \uparrow f_{\alpha}(K^{\vee})$$

using Lemma 3.7. Hence, the elements b_{α}^1 fulfil (*). The proof for c_{α}^1 is analogous. □

3.9 Theorem. *Let K be a Kleene algebra. The following conditions are equivalent:*

(1) *K is affine complete;*
(2) *$K^{\vee}$ is affine complete in K;*
(3) *$K^{\wedge}$ is affine complete in K;*
(4) *$K^{\vee}$ does not contain proper Boolean intervals and for every proper almost principal filter F in $K^{\vee}$ there exists $b \in K$ such that $F = K^{\vee} \cap \uparrow b$.*
(5) *$K^{\wedge}$ does not contain proper Boolean intervals and for every proper almost principal ideal I in $K^{\wedge}$ there exists $c \in K$ such that $F = K^{\wedge} \cap \downarrow c$.*

Proof. The existence of the dual automorphism $'$ for the lattice K yields that the conditions (2) and (3) and similarly the conditions (4) and (5) are equivalent. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) follow from Lemmas 2.1 and 2.3. So we have to prove only the implication (4) $\Longrightarrow$ (1). Let K be a Kleene algebra satisfying (4) and $f : K^n \longrightarrow K$ a compatible function. Hence, we have the constants b_α, c_α that satisfy (*) and 3.8(i),(ii). For $\alpha \in P$ we define polynomials C_α, D_α by the rule

$$C_\alpha = \bigvee_{j\in\alpha_0} x_j \vee \bigvee_{j\in\alpha_1} x'_j; \qquad D_\alpha = \bigvee_{j\in\underline{n}\setminus\alpha_1} x_j \vee \bigvee_{j\in\underline{n}\setminus\alpha_0} x'_j.$$

Let us set

$$p(x_1,\dots,x_n) = \bigwedge_{\alpha\in P} (c_\alpha \vee C_\alpha) \wedge \bigwedge_{\alpha\in P} (b_\alpha \vee D_\alpha).$$

To prove that p represents f, it suffices to show that $p(x_1,\dots,x_n) = f(x_1,\dots,x_n)$ for $x_1,\dots,x_n \in \{0,s,1\}$, where s is an arbitrary element of $K^\vee$. Without loss of generality, $x_1 = \dots = x_k = 0$, $x_{k+1} = \dots = x_l = s$ and $x_{l+1} = \dots = x_n = 1$. Let us denote $\beta = (\underline{k}, \underline{n}\setminus\underline{l})$, $\gamma = (\underline{k}, \underline{n}\setminus\underline{k})$. If $k = l$ then $p(x_1,\dots,x_n) = c_\beta = f(x_1,\dots,x_n)$ by 3.8. Suppose that $k < l$. We claim that

$$p(x_1,\dots,x_n) = c_\beta \wedge (b_\beta \vee s) \wedge (b_\gamma \vee s').$$

Clearly, $p(x_1,\dots,x_n) \le c_\beta \wedge (b_\beta \vee s) \wedge (b_\gamma \vee s')$, because $C_\beta = 0$, $D_\beta = s$ and $D_\gamma = s'$. The other inequality follows from the facts that

$$c_\alpha \vee C_\alpha \ge \begin{cases} C_\alpha = 1 & \text{if } \alpha_0 \not\subseteq \underline{l} \text{ or } \alpha_1 \not\subseteq \underline{n}\setminus\underline{k}; \\ c_\beta & \text{if } \alpha \le \beta; \\ c_\alpha \vee s' \ge c_\gamma \vee s' = b_\gamma \vee s' & \text{if } \gamma \ge \alpha \not\le \beta; \\ c_\alpha \vee s \ge c_{\alpha\cup\beta} \vee s \ge b_{\alpha\cup\beta} \vee s \ge b_\beta \vee s & \text{otherwise} \end{cases}$$

and

$$b_\alpha \vee D_\alpha \ge \begin{cases} D_\alpha = 1 & \text{if } \beta \not\le \alpha; \\ b_\alpha \vee s' = b_\gamma \vee s' & \text{if } \beta \le \alpha = \gamma; \\ b_\alpha \vee s \ge b_\beta \vee s & \text{if } \beta \le \alpha \ne \gamma. \end{cases}$$

We wish to show that $f(x_1,\dots,x_n) = f_\beta(s) = p(x_1,\dots,x_n)$. We have $c_\beta \ge c_\gamma = f_\beta(1)$ and also $c_\beta \ge f_\beta(s') \wedge s'$, thus, by 3.1, $c_\beta \ge (f_\beta(1) \vee f_\beta(s')) \wedge (f_\beta(1) \vee s') \ge f_\beta(s)$. Further, $b_\beta \vee s = f_\beta(s) \vee s \ge f_\beta(s)$ and $b_\gamma \vee s' = f_\beta(1) \vee s' \ge f_\beta(s)$ by 3.1. Hence, $f_\beta(s) \le p(x_1,\dots,x_n)$.

By the distributivity (using inequalities $b_\beta \le b_\gamma = c_\gamma \le c_\beta$) we can write

$$p(x_1,\dots,x_n) = b_\beta \vee (b_\gamma \wedge s) \vee (c_\beta \wedge s').$$

Using the equalities $b_\beta \vee s = f_\beta(s) \vee s$, $c_\beta \wedge s' = f_\beta(s') \wedge s'$ and the inequalities from 3.1 we have $b_\beta \le (f_\beta(s) \vee s) \wedge f_\beta(1) = (f_\beta(s) \wedge f_\beta(1)) \vee (s \wedge f_\beta(1)) \le f_\beta(s)$, $b_\gamma \wedge s = f_\beta(1) \wedge s \le f_\beta(s)$ and $c_\beta \wedge s' \le f_\beta(s)$. Hence, $p(x_1,\dots,x_n) \le f_\beta(s)$. □

3.10 Examples.
(1) Let $K_1 = \{(-\infty,-\infty)\} \cup R \times R \cup \{(\infty,\infty)\}$ be the Kleene algebra with the complementation defined by $(x,y)' = (-x,-y)$. Then $K_1^\vee = \{(x,y) \in R \times R \mid x \ge 0, y \ge 0\} \cup \{(\infty,\infty)\}$ is obviously an affine complete distributive lattice by 2.4(1). Hence $K_1^\vee$ is affine complete in the lattice K_1 and by 3.9 (or straightforward by 3.5) the Kleene algebra K_1 is affine complete.

Note that for the same reason, the Kleene algebra $K_1^\wedge \oplus K_1^\vee$, where $\oplus$ means the linear sum, is affine complete. In general, for every affine complete distributive lattice D, the Kleene algebra $D \oplus D^d$ is affine complete where D^d is a dual of D and the complementation operation is the antiisomorphism between D and D^d.

(2) Let $K_2 = K_1 \setminus \{(0,y) \mid y \in R\}$ be a subalgebra of K_1. Then $K_2^\vee = \{(x,y) \in R \times R \mid x > 0, y \geq 0\} \cup \{(\infty,\infty)\}$ is not, according to 2.4(3), an affine complete lattice because $F = \{(x,y) \in R \times R \mid x > 0, y \geq 1\} \cup \{(\infty,\infty)\}$ is a proper almost principal filter in $K_2^\vee$ without a smallest element. One can verify that the unary function $g : K_2^\vee \to K_2^\vee$ given by $g(x) = \min\{y \in F \mid x \leq y\}$ is a compatible function of the lattice $K_2^\vee$ but cannot be represented by a polynomial function of $K_2^\vee$ (see a similar verification in [12; 2.2]). However, note that there is an element b in K_2, for example, $b = (-1,1)$ such that $F = K_2^\vee \cap \uparrow b$. It can easily be seen that the condition (4) of 3.9 is satisfied, hence again, $K_2^\vee$ is affine complete in the lattice K_2 and the Kleene algebra K_2 is affine complete.

(3) Let $K_3 = K_2 \setminus \{(x,y) \in R \times R \mid x \cdot y < 0\}$ be a Kleene subalgebra of K_2. Then $K_3^\vee = \{(x,y) \in R \times R \mid x > 0, y \geq 0\} \cup \{(\infty,\infty)\} = K_2^\vee$ is again not an affine complete lattice. But note that for the almost principal filter without a smallest element F defined in (2) there is now no element $b \in K_3$ such that $F = K_3^\vee \cap \uparrow b$. Hence $K_3^\vee$ is not affine complete in the lattice K_3 and the Kleene algebra K_3 is not affine complete. It can be verified that the unary function $f : K_3 \to K_3$ given by $f(x) = \min\{y \in F \mid x \vee x' \leq y\}$ is a compatible function of the Kleene algebra K_3 but cannot be represented by a polynomial of K_3.

However, by 3.4 it is clear that K_3 is a locally affine complete Kleene algebra.

(4) Every finite Kleene algebra which is not a Boolean algebra is not affine complete. □

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DUALITY OF BOUNDED DISTRIBUTIVE q-LATTICES

Ivan Chajda and Miroslav Ploščica

Abstract. By a q-lattice is meant an algebra with two binary operations satisfying all normal lattice identities. We establish a duality in the sense of B. Davey and H. Werner for the quasivarieties of constant q-lattices and bounded distributive q-lattices.

1. Introduction

Let p, q be terms of the same similarity type. An identity $p = q$ is called *normal* (see [3], [7], [8], [9]) if it is either of the form $x = x$ (x is a variable) or none of p, q is equal identically to a single variable. So the lattice idempotence or absorption are not normal identities.

An algebra $\mathcal{A} = (A; \vee, \wedge)$ of type $(2,2)$ is called a *q-lattice* if it satisfies all normal identities of lattices. In fact, see [1], [2], [3], $\mathcal{A}$ is a q-lattice if it satisfies the following identities:

(commutativity)	$x \vee y = y \vee x$	$x \wedge y = y \wedge x$
(associativity)	$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(weak idempotence)	$x \vee (y \vee y) = x \vee y$	$x \wedge (y \wedge y) = x \wedge y$
(weak absorption)	$x \vee (x \wedge y) = x \vee x$	$x \wedge (x \vee y) = x \wedge x$
(equalization)	$x \vee x = x \wedge x\,.$	

A q-lattice $\mathcal{A}$ is *distributive* if it satisfies the distributive identity:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

(which is equal to its dual similarly as in the case of lattices). In every q-lattice $\mathcal{A} = (A; \vee, \wedge)$ we can introduce a binary relation Q as follows:

$$(a, b) \in Q \quad \text{if and only if} \quad a \wedge b = a \wedge a\,.$$

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It is easy to show that it is equivalent to $a \vee b = b \vee b$ and, moreover, Q is a *quasiorder* on A (i.e. a reflexive and transitive relation). A q-lattice $\mathcal{A}$ is a lattice if and only if Q is an order on A; in such a case, Q is the lattice order of $\mathcal{A}$.

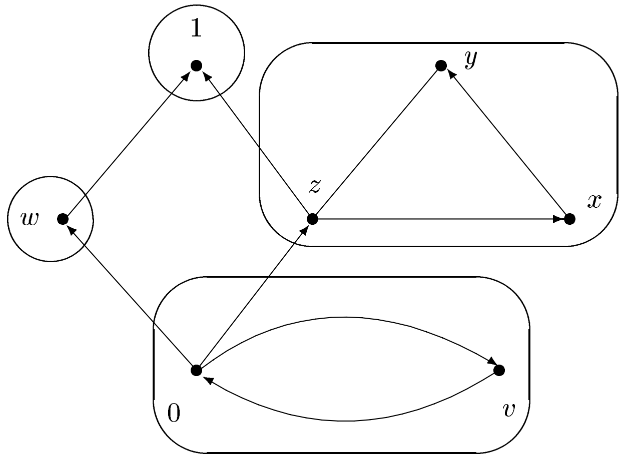

Fig. 1

Example. The following quasiordered set $\{0, x, y, z, v, w, 1\}$ is a distributive q-lattice which is not a lattice (the fact $(a, b) \in Q$ is visualized in Fig. 1 by a connected path of arrows from a to b):

Here e.g. $(0, v) \in Q$ and $(v, 0) \in Q$, $(w, 1) \in Q$, $(1, w) \notin Q$ etc. The operation tables for $\vee$ and $\wedge$ are as follows:

$\wedge$	0	v	w	x	y	z	1
0	0	0	0	0	0	0	0
v	0	0	0	0	0	0	0
w	0	0	w	0	0	0	w
x	0	0	0	z	z	z	z
y	0	0	0	z	z	z	z
z	0	0	0	z	z	z	z
1	0	0	w	z	z	z	1

	0	v	w	x	y	z	1
0	0	0	w	z	z	z	1
v	0	0	w	z	z	z	1
w	w	w	w	1	1	1	1
x	z	z	1	z	z	z	1
y	z	z	1	z	z	z	1
z	z	z	1	z	z	z	1
1	1	1	1	1	1	1	1

An element b of a q-lattice $\mathcal{A} = (A; \vee, \wedge)$ is called an *idempotent* if $b \vee b = b$ (or, equivalently, $b \wedge b = b$). The set of all idempotents of $\mathcal{A}$ is called the *skeleton* of $\mathcal{A}$ and it is denoted by $Sk\,\mathcal{A}$. It is easy to see that $Sk\,\mathcal{A}$ is the maximal sublattice of $\mathcal{A}$.

Evidently, the restriction of Q onto $Sk\,\mathcal{A}$ is the lattice order of $Sk\,\mathcal{A}$. Moreover, see [1], the q-lattice $\mathcal{A}$ is distributive if and only if $Sk\,\mathcal{A}$ is a distributive lattice.

Further, we introduce a binary relation ψ on A by $(a,b) \in \psi$ iff $(a,b) \in Q$ and $(b,a) \in Q$. Then (see [4] or [3]), ψ is a congruence on $\mathcal{A}$ and $\mathcal{A}/\psi \cong Sk\,\mathcal{A}$. The congruence classes of ψ are called *cells* of $\mathcal{A}$.

In the foregoing Example, the elements $0, w, z, 1$ are idempotents and the skeleton $Sk\,\mathcal{A}$ is the four element distributive lattice $\{0, z, w, 1\}$. $\mathcal{A}$ has four cells and the congruence ψ is shown in Fig. 1.

Hence, every cell of $\mathcal{A}$ contains exactly one idempotent (idempotents are the only results of operations). If d is the idempotent of a cell C of $\mathcal{A}$, then $x \vee y = d = x \wedge y$ for any $x, y \in C$. For more details of q-lattices, see e.g. [1], [2] and [4].

Let us recall some necessary concepts of duality theory given by B. Davey and H. Werner [6]. Let $\mathcal{V} = ISP(\underline{P})$ be a quasivariety generated by a non–trivial finite algebra $\underline{P} = (P; F)$. Let $\underset{\sim}{P} = (P; G, H, R, \tau)$ where τ is the discrete topology on P and G is the set of operations, H is a set of partial operations and R is a set of relations on P. Suppose that all those operations, partial operations and relations are subalgebras of appropriate powers of $\underline{P}$. In this case, $\underset{\sim}{P}$ is called *algebraic over* $\underline{P}$.

Let $\mathcal{W} = IS_cP(\underset{\sim}{P})$ be the class of all topological structures of the same type as $\underset{\sim}{P}$ which are isomorphic (i.e. simultaneously isomorphic and homeomorphic) to a closed substructure of a power of $\underset{\sim}{P}$. For every $\mathcal{A} \in \mathcal{V}$ the set $D(\mathcal{A})$ of all homomorphisms $\mathcal{A} \to \underline{P}$ is a closed substructure of $\underset{\sim}{P}^A$, hence $D(\mathcal{A}) \in \mathcal{W}$. Similarly, for each $X \in \mathcal{W}$ the set $E(X)$ of all morphisms $X \to \underset{\sim}{P}$ (i.e. continuous maps that preserve G, H and R) is a subalgebra of $\underline{P}^X$, hence $E(X) \in \mathcal{V}$. (See Lemmas 1.1 and 1.2 in [5].) We have thereby defined two contravariant hom-functors

$$D:\ \mathcal{V} \longrightarrow \mathcal{W}, \qquad E:\ \mathcal{W} \longrightarrow \mathcal{V},$$

which are adjoint to each other. Further, for every $a \in A$, the *evaluation mapping* $e_a : D(\mathcal{A}) \to \underset{\sim}{P}$ given by

$$e_a(x) = x(a) \quad \text{for each} \quad x \in D(A)$$

is a morphism. Similarly, the evaluation mapping $\varepsilon_x : E(X) \to \underline{P}$ given by $\varepsilon_x(\alpha) = \alpha(x)$ for each $\alpha \in E(X)$ is a homomorphism for each $x \in X$. The natural maps $e:\ \mathcal{A} \to ED(\mathcal{A})$ and $\varepsilon:\ X \to DE(X)$ given by evaluation (i.e. $e(a) = e_a$, $\varepsilon(x) = \varepsilon_x$) are embeddings for every $\mathcal{A} \in \mathcal{V}$, $X \in \mathcal{W}$.

Definition. If for all $\mathcal{A} \in \mathcal{V}$ the map e is an isomorphism (equivalently: if evaluation mappings are the only morphisms $D(\mathcal{A}) \to \underset{\sim}{P}$), we say that $\underset{\sim}{P}$ *yields a duality* on $\mathcal{V}$. If moreover, the map ε is an isomorphism for each $X \in \mathcal{W}$, the duality is called *full*.

Our aim is to establish a duality of this type for the quasivariety of all bounded distributive q-lattices. We shall use the well-known Priestley duality for bounded distributive lattices (see [11], [12], [13]).

2. Constant q-lattices

A q-lattice $\mathcal{C}$ is called a *constant q-lattice* if it contains exactly one cell, i.e. if $\mathcal{A}$ consists of one cell which equals to A. Hence, a constant q-lattice has the unique idempotent 0 and $x \vee y = 0 = x \wedge y$ for all elements x, y of $\mathcal{A}$.

We consider constant q-lattices as algebras with binary operations $\vee$, $\wedge$ and a nullary operation 0. It is easy to see that constant q-lattices form a quasivariety (in fact, a variety) $\mathcal{V}^* = ISP(\underline{B})$, where $\underline{B}$ is a two-element constant lattice defined on the set $B = \{0, c\}$.

Let us define $\underset{\sim}{B} = (B, \underline{\vee}, \underline{\wedge}, 0, \tau)$, where 0 is a nullary operation, τ is a discrete topology and $\underline{\vee}, \underline{\wedge}$ are the lattice operations derived from the ordering $0 < c$.

It is easy to see that the structure of $\underset{\sim}{B}$ is algebraic over $\underline{B}$. (Notice that $\rho \subseteq B^n$ is a subalgebra of $\underline{B}^n$ iff $(0, 0, ...0) \in \rho$.) Hence, for any $\mathcal{C} = (C; \vee, \wedge, 0) \in \mathcal{V}^*$, the dual space $D(\mathcal{C})$ is the set of all homomorphisms $\mathcal{C} \to \underline{B}$ with the structure inherited from $\underset{\sim}{B}$. This dual space is easy to describe. A map $\mathcal{C} \to \underline{B}$ is a homomorphism iff it preserves 0. Clearly, $(D(\mathcal{C}), \underline{\vee}, \underline{\wedge})$ is isomorphic to the Boolean lattice of all functions $(C \backslash \{0\} \to B)$. (Equivalently, the Boolean lattice of all subsets of $C \backslash \{0\}$.) The space $D(\mathcal{C})$ inherits its topology from the usual product topology of $\underset{\sim}{B}^C$. There is a close connection between this topology and the lattice operations $\underline{\vee}, \underline{\wedge}$. We need the following fact taken from [10].

Lemma 2.1. *Let a sublattice L of $\underset{\sim}{B}^X$ be topologically closed. Then $\bigwedge A \in L$, $\bigvee A \in L$ for every $\emptyset \neq A \subseteq L$.*

Proof. Let $a = \bigwedge A$. Clearly, for $x \in X$, $a(x) = 0$ iff $b(x) = 0$ for some $b \in A$. To prove that $a \in L$, it suffices to show that a belongs to the topological closure of L. The base of open sets of the topology of $\underset{\sim}{B}^X$ consists of all sets of the form

$$S = \{p \in \underset{\sim}{B}^X \mid p(x_1) = \cdots = p(x_m) = 0,\ p(y_1) = \cdots = p(y_n) = c\},$$

where $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$. Suppose that such a set S contains a, hence $a(x_1) = \cdots = a(x_m) = 0$, $a(y_1) = \cdots = a(y_n) = c$. We need to show that $S \cap L \neq \emptyset$. There exist $b_1, \ldots, b_m \in A$ such that $b_i(x_i) = 0$ and $b_i(y_1) = \cdots = b_i(y_n) = c$. If $m = 0$, then clearly $z \in S \cap L$ for arbitrary $z \in A$. If $m \geq 1$, we set $z = \bigwedge_{i=1}^m b_i$. Since the lattice operations in $\underset{\sim}{B}^X$ are pointwise, we have $z \in S$. Since L is a sublattice, we have $z \in L$. Hence, $S \cap L \neq \emptyset$.

Similarly we can prove that $\bigvee A \in L$. $\square$

For any $a \in C$ we have $h_a \in D(\mathcal{C})$ defined by $h_a(a) = c$ and $h_a(x) = 0$ for every $x \neq a$. It is easy to see that this map is an atom in $D(\mathcal{C})$ and every atom has this form.

Theorem 2.2. *The structure $\underset{\sim}{B} = (B, \underline{\vee}, \underline{\wedge}, 0, \tau)$ yields a duality on $\mathcal{V}^*$.*

Proof. Let $\mathcal{C} \in \mathcal{V}$. Let $\delta :\ D(\mathcal{C}) \to \underset{\sim}{B}$ be a morphism. We need to show that δ is the evaluation map for some $a \in C$. If δ is constant 0 then $\delta = e_0$. Let δ be non-constant. Since δ is a lattice homomorphism and $D(\mathcal{C})$ is a Boolean algebra, the set $U = \delta^{-1}(c)$ must be an ultrafilter. Since δ is continuous, the set U is closed. By 2.1 used for $A = U = L$, the ultrafilter U must have a smallest element. This smallest element is some atom h_a of $D(\mathcal{C})$. Therefore, for $h \in D(\mathcal{C})$ we have $\delta(h) = c$ iff $h \in U$ iff $h_a \leq h$ iff $h(a) = c$, which shows that $\delta = e_a$. $\square$

This duality is not full in the sense of [5] because every dual $D(\mathcal{C})$ is an atomic Boolean algebra but $IS_cP(\underset{\sim}{B})$ contains also non–Boolean lattices. It is worth mentioning that our duality is very similar to the duality for sets in [5].

3. Bounded q-lattices

A q-lattice $\mathcal{A}$ is called *bounded* if there exist elements 0 and 1 in $\mathcal{C}$ such that

$$x \wedge 0 = 0 \qquad \text{and} \qquad x \vee 1 = 1$$

are identities of $\mathcal{A}$ and either $0 \neq 1$ or $card\, A = 1$. Let us note that contrary to the case of lattices, it implies neither $x \vee 0 = x$ nor $x \wedge 1 = x$. Further, in a bounded q-lattice, $(0, a) \in Q$ and $(a, 1) \in Q$ for each $a \in A$. Let us note that it can happen also $(b, 0) \in Q$ or $(1, c) \in Q$ for some $b, c \in A$.

We consider bounded distributive q-lattices as algebras with binary operations $\wedge, \vee$ and nullary operations $0, 1$. They form the quasivariety $\mathcal{V} = ISP(\underline{P})$, where $\underline{P}$ is the four element bounded q-lattice visualized in Fig. 2. This follows from the fact that 2-element constant lattice and 2-element lattice are the only subdirectly irreducible distributive q-lattices. (See [4].)

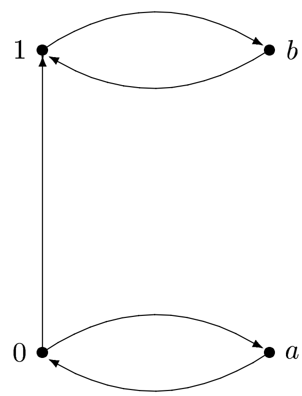

Fig. 2

Let us define the partial orderings $\leq, \leq_a, \leq_b, \leq_{ab}$ and the binary relation T on the set $P = \{0, 1, a, b\}$ by the following rules:
$x \leq y$ iff $x = y$ or $(x, y) = (0, 1)$ or $(x, y) = (a, b)$,
$x \leq_a y$ iff $x = y$ or $(x, y) = (0, a)$,
$x \leq_b y$ iff $x = y$ or $(x, y) = (1, b)$,
$x \leq_{ab} y$ iff $x \leq_a y$ or $x \leq_b y$,
$(x, y) \in T$ iff $\{x, y\} \subseteq \{0, 1\}$ or $\{x, y\} \subseteq \{a, b\}$.

Now we set $\underset{\sim}{P} = (P, \leq, \underline{\vee}, \underline{\wedge}, \leq_a, \leq_b, T, \{0, 1\}, \tau)$, where $\underline{\vee}, \underline{\wedge}$ are partial lattice operations determined by $\leq_{ab}$, $\{0, 1\}$ is a unary relation and τ is the discrete topology.

It is easy to verify that the structure of $\underset{\sim}{P}$ is algebraic over $\underline{P}$. Now we describe the dual space of a bounded distributive q-lattice $\mathcal{C} = (C; \vee, \wedge, 0, 1)$. Let N_C denote the set of non-idempotents of $\mathcal{C}$. Let $\mathcal{C}^*$ be the constant q-lattice defined on the set $\{0\} \cup N_C$ with a new idempotent $0 \notin N_C$. Every homomorphism $f : \mathcal{C} \to \underline{P}$ must map idempotents of $\mathcal{C}$ into $\{0, 1\}$. The restriction $f|_{Sk\, C}$ is a lattice homomorphism.

Let us introduce an equivalence relation Φ on $D(\mathcal{C})$ by $(f,g) \in \Phi$ iff $f|_{Sk\,\mathcal{C}} = g|_{Sk\,\mathcal{C}}$. Let $[f]_\Phi$ denote the equivalence class containing f. It is easy to see that $f \underline{\vee} g$ and $f \underline{\wedge} g$ are defined iff $(f,g) \in \Phi$. In fact, every Φ-equivalence class with $\underline{\vee}$, $\underline{\wedge}$ is a Boolean lattice. Every such class contains a unique *skeletal* homomorphism, i. e. homomorphism that maps whole $\mathcal{C}$ into $\{0,1\}$. Every such skeletal homomorphism is the least element in its equivalence class (with respect to $\underline{\vee}$, $\underline{\wedge}$). Let $D_S(\mathcal{C})$ denote the set of all skeletal members of $D(\mathcal{C})$

Lemma 3.1. *$W = ([h]_\Phi; \underline{\vee}, \underline{\wedge}, h, \tau|_W)$ is the dual space of the constant q-lattice $\mathcal{C}^*$ for every skeletal $h \in D(\mathcal{C})$ (in the sense of the previous section).*

Proof. To every $f \in [h]_\Phi$ we assign $f^* : \mathcal{C}^* \to \underline{B}$ by $f^*(0) = 0$ and (for $x \in N_C$) $f^*(x) = 0$ if $f(x) \in \{0,1\}$ and $f^*(x) = c$ if $f(x) \in \{a,b\}$. This defines an isomorphism of W and the dual space of $\mathcal{C}^*$. □

Similarly, the order relations $\leq_a$, $\leq_b$ compare only elements of the same Φ-equivalence class. On the other hand, the order relation $\leq$ compares only elements of different classes.

Lemma 3.2. *$V = (D_S(\mathcal{C}); \leq, \tau|_V)$ is isomorphic to the Priestley space of the lattice $Sk\,\mathcal{C}$.*

Proof. For every $f \in D_S(\mathcal{C})$ we define $f^* : Sk\,\mathcal{C} \to \{0,1\}$ by $f^* = f|_{Sk\,\mathcal{C}}$. This defines the required isomorphism. □

Hence, we have the following picture of $D(\mathcal{C})$. In the Priestley space of $Sk\,\mathcal{C}$, every point is replaced by the Boolean lattice representing the constant q-lattice $\mathcal{C}^*$. Besides that, we have relations $\leq_a, \leq_b, T$ and $\{0,1\}$, whose role will be explained in the sequel.

Lemma 3.3. *Let $h, k \in D(\mathcal{C})$. Then there is exactly one $g \in D(\mathcal{C})$ such that $(g,h) \in \Phi$, $(g,k) \in T$.*

Proof. If $x \in Sk\,\mathcal{C}$ then we set $g(x) = h(x)$, to ensure that $(g,h) \in \Phi$. Let $x \in N_C$. We set

$$g(x) = \begin{cases} 0 & \text{if } k(x) \in \{0,1\} \text{ and } h(x) \in \{0,a\} \\ 1 & \text{if } k(x) \in \{0,1\} \text{ and } h(x) \in \{1,b\} \\ a & \text{if } k(x) \in \{a,b\} \text{ and } h(x) \in \{0,a\} \\ b & \text{if } k(x) \in \{a,b\} \text{ and } h(x) \in \{1,b\}. \end{cases}$$

It is clear that g has the required properties. □

Theorem 3.4. *Let $\mathcal{V}$ be a quasivariety of all bounded distributive q-lattices. Then $\underset{\sim}{P}$ yields a duality on $\mathcal{V}$.*

Proof. Let $\delta \in ED(\mathcal{C})$. We need to show that δ is an evaluation map. The preservation of the unary relation $\{0,1\}$ means that δ must map skeletal members of $D(\mathcal{C})$ into $\{0,1\}$. By the Priestley duality and 3.2, there is an idempotent $z \in C$ such that $\delta(h) = (h|_{Sk\,\mathcal{C}})(z) = h(z)$ for every skeletal h.

Now, let $h \in D_S(\mathcal{C})$ and let $\mathcal{C}^*$ and W be as in 3.1. The morphism $\delta : D(\mathcal{C}) \to \underset{\sim}{P}$ induces the morphism $\delta' : W \to \underset{\sim}{B}$ by the rule $\delta'(k) = 0$ if $\delta(k) \in \{0,1\}$ and

$\delta'(k) = c$ otherwise. By 3.1, δ' must be an evaluation map, i.e. $\delta' = e_x$ for some $x \in \mathcal{C}^*$. Hence, $\delta'(k) = k^*(x)$ for every $k \in [h]_\Phi$. (See the proof of 3.1.)

If $x \in N_C$, we have

$$\delta'(k) = \begin{cases} 0 \text{ if } k(x) \in \{0,1\} \\ c \text{ otherwise.} \end{cases}$$

In other words,

$$\delta(k) \in \{0,1\} \text{ iff } k(x) \in \{0,1\}. \qquad (*)$$

Equivalently, $(\delta(k), k(x)) \in T$. We claim that this is true for arbitrary $k \in D(\mathcal{C})$, not only for $k \in [h]_\Phi$. Indeed, by 3.3 for every $k \in D(\mathcal{C})$ there is $g \in [h]_\Phi$ with $(g,k) \in T$. Then clearly $(g(x), k(x)) \in T$, $(g(x), \delta(g)) \in T$ and also, since δ preserves T, $(\delta(g), \delta(k)) \in T$. This implies that $(\delta(k), k(x)) \in T$.

Similarly, if $x = 0$, we have $\delta(k) \in \{0,1\}$, which also holds for every $k \in D(\mathcal{C})$.

Now we claim that $\delta = e_x$ if $x \in N_C$ and $\delta = e_z$ if $x = 0$. First we settle the case $x = 0$.

Let $x = 0$ and $k \in D(\mathcal{C})$. Then there is a skeletal $h \in D(\mathcal{C})$ with $(k,h) \in \Phi$. We have $\delta(h), \delta(k) \in \{0,1\}$. Since $h \underline{\vee} k = k$, also $\delta(h) \underline{\vee} \delta(k) = \delta(k)$ which is possible only if both $\delta(h), \delta(k)$ equal to 0 or both of them equal to 1. Since h and k coincide in idempotents, we have

$$\delta(k) = \delta(h) = h(z) = k(z),$$

which was to prove.

Suppose now that $x \in N_C$. Next we prove that x belongs to the cell containing z. For contradiction, suppose that this is not the case. Then there exists $k \in D_S(\mathcal{C})$ such that $k(x) \neq k(z)$. Without loss of generality, $k(x) = 0$, $k(z) = 1$. Let us define $k_1 \in D(\mathcal{C})$ by the rule that $k_1(x) = a$ and $k_1(y) = k(y)$ for all $y \neq x$. Then $k \leq_a k_1$, therefore $\delta(k) \leq_a \delta(k_1)$ which is a contradiction, because $\delta(k) = k(z) = 1$ and $\delta(k_1) \in \{a, b\}$ (since $k_1(x) = a$ and $(\delta(k_1), k_1(x)) \in T$).

Now let $k \in D(\mathcal{C})$ be arbitrary and let $h \in D_S(\mathcal{C})$ be such that $(k,h) \in \Phi$. Again we have $\delta(h) \underline{\vee} \delta(k) = \delta(k)$. We distinguish four cases.

If $\delta(k) = 0$ then $\delta(h) = 0$ (h is skeletal), hence $0 = h(z) = h(x \vee x) = h(x) \vee h(x) = h(x)$ and therefore $k(x) \in \{0, a\}$ (from $(h,k) \in \Phi$). On the other hand, from (*) we have $k(x) \in \{0,1\}$, hence $k(x) = 0 = \delta(k)$.

If $\delta(k) = a$ then $\delta(h) = 0$, which implies $k(x) \in \{0, a\}$. From (*) we have $k(x) \in \{a, b\}$, hence $k(x) = a = \delta(k)$.

If $\delta(k) = 1$ then $\delta(h) = 1$, which implies $h(x) = 1$ and $k(x) \in \{1, b\}$. From (*) we have $k(x) \in \{0,1\}$, hence $k(x) = 1 = \delta(k)$.

Finally, if $\delta(k) = b$ then we get $\delta(h) = 1$, $h(x) = 1$ and $k(x) \in \{1, b\}$. From (*) we have $k(x) \in \{a, b\}$, hence $k(x) = b = \delta(k)$.

Thus, $\delta(k) = k(x)$ holds in all cases, which means that $\delta = e_x$. The proof is complete. $\square$

Let us show how the duality works for the q-lattice $\mathcal{A}$ on Figure 1.

Evidently, $D(\mathcal{A})$ consists of exactly 16 homomorphisms given by the following table:

	0	v	w	z	x	y	1
h_0	0	0	0	1	1	1	1
h_1	0	0	0	1	1	b	1
h_2	0	0	0	1	b	1	1
h_3	0	0	0	1	b	b	1
h_4	0	a	0	1	1	1	1
h_5	0	a	0	1	1	b	1
h_6	0	a	0	1	b	1	1
h_7	0	a	0	1	b	b	1
h_8	0	0	1	0	0	0	1
h_9	0	0	1	0	0	a	1
h_{10}	0	0	1	0	a	0	1
h_{11}	0	0	1	0	a	a	1
h_{12}	0	a	1	0	0	0	1
h_{13}	0	a	1	0	0	a	1
h_{14}	0	a	1	0	a	0	1
h_{15}	0	a	1	0	a	a	1
	e_0	e_v	e_w	e_z	e_x	e_y	e_1

Clearly h_0, h_8 are skeletal homomorphisms, i.e. $D_S(\mathcal{A}) = \{h_0, h_8\}$. This corresponds to the fact that $Sk\,\mathcal{A}$ is the four element Boolean lattice and its Priestley space is the two–element antichain.

The dotted lines denote $\leq_b$ and the solid lines denote $\leq_a$. The equivalence relation T consists of all pairs (h_i, h_{i+8}), $i = 0, \dots, 7$. The equivalence relation Φ has two equivalence classes, which are isomorphic to the dual of the constant q-lattice $\mathcal{A}^*$.

The dual space $D(\mathcal{A})$ looks as shown in Fig. 3.

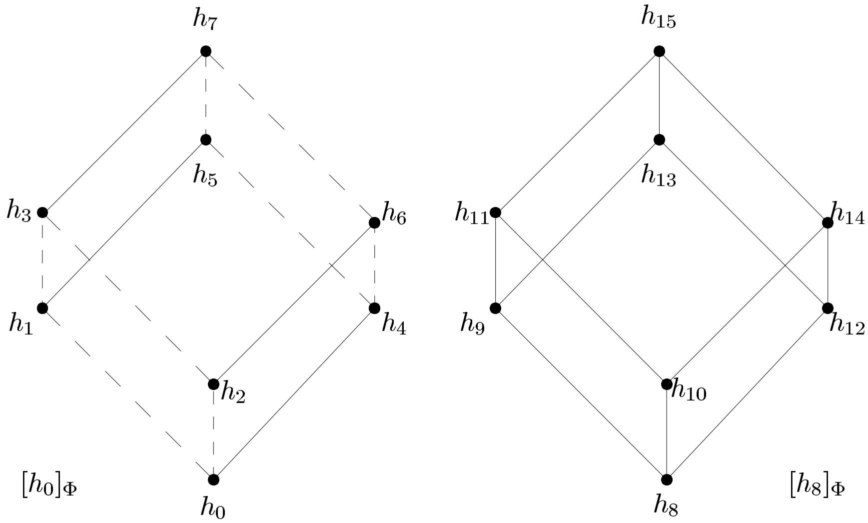

Fig. 3

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HISTORY OF THE NUMBER OF FINITE POSETS

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Abstract. In this paper we introduce the survey of all main known results on the number of finite partially ordered sets. We also present the similar and connected problems. The historical review of related works is also included. In this context there are introduced author's works and results in this branch.

1. Introduction

In the first section we remind the main basic notions and some their relations. Let us denote by N the set of all positive integers and put $N_0 := N \cup \{0\}$. Further, let A be a finite n-element set, $n \in N_0$. By $|A|$ we shall denote the number of all elements of A. As usual, a binary relation ρ on A is a subset of $A \times A$. We define:

Definition 1. A binary relation ρ on A is called

(1) reflexive if $\forall x \in A : [x, x] \in \rho$,
(2) symmetric if $\forall x, y \in A : [x, y] \in \rho \Rightarrow [y, x] \in \rho$,
(3) antisymmetric if $\forall x, y \in A : [x, y] \in \rho \wedge [y, x] \in \rho \Rightarrow x = y$,
(4) transitive if $\forall x, y, z \in A : [x, y] \in \rho \wedge [y, z] \in \rho \Rightarrow [x, z] \in \rho$.

A binary relation ρ is called a quasi-order if it is reflexive and transitive. Furthermore, ρ is called an equivalence if it is reflexive, symmetric and transitive and finally ρ is called a partial order or an ordering if it is reflexive, antisymmetric and transitive. A partially ordered set (A, ρ) or poset, for short, is a set A together with a partial order ρ. We also call (A, ρ) a labelled poset.

Definition 2. A topology on A is a family τ of subsets of A such that

(1) $\emptyset \in \tau \wedge A \in \tau$,
(2) $\forall X, Y \in \tau : X \cup Y \in \tau$,
(3) $\forall X, Y \in \tau : X \cap Y \in \tau$.

The elements of τ are called open sets. A topology is said to be T_0 if for all a, b in A such that $a \neq b$ there exists an open set containing one of a, b but not the other.

At the very beginning we recall an important fact that there is a connection between binary relations on A and topologies on A. In 1937 P. S. Alexandrov [1] and also G. Birkhoff [2] observed that there is the one-to-one correspondence between topologies on A and quasi-orders on A, and furthermore, there is the one-to-one correspondence between partial orders on A and T_0-topologies on A (see [2], 3.ed, p.117).

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Definition 3. A partition of a finite n-set A is a collection $\mathscr{P} = \{A_1, \ldots, A_k\}$ of subsets of A, where $1 \leqq k \leqq n$, such that

(1) $\forall i \in \{1, \ldots, k\} : A_i \neq \emptyset$,
(2) $\forall i, j \in \{1, \ldots, k\}, i \neq j : A_i \cap A_j = \emptyset$,
(3) $A_1 \cup \cdots \cup A_k = A$.

We call A_i blocks of $\mathscr{P}$ and we say that $\mathscr{P}$ has k blocks. Then we define $S(n,k)$ to be the number of partitions of an n-set into k blocks. $S(n,k)$ is called a Stirling number of the second kind. By convention, we put $S(0,0) = 1$. Furthermore, the total number of partitions of an n-set A is called a Bell number and is denoted by $B(n)$. Thus we have the relation $B(n) = \sum_{k=1}^{n} S(n,k)$.

Now we remind further important and well-known result on the set-partitions and equivalence relations on a set A. We have the following assertion: There is a one-to-one correspondence between the set of all partitions of an n-set A and the set of all equivalence relations on A. Consequently, the Bell number $B(n)$ is the number of all equivalence relations on an n-set A. This correspondence is given in such a way that the elements which are equivalent lie in the same block.

Definition 4. Let ρ be an ordering on A and σ be an ordering on B. We say that two posets (A, ρ) and (B, σ) are isomorphic if there is an order-preserving bijection $f : A \to B$ whose inverse is order-preserving as well (i.e. $\forall x, y \in A : [x, y] \in \rho \Leftrightarrow [f(x), f(y)] \in \sigma$). This isomorphism decomposes the set of all posets on A into blocks, which we call non-isomorphic posets or also unlabelled posets.

Finally, it is necessary to recall the following notions of a partition and a composition of an integer n.

Definition 5. A partition of an integer $n \in N$ is a sequence $(x_1, \ldots, x_k) \in N^k$, where $1 \leqq k \leqq n$, such that $x_1 + \cdots + x_k = n$ and $x_1 \geqq \cdots \geqq x_k$. A composition of n is a sequence $(x_1, \ldots, x_k) \in N^k$, where $1 \leqq k \leqq n$, such that $x_1 + \cdots + x_k = n$. If exactly k summands appear in a partition, we call it a k-partition. Analogously, a composition of n in which exactly k summands occur, is called a k-composition.

It is known that there is a bijection between all k–compositions of an integer n and $(k-1)$–subsets of $\{1, 2, \ldots, n-1\}$. Hence there are $\binom{n-1}{k-1}$ k–compositions and 2^{n-1} compositions of n. On the other hand, it is not possible to count the number of partitions so easily. All the same, there are more ways to enumerate these numbers (see e.g. our paper [27]).

2. Counting the binary relations

One of the basic problems from the combinatorical analysis is to find the number of all configurations of the specific type. For example, to find the number of all binary relations, the number of set-partitions, the number of topologies and so on. It is well-known that the number of all binary relations on an n-set A is equal to 2^{n^2}. Quite easily we can count the numbers of reflexive, symmetric and antisymmetric relations. Let $\mathscr{R}(A)$ be the set of all reflexive relations on A, $\mathscr{S}(A)$ the set of all symmetric relations on A, $\mathscr{A}(A)$ the set of all antisymmetric relations on A and let

$\mathscr{T}(A)$ denote the set of all transitive relations on A. Thus we have:

$$|\mathscr{R}(A)| = 4^{\binom{n}{2}}, \tag{1}$$

$$|\mathscr{S}(A)| = 2^{\binom{n+1}{2}}, \tag{2}$$

$$|\mathscr{A}(A)| = 2^n \cdot 3^{\binom{n}{2}}, \tag{3}$$

$$|\mathscr{R}(A) \cap \mathscr{S}(A)| = 2^{\binom{n}{2}}, \tag{4}$$

$$|\mathscr{R}(A) \cap \mathscr{A}(A)| = 3^{\binom{n}{2}}, \tag{5}$$

$$|\mathscr{A}(A) \cap \mathscr{S}(A)| = 2^n . \tag{6}$$

These formulas can be deduced by means of elementary combinatorical techniques (i.e. by means of the rules of sum and product). The problem of finding these numbers is often submitted as an exercise. But the difficulties begin when we start to engage with counting binary relations which have the property of transitivity. For the number of equivalences and their classes we still have reasonable formulas. It is easy to verify that $S(n,k) = 0$ if $k > n$, $S(n,0) = 0$, $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$, $S(n,n-1) = \binom{n}{2}$, $S(n,n) = 1$. Now we introduce a short survey of possibilities how to count these numbers. We have the following formulas:

$$S(n,k) = kS(n-1,k) + S(n-1,k-1), \tag{7}$$

$$S(n,k) = \sum_{i=1}^{n-1} \binom{n-1}{i} S(i,k-1), \tag{8}$$

$$S(n,k) = (k!)^{-1} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n, \tag{9}$$

$$S(n,k) = \sum_{x_1+\cdots+x_k=n} 1^{x_1-1} 2^{x_2-1} \cdots k^{x_k-1}, \tag{10}$$

where the sum extends over all k-compositions of an integer n. Moreover, for the Bell numbers $B(n)$ we have the recursion

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k). \tag{11}$$

The Bell numbers can be also computed by means of the scheme (12), analogously to the Pascal triangle for counting binomial coefficients.

$$
\begin{array}{ccccccccc}
1 & & 1 & & 2 & & 5 & & 15 \\
 & 2 & & 3 & & 7 & & * & \\
 & & 5 & & 10 & & * & & \\
 & & & 15 & & * & & & \\
 & & & & * & & & &
\end{array}
\tag{12}
$$

In the following section we shall continue in our list of enumerative combinatorical results on binary relations with the property of transitivity. We shall also pay an attention to the numbers of such relations. We shall concentrate especially on our basic subject, which is the number of finite posets.

3. Counting finite posets

Our main combinatorical counting problem is the following: How many posets are there on n elements (enumeration of labelled posets) and what is the number of their isomorphism classes (enumeration of unlabelled posets)? These problems are unsolved up till now. No reasonable explicit or recursive formula for these numbers is still known. By reasonable we mean that the number of involved operations is of considerably smaller order than the numbers which we want to compute. The enumeration of all finite posets is a long-standing open problem. G. Birkhoff with his well-known book Lattice theory (see [2], third ed., pages 4 and 19) was in 1948 one of the first who formulated this problem. We quote: "Let $G(n)$ denote the number of nonisomorphic posets of n elements and $G^*(n)$ denote the number of different partial orderings of n elements. Compute for small n, and find asymptotic estimates and bounds for the rates of growth of the functions $G(n)$ and $G^*(n)$."

Now we make a short remark on the notation of the number of posets. This notation did still not stabilise. We already know the notation from Birkhoff's book. Several authors used this notation too, but it did not root. There was used the whole range of notations for the number of labelled posets up till now. For example $G^*(n)$ in [2], $H(n)$ in [46], d_n in [13], $A_0(n)$ in [22], $T_0(n)$ in [3], γ_n in [24], p_n in [26] and P_n in [23]. Next, in the case of numbers of unlabelled posets we have the similar situation. In this paper we shall use the following notation: p_n will denote the number of all partial orders of an n-element set A and P_n the number of non-isomorphic posets on A. Now we introduce the fundamental results on the number of posets.

For p_n we have the following explicit expression (see M. Erné, [23])

$$
p_n = \sum_{m=0}^{2^{n^2}-1} \prod_{i=0}^{n-1} x^m_{ni+i} \prod_{j=0}^{n-1} (1 - x^m_{ni+j} x^m_{nj+i}) \prod_{k=0}^{n-1} (1 - x^m_{ni+j} x^m_{nj+k} (1 - x^m_{ni+k})),
\tag{13}
$$

where

$$
x_i^m = \lfloor 2^{-i} m \rfloor - 2 \lfloor 2^{-i-1} m \rfloor
\tag{14}
$$

is the i-th digit in the binary expansion of m ($0 \leqq i \leqq n^2$). Each binary relation on the set $\{0, \dots, n-1\}$ is represented by one summand, where m runs from 0 to

$2^{n^2}-1$. The value of x^m_{ni+j} is 1 if i and j are related, and 0 otherwise. The product over i codes reflexivity, that over j antisymmetry, and that over k transitivity. Thus the whole product is 1 iff the relation is a partial order, and it is 0 in all other cases. The formula (13) is evidently not practical for computing p_n.

In 1966 L. Comtet [13] introduced the important and often used and occured formula

$$p_n = \sum_{(x_1,\dots,x_m)} \frac{n!}{x_1!\dots x_m!} V(x_1,\dots,x_m), \tag{15}$$

where the sum is taken over all compositions of the number n and $V(x_1,\dots,x_m)$ is the number of certain posets with respect to the composition $x_1+\dots+x_m=n$. This formula was in 1979 rediscovered by Z. I. Borevich. More exactly, $V(x_1,\dots,x_m)$ is the number of all so called V–nets of the type $(x_1,\dots,x_m)$. The detailed definition and explanation of this concept can be founded in [3]. In [4] and [5] Borevich derived a special case of $V(x_1,\dots,x_m)$ and determined some values of p_n. He also proved that all values $V(x_1,\dots,x_m)$ are odd numbers.

The following enumerative results show that nearly all the problems on finding the number of binary relations, where transitivity is one of the properties, can be converted to finding the values of p_n. Above all, it is possible to show that it holds:

$$|\mathscr{A}(A)\cap\mathscr{T}(A)| = 2^n p_n, \tag{16}$$

$$|\mathscr{S}(A)\cap\mathscr{T}(A)| = B(n+1). \tag{17}$$

Now we introduce the following notation. Let t_n be the number of all transitive relations on an n-set A and q_n be the number of all quasiorders on A. In 1967 Evans, Harary and Lynn [24] derived a formula relating the number of all quasi-orders on a set of n elements and the number of all partial orders or equivalently the number of topologies on an n-set and T_0-topologies. In particular, they proved the following formula

$$q_n = \sum_{k=1}^{n} S(n,k)p_k. \tag{18}$$

This result contributes to the intensive interest in the number of posets. In spite of the fact that neither the explicit nor recursive formula is still known, there was discovered the asymptotic estimate for p_n. The significant results in this area were presented in 1970 by D. J. Kleitman and B. L. Rothschild. In [30] they deduced the formula

$$\log_2 p_n = \frac{1}{4}n^2 + o(n^2). \tag{19}$$

Furthermore, in 1975 the same authors proved in [31] that

$$p_n = (1+O(\frac{1}{n}))\sum_{i=1}^{n}\sum_{j=1}^{n-i}\binom{n}{i}\binom{n-i}{j}(2^i-1)^j(2^j-1)^{n-i-j}. \tag{20}$$

This asymptotic formula was in 1981 simplified by K. H. Kim and F. W. Roush [26]. From further works which concern about the asymptotic behaviour of p_n we remark at least the papers of J. L. Davison [19] and D. Dhar [20].

In 1974 M. Erné, [22], showed that quasiorders are asymptotically posets, i.e.

$$\frac{q_n}{p_n} \to 1 \quad \text{for} \quad n \to \infty. \tag{21}$$

In 1992 we have derived in [29] the following formula (22) for the number of transitive relations, which enables to compute numbers t_n, if the values p_n are known:

$$t_n = \sum_{k=1}^{n} \alpha_k(n) p_k, \quad \text{where} \quad \alpha_k(n) = \sum_{s=0}^{k} \binom{n}{s} S(n-s, k-s). \tag{22}$$

In particular, using (22) we have computed the numbers t_n for $n \leqq 14$. Number t_{14} constitutes currently the greatest known value of the sequence t_n and exceeds 10^{28}. In [29] we have also proved the asymptotic formula

$$\frac{t_n}{2^n p_n} \to 1 \quad \text{for} \quad n \to \infty. \tag{23}$$

In 1987 H. J. Prömel [40] proved that the number of unlabelled structures is asymptotically $1/n!$ times the same labelled quantity. The paper [40] contains a short proof of this fact for all classes of structures whose logarithm approaches a quadratic in the size parameter n. In particular, for posets we have

$$\frac{p_n}{n! P_n} \to 1 \quad \text{for} \quad n \to \infty. \tag{24}$$

This problem was in 1981 introduced by K. H. Kim and F. W. Roush (see [26], Problem 3). At the end of this section we mention the solitary and very interesting result of Z. I. Borevich on the residual periodicity of the sequence p_n. In the period 1979-1982 Borevich published papers [6], [7] and [8], where he proved the following assertions. Let $m = \mathfrak{p}$ be an arbitrary prime number. Then the sequence $\{p_n \mod m\}_{n=1}^{\infty}$ is periodical and the length of its period is equal to $\mathfrak{p} - 1$. If $m = \mathfrak{p}^a$, where $a \in N$, then $\{p_n \mod m\}_{n=1}^{\infty}$ is periodical from $n \geqq \mathfrak{p}^{a-1}$ and the length of its period is equal to $\varphi(\mathfrak{p}^a) = \mathfrak{p}^a - \mathfrak{p}^{a-1}$. Furthermore, if $m = \mathfrak{p}_1 \dots \mathfrak{p}_k$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are the different primes, then the sequence $\{p_n \mod m\}_{n=1}^{\infty}$ is periodical and the length of its period is equal to the least common multiple of the numbers $\mathfrak{p}_1 - 1, \dots, \mathfrak{p}_k - 1$. Finally in the general case it holds: Let m be an arbitrary positive integer. Then there exists an index n_0 from which the sequence $\{p_n \mod m\}_{n=1}^{\infty}$ is periodical. Specially, the periodicity of the last figures of the sequence p_n follows from Borewich's results. This visible fact can be seen in Table 1 (see also the author's work [29]).

4. The history of the known values of p_n

As we remarked, the original stimulation for computing the values p_n for small n came from G. Birkhoff in 1948. First, finding the values p_2, p_3 and p_4 is not difficult. This problem was often submitted to the reader as an exercise. In 1966 L. Comtet has found in [13] the values p_5 and p_6. In 1967 J. W. Evans, F. Harary and M. S. Lynn have in [24] found the values p_n for $n \leqq 7$. Counting and verifying these values was left to the reader in the third edition of the Birkhoff's book [2].

The computer enumeration of the values p_n was based on the matrix representation of binary relations. Binary relations on a set of n elements can be represented as $n \times n$ matrices of zeros and ones. If ρ is a binary relation on a set of elements $x_1, \dots, x_n$, then we associate to ρ the matrix $M = (m_{i,j})$ such that $m_{i,j} = 1$ if $[x_i, x_j] \in \rho$ and $m_{i,j} = 0$ if $[x_i, x_j] \notin \rho$. This gives a one-to-one correspondence between binary relations and $n \times n$ matrices of zeros and ones (see [24], [35] and [48]). The matrix representation of a partial order was given in 1972 by K. K. H. Butler [9]. However, the basic idea can be already found even in the paper [48] by H. Sharp from 1966. It holds that the $n \times n$ matrix of zeros and ones represents a partial order on some n-set iff it is nonsingular and idempotent.

In 1974 M. Erné has published the important paper [22], where he has computed the values p_n up to $n \leqq 9$. Further significant results were obtained in 1977 in the paper [18] of S. K. Das, where the values p_n up to $n \leqq 11$ are computed. At that time this work presented the full list of values p_n. In this historical review it is necessary to underline the works of the soviet mathematicians in the period 1978-1982. In the papers [4] and [5] from 1978 and 1979 Z. I. Borevich, V. I. Rodionov and their coauthors computed the values p_9 and p_{10}. But at that time these values were already known. Further, Rodionov in [44] and [45] independently resumed the common works with Borevich. In 1982 he computed the values p_{11} and p_{12}. Now we introduce the table of the numbers p_n by [23] up to $n = 14$.

Table 1. The numerical values p_n for $n \leq 14$.

$p_1 =$	1	(Folklore)	
$p_2 =$	3	(Folklore)	
$p_3 =$	19	(Folklore)	
$p_4 =$	219	(Folklore)	
$p_5 =$	4 231	(1966)	L. Comtet
$p_6 =$	130 023	(1966)	L. Comtet
$p_7 =$	6 129 859	(1967)	Evans, Harary and Lynn
$p_8 =$	431 723 379	(1967)	Evans, Harary and Lynn
$p_9 =$	44 511 042 511	(1974)	M. Erné
$p_{10} =$	6 611 065 248 783	(1977)	S. K. Das
$p_{11} =$	1 396 281 677 105 899	(1977)	S. K. Das
$p_{12} =$	414 864 951 055 853 499	(1982)	V. I. Rodionov
$p_{13} =$	171 850 728 381 587 059 351	(1991)	M. Erné and K. Stege
$p_{14} =$	98 484 324 257 128 207 032 183	(1991)	M. Erné and K. Stege

As late as 1991, after a long pause, M. Erné and K. Stege presented in [23] the values p_n up to $n \leqq 14$. Currently the number p_{14} constitutes the greatest known value of the sequence p_n. At the present time we can use computers for finding further values of p_n by means of contemporary known methods. But the necessary computing time constitutes the insuperable barrier. Interesting informations on the time-consuming computation of the values p_n can be found in [23].

5. The history of the known values of nonisomorphic posets P_n

The number of works which deal with the computation of values P_n for small n is much less than the number of works which deal with the computation of p_n. By G. Birkhoff, [2], the values P_n for $n \leqq 6$ were found by I. Rose and R. T. Sasaki. In 1981 N. P. Chaudhuri and A. A. J. Mohammed, [12], were concerned with finding a method for verifying the results of Rose and Sasaki. In their paper the verification for $n = 4$ is shown. The values P_n for $n \leqq 6$ can also be found in [46] by R. A. Rozenfeld from 1985. The value P_7 was discovered in 1972 by J. A. Wright in his PhD-thesis [50]. In 1977 S. K. Das found in [18] the value P_8. Seven years later in 1984 R. H. Möhring introduced the value P_9. Further progress came in 1990, when J. C. Culberson and G. J. E. Rawlins computed the numbers of non-isomorphic posets up to $n \leqq 11$. Further, in 1990 A. M. Kutin, [36], was also engaged in computing P_n. C. Chaunier and N. Lygerös found in 1991 the value P_{12} and finally the latest progress in the computation of P_n came in 1992 when the same authors computed in [14] the value P_{13}. Now we introduce the known values of P_n by C. Chaunier and N. Lygerös.

Table 2. The numerical values P_n for $n \leqq 13$.

$P_1 =$	1	(Folklore)	
$P_2 =$	2	(Folklore)	
$P_3 =$	5	(Folklore)	
$P_4 =$	16	(Folklore)	
$P_5 =$	63	(Folklore)	
$P_6 =$	318	(1967)	I. Rose and R. T. Sasaki
$P_7 =$	2 045	(1972)	J. Wright
$P_8 =$	16 999	(1977)	S. K. Das
$P_9 =$	183 231	(1984)	R. H. Möhring
$P_{10} =$	2 567 284	(1990)	J. C. Culberson and G. J. E. Rawlins
$P_{11} =$	46 749 427	(1990)	J. C. Culberson and G. J. E. Rawlins
$P_{12} =$	1 104 891 746	(1991)	C. Chaunier and N. Lygerös
$P_{13} =$	33 823 327 452	(1992)	C. Chaunier and N. Lygerös

In the following section we shall deal with the more special problem of finding the number of connected posets.

Let (A, ρ) be a partially ordered set, $x, y \in A$. We say that two elements x and y are comparable and we write $x \frown y$, if $[x, y] \in \rho$ or $[y, x] \in \rho$. For $x, y \in A$ we put $x \sim y$ iff there are $k \in N$ and k elements $x_1, \ldots, x_k \in A$ such that $x \frown x_1, \ldots, x_k \frown y$. The poset (A, ρ) is called connected, if for all $x, y \in A : x \sim y$. By c_n we shall denote the number of all connected posets on an n-set A. Furthermore, an isomorphism decomposes the set of all connected posets on A into blocks, which we call non-isomorphic connected posets. The number of all non-isomorphic connected posets on an n-set A will be denoted by C_n.

The first mention of the number of connected posets came probably in 1963 from R. A. Rankin [43]. In [43] there are introduced the values c_n for $n \leqq 4$. Next, 11 years later M. Erné [22] found the values c_n up to $n \leqq 9$. In 1991 the same author and K. Stege [23] presented these numbers for $n \leqq 14$ (see Table 3). Further, we have almost no references on the number C_n of non-isomorphic connected posets. Let us remark that G. Birkhoff [2] did not refer to the numbers c_n and C_n. In 1985 R. A. Rozenfeld [46] presented the numbers C_n for $n \leqq 6$. After this solitary paper we have computed in 1994 the values C_n up to $n \leqq 13$ (see Table 3), [28]. In [28] we have also derived the following formulas (25) and (26) (cf. also our paper [27]):

$$P_n = \frac{1}{n} \sum_{k=0}^{n-1} \alpha(n-k) P_k \quad \text{and} \quad \alpha(m) = \sum_{k|m} k C_k, \tag{25}$$

$$P_n = -\sum_{k=0}^{n-1} Q_{n-k} P_k \quad \text{and} \quad Q_n = \sum_{S} (-1)^{k_1 + \cdots + k_n} \binom{C_1}{k_1} \cdots \binom{C_n}{k_n}, \tag{26}$$

where the sum extends over the set S of all solutions $[k_1, \ldots, k_n] \in \{0, 1, \ldots, n\}^n$ of the linear Diophantine equation $1k_1 + 2k_2 + \cdots + nk_n = n$.

Table 3. Initial values of the connected posets c_n and C_n.

$c_1 =$	1	$C_1 =$	1
$c_2 =$	2	$C_2 =$	1
$c_3 =$	12	$C_3 =$	3
$c_4 =$	146	$C_4 =$	10
$c_5 =$	3 060	$C_5 =$	44
$c_6 =$	101 642	$C_6 =$	238
$c_7 =$	5 106 612	$C_7 =$	1 650
$c_8 =$	377 403 266	$C_8 =$	14 512
$c_9 =$	40 299 722 580	$C_9 =$	163 341
$c_{10} =$	6 138 497 261 882	$C_{10} =$	2 360 719
$c_{11} =$	1 320 327 172 853 172	$C_{11} =$	43 944 974
$c_{12} =$	397 571 105 288 091 506	$C_{12} =$	1 055 019 099
$c_{13} =$	166 330 355 795 371 103 700	$C_{13} =$	32 664 484 238
$c_{14} =$	96 036 130 723 851 671 469 482		

7. The special classes of order structures

In the last section we make a short remark on enumeration in the special classes of order structures. In the scientific literature it was studied the whole range of ordered structures such as graded posets, interval orders, lattices, semiorders, series-parallel posets, tiered posets, two-dimensional posets and weak orders. We mention here only one particular. A lattice is a partially ordered set in which every pair of elements has the least upper bound (join) and the greatest lower bound (meet). G. Birkhoff already formulated in [2] the problem to find the number of all n-element lattices. For the number of lattices we have a similar situation as for the number of posets. No explicit or recursive formula is known. In 1979 S. Kyuno [37] described the algorithm for constructing Hasse diagrams of all n-element lattices and also found the number of lattices for $n \leqq 8$. As late as 1994 Y. Koda [34] computed these numbers up to $n \leqq 13$. In 1971 W. Klotz and L. Lucht [33] found the lower bound and in 1980 D. J. Kleitman and K. J. Winston [32] the upper bound for the number of lattices. The aim of this short section was to show that the enumeration problem of posets is not solitary and that there exists a whole family of similar problems. The survey of the enumeration problems in further classes of ordered structures together with the main results can be found e.g. in the paper [21] by M. El-Zahar.

At the end of this paper we present the survey of the main works related to our topic. Of course, this bibliography collection is not complete. The comprehensive resource of references can be found in the papers [6], [23] and [26].

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GREATEST COMMON SUBGROUP AND SMALLEST COMMON SUPERGROUP OF TWO FINITE GROUPS AND RELATED METRICS ON A SYSTEM OF FINITE GROUPS

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Abstract. The paper deals with metrics on a system of finite groups which are defined by the greatest common subgroup and the smallest common supergroup of two finite groups. An interesting result is obtained for groups S_4 and D_{12}.

INTRODUCTION

Metrics on systems of graphs and posets were investigated in papers [1], [7] and [4], [6] respectively. Paper [3] of A. Haviar investigated four metrics on a system of finite universal algebras. The present paper studies metrics on a system of finite groups which correspond to the substructure and superstructure metric of A. Haviar.

Two groups are considered to be near if they contain a large isomorphic subgroup. Alternatively, two groups are considered to be near if they are embedable into a small group.

1. GREATEST COMMON SUBGROUP AND SMALLEST COMMON SUPERGROUP OF TWO FINITE GROUPS

Definition 1.1: Let G_1 and G_2 be finite groups. The symbol $m(G_1, G_2)$ denotes the maximal order of a group G such that G_1 and G_2 contain subgroups K_1 and K_2 isomorphic to G. The symbol $M(G_1, G_2)$ denotes the minimal order of a group H containing subgroups H_1 and H_2 isomorphic to G_1 and G_2 respectively.

The product of two elements a and b of a group G will be denoted simply ab. If A and B are subset of a group, then the symbol AB donotes the set of all products ab, where $a \in A$ and $b \in B$. The unity of any group will be denoted by e. The symbol $|A|$ denotes the cardinality of a set A.

In the whole paper we shall use the following obvious lemma.

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Lemma 1.1: Let G be a group, G_1 and G_2 be finite subgroups of G.

Then $|G_1G_2| = \frac{|G_1|\cdot|G_2|}{|G_1\cap G_2|}$.

Proposition 1.2 For any two finite groups G_1 and G_2 the following inequalities hold.

(i) $1 \leq m(G_1, G_2) \leq$ g.c.d.$(|G_1|, |G_2|)$ and $m(G_1, G_2)$ is a common divisor of $|G_1|$ and $|G_2|$.

(ii) s.c.m. $(|G_1|, |G_2|) \leq M(G_1, G_2) \leq |G_1| \cdot |G_2|$ and $M(G_1, G_2)$ is a common multiple of $|G_1|$ and $|G_2|$.

(iii) $M(G_1, G_2) \geq \frac{|G_1|\cdot|G_2|}{m(G_1,G_2)}$

Proof: Parts (i) and (ii) are obvious. We shall prove (iii). Let H be a group of the minimal order containing subgroups H_1 and H_2 isomorphic to G_1 and G_2 respectively. Without loss of generality we may assume that $G_1 = H_1$ and $G_2 = H_2$. Then $G_1 \cap G_2$ is a subgroup of G_1 and G_2 which means $m(G_1, G_2) \geq |G_1 \cap G_2|$. Since G_1G_2 is a subset of H, we obtain

$$M(G_1, G_2) = |H| \geq |G_1G_2| = \frac{|G_1|\cdot|G_2|}{|G_1\cap G_2|} \geq \frac{|G_1|\cdot|G_2|}{m(G_1,G_2)}.$$

It completes the proof.

The symbol $D_n (n > 2)$ denotes the dihedral group, i.e. the symmetry group of a regular polygon with n edges. This group is generated by the elements r and t satisfying relations $r^n = t^2 = e$ and $trt = r^{-1}$. The element r is a rotation through angle $\frac{2\pi}{n}$ and t is an axial symmetry.

The symbol S_n denotes the group of all permutations of the set $\{1, ..., n\}$. It is easy to see that the cycle $\rho = (12...n)$ and the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & n & n-1 & \dots & 3 & 2 \end{pmatrix}$$

generate a subgroup of S_n isomorphic to D_n. For $n = 3$ the groups S_n and D_n are isomorphic.

Lemma 1.3: Let j and n be coprime integers, $1 \leq j \leq n-1$ and $0 \leq k \leq n-1$. There is a unique automorphism $\psi : D_n \to D_n$ such that $\psi(r) = r^j$ and $\psi(t) = r^k t$. Conversely, any automorphism $\psi : D_n \to D_n$ has such a form.

Proof: Under the above conditions about j and k the elements $\rho = r^j$ and $\tau = r^k t$ satisfy the same relations as r and t. Therefore, formulas $\psi(r) = r^j$ and $\psi(t) = r^k t$ define an automorphism. Let $\psi : D_n \to D_n$ be an automorphism. The order of the element $\psi(r)$ is n, so $\psi(r) = r^j$, where j and n are coprime integers and $1 \leq j \leq n-1$. The order of the element $\psi(t)$ is 2 and this element does not commute with $\psi(r) = r^j$. So, $\psi(t) = r^k t$, where $0 \leq k \leq n-1$.

If a natural k is a divisor of n, i.e. $n = jk$ for some natural j, then the elements $s = r^j$ and t satisfy relations $s^k = t^2 = e$ and $tst = s^{-1}$ and they generate a subgroup which may be identified with D_k. In this situation we shall assume $D_k \subset D_n$. The following lemma may be generalised, but we shall use only this special case.

Lemma 1.4: For any automorphism $\psi : D_4 \to D_4$ there is an automorphic extension $\varphi : D_{12} \to D_{12}$.

Proof: In this situation $n = 12, k = 4, j = 3$ and $s = r^3$. By Lemma 1.3., we have $\psi(s) = s$ or $\psi(s) = s^3$ and $\psi(t) = s^k t$, where $0 \leq k \leq 3$. For the definition of an automorphism $\varphi : D_{12} \to D_{12}$ it is sufficient to define $\varphi(t)$ and $\varphi(r)$. Put $\varphi(t) = \psi(t)$. If $\psi(s) = s$, then put $\varphi(r) = r$. In this case $\varphi(s) = \varphi(r^3) = (\varphi(r))^3 = r^3 = s = \psi(s)$. If $\psi(s) = s^3$, then put $\varphi(r) = r^{11}$. In this case $\varphi(s) = \varphi(r^3) = (\varphi(r))^3 = (r^{11})^3 = r^{33} = r^9 = s^3 = \psi(s)$. So, φ is an extension of ψ.

Lemma 1.5: There is a group H of order 96 which contains subgroups H_1 and H_2 isomorphic to S_4 and D_{12}.

Proof: Let C_4 be a subgroup of D_{12} generated by the element $s = r^3$ and D_3 be a subgroup generated by the elements $p = r^4$ and t. Then $|C_4 \cap D_3| = 1$ and $|C_4 D_3| = \frac{|C_4| \cdot |D_3|}{|C_4 \cap D_3|} = 24$ which means $C_4 D_3 = D_{12}$. Note that $xsx^{-1} = s$, when x is a rotation and $xsx^{-1} = s^{-1}$, when x is an axial symmetry. So, C_4 is a normal subgroup of D_{12} which is an internal semidirect product of C_4 and D_4, [5, *p*.27]. Using this fact and isomorphism of D_3 and S_3, it may be easily shown that D_{12} is isomorphic to the Cartesian product $C_4 \times S_3$ with the group operation defined by the formula $[x, \sigma][y, \tau] = [xy^{sgn\sigma}, \sigma\tau]$, where $x, y \in C_4, \sigma, \tau \in S_3$ and sgn σ denotes the sign of a permutation $\sigma \in S_3$. Replacing S_3 by S_4 in the above construction, we obtain the required group H.

The following theorem is the main result of this paper.

Theorem 1.6: Let $G_1 = S_4$ and $G_2 = D_{12}$. Then $m(G_1, G_2) = 8$ and $M(G_1, G_2) = 96$. It means that the inequality $M(G_1, G_2) \geq \frac{|G_1| \cdot |G_2|}{m(G_1, G_2)}$ can not be replaced by the equality $M(G_1, G_2) = \frac{|G_1| \cdot |G_2|}{m(G_1, G_2)}$.

Proof: We shall show the equality $m(G_1, G_2) = 8$. Clearly, both groups G_1 and G_2 contain a subgroup isomorphic to D_4 of order 8. It means $m(G_1, G_2) \geq 8$. The group D_{12} is generated by the elements r and t satisfying relations $r^{12} = e = t^2$ and $trt = r^{-1}$. The group S_4 contains only permutations of the form $(ijkl), (ijk), (ij), (kl)$ and (ij) the orders of which are $4, 3, 2$ and 2 respecively. On the other hand the group D_{12} contains the element r, the order of which is 12. The groups G_1 and G_2 are not isomorphic which means $m(G_1, G_2) \neq 24$. We shall show $m(G_1, G_2) \neq 12$. Let G be a subgroup of D_{12} with $|G| = 12$. Then the index of G is 2. So, G is a normal subgroup of D_{12} and the order of the factor group D_{12}/G is 2 which particulary means $r^2 \in G$. The order of r^2 is 6 and S_4 does not contain such elements. So, S_4 does not contain a subgroup isomorphic to G. It means $m(G_1, G_2) \neq 12$. It proves $m(G_1, G_2) = 8$. Proposition 1.2 implies $M(G_1, G_2) \geq 72$. We shall show $M(G_1, G_2) \neq 72$. Let H be a hypothetical group of order 72 which contains subgroups H_1 and H_2 isomorphic to S_4 and D_{12}. So, there are monomorphisms $\varphi_1 : S_4 \to H$ and $\varphi_2 : D_{12} \to H$ with

$\varphi_1(S_4) = H_1$ and $\varphi_2(D_{12}) = H_2$. Assume that S_4 is group of all permutation of the set $\{A, B, C, D\}$. The elements $s = r^3$ and t generate a subgroup of D_{12} which is identified with D_4. Let $\psi : D_4 \to S_4$ be a monomorphism defined by the formulas $\psi(s) = (ABCD)$ and $\psi(t) = (BD)$. Denote by $a = \varphi_1(ABC), x = \varphi_1(ABCD)$ and $y = \varphi_1(BD)$. Since $(ABC)^3 = e$ and $(ABC)(ABCD)(ABC) = (BD)$, we have $a^3 = e$ and $axa = y$ which means $ax = ya^2$. There is an element $b \in H$ such that $b \neq e, ab = ba, xb = bx, by = yb^2$ and $b^2y = yb$. This is a contradiction, because $ya^2b = axb = abx = bax = bya^2 = yb^2a^2$ which means $a^2b = b^2a^2 = a^2b^2$ and $e = b$. We shall show the existence of such an element $b \in H$. The images $\varphi_1(\psi(D_4))$ and $\varphi_2(D_4)$ are Sylow subgroups of order 8 in H. By Sylow theorem, they are conjugated by an inner automorphism in H, [5, p.39]. So, without loss of generality we may assume $\varphi_1(\psi(D_4)) = \varphi_2(D_4)$. Denote by $f = \varphi_2^{-1} \circ \varphi_1 \circ \psi$. The mapping f is an automorphism of D_4 and by Lemma 1.4., there is an automorphic extension $\varphi : D_{12} \to D_{12}$ of f. Now, the monomorphism $\varphi_2 \circ f : D_{12} \to H$ is an extension of $\varphi_1 \circ \psi : D_4 \to H$. Replacing φ_2 by $\varphi_2 \circ f$, we may assume that $\varphi_2(z) = \varphi_1(\psi(z))$ for any $z \in D_4$. The element $a = \varphi_1(ABC)$, generates a subgroup K_1 of H with $|K_1| = 3$. Since $|H| = 72$, the subgroup K_1 is contained in some Sylow group K of order 9, [5, p.39]. Denote by $G = \varphi_1(\psi(D_4)) = \varphi_2(D_4)$. Since $|G| = 8$, we have $|G \cap K| = 1$ and $|GK| = 72$ which means $GK = H$. Therefore, $H_2K = H$ and $|H_2 \cap K| = \frac{|H_2|\cdot|K|}{|H_2K|} = 3$. Put $K_2 = H_2 \cap K$. It is a subgroup of H_2 of order 3. The group D_{12} contains a unique group of order 3, it is a subgroup C_3 generated by the element $p = r^4$. It means $K_2 = \varphi_2(C_3)$. The element $b = \varphi_2(p) \neq e$ has the required properties. The group K is commutative, because any group of the order p^2 is commutative, [5, p.39]. It proves $ab = ba$. Since $ps = sp$ and $\varphi_2(s) = \varphi_1(\psi(s)) = \varphi_1(ABCD)) = x$, we have $xb = bx$. Finally, relations $by = yb^2$ and $b^2y = yb$ follow from relations $p^3 = t^2 = e, tpt = p^{-1}$ and $\varphi_2(t) = \varphi_1(\psi(t)) = \varphi_1(BD) = y$. The proof of $M(G_1, G_2) \neq 72$ is complete. The following multiple of 24 is 96. Now, $M(G_1, G_2) = 96$ by Lemma 1.5.

2.SUBGROUP METRICS

For two finite groups G_1 and G_2 put $d(G_1, G_2) = |G_1| + |G_2| - 2m(G_1, G_2)$.

Proposition 2.1: The function d is a metric, i.e. for any finite groups G_1, G_2 and G_3

(i) $d(G_1, G_2) \geq 0$ and $d(G_1, G_2) = 0$ if and only if G_1 and G_2 are isomorphic

(ii) $d(G_1, G_2) = d(G_2, G_1)$

(iii) $d(G_1, G_3) \leq d(G_1, G_2) + d(G_2, G_3)$ and the equality appears only in the case when G_2 is isomorphic to a subgroup of G_1 or G_3.

Proof: Parts (i) and (ii) are obvious. Let H_1 and H_2 be isomorphic subgroups of G_1 and G_2 respectively for which $|H_1| = |H_2| = m(G_1, G_2)$ and $\varphi : H_2 \to H_1$ be the corresponding isomorphism. Similarly, let K_2 and K_3 be isomorphic subgroups of G_2 and G_3 respectively for which $|K_2| = |K_3| = m(G_2, G_3)$ and $\psi : K_2 \to K_3$ be the corresponding isomorphism. Obviously, the groups $\varphi(H_2 \cap K_2)$ and $\psi(H_2 \cap K_2)$ are isomorphic which implies $m(G_1, G_3) \geq |H_2 \cap K_2| = |H_2| + |K_2| - |H_2 \cup K_2| \geq m(G_1, G_2) + m(G_2, G_3) - |G_2|$.

Therefore, $d(G_1, G_3) = |G_1| + |G_3| - 2m(G_1, G_3) \leq |G_1| + |G_3| + 2|G_2| -$

$2m(G_1,G_2)-2m(G_2,G_3)=d(G_1,G_2)+d(G_2,G_3)$. The equality appears if and only if $|H_2\cup K_2|=|G_2|$ which is possible only in the case $H_2=G_2$ or $K_2=G_2$. In the opposite case we should have $|H_2\cup K_2|=|H_2|+|K_2|-|H_2\cap K_2|\le\frac{1}{2}|G_2|+\frac{1}{2}|G_2|-1<|G_2|$.

Proposition 2.2: For any two finite groups G_1 and G_2
(i) $d(G_1,G_2)=|G_1|+|G_2|-2$ if the orders are coprime
(ii) $d(G_1,G_2)\le|G_1|+|G_2|-2p$ if the orders are not coprime and p is the greatest prime number dividing the orders $|G_1|$ and $|G_2|$.

Clearly, the metric d is unbouded. So, for any two finite groups G_1,G_2 put: $\delta(G_1,G_2)=1-\frac{m(G_1,G_2)}{max(|G_1|,|G_2|)}$.

Proposition 2.3: The function δ is a metric which attends values in the interval $<0,1)$. If $|G_1|=|G_2|=n$, then $d(G_1,G_2)=2n\delta(G_1,G_2)$.
Proof: The triangle inequality is obvious if G_2 is isomorphic to G_1 or G_3. In the opposite case $m(G_1,G_2)\le\frac{1}{2}max(|G_1|,|G_2|)$ and $m(G_2,G_3)\le\frac{1}{2}max(|G_2|,|G_3|)$. Therefore $\delta(G_1,G_3)<1=\frac{1}{2}+\frac{1}{2}\le\delta(G_1,G_2)+\delta(G_2,G_3)$. The other properties are obvious.

3.SUPERGROUP METRIC

Copying the superstructurre metric of [3], we define

$$\rho(G_1,G_2)=2M(G_1,G_2)-|G_1|-|G_2|$$

.

Example 3.1: Let $|G_1|=5,|G_2|=2$ and $|G_3|=3$. Then $M(G_1,G_3)=15,M(G_1,G_2)=10,M(G_2,G_3)=6,\rho(G_1,G_3)=22,\rho(G_1,G_2)=13,\rho(G_2,G_3)=7$ and $\rho(G_1,G_3)>\rho(G_1,G_2)+\rho(G_2,G_3)$. So, the function ρ is not a metric.

Example 3.2: Let $G_1=C_8,G_2=C_4\times C_2$ and $G_3=C_2\times C_2\times C_2$, where C_n denotes the cyclic group of order n. Then $m(G_1,G_3)=2,m(G_1,G_2)=4=m(G_2,G_3)$. By Proposition 1.2., we have $M(G_1,G_3)\ge 32,M(G_1,G_2)\ge 16$ and $M(G_2,G_3)\ge 16$. Using the direct products $C_2\times C_2\times C_8,C_2\times C_8$ and $C_2\times C_2\times C_4$, we obtain $M(G_1,G_3)=32,M(G_1,G_2)=(G_2,G_3)=16,\rho(G_1,G_3)=48,\rho(G_1,G_2)=\rho(G_2,G_3)=16$ and $\rho(G_1,G_3)>\rho(G_1,G_2)+\rho(G_2,G_3)$. Thus, the function ρ is not a metric on the system of all groups of order 8.

Part (iii) of Proposition 1.2. may be rewrite as $m(G_1,G_2)\ge\frac{|G_1|\cdot|G_2|}{M(G_1,G_2)}$. Now, the right side may be considered as an alternative of the left side and we define supergroup alternatives of subgroups metrics d and δ

$$d_1(G_1,G_2)=|G_1|+|G_2|-2\frac{|G_1|\cdot|G_2|}{M(G_1,G_2)}$$

$$\delta_1(G_1,G_2)=1-\frac{min(|G_1|,|G_2|)}{M(G_1,G_2)}$$

The proof of the next proposition is similar to the proof of 2.3.

Proposition 3.3: The function δ_1 is a metric which attends values in the interval $< 0, 1)$. If $|G_1| = |G_2| = n$, then $d_1(G_1, G_2) = 2n\delta_1(G_1, G_2)$.

Collolary 3.4: The function d_1 is a metric on a system of all groups of order n.

Example 3.1: Let $G_1 = S_4, G_2 = D_4$ and $G_3 = D_{12}$. Then by Theorem 1.6., $M(G_1, G_3) = 96, M(G_1, G_2) = 24 = M(G_2, G_3), d_1(G_1, G_3) = 36, d_1(G_1, G_2) = 16 = d_1(G_2, G_3)$ and $d_1(G_1, G_3) > d_1(G_1, G_2) + d_1(G_2, G_3)$. So, the function d_1 is not a netric on the system of all groups.

Proposition 1.2. and Theorem 1.6. imply

Theorem3.5: For any finite groups G_1 and G_2

$$d_1(G_1, G_2) \leq d(G_1, G_2)$$

$$\delta_1(G_1, G_2) \leq \delta(G_1, G_2)$$

If $G_1 = S_4$ and $G_2 = D_{12}$ then the inequalities are strict.

All metrics considered in the present paper are not interesting from the topological point of view because they induce the discrete topology on any set of groups which does not contain isomorphic groups. These metrics are only number characteristics which express the degree of relationship of two groups. The same may be said about cited papers [1],[3],[4],[6] and[7].

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JUBILEE: THE SIXTIETH BIRTHDAY OF PROFESSOR TIBOR KATRIŇÁK

Miroslav Haviar and Pavol Zlatoš

Professor RNDr. Tibor Katriňák, DrSc., one of the leading personalities of Slovak mathematics, celebrated his sixtieth birthday this year.

Tibor Katriňák was born on March 23, 1937 in Košice. He attended a grammar school in Spišská Nová Ves. In 1955–60 he studied Mathematics at the Faculty of Natural Sciences of the Comenius University in Bratislava. After graduating in 1960 he commenced and followed the academic career at this faculty and, since 1980, at the newly established Faculty of Mathematics and Physics of the Comenius University. He received his CSc. (Candidate of Sciences, the former Czechoslovak equivalent of PhD) degree from the Comenius University in 1965. The academic year 1967–68 he spent as a *Humboldt-Stiftung* fellow with the Department of Mathematics of the Univeristy in Bonn, Germany, and became an associate professor in 1968, after his come back to Bratislava. He was awarded the degree DrSc. (Doctor of Sciences) in 1980, and, only after the turn, in 1990 he was promoted a full professor at the Department of Algebra and Number Theory of the Faculty of Mathematics and Physics.

Professor Katriňák is a world-recognized authority in the fields of Lattice Theory and Universal Algebra. Together with his great teacher and, later on, the closest colleague Professor Milan Kolibiar they have been the central personalities of the 'Bratislava School of Algebra' since the late sixties. He contributed a good deal to the good name of Slovak and Czechoslovak mathematics, as well as to the ranking of the Comenius University.

His primary research interest has been the study of lattices and semilattices, in particular the lattices and semilattices with pseudocomplementation. The theory of pseudocomplemented lattices (p-algebras) and pseudocomplemented semilattices, which has its origin in the study of non-classical logics, became a vital branch of lattice theory since the early sixties and Katriňák's contribution to this theory was enormous. His papers devoted to the theory of pseudocomplemented lattices and semilattices were cited, for example, in the monographs [S 69], [G 71], [Bl-J 72], [B-D 74], [Je 76], [G 78] (20 papers!), [S 82], [Bl-V 94] and in hundreds of papers. He became famous by characterizations of various classes of pseudocomplemented lattices and semilattices by means of triples of simpler structures associated with each member and by a systematic treatment of the triple constructions.

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There are three types of triple constructions for pseudocomplemented semilattices and lattices in the literature: the first one, discovered by W. Nemitz [N 65] for Heyting semilattices, was improved by P. V. R. Murty and V. V. R. Rao [Mu-R 74]. Their version, which is described, e.g., in C. C. Chen and G. Grätzer [C-G 69] for Stone algebras, was later on extended and modified to work for all distributive pseudocomplemented lattices and Heyting algebras by T. Katriňák [21], [18]. The third one originated in the papers [12], [23] by T. Katriňák and was later developed and modified by W. Cornish [Co 74], P. Mederly [M 74], J. Schmidt [JS 75], and again by T. Katriňák and P. Mederly in [30], [46] and, in particular, in [65] where the previous triple methods were generalized to the largest possible class of "decomposable" pseudocomplemented semilattices.

A *pseudocomplemented semilattice* (PCS) is a bounded meet-semilattice $(S; \wedge, 0, 1)$, such that for every $a \in S$ there exists the *pseudocomplement* $a^\star$ of a, defined by $a^\star = \max\{x \in S \mid x \wedge a = 0\}$. A PCS which is even a lattice is called a *pseudocomplemented lattice* (PCL). A *p-algebra* $(S; \wedge, \vee, ^\star, 0, 1)$ is a PCL with the pseudocomplement operation included into its signature. A p-algebra is *distributive* or *modular* if its underlying lattice has the respective property. An element a of a PCS S is said to be *closed* if $a = a^{\star\star}$ and an element $d \in S$ is called *dense* if $d^\star = 0$. The sets of all closed and dense elements of S are denoted by $B(S)$ and $D(S)$, respectively. $B(S)$ is a Boolean algebra and $D(S)$ is a semilattice with 1, and even a lattice filter in S in case S is a PCL. Unfortunately, these two "substructures" associated with S do not entirely characterize S. However, in the late sixties W. Nemitz and, independently, C. C. Chen and G. Grätzer showed that, under certain conditions, a third bit of information, namely a kind of a connective map $\varphi(S) : B(S) \to D(S)$, is sufficient to characterize S by means of the triple $(B(S), D(S), \varphi(S))$. This gave rise to the "triple methods" in the theory of PCS's, PCL's and p-algebras, ellaborated mainly by T. Katriňák.

We shall continue with some concepts and results of [65]. A PCS S is said to be *decomposable* if for every $x \in S$ there exists a $d \in D(S)$ such that $x = x^{\star\star} \wedge d$. In a decomposable PCS S one can define, for every $a \in B(S)$, a semilattice congruence relation $\theta_a(S)$ on $D(S)$ by $x \equiv y\ (\theta_a(S))$ iff $a^\star \wedge x = a^\star \wedge y$. The map $a \mapsto \theta(S)(a) = \theta_a(S)$ is a $(0,1)$-isotone map from $B(S)$ into $\mathrm{Con}(D(S))$ and $(B(S), D(S), \theta(S))$ is the *triple associated with* the decomposable PCS S. On the other hand, an abstract triple (B, D, θ) consists of a Boolean algebra B, a $\wedge$-semilattice D with 1 and a $(0,1)$-isotone map $\theta : B \to \mathrm{Con}(D)$. Two (abstract) triples (B, D, θ) and (B', D', θ') are *isomorphic* if there is an isomorphism of Boolean algebras $f : B \to B'$ and an isomorphism of semilattices $g : D \to D'$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\ \theta\ } & \mathrm{Con}(D) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \overline{g}} \\ B' & \xrightarrow{\ \theta'\ } & \mathrm{Con}(D') \end{array}$$

commutes. (Here $\overline{g} : \mathrm{Con}(D) \to \mathrm{Con}(D')$ stands for the isomorphism of of congruence lattices induced by g.)

The essence of the generalized triple method presented in [65] lies in the following two results:

1. Two decomposable PCS's are isomorphic if and only if their associated triples are isomorphic.
2. Let (B, D, θ) be an (abstract) triple. Then one can construct a decomposable PCS S such that its associated triple $(B(S), D(S), \theta(S))$ is isomorphic to (B, D, θ).

We note that the idea of decomposing a PCS S into triple using congruence relations on $D(S)$ occured already in Katriňák's most cited paper [12] and that especially his paper [23] brought a new idea which later on led to this third type of triple constructions. Other important achievements of [65] can be summarized as follows:

3. Connections between all previous triple constructions were clarified.
4. It was shown that all the previously studied decomposable PCS's were "filter-decomposable", meaning that every congruence $\theta_a(S)$ was determined by the filter $D(S) \cap [a^\star)$.
5. A triple construction for a large class of the so-called *quasi-modular* PCL's, obtained by weakening the concept of modularity for PCL's to the quasi-modular identity

$$((x \wedge y) \vee z^{\star\star}) \wedge x = (x \wedge y) \vee (z^{\star\star} \wedge x),$$

was presented.

6. It was shown how homomorphisms and congruence relations of PCL's can be studied by means of triples.
7. Possible directions of further development of the topic were indicated.

In [22] the Stone and Post algebras of order n were studied. Almost the whole paper was taken into the monograph [Ba-D 74] (chapter X). G. Epstein and A. Horn [E-H 74] consider the Stone algebras of order n introduced in [22] to be one of the most interesting generalizations of the Post algebras which are known as algebraic models of multi-valued logics.

In [20] and [35] triples associated with the free Stone algebras with m generators are characterized. This solved the problems formulated in [C-G 69] and [G 71; Problem 54]. By 1982, all the known papers describing free p-algebras were concerned with distributive p-algebras. In [61] T. Katriňák extended these results giving a characterization of free algebras for the whole variety of p-algebras.

The other area of research interests of Professor Katriňák was the study of subdirectly irreducible algebras in certain varieties of p-algebras [19], [26], [28], [48], and the study of varieties of p-algebras [27], [66]. In [27] it is shown that the lattice of lattice varieties can be embedded into the lattice of all varieties of p-algebras, answering a problem formulated in [G 71] and explaining the major difficulties one meets when dealing with p-algebras.

A series of papers of T. Katriňák is devoted to the study of double p-algebras, mainly to the properties of distributive double p-algebras [24], [31], [63], and to the constructions of regular double p-algebras [32] and modular double p-algebras [44]. His results about injective double Stone algebras [33] were used by R. Beazer, B. Davey, A. Romanowska, A. Urquhart and many others. Representations by congruence lattices of distributive p-algebras are investigated in [42] and [84]. Many results of T. Katriňák concern characterizations of lattices and algebras whose congruence lattices belong to some variety of p-algebras [70], [72], [73], [77]. Characterizations of projective p-algebras in the classes of distributive p-algebras and all p-algebras are given in the papers [87] and [78], respectively.

So far we have focused mainly on research activities of Professor Katriňák. But it is important to say that he has also been an inspired teacher who instructed and positively influenced the careers of many Slovak mathematicians of the middle and young generations. He supervised seven PhD students and many others within the annual "student scientific competition".

Professor Tibor Katriňák has accepted many responsibilities within the Faculty of Mathematics and Physics, Comenius University and both the Czecho-Slovak and the Slovak mathematical community. This has been particularly invaluable after 1989 when, also within the mathematical community, not many poeple were ready "to give more than they receive". T. Katriňák has devoted a lot of time and energy serving in many ways the mathematical community in Slovakia. He did a lot of professional and organizing work as a chairman of the Committees for candidate and doctoral dissertations in Algebra and Number Theory, as a member of the Slovak Grant Agency for Mathematical and Physical Sciences, member of the Scientific Boards both at the University and Faculty levels, Editor in Chief of the journals Acta Mathematica Universitatis Comenianae (AMUC) and Mathematica Slovaca, as a member of editorial boards and a reviewer for other mathematical journals, organizer and co-organizer of several Summer Schools in Algebra, and we could follow by many other less official responsibilities like, for example, the responsibility for the faculty library. By his unselfishness, friendliness and willingness to help or offer an advice, by his commitment to serve the mathematical community, he nobly continues in the work and mission of his great teacher and close friend, the late Professor Milan Kolibiar.

It remains to conclude by saying that on the occasion of his 60th birthday, the entire Slovak and Czech mathematical community wishes Tibor Katriňák good health and a lot of success in his scientific, pedagogical and organizational work, as well as in his personal life.

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