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GAUGE-NATURAL TRANSFORMATIONS OF SOME COTANGENT BUNDLES

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Dedicated to Anton Dekrét on the occasion of his 65-th birthday

Abstract. For an arbitrary vector bundle $E \to M$, we determine all gauge-natural transformations $T^*E \to T^*E^*$. In order to describe the result geometrically, we characterize some properties of T^*E in terms of an original approach to the concept of double vector bundle.

In [4] and [7], some important relations between the cotangent bundle of the tangent bundle T^*TM and the cotangent bundle of the cotangent bundle T^*T^*M of a manifold M were studied and applied. In the present paper we discuss the "pure" case of the cotangent bundle T^*E of any vector bundle $E \to M$ and the cotangent bundle T^*E^* of its dual E^*. First of all we show that T^*E with the canonical projections to E and E^* has the structure of double vector bundle. The origin of such a concept goes back to [3], [6], [9], but our approach in Section 2 is new and we find it very simple. In Section 3 we construct a canonical isomorphism $\varepsilon : T^*E \to T^*E^*$. In the next section we use the viewpoint of the theory of gauge-natural bundles, [1], [2], and we determine all gauge-natural (in other words: geometrical) transformations $T^*E \to T^*E^*$. In Section 5 we clarify how the canonical isomorphism ε relates a linear vector field on E and the dual vector field on E^*. All manifolds and maps are assumed to be infinitely differentiable.

1. Two vector bundle structures on T^*E

Consider a vector bundle $p : E \to M$ and its dual bundle $p^* : E^* \to M$. Write $q_1 : T^*E \to E$ for the cotangent bundle of E. According to [2], p.227, there exists another projection $q_2 : T^*E \to E^*$. For every linear map $w : T_yE \to \mathbb{R}$, $y \in E$, we define $q_2(w)$ to be the restriction of w to the vertical tangent space V_yE, which is canonicaly identified with the fiber E_x, $x = py$. Let x^i, y^p be some linear fiber coordinates on E and x^i, z_p be the the dual coordinates on E^*. The additional coordinates on T^*E are given by $w = v_i dx^i + u_p dy^p$. Then the coordinate form of q_1 or q_2 is

$$(1) \qquad q_1(x^i, y^p, u_p, v_i) = (x^i, y^p), \qquad q_2(x^i, y^p, u_p, v_i) = (x^i, u_p)$$

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We are going to show that $q_2 : T^*E \to E^*$ is a vector bundle too. We shall use the well-known fact there are two vector bundle structures $\pi_E : TE \to E$ and $Tp : TE \to TM$ on TE, [2]. If $X^i = dx^i$, $Y^p = dy^p$ are the additional coordinates on TE, then $Tp(x^i, y^p, X^i, Y^p) = (x^i, X^i)$. The vector addition and the multiplication by reals on Tp have the following form

$$(x^i, y_1^p, X^i, Y_1^p) + (x^i, y_2^p, X^i, Y_2^p) = (x^i, y_1^p + y_2^p, X^i, Y_1^p + Y_2^p),$$
$$k(x^i, y^p, X^i, Y^p) = (x^i, ky^p, X^i, kY^p). \tag{2}$$

Consider $w_1 \in T^*_{y_1}E$ and $w_2 \in T^*_{y_2}E$ satisfying $q_2(w_1) = q_2(w_2)$. Let us decompose, with respect to Tp, a vector $Y \in T_{y_1+y_2}E$ as $Y_1 + Y_2$, $Y_1 \in T_{y_1}E$, $Y_2 \in T_{y_2}E$, $Tp(Y) = Tp(Y_1) = Tp(Y_2)$. Then we define $w_1 + w_2 \in T^*_{y_1+y_2}E$ by

$$(w_1 + w_2)(Y) = w_1(Y_1) + w_2(Y_2) \tag{3}$$

The correctness follows from the coordinate form of (3). We have $Y = (x^i, y_1^p + y_2^p, X^i, Y^p)$, $Y_1 = (x^i, y_1^p, X^i, Y_1^p)$, $Y_2 = (x^i, y_2^p, X^i, Y_2^p)$ with $Y^p = Y_1^p + Y_2^p$ and $w_1 = (x^i, y_1^p, u_p, v_{1i})$, $w_2 = (x^i, y_2^p, u_p, v_{2i})$. Then $w_1(Y_1) + w_2(Y_2) = u_p Y_1^p + v_{1i}X^i + u_p Y_2^p + v_{2i}X^i = u_p Y^p + (v_{1i} + v_{2i})X^i$. Moreover, the latter expression implies

$$(x^i, y_1^p, u_p, v_{1i}) + (x^i, y_2^p, u_p, v_{2i}) = (x^i, y_1^p + y_2^p, u_p, v_{1i} + v_{2i}) \tag{4}$$

Further, if $w \in T^*_yE$, $0 \neq k \in \mathbb{R}$ and $Y \in T_{ky}E$, we construct $\frac{1}{k}Y$ with respect to the vector bundle structure Tp. Hence $\frac{1}{k}Y \in T_yE$ and we set

$$(kw)(Y) = kw(\frac{1}{k}Y) \tag{5}$$

For $k = 0$, we consider the restriction T_0E of E to the zero section $0 : M \to E$ and the tangent map $T0 : TM \to TE$. We have the following decomposition $T_0E = TM \times_M E$. We set $pr_1 Y = Tp(Y)$. Then $Y - T0(pr_1 Y)$ is a vertical vector, which is identified with $pr_2 Y \in E$. Now we define $0w \in T^*_{0(x)}E$, $x = py$, by

$$(0w)(Y) = (q_2 w)(pr_2 Y), \qquad Y \in T_{0(x)}E. \tag{6}$$

In coordinates, one finds easily

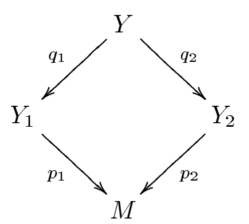
$$k(x^i, y^p, u_p, v_i) = (x^i, ky^p, u_p, kv_i), \qquad k \in \mathbb{R} \tag{7}$$

Clearly, (4) and (7) imply, that $q_2 : T^*E \to E^*$ is a vector bundle.

2. Double vector bundles

We define a fibered square to be a commutative diagram

(8)



in which all arrows are fibered manifolds. If there is no danger of confusion, we shall write Y for (8). The diagonal map in (8) will be denoted by $q : Y \to M$. If $(\bar{Y}, \bar{Y}_1, \bar{Y}_2, \bar{M}, \bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2)$ is another fibered square, then a morphism $Y \to \bar{Y}$ means a quadruple of maps $f : Y \to \bar{Y}$, $f_1 : Y_1 \to \bar{Y}_1$, $f_2 : Y_2 \to \bar{Y}_2$, $f_0 : M \to \bar{M}$ such that all pairs (f, f_1), (f, f_2), (f_1, f_0), (f_2, f_0) are fibered manifold morphisms. Hence we have a category $\mathcal{FS}$ of fibered squares.

A fibered square will be called a vector bundle square, if all arrows in (8) are vector bundles. For example, both $(T^*E, E, E^*, M, q_1, q_2, p, p^*)$ and $(TE, E, TM, M, \pi_E, Tp, p, \pi_M)$ are vector bundle squares. An $\mathcal{FS}$-morphism (f, f_1, f_2, f_0) of two vector bundle squares is said to be linear, if all pairs (f, f_1), (f, f_2), (f_1, f_0), (f_2, f_0) are vector bundle morphisms. Hence we obtain a category $\mathcal{VBS}$ of vector bundle squares.

Consider a manifold M and three vector spaces V_1, V_2, V_3. Put

$$(9) \qquad Y = M \times V_1 \times V_2 \times V_3, \qquad Y_1 = M \times V_1, \qquad Y_2 = M \times V_2.$$

Then we have canonical vector bundle structures on $Y \to Y_1$, $Y \to Y_2$, $Y_1 \to M$ and $Y_2 \to M$. We shall say that $Y = M \times V_1 \times V_2 \times V_3$ is a trivial double vector bundle.

Definition 1. *A vector bundle square* (8) *will be called a double vector bundle, if for every $x \in M$ there exists its neighbourhood $U \subset M$ such that $q^{-1}(U)$ is $\mathcal{VBS}$-isomorphic to a trivial double vector bundle.*

A morphism between two double vector bundles is an $\mathcal{VBS}$-morphism. Thus, we obtain a category $\mathcal{DVB}$ of double vector bundles.

Denote by $HY \subset Y$ the set of elements which are projected by q_1 into a zero vector in Y_1 and by q_2 into a zero vector in Y_2. By J. Pradines, [5], [6], HY is called the heart of the double vector bundle Y. In the trivial case we have $H(M \times V_1 \times V_2 \times V_3) = M \times V_3$. This implies that even in the general case both vector bundle structures q_1 and q_2 coincide on HY and $HY \to M$ is a vector bundle.

If $D \subset \mathbb{R}^m$ is an open subset and V is a vector space, then Section 1 implies

$$T^*(D \times V) = D \times V \times V^* \times \mathbb{R}^{m*}.$$

Quite similarly,

$$T(D \times V) = D \times V \times \mathbb{R}^m \times V.$$

Then one verifies easily

Proposition 1. *For every vector bundle E, both TE and T^*E are double vector bundles.*

We remark that a direct characterization of double vector bundles in terms of the underlying vector bundle structures can be deduced directly from the results of [8].

3. The canonical isomorphism $T^*E \to T^*E^*$

We are going to construct a canonical map $\varepsilon : T^*E \to T^*E^*$. Consider the evaluation map $e : E \times_M E^* \to \mathbb{R}$. Its differential

$$de : TE \times_{TM} TE^* \to \mathbb{R}$$

is the second component of the tangent map $Te : TE \times_{TM} TE^* \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$.

Proposition 2. *For every covector $C \in T^*_y E$, there exists a unique element $z \in E^*$ satisfying $py = p^*z$ and a unique covector $\varepsilon C \in T^*_z E^*$ such that every vectors $A \in T_y E$ and $B \in T_z E^*$ over the same vector $Tp(A) = Tp^*(B) \in TM$ satisfy*

$$de(A, B) = C(A) - (\varepsilon C)(B) \tag{10}$$

Proof. The coordinate form of the evaluation map e is $y^p z_p$, so that de is of the form

$$z_p dy^p + y^p dz_p.$$

Consider $C = (x^i, y^p, u_p, v_i)$, $A = (x^i, y^p, a^i, a^p)$, $B = (x^i, g_p, a^i, b_p)$ and write $D = (x^i, g_p, h^p, k_i)$. Then the condition $de(A, B) = C(A) - D(B)$ reads

$$g_p a^p + y^p b_p = v_i a^i + u_p a^p - h^p b_p - k_i a^i.$$

Since a^i, a^p and b_p are arbitrary, the unique $D = \varepsilon C$ is of the form

$$g_p = u_p, \qquad h^p = -y^p, \qquad k_i = v_i. \tag{11}$$

Definition 2. *The map* (11) *from Proposition 2 will be called the canonical isomorphism $\varepsilon : T^*E \to T^*E^*$.*

Clearly, ε is an isomorphism of double vector bundles.

4. All gauge-natural transformations $T^*E \to T^*E^*$

Using the viewpoint of the theory of gauge-natural bundles, [1], [2], we can say that $\varepsilon : T^*E \to T^*E^*$ is a gauge-natural transformation. We are going to determine all gauge-natural transformations $T^*E \to T^*E^*$. For every $w \in T^*E$, we have $q_1(w) \in E$, $q_2(w) \in E^*$ and $p_1(q_1(w)) = p_2(q_2(w))$, so that we can evaluate $\langle q_1(w), q_2(w)\rangle \in \mathbb{R}$. Write $\eta \mapsto (k)_1\eta$ or $\eta \mapsto (k)_2\eta$, $k \in \mathbb{R}$, $\eta \in T^*E^*$, for the scalar multiplication with respect to the first or second vector bundle structure on T^*E^*, respectively.

Proposition 3. *All gauge-natural transformations $T^*E \to T^*E^*$ are of the form*

$$w \mapsto A(\langle q_1(w), q_2(w)\rangle)_1 B(\langle q_1(w), q_2(w)\rangle)_2 \varepsilon(w), \tag{12}$$

where $A(t)$ and $B(t)$ are two arbitrary smooth functions of one variable.

Proof. As remarked in [2], p.409, the category $\mathcal{VB}_{m,n}$ of vector bundles with m-dimensional bases and n-dimensional fibers and their local isomorphisms is naturally equivalent to the category $\mathcal{PB}_m(GL(n))$ of $GL(n)$-principal bundles with m-dimensional bases and their local isomorphisms. Consider the trivial vector bundle

$\mathbb{R}^m \times \mathbb{R}^n$ and write $S = (T^*(\mathbb{R}^m \times \mathbb{R}^n))_0$ and $Z = (T^*(\mathbb{R}^m \times \mathbb{R}^{n*}))_0$, $0 \in \mathbb{R}^m$. Then both S and Z are $W^1_m GL(n)$-spaces and all gauge-natural transformations $T^*E \to T^*E^*$ are in the bijection with the $W^1_m GL(n)$-equivariant maps $S \to Z$, [2], Chapter XII.

On S, we have the coordinates y^p, u_p, v_i from Section 1. Every element of $T^*(\mathbb{R}^m \times \mathbb{R}^{n*})$ can be written in the form $\xi_i dx^i + \mu^p dz_p$, so that z_p, μ^p, ξ_i are the corresponding coordinates on Z. The elements of $W^1_m GL(n)$ are of the form (a^i_j, a^p_q, a^p_{qi}), $\det a^i_j \neq 0$, $\det a^p_q \neq 0$, [2], p.153. Using standard evaluations, one finds easily the following actions of $W^1_m GL(n)$ on S and Z

$$\bar{y}^p = a^p_q y^q, \qquad \bar{u}_p = \tilde{a}^q_p u_q, \qquad \bar{v}_i = \tilde{a}^j_i v_j - \tilde{a}^j_i \tilde{a}^p_r a^r_{qj} u_p y^q \tag{13}$$

$$\bar{z}_p = \tilde{a}^q_p z_q, \qquad \bar{\mu}^p = a^p_q \mu^g \qquad \bar{\xi}_i = \tilde{a}^j_i \xi_j + \tilde{a}^j_i \tilde{a}^p_r a^r_{qj} z_p \mu^q \tag{14}$$

where $\tilde{a}^i_j$ or $\tilde{a}^p_q$ is the inverse matrix to a^i_j or a^p_q, respectively. To determine all $W^1_m GL(n)$-equivariant maps $f : S \to Z$, we shall use the methods from [2]. Thus, let us start from an arbitrary map

$$z_p = f_p(y, u, v), \qquad \mu^p = f^p(y, u, v), \qquad \xi_i = f_i(y, u, v). \tag{15}$$

Consider first the equivariancy with respect to the canonical injection $GL(m) \times GL(n) \to W^1_m GL(n)$, which is characterized by $a^p_{qi} = 0$. Using the homotheties in $GL(m)$, we find that f_p and f^p are independent of v and f_i is linear in v_i. The equivariancy with respect to the whole group $GL(m)$ yields

$$f_i = g(y, u) v_i. \tag{16}$$

Then we consider equivariancy with respect to $GL(n)$. The tensor evaluation theorem, [2], Section 26, yields

$$f^p = -A(y^q u_q) y^p, \qquad f_p = B(y^q u_q) v_p. \tag{17}$$

Consider now the equivariancy with respect to the kernel K of the canonical projection $W^1_m GL(n) \to GL(m) \times GL(n)$, which is characterized by $a^i_j = \delta^i_j$, $a^p_q = \delta^p_q$. This yields

$$a^p_{qi} A u_p B y^q = a^p_{qi} g u_p y^q.$$

Thus,

$$g(y, u) = A(y^p u_p) B(y^q u_q) \tag{18}$$

Clearly, (16)-(18) is the coordinate form of (12). $\square$

5. Linear vector fields

In general, every vector field $X : N \to TN$ on a manifold N defines a function $\widetilde{X} : T^*N \to \mathbb{R}$, $\widetilde{X}(z) = \langle X(x), z\rangle$, $z \in T^*_x N$. Conversely, every function $f : T^*N \to \mathbb{R}$ linear on each fiber is of the form $f = \widetilde{X}$ for a vector field $X : N \to TN$.

Consider a linear vector field $X : E \to TE$, [2], p.379. Its coordinate form is

$$X^i(x)\frac{\partial}{\partial x^i} + X^p_q(x)y^q\frac{\partial}{\partial y^p}.$$

Using flows, one constructs the dual vector field $X^* : E^* \to TE^*$, [2], p.380, whose coordinate expression is

$$X^i(x)\frac{\partial}{\partial x^i} - X^q_p(x)z_q\frac{\partial}{\partial z_p}.$$

Proposition 4. *For every linear vector field $X : E \to TE$, we have*

$$\widetilde{X} = \widetilde{X^*} \circ \varepsilon.$$

Proof. In the coordinates of the proof of Proposition 2, $\widetilde{X} = X^i(x)v_i + X^p_q(x)y^q u_p$, $\widetilde{X^*} = X^i(x)k_i - X^p_q(x)g_p h^q$. Then (11) yields our claim.

6. Remark

It can be expected from the trivialization $T(D \times V^*) = D \times V^* \times \mathbb{R}^m \times V^*$, that there is no natural isomorphism $TE \to TE^*$. To confirm it rigorously, one can determine all gauge-natural transformations $TE \to TE^*$. Using the basic methods from [2], one obtains easily the following result. Consider the projection $Tp : TE \to TM$, the homothetic transformation $k_M : TM \to TM$, $v \mapsto kv$, $k \in \mathbb{R}$, and the tangent map $T0 : TM \to TE^*$ of the zero section $0 : M \to E^*$. All gauge-natural transformations $TE \to TE^*$ are of the form

$$T0 \circ k_M \circ Tp : TE \to TE^*, \qquad k \in \mathbb{R}.$$

Clearly, none of them is an isomorphism.

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