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EXISTENCE OF INVARIANT TORI OF CRITICAL DIFFERENTIAL EQUATION SYSTEMS DEPENDING ON MORE-DIMENSIONAL PARAMETER. PART I.

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Dedicated to Anton Dekrét on the occasion of his 65-th birthday

ABSTRACT. In the paper a system of differential equations depending on more-dimensional parameter with the matrix of the first linear approximation P having pure imaginary eigenvalues while the others do not lie on the imaginary axis is studied. Conditions under which such a system has invariant tori are presented (section 1). In sections 2, 3 the cases when P has one and two pairs of pure imaginary eigenvalues are investigated. In Part II the cases with three and four pairs of pure imaginary eigenvalues will be analysed.

Introduction

In the monograph [1] Yu. N. Bibikov studies the system of differential equations depending on a small non-negative parameter μ:

$$\dot{x} = X(x,\mu) + X^*(x,\mu) \ , \tag{1}$$

where $x = (x_1, \dots, x_n)$, $X(x,\mu)$ - a vector polynomial with respect to x, μ, $X(0,0) = 0$, $X^*(x,\mu) : \mathbb{M} \to \mathbb{R}^n$, $M = \{(x,\mu) : ||x|| < K, \ 0 \le \mu < L\}$ - a continuous vector function with the property:

$$X^*(\sqrt{\mu}x, \mu) = (\sqrt{\mu})^{3p+2}\tilde{X}(x,\mu) \ ,$$

p - a natural number, $\tilde{X}(x,\mu)$ - a function of the class $C^{10}_{x\mu}(\mathbb{M})$. It is supposed that the spectrum of the linear approximation matrix P of the polynomial $X(x,\mu)$ consists of m pairs of pure imaginary eigenvalues while the others have non-zero real parts. Yu. N. Bibikov found conditions under which to every small parameter μ there exists an invariant manifold of the system (1) that is homeomorphic with a torus. He also presents in [1] an idea how these results can be utilized in the case when the parameter μ is m-dimensional one, where m is the number of the pairs of pure imaginary eigenvalues of the matrix P.

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In applications the dimension of the parameter μ is not a function of the number of pure imaginary eigenvalues of P but it follows from the character of a process which is described by the considered system. Therefore it is worth studying the system (1) which depend on the more-dimensional parameter μ with an arbitrary dimension.

In this article the system (1) is investigated on the domain:

$$(2)\ \ \mathbb{M} = \{(x,y) : x = (x_1,\dots,x_n),\ \mu = (\mu_1,\dots,\mu_d),\ d \geq 1,\ ||x|| < K,\ ||\mu|| < L\}$$

(in the whole article Euclidean norm is used).

Let us take an arbitrary parameter $\mu \in \mathbb{M}$. Consider the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 \leq \varepsilon < L,\ \mu_0 = \frac{\mu}{||\mu||}\}$ (index "o" at parameters μ will always have this meaning). The system (1) depending on parameters $\mu \in \delta(\mu_0)$ has the form:

$$(3) \qquad \dot{x} = X(x,\varepsilon\mu_0) + X^*(x,\varepsilon\mu_0),\ 0 \leq \varepsilon < L\ .$$

The system (3) is the system of differential equations depending on one-dimensional non-negative parameter ε. It means the system (3) is the system of the kind (1) which was studied in [1]. Such an access enables to investigate the system (1) on the domain (2) and utilize the results achieved in [1]. Doing it the problem of determining subsets of the set $\mathbb{M}$ with respect to μ on which invariant manifolds of the system (1) exist arises.

In section 1 preliminary transformations of the system (1) depending on parameters μ from the domain (2) are performed enabling to utilize the results from [1]. In sections 2, 3 the cases when the matrix P has one and two pairs of pure imaginary eigenvalues are studied.

1. The existence of invariant tori

Consider the system of differential equations

$$(1.1) \qquad \dot{x} = X(x,\mu) + X^*(x,\mu)\ ,$$

where $x = (x_1,\dots,x_n)$, $\mu = (\mu_1,\dots,\mu_d)$, $\dot{x} = \frac{dx}{dt}$, $X(x,\mu)$ - a vector polynomial with respect to x,μ, $X(0,0) = 0$, $X^*(x,\mu) : \mathbb{M} \to \mathbb{R}^n$, $\mathbb{M} = \{x,\mu) : ||x|| < K,\ ||\mu|| < L\}$ - a continuous function with the property:

$$(1.2) \qquad X^*(\sqrt{\varepsilon}x,\varepsilon\mu_0) = (\sqrt{\varepsilon})^{3p+2}\tilde{X}(x,\varepsilon,\mu_0),$$

$0 \leq \varepsilon < L$, $\mu \in \mathbb{M}$, p - a natural number, $\tilde{X}(x,\varepsilon,\mu_0)$ - a continuous function with respect to x,ε,μ_0 of the class $C^1_x(\mathbb{M})$.

We suppose that the matrix $P = \frac{\partial X(0,0)}{\partial x}$ has m pairs of pure imaginary eigenvalues $\pm i\lambda_1,\dots,\pm i\lambda_m$ and the others $\lambda_{2m+1},\dots,\lambda_n$ have non-zero real parts. Further we suppose that $\det P \neq 0$.

Note 1.1. The requirements on the functions $X(x,\mu)$, $X^*(x,\mu)$ in (1.1) are not very limiting as every system $\dot{x} = f(x,\mu)$, $f(x,\mu) \in C^{3p+3}(\mathbb{M})$, $f(0,0) = 0$, can be expressed in the form (1.1). For that it is sufficient to introduce the function $f(x,\mu)$ in the form of the Taylor polynomial with the Lagrange form of the remainder. In this case $X(x,\mu) = \sum_{k=0}^{N} X_k(x_1,\dots,x_n)\, \mu_1^{k_1} \dots \mu_d^{k_d}$, $k = k_1 + \dots + k_d$, N - the whole part of the number $\frac{3p+1}{2}$, $X_k(x_1,\dots,x_n)$ - polynomials of the degree not higher then $3p+1-2k$.

Let us denote $F(x,\mu) = X(x,\mu) + X^*(x,\mu)$. In the power of (1.2) $F(0,0) = 0$. This means that the origin $(x,\mu) = (0,0)$ is the state of equilibrium of the system (1.1). Since

$$\frac{\partial X^*(x,\mu)}{\partial x} = \frac{\partial X^*(\sqrt{\varepsilon}y, \varepsilon\mu_0)}{\partial x} = \frac{\partial}{\partial x}\left[(\sqrt{\varepsilon})^{3p+2}\tilde{X}(y,\varepsilon,\mu_0)\right] =$$
$$= \frac{\partial}{\partial y}\left[(\sqrt{\varepsilon})^{3p+2}\tilde{X}(y,\varepsilon,\mu_0)\right]\cdot\frac{\partial y}{\partial x} = (\sqrt{\varepsilon})^{3p+1}\frac{\partial}{\partial y}\tilde{X}(y,\varepsilon,\mu_0)\ ,$$

we have:

$$\left|\frac{\partial F(0,0)}{\partial x}\right| = \left|\frac{\partial X(0,0)}{\partial x} + \frac{\partial X(0,0)}{\partial x}\right| = |P| \neq 0\ .$$

Using Implicit Function Theorem on the function $F(x,\mu)$ we get that in a small neighbourhood $O(0)$ of the origin $\mu = 0$ there exists a function $x = \psi(\mu)$ with the following properties:

1. $\psi(0) = 0$
2. $F[\psi(\mu),\mu] = 0$ for $\mu \in O(0)$.

We see that to every small enough parameter $\mu^* \in \mathbb{M}$ there exists the state of equilibrium of the system (1.1) $x^* = \psi(\mu^*)$. It will be shown that to such a μ^* there exists also under certain conditions an invariant manifold of the system (1.1) which is homeomorphic with a torus. When such a situation realizes we say that at $\mu = 0$ the bifurcation of an invariant torus arises from the state of equilibrium $x = 0$.

Lemma 1.1. *System (1.1) can be reduced by the transformation*

$$x = S\xi + T\mu\ , \tag{1.3}$$

where $\xi = col(y,\bar{y},z)$, $y = col(y_1,\dots,y_m)$, y *- the complex conjugate vector to* y *(in the article the symbol "*$\bar{a}$*" always means the complex conjugate expression to* a*,* $z = col(z_1,\dots,z_{n-2m})$, S *- a regular* $n\times n$*-matrix,* T *-* $n\times d$*-matrix, to the system*

$$\begin{aligned} \dot{y} &= i\lambda y + Y(y,\bar{y},z,\mu) + Y^*(y,\bar{y},z,\mu) \\ \bar{y}^{\cdot} &= i\lambda\bar{y} + \bar{Y}(y,\bar{y},z,\mu) + \bar{Y}^*(y,\bar{y},z,\mu) \\ \dot{z} &= Jz + Z(y,\bar{y},z,\mu) + Z^*(y,\bar{y},z,\mu)\ , \end{aligned} \tag{1.4}$$

where $\lambda = diag(\lambda_1,\dots,\lambda_m)$, J *- a Jordan canonical lower matrix,* $Y,\bar{Y},Z$ *- vector polynomials without scalar and linear terms,* $Y^*,\bar{Y}^*,Z^*$ *- continuous functions having the property (1.2), i.e. for example*

$$Y(\sqrt{\varepsilon}y,\sqrt{\varepsilon}\bar{y},\sqrt{\varepsilon}z,\ \varepsilon\mu_0) = (\sqrt{\varepsilon})^{3p+2}\tilde{Y}(y,\bar{y},z,\varepsilon,\mu_0)\ ,$$

$\tilde{Y}$ - *a continuous function of the class* $C^1_{y,\bar{y},z}$ *in a neighbourhood of the point* $y = 0, z = 0, 0 \le \varepsilon < L,\ \mu \in \mathbb{M}$. *The second equation in (1.4) is conjugated to the first one in (1.4) and can be gained from this by the change* y *for* $\bar{y}$, $\bar{y}$ *for* y *and* i *for* $-i$. *Further equations which will be conjugated to another ones will not be written.*

Proof. Expressing (1.1) in the form

$$\dot{x} = Px + Qx + X^1(x, \mu) + X^*(x, \mu) \tag{1.5}$$

and putting (1.3) into (1.5) we get:

$$S\dot{\xi} = P(S\xi + T\mu) + Q\mu + X^1(S\xi + T\mu, \mu) + X^*(S\xi + T\mu, \mu)\ .$$

From this we have:

$$\dot{\xi} = S^{-1}PS\xi + (S^{-1}PT + S^{-1}Q)\mu + S^{-1}X^1 + S^{-1}X^*\ . \tag{1.6}$$

If the matrices S, T are taken in the way to get: $S^{-1}PS = diag(i\lambda, -i\lambda, J),\ T = -P^{-1}Q$, then (1.6) gives the system (1.4). The proof is over.

Consider now the system

$$\begin{aligned} \dot{y} &= i\lambda y + Y(y, \bar{y}, z, \mu) \\ \dot{z} &= Jz + Z(y, \bar{y}, z, \mu)\ , \end{aligned} \tag{1.7}$$

which is gained from the system (1.4) by taking away the functions Y^*, Z^*.

Lemma 1.2. *Let the eigenvalues* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *of the matrix* P *satisfy the condition:*

$$q_1\lambda_1 + \cdots + q_m\lambda_m \neq 0 \quad \text{for} \quad 0 < |q| \le 3p + 2\ , \tag{1.8}$$

$|q| = |q_1| + \cdots + |q_m|,\ q_i$ *- integer numbers,* $i = 1, \ldots, m$.

There exists a polynomial transformation

$$\begin{aligned} y &= u + h(u, \bar{u}, \mu) \\ z &= v + g(u, \bar{u}, \mu)\ , \end{aligned} \tag{1.9}$$

where $u = (u_1, \ldots, u_m),\ v = (v_1, \ldots, v_{n-2m}),\ h, g$ are polynomials without scalar and linear terms, that reduces the system (1.7) to the system

$$\begin{aligned} \dot{u} &= i\lambda u + uU(u \cdot \bar{u}, \mu) + U^0(u, \bar{u}, v, \mu) + U^*(u, \bar{u}, v, \mu) \\ \dot{v} &= Jv + V^0(u, \bar{u}, v, \mu) + V^*(u, \bar{u}, v, \mu)\ , \end{aligned} \tag{1.10}$$

where $U(u\cdot\bar{u},\mu)$ - a vector polynomial with respect to $u\cdot\bar{u},\mu$ without scalar terms, $U^0(u,\bar{u},0,\mu)=0$, $V^0(u,\bar{u},0,\mu)=0$, U^*,V^* have the property (1.2).

Proof. Differentiating (1.9) with respect to t and taking into account (1.7) and (1.10) we obtain:

$$i\lambda(u+h)+Y(u+h,\bar{u}+\bar{h},v+g,\mu)=i\lambda u+uU+U^0+U^*+$$
$$+\frac{\partial h}{\partial u}(i\lambda u+uU+U^0+U^*)+\frac{\partial h}{\partial \bar{u}}(-i\lambda\bar{u}+\bar{u}\bar{U}+\bar{U}^0+\bar{U}^*)$$
$$J(v+g)+Z(u+h,\bar{u}+\bar{h},v+g,\mu)=Jv+V^0+V^*+\frac{\partial g}{\partial u}(i\lambda u+uU+U^0+U^*)+$$
$$+\frac{\partial g}{\partial \bar{u}}(-i\lambda\bar{u}+\bar{u}\bar{U}+\bar{U}^0+\bar{U}^*)\ .$$

Giving away expressions with the property (1.2) and putting $v=0$ we get from these equations:

$$i\lambda u\frac{\partial h}{\partial u}-i\lambda\bar{u}\frac{\partial h}{\partial \bar{u}}-i\lambda h=Y(u+h,\bar{u}+\bar{h},g,\mu)-uU\frac{\partial h}{\partial u}-\bar{u}\bar{U}\frac{\partial h}{\partial \bar{u}}-uU$$

(1.11)

$$i\lambda u\frac{\partial g}{\partial u}-i\lambda\bar{u}\frac{\partial g}{\partial \bar{u}}-Jg=Z(u+h,\bar{u}+\bar{h},g,\mu)-uU\frac{\partial g}{\partial u}-\bar{u}\bar{U}\frac{\partial g}{\partial \bar{u}}\ .$$

Expressing the polynomials h,g in the form of the sum of vector homogenous polynomials $h^{(s)},g^{(s)},s$ - the degree, we get from (1.11) that $h^{(s)},g^{(s)}$ are determined by the equations:

$$i\lambda u\frac{\partial h^{(s)}}{\partial u}-i\lambda\bar{u}\frac{\partial h^{(s)}}{\partial \bar{u}}-i\lambda h^{(s)}=P^{(s)}(h^{(i)},g^{(j)})-(uU)^{(s)}$$

(1.12)

$$i\lambda u\frac{\partial g^{(s)}}{\partial u}-i\lambda\bar{u}\frac{\partial g^{(s)}}{\partial \bar{u}}=R^{(s)}(h^{(i)},g^{(j)}),\ i<s,j<s\ .$$

We see that if we calculate $h^{(s)},g^{(s)}$ in the direction of arising s then the functions $P^{(s)},R^{(s)}$ will be known for every s. For the coefficients $h_k^{(q,\tilde{q},r)}$, $g_k^{(q,\tilde{q},r)}$, $q=(q_1,\dots,q_m)$, $\tilde{q}=(\tilde{q}_1,\dots,\tilde{q}_m)$, $r=(r_1,\dots,r_d)$ of the polynomials $h^{(s)}=col(h_1^{(s)},\dots,h_m^{(s)})$, $g^{(s)}=col(g_1^{(s)},\dots,g_{n-2m}^{(s)})$ we get from (1.12) the equations:

(1.13)

$$i\left[\sum_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_k\right]h_k^{(q,\tilde{q},r)}=P_k^{(q,\tilde{q},r)}-U_k^{(q,\tilde{q},r)},\ k=1,\dots,m$$

(1.14)

$$i\left[\sum_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_{2m+l}\right]g_l^{(q,\tilde{q},r)}=R_l^{(q,\tilde{q},r)},\ l=1,\dots,n-2m.$$

When $(q,\tilde{q},r)$ is such a set that $q_j=\tilde{q}_j,\ q_k=\tilde{q}_k+1,\ j=1,\dots,m,\ j\neq k$, then $\sum_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_k=0$ in (1.13). In this case we put $U_k^{(q,\tilde{q},r)}=P_k^{(q,\tilde{q},r)}$ and $h_k^{(q,\tilde{q},r)}=0$. For other sets $(q,\tilde{q},r)$ in the power of (1.8) $\sum_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_k\neq 0$. In these cases we put $U_k^{(q,\tilde{q},r)}=0$. Then the corresponding coefficient $h_k^{(q,\tilde{q},r)}$ is determined by equation (1.13) uniquely. The coefficients $g_l^{(q,\tilde{q},r)}$ in (1.14) are determined uniquely for every set of $(q,\tilde{q},r)$ as $\sum_{j=1}^{m}(q_j-\tilde{q}_j)\lambda_j-\lambda_{2m+l}\neq 0$ since $Re\lambda_{2m+l}\neq 0,\ l=1,\dots,n-2m$. The proof is over.

Let us perform the transformation (1.9) on the system (1.4). We again get system (1.10) but this time with another functions U^*,V^* having again the property (1.2). Introducing into this system polar coordinates

$$u=\rho e^{i\varphi},\quad \bar{u}=\rho e^{-i\varphi}\ , \tag{1.15}$$

$\rho=col(\rho_1,\dots,\rho_m)$, $\varphi=col(\varphi_1,\dots,\varphi_m)$, $e^{i\varphi}=col(e^{i\varphi_1},\dots,e^{i\varphi_m})$, we get:

$$\begin{aligned}\dot{\rho}&=\rho F(\rho^2,\mu)+F^0(\rho,\varphi,v,\mu)+F^*(\rho,\varphi,v,\mu)\\ \dot{\varphi}&=\lambda+\Phi(\rho^2,\mu)+\rho^{-1}[\Phi^0(\rho,\varphi,v,\mu)+\Phi^*(\rho,\varphi,v,\mu)]\\ \dot{v}&=Jv+V^0(\rho e^{i\varphi},\rho e^{-i\varphi},v,\mu)+V^*(\rho e^{i\varphi},\rho e^{-i\varphi},v,\mu)\ ,\end{aligned} \tag{1.16}$$

where $\rho^2=(\rho_1^2,\dots,\rho_m^2)$, $\rho^{-1}=(\rho_1^{-1},\dots,\rho_m^{-1})$, $F=ReU(\rho^2,\mu)$, $\Phi=ImU(\rho^2,\mu)$, $F^0+F^*=Ree^{-i\varphi}[U^0(\rho e^{i\varphi},\ \rho e^{-i\varphi},v,\mu)+U^*(\rho e^{i\varphi},\ \rho e^{-i\varphi},v,\mu)]$, $\Phi^0+\Phi^*=Ime^{-i\varphi}[U^0(\rho e^{i\varphi},\rho e^{-i\varphi},v,\mu)+U^*(\rho e^{i\varphi},\rho e^{-i\varphi},v,\mu)]$, $F^0(\rho,\varphi,0,\mu)=0$, $\Phi^0(\rho,v,0,\mu)=0$, $F^*(\sqrt{\varepsilon}\rho,\varphi,\sqrt{\varepsilon}v,\varepsilon\mu_0)=(\sqrt{\varepsilon})^{3p+2}\tilde{F}(\rho,\varphi,v,\varepsilon,\mu_0)$, $\Phi^*(\sqrt{\varepsilon}\rho,\varphi,\sqrt{\varepsilon}v,\varepsilon\mu_0)=\sqrt{\varepsilon}^{3p+2}\tilde{\Phi}(\rho,\varphi,v,\varepsilon,\mu_0)$, $\tilde{F}$, $\tilde{\Phi}$ - continuous functions with respect to all variables of the class $C^1_{\rho,\varphi,v}$. All functions in (1.16) depending on φ are 2π-periodic with respect to all components of the vector φ.

Denote the linear parts of the function $F(\rho^2,\mu)$ by the expression $B\rho^2+C\mu$, where

$$B=\begin{pmatrix}B_{11}&\dots&B_{1m}\\ \dots&\dots&\dots\\ B_{m1}&\dots&B_{mm}\end{pmatrix}\ ,\qquad C=\begin{pmatrix}C_{11}&\dots&C_{1o}\\ \dots&\dots&\dots\\ C_{m1}&\dots&C_{mo}\end{pmatrix}\ .$$

The equation

$$B\rho^2+C\mu=0 \tag{1.17}$$

is called the bifurcation equation of system (1.16).

Let us suppose that $\det B\neq 0$ and that at least one element of the matrix C is different from zero.

Take an arbitrary $\mu\in\mathbb{M}$. The bifurcation equation (1.17) on the beam $\delta(\mu_0)=\{\varepsilon\mu_0:\ 0\le\varepsilon<L\}$ has the form:

$$B\rho^2+\varepsilon C\mu_0=0\ .$$

Solving this equation with respect to ρ^2 we have:

$$\rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0) \ ,$$

where $\alpha^2(\mu_0) = col[\alpha^2(\mu_0), \dots, \alpha_m^2(\mu_0)] = \Lambda\mu_0$,

$$\Lambda = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1o} \\ \dots & \dots & \dots \\ \alpha_{m1} & \dots & \alpha_{mo} \end{pmatrix} .$$

We say that the bifurcation equation (1.17) satisfies the condition of positiveness at $\mu \in \mathbb{M}$ if $\alpha^2(\mu_0)$ is positive at every component $\alpha_k^2(\mu_0)$, $k = 1, \dots, m$. Let $\mathcal{DP}$ denote the subset of all parameters $\mu \in \mathbb{M}$ at which the bifurcation equation satisfies the condition of positiveness. We shall call this subset $\mathcal{DP}$ the domain of positiveness of the bifurcation equation (1.17).

Lemma 1.3. *The domain of positiveness $\mathcal{DP}$ of the bifurcation equation (1.17) is an open cone with the apex at the origin $\mu = 0$ consisting of beams $\delta(\mu_0) = \{\varepsilon\mu_0 : \mu \in \mathbb{M}, 0 < \varepsilon < L, \alpha_k^2(\mu_0) > 0, k = 1, \dots, m\}$.*

Proof. Consider an arbitrary $\mu^* \in \mathcal{DP}$ and take an arbitrary $\mu \in \delta(\mu_0^*)$, $\mu = \varepsilon\mu_0^*$, $\varepsilon = ||\mu||$. As $\alpha^2(\mu_0) = col[\alpha_1^2(\mu_0), \dots, \alpha_m^2(\mu_0)]$ and $\alpha_k^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{k1}\mu_1 + \dots + \alpha_{kd}\mu_d) = \frac{1}{||\mu||}(\alpha_{k1}\varepsilon\frac{\mu_1^*}{||\mu^*||} + \dots + \alpha_{kd}\varepsilon\frac{\mu_d^*}{||\mu^*||}) = \frac{1}{||\mu^*||}(\alpha_{k1}\mu_1^* + \dots + \alpha_{kd}\mu_d^*) = \alpha_k^2(\mu_0^*) > 0$, $k = 1, \dots, m$, we get that $\delta(\mu_0^*) \subset \mathcal{DP}$. This means that $\mathcal{DP}$ is a cone. We need to show yet that to this μ^* there exists such $\sigma > 0$ that the sphere $O_\sigma(\mu^*) \subset \mathcal{DP}$. As $\mu^* \in \mathcal{DP}$ so $\alpha_k^2(\mu_0^*) = \nu_k > 0$, $k = 1, \dots, m$. Take an arbitrary μ from a sphere $O_\sigma(\mu^*)$, $\mu \neq \mu^*$. Then $\alpha_k^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{k1}\mu_1 + \dots + \alpha_{kd}\mu_d) = \frac{1}{||\mu||}[\alpha_{k1}(\mu_1^* + \sigma_1) + \dots + \alpha_{kd}(\mu_d^* + \sigma_d)]$, $-\sigma < \sigma_j < \sigma$, $j = 1, \dots, d$, $k = 1, \dots, m$. From this equation we have:

$$\begin{aligned} &\alpha_k^2(\mu_0) > \frac{1}{||\mu^*|| + \sigma}(\alpha_{k1}\mu_1^* + \dots + \alpha_{kd}\mu_d^*) - \frac{1}{||\mu^*|| - \sigma}d\alpha\sigma \ , \\ &\alpha = \max\{|\alpha_{kl}|\}, \ k = 1, \dots, m; \ l = 1, \dots, d \ . \end{aligned}$$

If we take $\sigma = \frac{||\mu^*||}{s}$ then we get from the last inequality:

$$\alpha_k^2(\mu_0) > \frac{s}{s+1}\alpha_k^2(\mu_0^*) - \frac{d\alpha}{s-1} > \frac{s}{s+1}\nu - \frac{d\alpha}{s-1} > 0$$

for big enough natural number s, $\nu = \min\{\nu_1, \dots, \nu_m\}$, $k = 1, \dots, m$. The proof is over.

Let us take an arbitrary $\mu \in \mathcal{DP}$. On the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 \leq \varepsilon < L\}$ the system (1.16) has the form:

$$\begin{aligned} &\dot\rho = \rho F(\rho^2, \varepsilon\mu_0) + F^0(\rho, \varphi, v, \varepsilon\mu_0) + F^*(\rho, \varphi, v, \varepsilon\mu_0) \\ &\dot\varphi = \lambda + \Phi(\rho^2, \varepsilon\mu_0) + \rho^{-1}[\Phi^0(\rho, \varphi, v, \varepsilon\mu_0) + \Phi^*(\rho, \varphi, v, \varepsilon\mu_0)] \\ &\dot v = Jv + V^0(\rho e^{i\varphi}, \rho e^{-i\varphi}, v, \varepsilon\mu_0) + V^*(\rho e^{i\varphi}, \rho e^{-i\varphi}, v, \varepsilon\mu_0) \ . \end{aligned} \tag{1.18}$$

The system (1.18) is the system of differential equations depending on one-dimensional non-negative parameter ε with the bifurcation equation satisfying the condition of positiveness. As it was shown in [1] the system (1.18) can be reduced introducing new variables x_1, φ_1, v_1 by the relations $\rho = \sqrt{\varepsilon}[\alpha(\mu_0) + x_1], \varphi = \varphi_1, \ v = \sqrt{\varepsilon} v_1$ to the system

$$
\begin{aligned}
\dot{x}_1 &= \varepsilon X_1(x_1, \varepsilon, \mu_0) + \sqrt{\varepsilon} X_1^0(x_1, \varphi_1, v_1, \varepsilon, \mu_0) + \\
&\quad + (\sqrt{\varepsilon})^{3p+1} \tilde{X}_1(x_1, \varphi_1, v_1, \varepsilon, \mu_0) \\
\dot{\varphi}_1 &= \lambda_1(\varepsilon) + \varepsilon \Phi_1(x_1, \varepsilon, \mu_0) + \sqrt{\varepsilon} \Phi_1^0(x_1, \varphi_1, v_1, \varepsilon, \mu_0) + \\
&\quad + (\sqrt{\varepsilon})^{3p+1} \tilde{\phi}_1(x_1, \varphi_1, v_1, \varepsilon, \mu_0) \\
\dot{v}_1 &= J v_1 + \sqrt{\varepsilon} V_1^0(x_1, \varphi_1, v_1, \varepsilon, \mu_0) + (\sqrt{\varepsilon})^{3p+1} \tilde{V}_1(x_1, \varphi_1, v_1, \varepsilon, \mu_0) \ ,
\end{aligned} \tag{1.19}
$$

where X_1, Φ_1 - vector polynomials, $X_1(0,0,\mu_0) = 0, \ \Phi_1(0,\varepsilon,\mu_0) = 0, \ \lambda(0) = \lambda, \ X_1^0, \Phi_1^0, V_1^0, \ \tilde{X}_1, \tilde{\Phi}_1, \tilde{V}_1$ - continuous functions in all variables of the class $C^1_{x_1,\varphi_1,v_1}$ on the domain $\mathbb{M}_1 = \{(x_1, \varphi_1, v_1, \varepsilon, \mu) : ||x_1|| < K_1, \ ||v_1|| < K_1, \ \varphi_1 \in \mathbb{R}^m, \ 0 \leq \varepsilon < L, \ \mu \in \mathcal{DP}\}, \ X_1^0, \ \Phi_1^0, \ V_1^0$ vanishing at $v_1 = 0$ and

$$
P_1(\mu) = \frac{\partial X_1(0,0,\mu_0)}{\partial x_1} = 2[\text{diag } \alpha(\mu_0)] B [\text{diag } \alpha(\mu_0)] \ . \tag{1.20}
$$

We say that $P_1(\mu)$ is non-critical at $\mu \in \mathcal{DP}$ if its eigenvalues do not lie on the imaginary axis and is critical at $\mu \in \mathcal{DP}$ if it has at least one pair of pure imaginary eigenvalues while the others have non-zero real parts. Let $\mathcal{DC}$ denote the subset of all parameters $\mu \in \mathcal{DP}$ at which the matrix $P_1(\mu)$ is critical. We shall call this subset $\mathcal{DC}$ the domain of criticalness of the bifurcation equation (1.17).

Theorem 1.1. *To every $\mu \in \mathcal{DP} \setminus \mathcal{DC}$ of the bifurcation equation (1.17) there exists the invariant manifold of the system (1.19) which is defined by the equations*

$$
\begin{aligned}
x_1 &= ||\mu|| \eta_1(\varphi_1, ||\mu||, \mu_0) \\
v_1 &= ||\mu||^2 \Theta_1(\varphi_1, ||\mu||, \mu_0) \ ,
\end{aligned} \tag{1.21}
$$

where $\eta_1(\varphi_1, ||\mu||, \mu_0), \ \Theta_1(\varphi_1, ||\mu||, \mu_0)$ are continuous functions 2π-periodic in all components of $\varphi_1, \ \varphi_1 \in \mathbb{R}^m, \ 0 \leq ||\mu|| < L, \ \mu \in \mathcal{DP} \setminus \mathcal{DC}$. The natural number p in (1.2) can be taken $p = 1$.

Proof. Consider an arbitrary $\mu \in \mathcal{DP} \setminus \mathcal{DC}$. The parameter μ lies on the beam $\delta(\mu_0) = \{\varepsilon \mu_0 : \ 0 \leq \varepsilon < L\}$. On this beam the system (1.16) can be reduced to the system (1.19). According to Theorem from section 3 of Chapter 1 in [1] there exists to every $\varepsilon, \ 0 < \varepsilon < L$ (in the case of necessity L is taken smaller) the invariant manifold

$$
\begin{aligned}
x_1 &= \varepsilon \eta_1(\varphi_1, \varepsilon, \mu_0) \\
v_1 &= \varepsilon^2 \Theta_1(\varphi_1, \varepsilon, \mu_0) \ ,
\end{aligned}
$$

where η_1, Θ_1 are continuous functions 2π-periodic in all components $\varphi_1, \ \varphi_1 \in \mathbb{R}^m, \ 0 \leq \varepsilon < L, \ p$ can be $p = 1$. In our case $\varepsilon = ||\mu||$. The proof is over.

2. One pair of pure imaginary eigenvalues

Suppose that the eigenvalues of the matrix P of the system (1.1) are: $\pm i\lambda,\ \lambda_3, \dots, \lambda_n$, $Re\lambda_k \neq 0,\ k = 3, \dots, n$.

The bifurcation equation (1.17) of system (1.16) is:

$$B\rho^2 + C\mu = 0\ , \tag{2.1}$$

where $B \in \mathbb{R}, C = (C_1, \dots, C_d),\ C_k \in \mathbb{R},\ k = 1, \dots, d.$

We suppose that $B \neq 0$ and the vector C has at least one element different from zero.

Theorem 2.1. *If the matrix P of the system (1.1) has one pair of pure imaginary eigenvalues and the others have non-zero real parts then:*

1. *$\mathcal{DP}$ of the bifurcation equation (2.1) is the whole half-sphere of the sphere $O = \{\mu = (\mu_1, \dots, \mu_d)\ :\ 0 < ||\mu|| < L\}$ which is determined by the hyperplane $C_1\mu_1 + \dots + C_d\mu_d = 0$ and by a point $\mu^* \in O$ at which $-\frac{1}{B}(C_1\mu_1^* + \dots + C_d\mu_d^*) > 0$.*
2. *$\mathcal{DC}$ of the bifurcation equation (2.1) is empty set.*

Proof. Let us take an arbitrary $\mu \in \mathbb{M}$. The bifurcation equation (2.1) has on the beam $\delta(\mu_0) = \{\varepsilon\mu_0 :\ 0 \le \varepsilon < L\}$ the form: $B\rho^2 + \varepsilon C\mu_0 = 0$. Solving this equation with respect to ρ^2 we get: $\rho^2 = \varepsilon\alpha^2(\mu_0)$, where $\alpha^2(\mu_0) = -\frac{1}{B||\mu||}(C_1\mu_1 + \dots + C_d\mu_d)$.

$\mathcal{DP}$ is the set of all $\mu \in \mathbb{M}$ at which

$$\alpha^2(\mu_0) = -\frac{1}{B||\mu||}(C_1\mu_1 + \dots + C_d\mu_d) > 0\ .$$

From this inequality the first assertion of Theorem 2.1 follows.

The matrix $P_1(\mu)$ of the system (1.19) has on $\mathcal{DP}$ according to (1.20) this form:

$$P_1(\mu) = 2[\text{diag}\ \alpha(\mu_0)]\ B[\text{diag}\ \alpha(\mu_0)] = 2\sqrt{-\frac{1}{B||\mu||}(C_1\mu_1 + \dots + C_d\mu_d)}\cdot$$
$$B\sqrt{-\frac{1}{B||\mu||}(C_1\mu_1 + \dots + C_d\mu_d)} = -\frac{2}{||\mu||}(C_1\mu_1 + \dots + C_d\mu_d) \neq 0$$

for all $\mu \in \mathcal{DP}$. The proof is over.

Consequence of Theorem 1.1 and Theorem 2.1. To every $\mu \in \mathcal{DP}$ of the bifurcation equation (2.1) there exists the invariant manifold of the system (1.19) of the kind (1.21).

3. Two pairs of pure imaginary eigenvalues

Suppose that the matrix P of the system (1.1) has two pairs of pure imaginary eigenvalues $\pm i\lambda_1,\ \pm i\lambda_2$ and the others $\lambda_5, \dots, \lambda_n$ have non-zero real parts.

The bifurcation equation (1.17) of the system (1.16) is:

$$B\rho^2 + C\mu = 0\ , \tag{3.1}$$

where

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\ , \qquad C = \begin{pmatrix} C_{11} & \dots & C_{1d} \\ C_{21} & \dots & C_{2d} \end{pmatrix}\ .$$

We suppose that $\det B \neq 0$ and the matrix C has at least one element different from zero.

Let us take an arbitrary $\mu \in \mathbb{M}$. The equation (3.1) has on the beam $\delta(\mu_0) = \{\varepsilon\mu_0 :\ 0 \le \varepsilon < L\}$ the form: $B\rho^2 + \varepsilon C\mu_0 = 0$. Solving this equation with respect to ρ^2 we get

$$\rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0)\ , \tag{3.2}$$

where

$$\alpha^2(\mu_0) = \begin{pmatrix} \alpha_1^2(\mu_0) \\ \alpha_2^2(\mu_0) \end{pmatrix} = \Lambda\mu_0, \quad \Lambda = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1d} \\ \alpha_{21} & \dots & \alpha_{2d} \end{pmatrix}\ .$$

The matrix $P_1(\mu)$ which is defined by (1.20) has the form:

$$P_1(\mu) = 2\begin{pmatrix} \alpha_1^2(\mu_0)B_{11} & \alpha_1(\mu_0)\alpha_2(\mu_0)B_{12} \\ \alpha_1(\mu_0)\alpha(\mu_0)B_{21} & \alpha_2^2(\mu_0)B_{22} \end{pmatrix}\ ,$$

where

$$\alpha_1(\mu_0) = \sqrt{\frac{1}{||\mu||}(\alpha_{11}\mu_1 + \dots + \alpha_{1d}\mu_d)}, \quad \alpha_2(\mu_0) = \sqrt{\frac{1}{||\mu||}(\alpha_{21}\mu_1 + \dots + \alpha_{2d}\mu_d)}\ .$$

Lemma 3.1. *The matrix $P_1(\mu)$ is critical at $\mu \in \mathcal{DP}$ only if the following two conditions are satisfied:*

$$\begin{aligned} &1. \qquad \det B > 0 \\ &2. \qquad a_1(\mu_0) = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22} = 0\ . \end{aligned} \tag{3.3}$$

Proof. The characteristic equation of the matrix $\frac{P_1(\mu)}{2}$ which is similar to $P_1(\mu)$ is:

$$\lambda^2 - a_1(\mu_0)\lambda + a_2(\mu_0) = 0\ , \tag{3.4}$$

where $a_1(\mu_0) = Tr\frac{P_1(\mu)}{2} = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22}$, $a_2(\mu_0) = \det\frac{P_1(\mu)}{2} = \alpha_1^2(\mu_0)\alpha_2^2(\mu_0)\cdot \det B$.

Comparing (3.4) with its expression by means of its pure imaginary roots we gain the conditions for $P_1(\mu)$ to have a pair of pure imaginary eigenvalues:

$$a_1(\mu_0) = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22} = 0, \quad a_2(\mu_0) = \alpha_1^2(\mu_0)\alpha_2^2(\mu_0)\det B > 0\ .$$

Taking into account that $\alpha_1^2(\mu_0) > 0,\ \alpha_2^2(\mu_0) > 0$ at every $\mu \in \mathcal{DP}$ we get the assertion of Lemma 3.1.

Theorem 3.1. *Let the rank $h(\Lambda)$ of the matrix Λ in (3.2) be 1. Then the following holds for $\mathcal{DP}$ and $\mathcal{DC}$ of the bifurcation equation (3.1):*

1. $\mathcal{DP} \neq \emptyset \Leftrightarrow \alpha_2 = k\alpha_1,\ k > 0.$
2. $\mathcal{DC} \neq \emptyset \Leftrightarrow \{(\det B > 0) \wedge [(B_{11} = B_{22} = 0) \vee (B_{11} = -kB_{22})]\}.$
3. *If* $\mathcal{DC} \neq \emptyset \Rightarrow \mathcal{DC} \equiv \mathcal{DP}.$

Proof. The domain of positiveness of the bifurcation equation (3.1) is determined by the inequalities:

$$\alpha_1^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d) > 0$$

(3.5)

$$\alpha_2^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d) > 0\ .$$

The first inequality in (3.5) is satisfied at all parameters $\mu \in \mathbb{M}$ which belong to that half-sphere of the sphere $O = \{\mu = (\mu_1, \ldots, \mu_d) :\ 0 < ||\mu|| < L\}$ which is determined by the hyperplane $\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d = 0$ and by a point $\mu^* \in O$ at which $\alpha_1^2(\mu_0^*) > 0$. As $h(\Lambda) = 1$ so there exists $k \in \mathbb{R}$ such that $\alpha_2 = k\alpha_1$. Using this we can express the second inequality in (3.5) in the form: $\frac{k}{||\mu||}(\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d) > 0$. From this inequality it follows that the parameters μ which satisfy the first inequality in (3.5) will also satisfy the second inequality in (3.5) only when $k > 0$. This gives the first assertion of Theorem 3.1.

Let $\mathcal{DC} \neq \emptyset$. Take an arbitrary $\mu \in \mathcal{DP}$. As $\alpha_2 = k\alpha_1,\ k > 0$, so $\alpha_2^2(\mu_0) = k\alpha_1^2(\mu_0)$. Therefore the conditions of criticalness (3.3) of the matrix $P_1(\mu)$ can be written in the form:

1. $\det B > 0$

(3.6) 2. $a_1(\mu_0) = \alpha_1^2(\mu_0)(B_{11} + kB_{22}) = 0\ .$

The equation (3.6) is satisfied only when $B_{11} = B_{22} = 0$ or $B_{11} = -kB_{22}$. From this equation also follows that when $B_{11} = B_{22} = 0$ or $B_{11} = -kB_{22}$ then (3.6) is satisfied at every $\mu \in \mathcal{DP}$. This gives the second and the third assertion of Theorem 3.1. The proof is over.

Theorem 3.2. *Let the rank $h(\Lambda)$ of the matrix Λ in (3.2) be 2. Then the following holds:*

1. $\mathcal{DP} \neq \emptyset$
2. $\mathcal{DC} \neq \emptyset \Leftrightarrow \{(\det B > 0) \wedge [(B_{11} = B_{22} = 0) \vee (B_{11}B_{22} < 0)]\}$
3. $\mathcal{DC} \equiv \mathcal{DP} \Leftrightarrow [(\det B > 0) \wedge (B_{11} = B_{22} = 0)].$

Proof. As $h(\Lambda) = 2$ then from the definition of the rank of a matrix follows that the dimension o of the parameter μ is at least 2, i.e. $o \geq 2$. The domain of positivenes $\mathcal{DP}$ of the equation (3.1) is determined by the inequalities

$$\frac{1}{||\mu||}(\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d) > 0$$

(3.7)

$$\frac{1}{||\mu||}(\alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d) > 0\ .$$

Expressing (3.7) in the form of equations we get:

$$\begin{aligned} &\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d - t_1 \qquad = 0 \\ &\alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d \qquad - t_2 = 0\ ,\ t_1 > 0, t_2 > 0\ . \end{aligned} \tag{3.8}$$

As the rank of the matrix of the system (3.8) is 2 this system has infinite number of solutions with $t_1 > 0, t_2 > 0$. Therefore the inequalities (3.7) have solutions $\mu^* = (\mu_1^*, \ldots, \mu_d^*)$. As parameters $\mu = \varepsilon\mu^*$ for $0 < \varepsilon < L$ also satisfy (3.7) so $\mathcal{DP} \neq \emptyset$. This gives the first assertion of Theorem 3.2.

Let $\mathcal{DC} \neq \emptyset$. The conditions of the criticalness of the matrix $P_1(\mu)$ are:

$$a_1(\mu_0) = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22}, \quad \det B > 0\ . \tag{3.9}$$

Let $\mu^* \in \mathcal{DC}$. It means that $a_1(\mu_0^*) = 0,\ \det B > 0$. But as at the same time $\mu^* \in \mathcal{DP}$ so $\alpha_1^2(\mu_0^*) > 0,\ \alpha_2^2(\mu_0^*) > 0$. From (3.9) it follows that $B_{11} = B_{22} = 0$ or $B_{11}B_{22} < 0$.

Let

$$(\det B > 0) \wedge [(B_{11} = B_{22} = 0) \vee (B_{11}B_{22} < 0)]\ . \tag{3.10}$$

$\mathcal{DC}$ is the set of parameters $\mu \in \mathcal{DP}$ satisfying the relations:

$$\begin{aligned} \alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d &> 0 \\ \alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d &> 0 \\ (\alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d)B_{11} + (\alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d)B_{22} &= 0\ . \end{aligned} \tag{3.11}$$

We shall show that under the assumptions (3.10) these relations have solutions. Expressing (3.11) in the form of equations we get:

$$\begin{aligned} \alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d - t_1 \qquad &= 0 \\ \alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d \qquad - t_2 &= 0 \\ (\alpha_{11}B_{11} + \alpha_{21})B_{22}\mu_1 + \cdots + (\alpha_{1d}B_{11} + \alpha_{2d}B_{22})\mu_d o &= 0\ . \end{aligned} \tag{3.12}$$

If $B_{11} = B_{22} = 0$ then the third equation in (3.12) is satisfied for every $\mu \in \mathcal{DP}$ and $\mathcal{DC} \equiv \mathcal{DP}$.

If $B_{11}B_{22} < 0$ then the system (3.12) can be reduced to the form:

$$\begin{aligned} \alpha_{11}\mu_1 + \cdots + \alpha_{1d}\mu_d - t_1 \qquad &= 0 \\ \alpha_{21}\mu_1 + \cdots + \alpha_{2d}\mu_d \qquad - t_2 &= 0 \\ B_{11}t_1 + B_{22}t_2 &= 0\ . \end{aligned} \tag{3.13}$$

As $h(\Lambda) = 2$ so the rank of the system (3.13) is 3. One of $d-1$ parameters of this system is t_2. For t_1 we get: $t_1 = -\frac{B_{22}}{B_{11}}t_2 > 0$ as $t_2 > 0$. So the system (3.13) has infinite number of solutions $(\mu_1^*, \ldots, \mu_d^*, t_1^*, t_2^*)$ with $t_1^* > 0, t_2^* > 0$. This means that at these solutions $\alpha_1^2(\mu_0^*) > 0,\ \alpha_2^2(\mu_0^*) > 0$ and $a_1(\mu_0^*) = 0$. Thus parameters

$\mu = \varepsilon\mu^*$, $0 < \varepsilon < L$, belong to $\mathcal{DC}$ and therefore $\mathcal{DC} \neq \emptyset$. This gives the second assertion of Theorem 3.2 and the relation: $[(\det B > 0) \wedge (B_{11} = B_{22} = 0)] \Rightarrow \mathcal{DC} \equiv \mathcal{DP}$.

Suppose that $\mathcal{DC} \equiv \mathcal{DP}$. Consider the parameters μ^*, μ^+ which are the solutions of (3.8) corresponding to the pairs $(t_1^* = 1, t_2^* = 1)$, $t_1^+ = 1, t_2^+ = 2)$ respectively and an arbitrary choice of the other parameters of (3.8). Then we have from (3.8):

$$\begin{aligned} \alpha_{11}\mu_1^* + \cdots + \alpha_{1d}\mu_d^* &= 1 \\ \alpha_{21}\mu_1^* + \cdots + \alpha_{2d}\mu_d^* &= 1 \end{aligned}$$

and also

$$\begin{aligned} \alpha_{11}\mu_1^+ + \cdots + \alpha_{1d}\mu_d^+ &= 1 \\ \alpha_{21}\mu_1^+ + \cdots + \alpha_{2d}\mu_d^+ &= 2\ . \end{aligned}$$

Take an arbitrary $\varepsilon_0, 0 < \varepsilon_0 < L$ and consider the parameters $\mu^*(\varepsilon_0) = \varepsilon_0\mu^* \in \mathcal{DP}$, $\mu^+(\varepsilon_0) = \varepsilon_0\mu^+ \in \mathcal{DP}$. According to the assumption $\mathcal{DC} \equiv \mathcal{DP}$ we have: $\mu^*(\varepsilon_0) \in \mathcal{DC}$, $\mu^+(\varepsilon_0) \in \mathcal{DC}$. Thus the conditions of criticalness at $\mu^*(\varepsilon_0)$, $\mu^+(\varepsilon_0)$ are satisfied what means:

$$(3.14) \qquad \begin{aligned} &\alpha_1^2[\mu_0^*(\varepsilon_0)]B_{11} + \alpha_2^2[\mu_0^*(\varepsilon_0)]B_{22} = 0 \\ &\alpha_1^2[\mu_0^+(\varepsilon_0)]B_{11} + \alpha_2^2[\mu_0^+(\varepsilon_0)]B_{22} = 0\ . \end{aligned}$$

As $\alpha_1^2[\mu_0^*(\varepsilon_0)] = \alpha_2^2[\mu_0^*(\varepsilon_0)] = \frac{1}{||\mu^*||}$ and $\alpha_1^2[\mu_0^+(\varepsilon_0)] = \frac{1}{||\mu^+||}$, $\alpha_2^2[\mu_0^+(\varepsilon_0)] = \frac{2}{||\mu^+||}$, the equations (3.14) have the form:

$$\begin{aligned} \frac{1}{||\mu^*||}B_{11} + \frac{1}{||\mu^*||}B_{22} &= 0 \\ \frac{1}{||\mu^+||}B_{11} + \frac{2}{||\mu^+||}B_{22} &= 0\ . \end{aligned}$$

But this system is satisfied only when $B_{11} = B_{22} = 0$. This gives the relation: $\mathcal{DC} \equiv \mathcal{DP} \Rightarrow [(\det B > 0) \wedge (B_{11} = B_{22} = 0)]$. The proof of Theorem 3.2 is over.

According to Theorem 1.1 to every $\mu \in \mathcal{DP} \setminus \mathcal{DC}$ there exists an invariant manifold (1.21) which is homeomorphic with an invariant torus. Suppose now that $\mu \in \mathcal{DC}$ of the bifurcation equation (3.1). This means that $P_1(\mu)$ is critical on the beam of parameters $\delta(\mu_0) = \{\varepsilon\mu_0 :\ 0 \leq \varepsilon < L\}$. On this beam the system

$$(3.15) \qquad \dot{x}_1 = \varepsilon X_1(x_1, \varepsilon, \mu_0)$$

which is gained from the first equation of (1.19) is two-dimensional system with the critical matrix $P_1(\mu) = \frac{\partial X_1(0,0,\mu_0)}{\partial x_1}$. Denote its eigenvalues $\pm i\lambda^1$. The system (3.15) is the system of the same character as the system $\dot{x} = X(x, \varepsilon, \mu_0)$ which is gained from the system (1.1) being expressed on the beam $\delta(\mu_0)$. As it was shown in [1] we can do on (1.19) the analogical sequence of transformations as it was done on the system (1.1). During this process we get the bifurcation equation

$$B_1\rho_1^2 + \varepsilon C_1(\mu_0) = 0, \quad B_1 \in \mathbb{R},\ C_1(\mu_0) \in \mathbb{R}\ . \tag{3.16}$$

If (3.16) satisfies the condition of positiveness, i.e. $\rho_1^2 = \varepsilon[-\frac{1}{B_1}C_1(\mu_0)] = \varepsilon\alpha^2(\mu_0)$, $\alpha^2(\mu_0)\ > 0$, the system (1.19) can be reduced to the system

$$\begin{aligned}
\dot{x}_2 &= \varepsilon^2 X_2(x_2,\varepsilon,\mu_0) + X_2^0(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)+\\
&\quad + (\sqrt{\varepsilon})^{3p}\tilde{X}_2(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)\\
\dot{\varphi}_{12} &= \lambda_1(\varepsilon) + \varepsilon^2\Phi_{12}(x_2,\varepsilon,\mu_0) + \Phi_{12}^0(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)+\\
&\quad + (\sqrt{\varepsilon})^{3p}\tilde{\Phi}_{12}(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)\\
\dot{\varphi}_{22} &= \varepsilon\lambda_2(\varepsilon) + \varepsilon^2\Phi_{22}(x_2,\varepsilon,\mu_0) + \Phi_{22}^0(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)+\\
&\quad + (\sqrt{\varepsilon})^{3p}\tilde{\Phi}_{22}(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)\\
\dot{v}_{12} &= Jv_{12} + V_{12}^0(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)+\\
&\quad + (\sqrt{\varepsilon})^{3p+1}\tilde{V}_{12}(x_2,\varphi_{12},\varphi_{22},v_{12},\varepsilon,\mu_0)\ ,
\end{aligned} \tag{3.17}$$

where $\dim x_2 = \dim\varphi_{22} = 1,\ \dim\varphi_{12} = 2,\ \dim v_{12} = n-4,\ \lambda_1(0) = \lambda,\ \lambda_2(0) = \lambda^1$ and the functions $X_2, \Phi_{12}, \Phi_{22}, X_2^0, \Phi_{12}^0, \Phi_{22}^0, V_{12}^0, \tilde{X}_2, \tilde{\Phi}_{12}, \tilde{\Phi}_{22}, \tilde{V}_{12}$ have the same character as the analogical functions in (1.19).

Denote $P_2(\mu) = \frac{\partial X_2(0,0,\mu_0)}{\partial x_2}$. It was shown in [1] that $P_2(\mu) = 2\alpha^2(\mu_0)B_1 = -2C_1(\mu_0)$. As the bifurcation equation (3.16) satisfies the condition of positiveness we have $P_2(\mu) \neq 0$ what means that $P_2(\mu)$ is non-critical. Therefore according to Theorem of section 3 Chapter 1 in [1] the following assertion is valid.

Theorem 3.3. *Let $\mu \in \mathcal{DC}$ of the bifurcation equation (3.1). If the bifurcation equation (3.16) satisfies at μ the condition of positiveness then p in (1.2) can be taken $p = 2$ and to this μ there exists the invariant manifold of the system (3.17) which is defined by the equations*

$$\begin{aligned}
x_2 &= ||\mu||\eta_2(\varphi_{12},\varphi_{22},||\mu||,\mu_0)\\
v_{12} &= ||\mu||^2\Theta_2(\varphi_{12},\varphi_{22},||\mu||,\mu_0)\ ,
\end{aligned}$$

where η_2, Θ_2 are continuous functions in all variables 2π-periodic at $\varphi_{12}, \varphi_{22}, \varphi_{12} \in \mathbb{R}^2$, $\varphi_{22} \in \mathbb{R}^1$, $0 \leq \varepsilon < L$.

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