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MINIMAL ECCENTRIC SEQUENCES WITH LEAST ECCENTRICITY THREE

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Dedicated to Anton Dekrét on the occasion of his 65-th birthday

Abstract. An eccentric sequence is called minimal if it has no proper eccentric subsequence with the same number of distinct eccentricities. All minimal eccentric sequences with least value 2 were found by R. Nandakumar (see [1]). In the paper it is shown that there are exactly 13 minimal eccentric sequences with least eccentricity three (see Theorem 5.1).

1. Introduction

In this paper we consider undirected connected graphs without loops and multiple edges. If G is a graph we denote by $V(G)$ the set of all its vertices and by $E(G)$ the set of all its edges. We write $|V(G)|$ for the cardinality of $V(G)$. The subgraph of G induced by a set of vertices $\{v_1, \dots, v_n\}$ will be denoted by $< v_1, \dots, v_n >$. Let $\deg v$ denote the degree of a vertex v and $d(u, v)$ denote the distance between vertices u, v. If u, v are vertices of a graph then uv is the edge which is incident with each of two vertices u and v. Let us denote by $\operatorname{diam} G$ the diameter of the graph G.

The eccentricity of a vertex $u \in V(G)$ is the integer

$$e_G(u) = \max\{d(u, v); v \in V(G)\}.$$

We write simply $e(u)$ when no confusion can arise. The eccentric sequence of a graph G is a list of the eccentricities of its vertices in nondecreasing order. Since often there are many vertices having the same eccentricity we will simplify the sequence by listing it as

$$e_1^{m_1}, e_2^{m_2}, \dots, e_s^{m_s}$$

where the e_i are the eccentricities for which $e_i < e_{i+1}$ and m_i is the multiplicity of e_i. For example 3^3, 4^3, 5^2 is the eccentricity sequence of the graph in Fig. 1.1 (at each vertex its eccentricity can be found).

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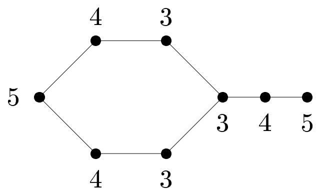


Fig. 1.1

Fig. 1.2

Isomorphic graphs have the same eccentric sequence (we will identify them). Obviously, the converse is not true.

A sequence of positive integers is called eccentric if there is a graph which realizes the considered sequence as the sequence of its eccentricities. An eccentric sequence is called minimal (by R. Nandakumar) if it has no proper eccentric subsequence with the same number of distinct eccentricities. For example, the eccentric sequence $3^3, 4^3, 5^2$ is not minimal since the graph in Fig. 1.2 has the eccentric sequence $3^2, 4^2, 5^2$.

Now we remind some well known properties of eccentric sequences (see [1]).

For every eccentric sequence $e_1^{m_1}, e_2^{m_2}, \ldots, e_s^{m_s}$ holds

1. $e_{i+1} = e_i + 1$ for $i = 1, 2, \ldots, s-1$, i.e. the $e_i's$ are consecutice positive integers,
2. $e_s \leq 2e_1$, i.e. the diameter is at most twice the radius.

The next assertions can be found in [2].

Theorem 1.1. *If $e_1^{m_1}, e_2^{m_2}, \ldots, e_s^{m_s}$ is an eccentric sequence then $m_i \geq 2$ for every $i \geq 2$.*

Theorem 1.2. *A sequence of positive integers is eccentric if and only if some its subsequence with the same number of distinct integers is eccentric.*

All minimal eccentric sequences with least value 2 were found by R. Nandakumar (see [1]). In this paper it is shown that there are exactly 13 minimal eccentric sequences with least eccentricity three, namely 3^6; $3^5, 4^2$; $3^4, 4^4$; $3^3, 4^6$; $3^2, 4^8$; $3, 4^{10}$; $3, 4^2, 5^{12}$; $3, 4^3, 5^9$; $3, 4^4, 5^7$; $3, 4^5, 5^4$; $3, 4^7, 5^2$; $3^2, 4^2, 5^2$; $3, 4^2, 5^2, 6^2$, (Theorem 5.1).

Now we will give some statements which we will use in the paper.

Lemma 1.3. *Let G be a graph and let $uv \in E(G)$. If $\deg v = 1$ and $e(u) \geq 2$ then $e(v) = e(u) + 1$.*

Proof. Any nontrivial path contained the vertex v also contains the vertex u.

Lemma 1.4. *If $uv \in E(G)$, $vw \in E(G)$, $\deg v = 2$, $\deg w = 1$, $e(u) \geq 3$ then $e(w) = e(v) + 1 = e(u) + 2$.*

Proof. It is obvious.

2. Eccentric sequences of type $3, 4^l$ and $3, 4^l, 5^r$, $l + r = 9$.

In this section we will show that there are only two 10-vertices graphs with eccentric sequences of type $3, 4^l$ or $3, 4^l, 5^r$. From this (by Theorem 1.2) it is possible to obtain some properties of minimal eccentric sequences (Corollary 2.5 - 2.7).

Lemma 2.1. *If the eccentric sequence of a graph G is $3, 4^l$ or $3, 4^l, 5^r$ then G contains the subgraph in Fig. 2.1 satisfying $e(v_4) = 3$ and*

(2.1) $$d(v_3, v_7) = d(v_1, v_5) = 4,$$

or the subgraph in Fig. 2.2 satisfying $e(v_4) = 3$ and

(2.2) $$d(v_4, v_1) = d(v_4, v_7) = d(v_4, w_3) = 3$$

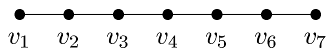


Fig. 2.1

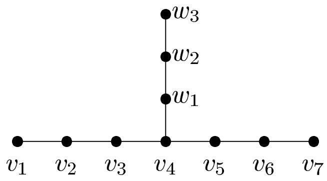


Fig. 2.2

Proof. Let v_4 be a vertex of G for which $e(v_4) = 3$. Since $e(v_4) = 3$ there is a path v_4, v_3, v_2, v_1 with the property $d(v_4, v_1) = 3$. Let $V_1 = V(G) - \{v_1, v_2, v_3, v_4\}$. If $d(v, v_4) \leq 2$ for any vertex $v \in V_1$ then $e(v_3) \leq 3$, a contradiction. If for any vertex $v \in V_1$ for which $d(v, v_4) = 3$ we would have that a shortest path between v, v_4 contains one of the vertices v_2, v_3 then again $e(v_3) \leq 3$, which is impossible. Hence the graph G contains a path v_4, v_5, v_6, v_7 satisfying $d(v_4, v_7) = 3$ that is disjoint with the path v_1, v_2, v_3. Thus the graph G contains the subgraph in Fig. 2.1.

If the distance between vertices from the set $V_2 = V(G) - \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and the vertex v_4 is at most two then $d(v_3, v_7) = d(v_1, v_5) = 4$ (because $e(v_3) = e(v_5) = 4$). Let there exist a vertex $w_3 \in V_2$ for which $d(v_4, w_3) = 3$. If for every vertex $x \in V_2$ satisfying $d(v_4, x) = 3$ every shortest path between vertices v_4, x contains at least one of the vertices v_2, v_3, v_5, v_6 then without loss of generality we can assume that (2.1) is satisfied. In the opposite case G contains the subgraph in Fig. 2.2 satisfying $e(v_4) = 3$ and (2.2). □

Lemma 2.2. *Let the eccentric sequence of a graph G be $3, 4^l$ or $3, 4^l, 5^r$ and $e(v_4) = 3$. Let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be a path for which $d(v_1, v_4) = d(v_7, v_4) = 3$ and let G_1, G_2 be subgraphs of the graph G satisfying the following conditions*

(j) $V(G_1) \cap V(G_2) \subset \{v_4\}$,
(jj) $V(G) - \{v_4\} \subset V(G_1) \cup V(G_2)$,
(jjj) $v_1 \in V(G_1)$, $v_7 \in V(G_2)$.

Then none of the following assertions holds

(a) *there is no edge uv with $u \in V(G_1) - \{v_4\}$ and $v \in V(G_2) - \{v_4\}$,*
(b) *$uv \in E(G)$, $u \in V(G_1) - \{v_4\}$, $v \in V(G_2) - \{v_4\}$ and $e_{G_1}(u) \leq 2$, $e_{G_2}(v) \leq 3$ (or $e_{G_1}(u) \leq 3$, $e_{G_2}(v) \leq 2$).*

Proof. In the case (a) a shortest path between vertices v_1, v_7 contains the vertex v_4 and it yields that $d(v_1, v_7) = d(v_1, v_4) + d(v_4, v_7) = 3 + 3 = 6$. Then $e(v_1) \geq 6$, a contradiction.

In the case (b) we have $e_G(v) \leq 3$, $v \neq v_4$ but v_4 is the only vertex with eccentricity less than four, a contradiction. □

Corollary 2.3. *There exists no graph satisfying all assumptions of Lemma 2.2 and simultaneously* $\operatorname{diam} G_1 \leq 2$, $\operatorname{diam} G_2 \leq 3$ *(or* $\operatorname{diam} G_1 \leq 3$, $\operatorname{diam} G_2 \leq 2$*).*

Theorem 2.4. *If the eccetric sequence of a graph G is $3, 4^l$ or $3, 4^l, 5^k$ and* $|V(G)| = 10$ *then G is the graph in Fig 2.3 or in Fig. 2.4.*

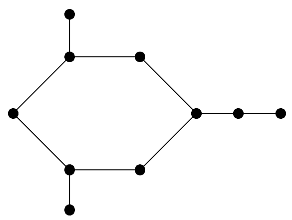


Fig. 2.3

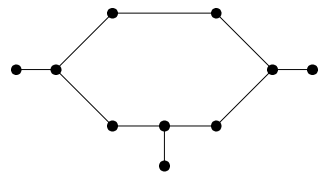


Fig. 2.4

Proof. Let $V(G) = \{v_1, \ldots, v_{10}\}$. We distinguish two cases (with respect to Lemma 2.1).

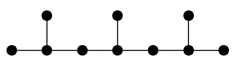
I. Let G contain a path $v_1, v_2, \ldots, v_7$ and let $e(v_4) = 3$ and (2.1) hold.

From (2.1) follows that the subgraph $\langle v_1, v_2, \ldots, v_7 \rangle$ has six edges. It is easy to see that neither $\deg v_1 = 1$ and $\deg v_2 = 2$ nor $\deg v_7 = 1$ and $\deg v_6 = 2$ is possible (see Lemma 1.4).

Since $e(v_4) = 3$ it is sufficient to consider the following cases with respect to symmetry, suitable denotation and to the number of vertices from the set $\{v_8, v_9, v_{10}\}$ which are adjacent to at least one of the vertices from the set $\{v_2, v_3, v_4, v_5, v_6\}$.

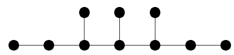
a) Each of the vertices v_8, v_9, v_{10} is adjacent to at least one vertex from the set $V' = \{v_2, v_3, v_4, v_5, v_6\}$.

$a_1)$ $\quad v_4v_8, v_2v_9, v_6v_{10} \in E(G)$



By Lemma 2.2 we have $\deg v_8 = 1$ (in the opposite case the graph G would not exist). Since any shortest path between v_1, v_7 can not contain the vertex v_4 we have that an edge of type v_iv_j, $i \in \{1, 2, 3, 9\}$, $j \in \{5, 6, 7, 10\}$ belongs to G. With respect to (2.1), $e(v_2) \geq 4$ and $e(v_6) \geq 4$ this is possible only for $i = 9$, $j = 10$, i.e. $v_9v_{10} \in E(G)$. Therefore, the graph G contains the subgraph in Fig. 2.4 with the eccentric sequence $3, 4^7, 5^2$. It is easy to check that if we add an edge to the graph in Fig. 2.4 we obtain a graph in which eccentricities at least two vertices are at most three.

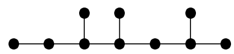
a_2) $v_4v_8, v_3v_9, v_5v_{10} \in E(G)$



Let the subgraphs $H_1 = \langle v_1, v_2, v_3, v_9 \rangle$ and $H_2 = \langle v_5, v_6, v_7, v_{10} \rangle$ have at least four edges (i.e. their diameters are at most two). The inequality $\deg v_8 > 1$ is impossible by Lemma 2.2. So, let $\deg v_8 = 1$. Since any shortest path between vertices v_1, v_7 can not contain v_4 and (1.2) holds we get $v_iv_j \in E(G)$, $i \in \{3, 9\}$, $j \in \{5, 10\}$. Hence $e(v_i) \leq 3$ (and $e(v_j) \leq 3$, too), a contradiction.

Thus (with respect to symmetry) we can assume that the subgraph H_1 has three edges. Since $\deg v_1 > 1$ or $\deg v_2 > 2$ we get (with respect to (2.1)) $v_2v_8 \in E(G)$. Since $\deg v_7 > 1$ or $\deg v_6 > 2$ we get a contradiction ($e(v_8) \leq 3$ or by Lemma 2.2).

a_3) $v_4v_8, v_3v_9, v_6v_{10} \in E(G)$



Let the graph $H_1 = \langle v_1, v_2, v_3, v_9 \rangle$ have at least four edges. By Lemma 2.2 we have again $\deg v_8 = 1$. With respect to (2.1) any shortest path between v_1, v_7 can not contain v_4 and thus an edge of type v_iv_j, $i \in \{2, 9\}$, $j \in \{5, 10\}$ belongs to $E(G)$. Hence $e(v_i) \leq 3$, a contradiction. Now, we can assume that $|E(H_1)| = 3$. Then $\deg v_1 = 1$ (by (2.1)). By Lemma 2.2 we get $\deg v_2 = 2$, a contradiction (see Lemma 1.4).

a_4) $v_4v_8, v_4v_9 \in E(G)$ and the vertex v_{10} is adjecent to at least one of the vertices v_2, v_3.

Deg $v_7 = 1$ (by (2.1)) and so $\deg v_6 > 2$ (by Lemma 1.4). Hence we have (with respect to (2.1) and Lemma 2.2) one of the cases a_1), a_3).

a_5) Let $v_6v_8 \in E(G)$ and each of the vertices v_9, v_{10} be adjacent to at least one of the vertices v_2, v_3.

This is impossible by Lemma 2.2.

a_6) Let $v_5v_8 \in E(G)$ and each of the vertices v_9, v_{10} be adjacent to at least one of the vertices v_2, v_3.

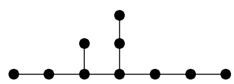
Since $\deg v_7 > 1$ or $\deg v_6 > 2$ we obtain (with respect to (2.1)) that the graph G does not exist by Lemma 2.2.

a_7) Each of the vertices v_8, v_9, v_{10} is adjacent to at least one vertex from the set $\{v_2, v_3, v_4\}$.

From (2.1) it follows $\deg v_7 = 1$. Since $\deg v_6 > 2$ we get one of the previous cases.

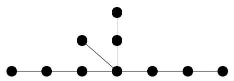
b) Let $v_8v_9 \in E(G)$ and each of the vertices v_8, v_{10} be adjacent to at least one vertex from the set $V' = \{v_2, \ldots, v_6\}$. We distinguish six subcases.

b_1) $v_4v_8, v_3v_{10} \in E(G)$



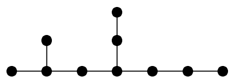
Since neither $\deg v_1 = 1$ and $\deg v_2 = 2$ nor $\deg v_7 = 1$ and $\deg v_6 = 2$ can holds, we have (by (2.1)) that $e(v_8) \leq 3$ or the graph G does not exist by Lemma 2.2, a contradiction.

$b_2)$ $v_4v_8, v_4v_{10} \in E(G)$



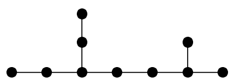
Similarly, since neither $\deg v_1 = 1$ and $\deg v_2 = 2$ nor $\deg v_7 = 1$ and $\deg v_6 = 2$ holds we have (by (2.1)) that either $e(v_8) \le 3$ or $e(v_{10}) \le 3$ holds or the graph G does not exist by Lemma 2.2, a contradiction.

$b_3)$ $v_4v_8, v_2v_{10} \in E(G)$



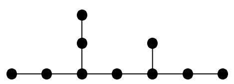
If $\deg v_7 > 1$ then $v_7v_9 \in E(G)$ by (2.1) and the graph G does not exist by Lemma 2.2. If $\deg v_6 > 2$ then $v_6v_{10} \in E(G)$ or the graph G does not exist by Lemma 2.2. If $v_6v_{10} \in E(G)$ we get that the graph G contains the subgraph in Fig. 2.3 with the eccentric sequence $3, 4^5, 5^4$. It is easy to check that if we add an edge to the graph in Fig. 2.5 we obtain a graph in which eccentricities at least two vertices are at most most three, a contradiction.

$b_4)$ $v_3v_8, v_6v_{10} \in E(G)$



Since $\deg v_1 > 1$ or $\deg v_2 > 2$ we obtain (with respect to (2.1)) that the graph G does not exist by Lemma 2.2.

$b_5)$ $v_3v_8, v_5v_{10} \in E(G)$



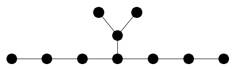
Since $\deg v_1 > 1$ or $\deg v_2 > 2$ the subgraph $\langle v_1, v_2, v_3, v_8, v_9\rangle$ has at least five edges (with respect to (2.1)). Since $\deg v_7 > 1$ or $\deg v_6 > 2$ the graph G does not exist by Lemma 2.2.

$b_6)$ $v_3v_8 \in E(G)$ and the vertex v_{10} is adjacent to at least one vertex from the set $\{v_2, v_3, v_4\}$.

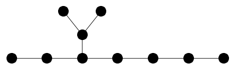
By (2.1) we have $\deg v_7 = 1$. If $\deg v_2 > 2$ then $e(v_8) \le 3$ or it takes place the case b_4).

c) It remains the next three cases:

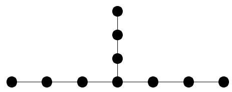
(c) $v_4v_8, v_8v_9, v_8v_{10} \in E(G)$



(d) $v_3v_8, v_8v_9, v_8v_{10} \in E(G)$



(e) $v_4v_8, v_8v_9, v_9v_{10} \in E(G)$



Since neither $\deg v_1 = 1$ and $\deg v_2 = 2$ nor $\deg v_7 = 1$ and $\deg v_6 = 2$ is possible we have $e(v_8) \leq 3$ or $e(v_9) \leq 3$ (with respect to (2.1)), a contradiction.

II. If it does not take place the case I then by Lemma 2.1 the graph G has the subgraph in Fig. 2.2 and without loss of generality we can asuume

$$d(v_3, w_3) = d(w_1, v_7) = d(v_5, v_1) = 4 \tag{2.3}$$

We first show that at least two of the vertices v_1, v_7, w_3 have degree one. On the contrary we may suppose (without loss of generality) that $\deg v_1 > 1$ and $\deg v_7 > 1$. This is possible only (with regard to (2.3)) if $v_1w_2 \in E(G)$ and $v_7v_2 \in E(G)$. Then we obtain $e(v_2) \leq 3$, a contradiction. So let $\deg v_1 = \deg w_3 = 1$. The inequality $d(v_1, w_3) \leq 5$ yields $d(v_2, w_2) \leq 3$. With respect to (2.3) we get that $v_2w_2 \notin E(G)$ and any shortest path between the vertices v_2, w_2 can contain neither vertex v_3 nor v_7. It can contain only the vertex w_1 and this is possible only if $v_2w_1 \in E(G)$. Since $e(v_2) = 4$ we get $\deg v_7 = 1$ and so $d(v_6, w_2) \leq 3$. As in the previous case we obtain $v_6w_2 \notin E(G)$ and moreover, any shortest path between vertices w_2, v_6 can not contain the vertex w_1. This path can contain only the vertex v_5 and this is possible only if $w_2v_5 \in E(G)$. Then $e(w_2) \leq 3$, a contradiction. □

Corollary 2.5. *The eccentric sequence* $3, 4^{10}$ *is minimal.*

Proof. The sequence $3, 4^{10}$ is the eccentric sequence of the graph in Fig. 2.5. According to Theorem 2.4 and Theorem 1.2 there is no graph with eccentric sequence $3, 4^l$, $l < 10$. □

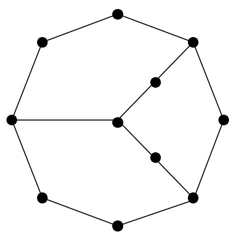


Fig. 2.5

Corollary 2.6. *The eccentric sequences* $3, 4^5, 5^4$ *and* $3, 4^7, 5^2$ *are minimal.*

Proof. The sequence $3, 4^5, 5^4$ is the eccentric sequence of the graph in Fig 2.3. The sequence $3, 4^7, 5^2$ is the eccentric sequence of the graph in Fig 2.4. Now we will show that there is no graph with eccentric sequence $3, 4^l, 5^r$, $l + r < 9$. On the contrary, let such graph G exist. According to Theorem 1.2 it is sufficient to consider the case $l + r = 8$. Let $e(v_4) = 3$ and let we obtain a graph G' from the graph G by adding a vertex $v \notin V(G)$ and an edge v_4v. Then, evidently, $e_{G'}(v) = 4$ and for any vertex $u \in V(G)$ it holds $e_G(u) = e_{G'}(u)$ (because $d_{G'}(u, v) \leq 4$). Then the graph G' has to be one of the graph in Fig. 2.3 or Fig. 2.4 but it is easily to see that G' is not any of them, a contradiction. □

Corollary 2.7. *A sequence* $3, 4^6, 5^r$ *is not a minimal eccentric sequence for any* r.

Proof. There is no graph with the eccentric sequence $3, 4^6, 5^3$ according to Theorem 2.4. Then it is sufficient to use Corollary 2.6. □

Minimal eccentric sequences with least eccentricity 2 can be found in [1], but with a mistake in Fig. 9.4. Therefore we will touch this case in the last part of this section.

Theorem 2.8. *The eccentric sequence* $2, 3^6$ *is minimal and it is realized by only the graph in Fig. 2.6.*

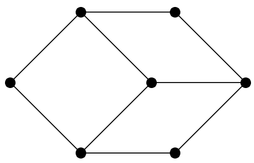


Fig. 2.6

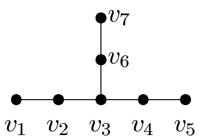


Fig. 2.7

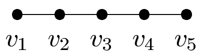


Fig. 2.8

Proof. In the same way as in the proof of Lemma 2.1 we can show that the graph G with the eccentric sequence $2, 3^k$ contains the subgraph in Fig. 3.4 where $e(v_3) = 2$ and $d(v_1, v_3) = d(v_5, v_3) = d(v_7, v_3) = 2$ or it contains the subgraph in Fig. 2.8 satisfying $e(v_3) = 2$ and $d(v_1, v_4) = d(v_2, v_5) = 3$ (if the vertices of G are suitable denoted).

I. In the first part of the proof we will show that the eccentric sequence $2, 3^6$ is minimal.

Suppose on the contrary that there exists a graph G with the eccentric sequence $2, 3^5$. It is obvious that the graph in Fig. 2.8 is a subgraph of G, where $e(v_3) = 2$ and $d(v_1, v_4) = d(v_2, v_5) = 3$.

Since $e(v_3) = 2$ it is sufficient to consider two possibilities for the vertex v_6 .

a)Let $v_2v_6 \in E(G)$. The equality $d(v_2, v_5) = 3$ implies that $\deg v_5 = 1$ and so $e(v_5) = e(v_4) + 1$ (by Lemma 1.3), a contradiction.

b) Let $v_3v_6 \in E(G)$. Since $e(v_2) = e(v_4) = 3$ we get $\deg v_1 > 1$ and $\deg v_5 > 1$ (by Lemma 1.3) and so $v_1v_6 \in E(G)$ and $v_5v_6 \in E(G)$, which gives $e(v_6) \leq 2$, a contradiction.

II. We will show that the eccentric sequence $2, 3^6$ is realized by only the graph in Fig. 2.6.

a) Let G contain the subgraph in Fig. 2.7 where $e(v_3) = 2$ and $d(v_1, v_3) = d(v_5, v_3) = d(v_7, v_3) = 2$.

Firstly, let $v_5v_7 \in E(G)$. Since $\deg v_1 > 1$ (by Lemma 1.3) we have $e(v_4) \le 2$ or $e(v_6) \le 2$, a contradiction.

Secondly, let $v_5v_7 \notin E(G)$. With respect to symmetry we can also assume that $v_1v_7 \notin E(G)$ and $v_1v_5 \notin E(G)$. Since $d(v_3, v_5) = 2$ and $\deg v_5 > 1$ (by Lemma 1.3) we can assume (without loss of generality) that $v_5v_6 \in E(G)$. From $\deg v_1 > 1$ and $e(v_6) = 3$ we have that $v_1v_4 \in E(G)$. From $\deg v_7 > 1$ and $e(v_4) = 3$ we have $v_2v_7 \in E(G)$. Hence the graph in Fig. 2.6 is a subgraph of G. It is easy to check that if we add an edge to the graph in Fig. 2.6 we obtain a graph in which eccentricities of at least two vertices are at most 2.

b) Let G contain a subgraph in Fig. 2.8 where $e(v_3) = 2$ and $d(v_1, v_4) = d(v_2, v_5) = 3$.

In this case the subgraph $\langle v_1, v_2, v_3, v_4, v_5\rangle$ has 4 edges. Since $e(v_3) = 2$, it is sufficient to consider (with respect to symmetry) the next four possibilities.

1) $v_2v_6, v_2v_7 \in E(G)$

Since $d(v_2, v_5) = 3$ we get $\deg v_5 = 1$ which is impossible (by Lemma 1.3).

2) $v_3v_6, v_3v_7 \in E(G)$

In this case a shortest path between v_1 and v_5 contains a vertex v_i, $i \in \{6, 7\}$ and so $e(v_i) \le 2$, a contradiction.

3) $v_2v_6, v_4v_7 \in E(G)$

If $v_1v_6 \notin E(G)$ we obtain that $e(v_4) \le 2$ (because $\deg v_1 > 1$ and $\deg v_6 > 1$ by Lemma 1.3), a contradiction. Now, let $v_1v_6 \in E(G)$. Any shortest path between v_1 and v_5 does not contain the vertex v_3 (because $d(v_1, v_5) \le 3$) and so there exists an edge $v_iv_j \in E(G)$, where $i \in \{1, 2, 6\}$, $j \in \{4, 5, 7\}$. This yields $e(v_j) \le 2$, a contradiction.

4) $v_2v_6, v_3v_7 \in E(G)$

Since $\deg v_5 > 1$ and $e(v_2) = 3$ we obtain $v_5v_7 \in E(G)$. Analogously as in the previous case it can be shown that $v_1v_6 \notin E(G)$. Since $\deg v_1 > 1$ and $d(v_1, v_4) = 3$ we have $v_1v_7 \in E(G)$. Similarly, $\deg v_6 > 1$ and $e(v_7) = 3$ imply $v_6v_4 \in E(G)$. Thus we obtain the graph in Fig. 2.6. $\square$

Remark. Theorem 2.8 implies that the graph corresponding to the eccentric sequence $2, 3^6$ in [1, Fig. 9.4] is wrong. Its eccentric sequence is $2, 3^4, 4^2$ and it is easy to see that it is not even a subgraph of the graph in Fig. 2.6.

3. Minimal eccentric sequences of type $3^k, 4^l$, $k \ge 2$.

Lemma 3.1. *Let G be a graph with an eccentric sequence $3^k, 4^l$, $k \ge 2$. Then G contains the subgraph in Fig. 3.1 satisfying $e(v_4) = 3$ and*

$$d(v_1, v_4) = d(v_4, v_7) = 3, \tag{3.1}$$

or the subgraph in Fig. 3.2 satisfying $e(v_3) = e(v_4) = 3$ and

$$d(v_1, v_4) = d(v_3, v_6) = 3 \tag{3.2}$$

(provided that the vertices of the graph G are suitable denoted).

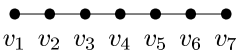


Fig. 3.1

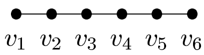


Fig. 3.2

Proof. Let $v_4 \in V(G)$, $e(v_4) = 3$ and let the graph G do not contain the graph in Fig. 3.1 satisfying (3.1). There exists a vertice $v_1 \in V(G)$ with $d(v_1, v_4) = 3$. Let v_1, v_2, v_3, v_4 be a path from v_1 to v_4 of the length 3. We denote

$$V' = V(G) - \{v_1, v_2, v_3, v_4\}.$$

Evidently, $V' \neq \emptyset$. If $v \in V'$ with $d(v, v_4) = 3$ then every shortest path from v to v_4 contains the vertex v_2 or v_3. Therefore, $d(v, v_3) \leq 2$. If $v \in V'$ with $d(v, v_4) = 1$ then again $d(v, v_3) \leq 2$. If also for each vertex $v \in V'$ with $d(v, v_4) = 2$ we have $d(v, v_3) \leq 2$ then $e(v_3) \leq 2$, which is impossible. Hence there exists a vertex v which satisfies $d(v, v_4) = 2$ and $d(v, v_3) = 3$. Consequently, the graph G contains the subgraph in Fig. 2 satisfying (3.2) (we put $v_6 = v$). □

Lemma 3.2. *Let G be a graph with an eccentric sequence $3^k, 4^l$ and let v_1, v_2, v_3, v_4, v_5, v_6 be a path of G such that $e(v_3) = e(v_4) = 3$ and (3.2) are satisfied. Let G_1 and G_2 be subgraphs of G satisfying the following three conditions:*

(k) $\quad V(G_1) \cap V(G_2) = \emptyset,$

(kk) $\quad V(G_1) \cup V(G_2) = V(G),$

(kkk) $\quad v_1, v_2, v_3 \in V(G_1), \quad v_4, v_5, v_6 \in V(G_2).$

We claim that

a) *there exists an edge* $uv \in E(G)$ *for which* $u \in V(G_1) - \{v_3\}$, $v \in V(G_2) - \{v_4\}$,

b) $uv \in E(G)$, $u \in V(G_1) - \{v_3\}$, $v \in V(G_2) - \{v_4\}$ *and* $\mathrm{diam}(G_1) \leq 3$, $\mathrm{diam}(G_2) \leq 2$ *give* $k \geq 3$,

c) $uv \in E(G)$, $u \in V(G_1) - \{v_3\}, v \in V(G_2) - \{v_4\}$ *and* $\mathrm{diam}(G_1) \leq 2$, $\mathrm{diam}(G_2) \leq 2$ *give* $k \geq 4$.

Proof. a) Suppose, contrary to our claim, that there is no edge uv with $u \in V(G_1) - \{v_3\}$, $v \in V(G_2) - \{v_4\}$. Then every shortest path between the vertices v_1 and v_6 contains the vertex v_3 or v_4, which implies $d(v_1, v_6) \geq 5$. This contradicts to our assumption ($e(v_1) \leq 4$).
b) It is clear that $e(u) \leq 3$.
c) It follows easily that $e(u) \leq 3$ and $e(v) \leq 3$. □

Lemma 3.3. *Let G be a graph with an eccentric sequence of type $3^k, 4^l$ and let $V(G) = \{v_1, v_2, \ldots, v_8\}$. If a circle $v_1, v_2, \ldots, v_m, v_1$ with $m \in \{4, 5, 6, 7\}$ is a subgraph of G and each vertex v_j, $j > m$, is adjacent to at least one vertex from the set $\{v_1, v_2, \ldots, v_m\}$, then $k \geq 4$.*

Proof. It is clear that $k \geq 5$ if $m = 7$. Let $m = 6$. At most one vertex of the circle $v_1, v_2, \ldots, v_m$ is at distance 4 from v_7 and at distance 4 from v_8. It implies that $l \leq 4$ and so $k \geq 4$.

Let $m \in \{4, 5\}$. It is easily seen that $d(v_i, v_j) \leq 3$ for each $i \in \{1, 2, \ldots, m\}$, $j \in \{m + 1, \ldots, 8\}$. It implies $e(v_i) \leq 3$ for each $i = 1, 2, \ldots, m$. □

Lemma 3.4. *Let G be a connected graph such that $|V(G)| = 7$ and let a circle of the length at least 6 be its subgraph. Then the eccentricities of at least 5 vertices of G do not exceed 3.*

Proof. It is immediate. □

Lemma 3.5. *A graph G with the eccentric sequence $3^2, 4^7$ does not exist.*

Proof. Suppose that there is a graph G with the eccentric sequence 3^2, 4^7. Let $V(G) = \{v_1, v_2, \ldots, v_9\}$. Suppose that the assertion is false. We will use Lemma 3.1 and hence we distinguish two cases.

I. Let the graph G contains a path $v_1, v_2, \ldots, v_7$ such that $e(v_4) = 3$ and the equalities (3.1) hold.

If $\deg v_1 = \deg v_7 = 1$ then (by Lemma 1.1) $e(v_2) = 3$ and $e(v_6) = 3$, a contradiction. Therefore we will suppose that at least one of the vertices v_1, v_7 has the degree at least 2. If we take symmetry and the equality $e(v_4) = 3$ into account, we see that it is sufficient to distinguish the following subcases.

a) Each of the vertices v_8, v_9 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4, v_5, v_6\}$.

a_1) The vertex v_8 is adjacent to v_2 or v_3 and the vertex v_9 to v_5 or v_6.
Without loss of generality we may assume that both vertices with the eccentricity 3 belong to the set $\{v_1, v_2, v_3, v_4, v_8\}$.

Firstly, suppose that $v_2v_8 \in E(G)$ or the subgraph $\langle v_1, v_2, v_3, v_8 \rangle$ of G has at least 4 edges. Any shortest path between v_1 and v_7 does not contain v_4 (with respect to (3.1)) hence there is an edge v_iv_j, $i \in \{1, 2, 3, 8\}$, $j \in \{5, 6, 7, 9\}$. It gives $e(v_j) \leq 3$, a contradiction.

Secondly, let $v_3v_8 \in E(G)$ and the subgraph $\langle v_1, v_2, v_3, v_8 \rangle$ has exactly 3 edges. The assumption $\deg v_1 > 1$ implies $e(v_5) \leq 3$, a contradiction. If $\deg v_1 = 1$ then $\deg v_2 \geq 3$ (by Lemma 1.4) and so $E(G)$ contains an edge v_2v_j, $j \in \{5, 6, 7, 9\}$. Thus $e(v_j) \leq 3$, a contradiction.

a_2) Each of the vertices v_8, v_9 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4\}$ and $v_4v_9 \notin E(G)$.

Firstly, let $\deg v_7 > 1$. In this case we have that eccentricities of at least two vertices from v_2, v_3, v_8, v_9 are at most 3, a contradiction. (Note that $v_4v_8 \in E(G)$ gives $v_7v_8 \notin E(G)$ because $d(v_4, v_7) = 3$.)

Secondly, $\deg v_7 = 1$ gives $e(v_6) = 3$ and $\deg v_6 > 2$ (by Lemma 1.3 and 1.2). Hence we have either the case a_1) or at least one of the inequalities $e(v_2) \leq 3$, $e(v_3) \leq 3$ holds, which is impossible.

a_3) Let $v_4v_8, v_4v_9 \in E(G)$.
The inequality $\deg v_1 > 1$ gives (with respect to (3.1)) $e(v_5) \leq 3$ and $e(v_6) \leq 3$, which is impossible. Similarly, $\deg v_7 > 1$ yields $e(v_2) \leq 3$ and $e(v_3) \leq 3$.

b) The vertex v_9 is not adjacent to any vertex from the set $\{v_2, v_3, v_4, v_5, v_6\}$.

In this case we have $v_8v_9 \in E(G)$ and it is sufficient to consider two possibilities.
b_1) Let $v_4v_8 \in E(G)$.
Without loss of generality we may assume that both vertices with the eccentricity 3 belong to the set $\{v_1, v_2, v_3, v_4, v_8, v_9\}$.

Firstly, if $\deg v_1 > 1$ we obtain $v_1v_9 \in E(G)$ (because $e(v_5) = 4$). By our assumption $e(v_7) = e(v_6) = 4$ and so $\deg v_7 > 1$. Therefore eccetricities of at least two vertices from the set $\{v_2, v_3, v_8, v_9\}$ are at most 3, a contradiction.

Secondly, if $\deg v_1 = 1$ then $e(v_2) = 3$ and $\deg v_2 > 2$ (by Lemmas 1.3 and 1.4). Since $e(v_8) = 4$ we obtain that $\deg v_9 > 1$ (by Lemma 1.3). Therefore, $v_7v_9 \in E(G)$ holds. From $\deg v_2 > 2$ we get $v_2v_j \in E(G)$, $j > 4$. This yields $e(v_j) \leq 3$, a contradiction.
b_2) Let $v_3v_8 \in E(G)$.
If $\deg v_7 > 1$ then eccentricities of at least two vertices from the set $\{v_2, v_3, v_8\}$ are at most 3. The equality $\deg v_7 = 1$ guarantees $e(v_6) = 3$ and $\deg v_6 > 2$. This forces $e(v_2) \leq 3$ or $e(v_8) \leq 3$. It again contradicts our assumption.

II. Let the graph G contain a path $v_1, v_2, \ldots, v_6$ (see Fig. 3.2) such that $e(v_3) = e(v_4) = 3$ and (3.2) hold.

From $e(v_2) = e(v_5) = 4$ we get $\deg v_1 > 1$ and $\deg v_6 > 1$ (by Lemma 1.3). Since we may assume that the case I does not take place and $e(v_3) = e(v_4) = 3$, it is sufficient to consider the next possibilities.

a) Each of the vertices v_7, v_8, v_9 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4, v_5\}$.
a_1) If at least one of the vertices v_3, v_4 is adjacent to no vertex from the set $\{v_7, v_8, v_9\}$ then we obtain a contradiction (by Lemma 3.2).
a_2) Let $v_3v_7, v_4v_8 \in E(G)$.
We may suppose that the vertex v_9 is adjacent to v_2 or v_3. Since $\deg v_6 > 1$ we have either $e(v_2) \leq 3$ or $v_8v_6 \in E(G)$. In the second case we again obtain a contradiction by Lemma 3.2.

b) Let $v_3v_7, v_7v_8 \in E(G)$ and let the vertex v_8 be adjacent to no vertex from the set $\{v_2, v_3, v_4, v_5\}$.
b_1) In each of the subcases $v_2v_9 \in E(G)$, $v_3v_9 \in E(G)$, $v_7v_9 \in E(G)$ we can obtain $e(v_2) \leq 3$ or $e(v_7) \leq 3$ (because $\deg v_6 > 1$).
b_2) Let $v_4v_9 \in E(G)$.
If $v_6v_9 \notin E(G)$ then $e(v_2) \leq 3$ or $e(v_7) \leq 3$ (because $\deg v_6 > 1$ by Lemma 1.3). If $v_6v_9 \in E(G)$ then $e(v_2) \leq 3$ (because $\deg v_1 > 1$ and $d(v_1, v_4) = 3$) or the graph G does not exist by Lemma 3.2.
b_3) Let $v_5v_9 \in E(G)$.
If the subgraph $\langle v_1, v_2, v_3, v_7, v_8 \rangle$ contains at least 5 edges using Lemma 3.2 we obtain a contradiction. Otherwise, we have $e(v_5) \leq 3$ (because $\deg v_1 > 1$ and $\deg v_8 > 1$ by Lemma 1.3). □

Lemma 3.6. *A graph with the eccentric sequence* $3^3, 4^5$ *does not exist.*

Proof. Suppose that there is a graph G with the eccentric sequence 3^3, 4^5. We again use Lemma 3.1 and hence the proof splits into two parts.

I. Let G contain a path $v_1, v_2, \ldots, v_7$ such that $e(v_4) = 3$ and the equalities (3.1) hold.

Since $e(v_4) = 3$ the vertex v_8 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4, v_5, v_6\}$. We consider two cases.

a) The subgraph $\langle v_1, v_2, \ldots, v_7\rangle$ of G contains at least 7 edges.

Evidently $d(v_1, v_7) \le 4$ and in accordance with Lemma 3.3 it is sufficient to consider two subcases.

a_1) $v_2v_5 \in E(G)$

Since $e(v_5) \ge 3$ we obtain $v_3v_8 \in E(G)$. In the case $\deg v_7 > 1$ we may use (3.1) and Lemma 3.3 and we get a contradiction. The equality $\deg v_7 = 1$ yields $\deg v_6 > 2$ (by Lemma 1.4) and we have again (by Lemma 3.3) a contradiction.

a_2) $v_3v_5 \in E(G)$

In this case $e(v_3) \le 3$ and $e(v_5) \le 3$. The case a_1) implies that $v_2v_5 \notin E(G)$ and with respect to symmetry also $v_3v_6 \notin E(G)$. Then using the Lemma 3.3 we get that every shortest path between v_1 and v_7 must contain the vertex v_8 and so $e(v_8) \le 3$.

b) The subgraph $\langle v_1, v_2, \ldots, v_7\rangle$ contains only 6 edges.

By Lemma 1.4 we have $v_1v_8 \in E(G)$ or $v_2v_8 \in E(G)$. Analogously, $v_6v_8 \in E(G)$ or $v_7v_8 \in E(G)$. Hence we get a contradiction (with Lemma 3.3 or $e(v_8) \le 2$).

II. The graph G contains a path $v_1, v_2, \ldots, v_6$ where $e(v_3) = e(v_4) = 3$ and (3.2) holds.

Since we can assume that the case I does not take place it is sufficient to consider the next possibilities.

a) Each of the vertices v_7, v_8 is adjacent to at least one vertex from the set $\{v_2, v_3, v_4, v_5\}$.

We distinguish two cases.

a_1) The subgraph $\langle v_1, v_2, \ldots, v_6\rangle$ contains at least 6 edges.

In this case we get $v_1v_6 \in E(G)$ or $v_2v_5 \in E(G)$ (in accordance with (3.2)), which is impossible by Lemma 3.3.

a_2) The subgraph $\langle v_1, v_2, \ldots, v_6\rangle$ contains only 5 edges.

The case $\deg v_6 = 1$ and $\deg v_5 = 2$ is impossible (by Lemma 1.4). If we take symmetry into account it is sufficient to distinguish the next subcases.

1) $v_2v_7, v_5v_8 \in E(G)$.

In this subcase we have that eccentricities of at least 4 vertices are at most 3 (by Lemma 3.2).

2) $v_2v_7, v_4v_8 \in E(G)$

Since $\deg v_6 > 1$ or $\deg v_5 > 2$ we can again get that eccentricities of at least 4 vertices are at most 3 (by Lemmas 3.2 and 3.3).

3) $v_2v_7, v_3v_8 \in E(G)$

With respect to (3.2) and Lemma 3.3 we have $\deg v_7 = 1$. Then $\deg v_6 > 2$ and we obtain a contradiction (by Lemmas 3.2 and 3.3).

4) $v_2v_7, v_2v_8 \in E(G)$

By Lemma 3.3 we obtain $\deg v_6 = 1$ and $\deg v_5 = 2$, a contradiction (by Lemma 1.4).

5) $v_3v_7, v_4v_8 \in E(G)$

If $\deg v_5 > 2$ or $\deg v_2 > 2$ we obtain one of previous cases. In the opposite case we have (by Lemma 1.4) $\deg v_1 > 1$, $\deg v_6 > 1$ and this is impossible by Lemma 3.2.

6) $v_3v_7, v_3v_8 \in E(G)$

With respect to the equalities (3.2) we get $\deg v_6 = 1$. If $\deg v_5 > 2$ we obtain the case 2). Otherwise we have a contradiction by Lemma 1.4.

b) The vertice v_8 is adjacent to no vertex from the set $\{v_2, v_3, v_4, v_5\}$.
With respect to symmetry we may assume that $v_3v_7, v_7v_8 \in E(G)$. We distinguish two possibilities.

b_1) A subgraph $\langle v_1, v_2, \dots, v_6\rangle$ has at least 6 edges.
In this case we get $v_1v_6 \in E(G)$ or $v_2v_5 \in E(G)$. By Lemma 3.3 we get $\deg v_8 = 1$ and $\deg v_7 = 2$. Accordingly to Lemma 1.4 it is impossible.

b_2) A subgraph $\langle v_1, v_2, \dots, v_6\rangle$ has 5 edges.

Firstly, let $v_4v_7 \notin E(G)$ and so $d(v_4, v_8) = 3$. With respect to the previous case we may assume that the subgraph $\langle v_3, v_4, v_5, v_6, v_7, v_8\rangle$ has 5 edges which implies $\deg v_6 = 1$ and $\deg v_5 = 2$, a contradiction (by Lemma 1.4).

Secondly, let $v_4v_7 \in E(G)$. By Lemma 3.2 we obtain that $\deg v_8 = 1$. Hence, a shortest path between v_1 and v_6 contains v_7 and so $e(v_7) \leq 2$ (because $v_1v_7 \notin E(G)$ and $v_6v_7 \notin E(G)$ by (3.2)). □

Lemma 3.7. *A graph G with the eccentric sequence $3^4, 4^3$ does not exist.*

Proof. On the contrary, let there exist such a graph G.

Similarly as at the previous proofs we distinguish two cases.

a) Let G contain a path $v_1, v_2, \dots, v_7$ where $e(v_4) = 3$ and (3.1) holds.

By (3.1) and Lemma 3.4 we deduce that $\deg v_7 = 1$. From this it follows that $\deg v_6 > 2$ (by Lemma 1.4). By Lemma 3.4 and the equalities (3.1) we obtain $v_iv_6 \in E(G)$ for some $i \in \{2, 3\}$. It implies $e(v_i) \leq 2$, a contradiction.

b) Let G contain a path $v_1, v_2, \dots, v_6$ where $e(v_3) = e(v_4) = 3$ and (3.2) holds.

It suffices to consider the next two cases.

b_1) $v_2v_7 \in E(G)$
Applying (3.2) and Lemma 3.4 gives $\deg v_6 = 1$. Therefore $e(v_5) > 2$ by Lemma 1.4. This clearly forces $v_5v_i \in E(G)$, $i \in \{2, 7\}$ (with respect to (3.2)) an so $e(v_i) \leq 2$, a contradiction.

b_2) $v_3v_7 \in E(G)$
We have $\deg v_6 = 1$ using Lemma 3.4 and the equalities (3.2). This gives $e(v_5) > 2$ by Lemma 1.4. Consequently, $v_5v_i \in E(G)$, $i \in \{2, 7\}$.

Firstly, if $v_5v_2 \in E(G)$ then $e(v_2) \leq 2$, a contradiction.

Secondly, if $v_5v_7 \in E(G)$ (and $v_5v_2 \notin E(G)$) then a shortest path between v_1 and v_6 contains v_7 and in consequence, $e(v_7) \leq 2$ (because $d(v_6, v_7) = 2$ and $d(v_1, v_6) \leq 4$). □

4. Minimal eccentric sequences of type $3, 4^l, 5^r$, $l + r > 9$.

In this section we will use the following basic idea.

Let G be a graph. Fix the vertex with the eccentricity 3 and denote it by s (a source). In this case we denote

$A := \{v \in V(G); d(s, v) = 1\}$,

$B := \{v \in V(G); d(s,v) = 2\}$,
$C := \{v \in V(G); d(s,v) = 3\}$.
If $v \in C$ ($v \in B$), there exists a vertice $u \in B$ ($u \in A$) such that $uv \in E(G)$. If $v \in A$ then $vs \in E(G)$. On the other hand, $vs \notin E(G)$ if $v \in B \cup C$ and $vu \notin E(G)$ if $v \in C$, $u \in A$. The distance of vertices u, v is at most 2 if $u, v \in A$, is at most 3 if $v \in A$, $u \in B$, respectively, etc. In the case $\deg s = 2$ we divide the vertices of G more carefully. In this case we put $A = \{a, a'\}$. Further, we will denote

by $b_1, b_2, \dots, b_n$ the vertices from the set $\{v \in B; d(v,a) = 1\}$,

by $b'_1, b'_2, \dots, b'_m$ the vertices from the set $\{v \in B; d(v,a') = 1\}$,

by $c_1, c_2, \dots c_p$ the vertices from the set C which are adjacent to at least one vertex from the set $\{b_1, b_2, \dots, b_n\}$,

by $c'_1, c'_2, \dots, c'_q$ the vertices from the set C which are adjacent to at least one vertex from the set $\{b'_1, b'_2, \dots, b'_m\}$.

It is possible that, $c_i = c'_j$ for some i, j. On the other hand, if $aa' \in E(G)$ then $e(a) \leq 3$, which is impossible in the considered case. Similarly, if $b_i = b'_j$ for some i, j then $e(b_i) \leq 3$.

Lemma 4.1. *Let G be a graph with the eccentric sequence $3, 4^l, 5^r$, where $l \leq 4$. Let $e(s) = 3$ and $\deg s \geq 3$ hold. Then*

a) $|C| > 2$,

b) $d(c_i, c_j) > 2$ for every $i \neq j$ if $|C| = 3$.

Proof. By our assumption $|A| \geq 3$ and $e(v) = 4$ if $v \in A$. Therefore the eccentricity of at most one vertex from B is 4 (because $l \leq 4$).

a) The inequality $|C| \geq 2$ follows by Lemma 2.1. Let $C = \{c_1, c_2\}$. If the eccentricity of a vertex $b \in B$ is 5 then $d(b, c_1) = 5$ or $d(b, c_2) = 5$. If a shortest path between c_1 and c_2 contains at least two vertices $b_i, b_j \in B$ then $e(b_i) \leq 4$ and $e(b_j) \leq 4$, a contradiction. Otherwise, $d(c_1, c_2) \leq 2$ and there exist edges $c_1 b_i, c_2 b_j \in E(G)$, $i \neq j$ (by Lemma 2.1) and so $e(b_i) \leq 4$ and $e(b_j) \leq 4$, a contradiction.

b) Let $C = \{c_1, c_2, c_3\}$ and let (on the contrary) $d(c_1, c_2) \leq 2$. It is sufficient to distinguish the following two cases.

b_1) $c_1 c_2 \in E(G)$

Firstly, let $d(c_1, c_3) \leq 2$ or $d(c_2, c_3) \leq 2$. By Lemma 2.1 there are edges $c_i b_j$, $c_s b_t$, $j \neq t$. Then we conclude that $e(b_j) \leq 4$ and $e(b_t) \leq 4$, which contradicts our assumption $l \leq 4$.

Secondly, let $d(c_1, c_3) > 2$ and $d(c_2, c_3) > 2$. In this case we consider a shortest path between c_1 and c_3

$c_1 = u_1, u_2, \dots, u_i, u_{i+1} = c_3$

and a shortest path between c_2 and c_3

$c_2 = v_1, v_2, \dots, v_j, v_{j+1} = c_3$.

Both of the paths contain at least two vertices from B. If one of the paths contains all vertices from C then the eccetricities of all vertices of this path from B are at most 4, a contradiction. For otherwise we have either $e(u_2) \leq 4$ and $e(v_2) \leq 4$ provided that $u_2 \neq v_2$ or $e(u_2) \leq 4$ and $e(u_i) \leq 4$ provided that $u_2 = v_2$.

b_2) $b_1 c_1, b_1 c_2 \in E(G)$

We distinguish three cases.

If $d(c_1,c_3) \le 3$ and $d(c_2,c_3) \le 3$ then there exists an edge $c_ib_j \in E(G)$, $j > 1$ (by Lemma 2.1) which implies $e(b_1) \le 4$ and $e(b_j) \le 4$, a contradiction.

If $d(c_1,c_3) > 3$ and $d(c_2,c_3) > 3$ we obtain a contradiction in the same way as in the second case of the part b_1).

Let $d(c_1,c_3) \le 3$ and $d(c_2,c_3) > 3$. Obviously, $e(b_1) \le 4$. Let

$c_2 = v_1, v_2, \ldots, v_i, v_{i+1} = c_3$

be a shortest path between the vertices c_2 and c_3. If this path contains b_1 or c_1 we have $e(v_i) \le 4$. Otherwise we get $e(v_2) \le 4$. These conclusions contradict $l \le 4$. □

Lemma 4.2. *If $b_ic_j \in E(G)$ and b_i is a cut vertex of G then $e(c_j) = e(b_i) + 1$.*

Proof. It is obvious that $e(c_j) \ge 3$. If $d(u,c_j) \ge 3$ then a shortest path between u and c_j contains the vertex b_i. □

Lemma 4.3. *A graph G with the eccentric sequence $3, 4^2, 5^r$ exists if and only if $r \ge 12$.*

Proof. In this case $A = \{a, a'\}$. Vertices from B adjacent to the vertex a will be denoted by $b_1, b_2, \ldots b_n$ and vertices of B adjacent to a' will be denoted by $b'_1, b'_2, \ldots, b'_m$, etc.

It is easily seen that $aa' \notin E(G)$ (in the opposite case we get $e(a) \le 3$ which is impossible). Further, $b_i \ne b'_j$ and $b_ib'_j \notin E(G)$ for each $i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,m\}$ (otherwise, we have $e(b_i) \le 4$ which is impossible). Analogously $c_i \ne c'_j$ for each $i \in \{1,2,\ldots p\}$, $j \in \{1,2,\ldots,q\}$. Since $d(c_1,c'_1) \le 5$ there exists an edge $c_ic'_j \in E(G)$, for some i, j.

We denote the subgraph $\langle a, b_1, \ldots, b_n, c_1, \ldots c_p\rangle$ by H. If $diam H \le 3$ we obtain $e(c'_j) \le 4$ (because $c_ic'_j \in E(G)$), a contradiction. Thus, we can suppose that $diam H = 4$. Now we are going to show that $|V(H)| \le 6$ is impossible.

On the contrary, suppose that $diam H = 4$ and $|V(H)| \le 6$. Without loss of generality we can assume that the following conditions are satisfied: $d_H(c_1,c_2) = 4$, a shortest path between c_1 and c_2 is c_1, b_1, a, b_2, c_2, $\deg_H c_1 = 1$ and if $p = 3$ (i.e. there is a vertex c_3) then $c_3b_2 \in E(H)$. If there is no edge of type $c_1c'_i$ then we have $e(b_1) = 4$ (because $e(c_1) = 5$), a contradiction. Analogously, if $\deg_H c_2 = 1$ then $c_2c'_k \in E(G)$ for some k. If $\deg_H c_2 > 1$ then either $c_2b_3 \in E(H)$ or $c_2c_3 \in E(G)$. If there is no edge of type $c_jc'_k$, $j \in \{2,3\}$ then we have $e(a) = 3$ (because $e(c_2) = 5$), a contradiction. Thus we may assume that $c_1c'_i, c_jc'_k \in E(G)$ for some i,j,k, $j > 1$. Let b'_t be a vertex adjacent to c'_k. Then, obviously, we get $e(b'_t) \le 4$, a contradiction. Thus, we conclude that $|V(H)| \ge 7$.

If $H' = \langle a', b'_1, \ldots, b'_m, c'_1, \ldots, c'_q\rangle$ then with respect to symmetry we get $|V(H')| \ge 7$, too. Therefore, the graph G has at least 15 vertices.

The graph with the eccentric sequence $3, 4^2, 5^{12}$ is in Fig.4.1. □

It is possible to show that the eccentric sequence $3, 4^2, 5^{12}$ is realized by only the graph in Fig. 4.1, but we will not deal with this point here.

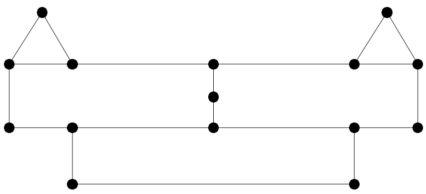


Fig. 4.1

Lemma 4.4. *A graph G with the eccentric sequence $3, 4^3, 5^8$ does not exist.*

Proof. Suppose on the contrary, that there is such a graph G. We distinguish two cases.

I. Let $\deg s = 2$.

Suppose that the vertices of the graph G are denoted as in the proof of Lemma 4.3. If there exists an edge of type $b_i b'_j$ then $e(b_i) \le 4$ and $e(b'_j) \le 4$, i.e. there exist at least 4 vertices with eccentricity at most 4, a contradiction. Since $d(c_1, c'_1) \le 5$, either $c_i = c'_j$ or G contains an edge of type $c_i c'_j$ for some i, j.

a) Let $c_i = c'_j$.

Without loss of generality we can assume that $c_1 = c'_1$ and the graph G contains the subgraph in Fig. 4.2.

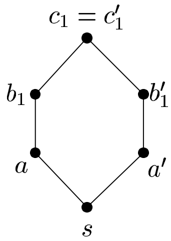


Fig. 4.2

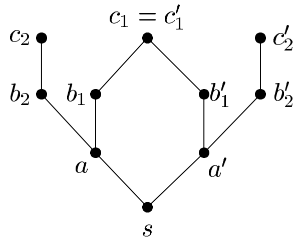


Fig. 4.3

Then $e(c_1) \le 4$ and so each vertex from the set $V(G) - \{s, a, a', c_1\}$ has the eccentricity 5. It implies that G contains the subgraph in Fig. 4.3 satisfying

$$\text{(f)} \qquad d(b_1, c'_2) = 5 = d(b'_1, c_2).$$

Since $d(c_2, c'_2) \le 5$ there exists a path

$c_2 = v_1, v_2, \ldots, v_s, v_{s+1} = c'_2$

of the length at most 5 (i.e. $s \le 5$). The path contains no vertex from the set $\{s, a, a', b_1, b'_1, c_1\}$ because (f) holds. For the same reason this path contains at least one vertex $v \notin \{b_2, b'_2, c_2, c'_2\}$. It follows easily that $e(b_2) \le 4$ or $e(b'_2) \le 4$, a contradiction.

b) Let $c_i c'_j \in E(G)$.

Without loss of generality we may assume that the graph in Fig. 4.4 is a subgraph of G.

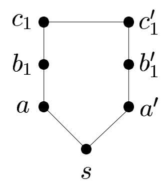


Fig. 4.4

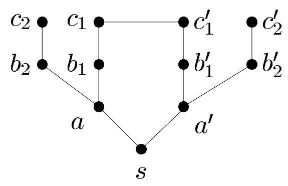


Fig. 4.5

Since at most one vertex from the set $\{b_1, c_1, b_1', c_1'\}$ has the eccentricity 4 the graph in Fig. 4.5 is a subgraph of G, too.

Besides the vertices of the graph in Fig. 4.5 the graph G has one another vertex w. Without loss of generality we may assume that $d(a, w) \leq 2$. The inequality $\deg c_2' > 1$ yields either $e(a) \leq 3$ or $e(b_1) \leq 4$ and $e(c_1) \leq 4$, a contradiction.

Let $\deg c_2' = 1$, i.e. $e(b_2') = 4$ (by Lemma 1.3). Then $\deg b_2' > 2$ by Lemma 1.4 and we conclude $e(c_1) \leq 4$ or $e(b_2) \leq 4$, a contradiction.

II. Let $\deg s = 3$.

Since $|A| = 3$ we have $|B| + |C| = 8$ and every vertex from the set $B \cup C$ has the eccentricity 5. Then, by Lemma 4.2, $\deg c_i \geq 2$ for every vertex $c_i \in C$. Lemma 4.1 implies that $|C| > 2$ and if $|C| = 3$ then $|B| \geq 6$ (a contradiction). Therefore, we can suppose that $|C| \geq 4$. We will distinguish three cases.

1) Let $|C| = |B| = 4$.

Denote $H = \langle c_1, c_2, c_3, c_4, b_1, b_2, b_3, b_4 \rangle$. We consider two possibilities.

a) The vertices c_1, c_2, c_3, c_4 belong to the same component H_1 of H.

If $V|H_1| = 8$ then there exists a vertex v for which $e_{H_1}(v) \leq 4$ (for example, we can take a vertex from the centre of a spanning tree of H_1). This clearly forces $e_G(v) \leq 4$, a contradiction.

Let $V|(H_1)| \leq 7$ and suppose that b_4 does not belong to $V(H_1)$. Since every vertex $c \in C$ is adjacent to a vertex $b \in B$ we may assume that $b_1c_1, b_1c_2 \in E(G)$. Clearly $e(b_1) = 5$ and so we may assume (without loss of generality) that $d(b_1, c_4) = 5$. This yields that a shortest path between b_1 and c_4 in the subgraph H_1 contains the vertices b_2, b_3, c_3. It implies either $e(b_2) \leq 4$ or $e(b_3) \leq 4$, a contradiction.

b) The vertices c_1, c_2, c_3, c_4 do not belong to the same component of H.

With respect to Lemma 4.2 they belong to only two components. It is sufficient to consider two subcases.

b_1) Let $H_1 = \langle b_1, b_2, c_1, c_2, c_3 \rangle$, $H_2 = \langle b_3, b_4, c_4 \rangle$.

A shortest path between c_1 and c_4 contains a vertex b_i, $i \in \{1, 2\}$, which implies $e(b_i) \leq 4$, a contradiction.

b_2) Let $H_1 = \langle b_1, b_2, c_1, c_2 \rangle$, $H_2 = \langle b_3, b_4, c_3, c_4 \rangle$. Consider two paths P_1 and P_2 from four shortest paths (in G) between vertices c_i and c_j, $i \in \{1, 2\}$, $j \in \{3, 4\}$, which contain a vertex b_k, $k \in \{1, 2\}$. If P_1 and P_2 have the same initial vertex then $e(b_k) \leq 4$. In the opposite case we have $e(b_m) \leq 4$, $m \in \{3, 4\}$, where b_m is a vertex of P_1 or P_2.

2) Let $|B| = 3$, $|C| = 5$.

We put $H = \langle c_1, c_2, c_3, c_4, c_5, b_1, b_2, b_3 \rangle$. The vertices c_1, c_2, c_3, c_4, c_5 belong to the same component H_1 of H (by Lemma 4.2). If $|V(H_1)| = 8$ then there exists a vertex v with $e_{H_1}(v) \leq 4$ and so $e_G(v) \leq 4$. If $|V(H)| < 8$ and the vertex b_3

does not belong to H_1 then $e(b_1) \le 4$, $e(b_2) \le 4$. In the both cases we get a contradiction.

3) Let $|B| = 2$, $|C| = 6$.

It is easy to check that we again get $e(b_1) \le 4$ and $e(b_2) \le 4$, which is impossible.

Since the set B contains at least two elements (by Lemma 2.1), the proof is complete. □

Lemma 4.5. *There is no graph G with the eccentric sequence $3, 4^4, 5^6$.*

Proof. Suppose, contrary to our claim, that there is such a graph G. Let $s \in V(G)$ be the vertex with the eccentricity 3. We again distinguish two cases.

I. Let $\deg s = 2$.

We will suppose that the vertices of G are denoted as in the proof of Lemma 4.3.

a) Suppose there exists an edge $b_i b'_j \in E(G)$.

We may assume that the graph G contains the subgraph in Fig. 4.6.

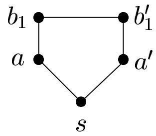


Fig. 4.6

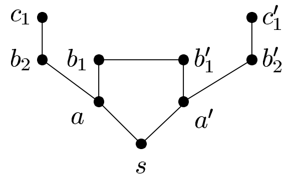


Fig. 4.7

Since $e(a) = e(a') = 4$, also the graph in Fig. 4.7 is a subgraph of G, where

$$(g) \qquad d(a, c'_1) = d(a', c_1) = 4.$$

Therefore, the eccentricity of every vertex from $V(G) - \{s, a, a', b_1, b'_1\}$ is 5. We may also assume that $d(b_2, c'_1) = 5$.

Besides the vertices of the graph in Fig. 4.7 the graph G has another vertices x, y. With respect to symmetry it is sufficient to consider the next cases.

a_1) Let $d(x, a') \le 2$ and $d(y, a') \le 2$. Then $\deg c_1 = 1$ and so $e(c_1) \ne e(b_2)$ (by Lemma 1.3) or $e(a') \le 3$, a contradiction.

a_2) Suppose that $d(a, x) \le 2$ and $d(a', y) \le 2$.

Let

$c_1 = v_1, v_2, \ldots, v_k = c'_1$

be a shortest path between vertices c_1 and c'_1. Since $d(b_2, c'_1) = 5$ and $d(c_1, c'_1) \le 5$ we obtain $v_2 \ne b_2$ and $k \in \{5, 6\}$. The equalities (g) and $e(b'_1) = 4$ yield that $v_2 = x$ and $v_3 = y$ and so $e(y) \le 4$, a contradiction.

b) Let $c_i = c'_j$ for some i, j.

We may assume that the graph G contains the subgraph in Fig. 4.2 and, moreover, only one vertex from $V(G) - \{s, a, a', c_1\}$ has the eccentricity 4.

Let $e(b'_1) = 5$. Then G contains the subgraph in Fig. 4.8, where $d(c_2, b'_1) = 5$.

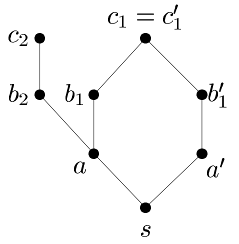


Fig. 4.8

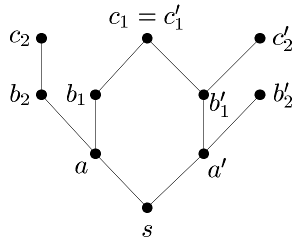


Fig. 4.9

It is sufficient to consider two possibilities.
b_1) Let G contain a subgraph in Fig. 4.3 satisfying $d(c_2', b_1) = 5$ and let $d(a', x) \leq 2$ for a remaining vertex x.

If $\deg c_2 = 1$ then $e(b_2) = 4$ and $\deg b_2 > 2$ (see Lemma 1.4). Hence, we obtain $e(b_1') \leq 4$ or $e(b_2') \leq 4$. If $\deg c_2 > 1$ then $d(b_1', c_2) < 5$. In both cases we have a contradiction.
b_2) In the opposite case to b1) the graph G contains the subgraph in Fig. 4.9 (because $e(b_1) = e(a) = 4$) and the eccentricity of every vertex from $V(G) - \{s, a, a', b_1, c_1\}$ is 5. Then clearly $\deg c_2 > 1$ (by Lemma 1.1). If the remaining vertex x satisfies $d(a', x) \leq 2$ we obtain $d(b_1', c_2) < 5$, a contradiction. Let $d(a, x) \leq 2$. If a shortest path P between c_2 and c_2' contains b_2 then we get $e(b_2) \leq 4$, a contradiction. In the opposite case we get $d(c_2, b_1') < 5$ or $e(a) \leq 3$, a contradiction.

c) Let $c_i c_j' \in E(G)$ for some i, j.

We may assume that the graph G contains the subgraph in Fig. 4.10 (otherwise the eccentricity each of the vertices b_1, b_1', c_1, c_1' is at most 4) and moreover $d(b_1', c_2) = 5$. We distinguish two subcases.
c_1) The graph G contains the subgraph in Fig. 4.5 satisfying $d(b_1, c_2') = 5$.

Let $\deg c_2 = \deg c_2' = 1$. If $b_2 b_2' \in E(G)$, then $e(a) \leq 3$. In the opposite case (i.e. $b_2 b_2' \notin E(G)$) we denote by v a vertex of a shortest path between b_2 and b_2' satisfying $v \neq b_2$, $v \neq b_2'$. Since $d(b_2, b_2') \leq 3$ (it follows from $d(c_2, c_2') \leq 5$) we have $v \neq s$ and $e(v) \leq 3$, a contradiction.

The inequality $\deg c_2 > 1$ implies $d(b_1', c_2) \leq 4$ (and similarly for c_2'), a contradiction.

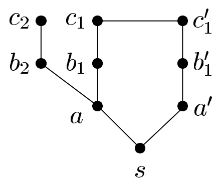


Fig. 4.10

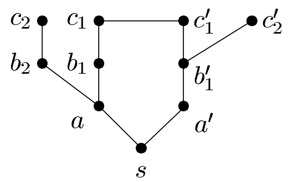


Fig. 4.11

c_2) In the opposite case to c_1) the graph G contains the subgraph in Fig. 4.11 (note that $b_1' c_2' \notin E(G)$ gives $e(a) \leq 3$) and the eccentricity of every vertex from $V(G) - \{s, a, a', b_1, c_1\}$ is 5.

Firstly, let $d(a', x) \leq 2$ for the remaining vertex x. The equality $\deg c_2 = 1$

yields $e(b_2) = 4$ (by Lemma 1.3), a contradiction. If $\deg c_2 > 1$ then $d(b_1', c_2) \leq 4$, a contradiction.

Secondly, let $d(a, x) \leq 2$. If $\deg c_2' = 1$ then $e(b_1') = 4$ (by Lemma 1.3), a contradiction. On the other hand, if $\deg c_2' > 1$ then $e(a) \leq 3$ or $e(b_1) \leq 3$, a contradiction.

II. Let $\deg s \geq 3$.

a) Suppose that $\deg s = 4$.

In this case we have $|B| + |C| = 6$ and the eccentricity of every vertex from $B \cup C$ is 5. Denote by H the subgraph of G induced by the set $B \cup C$. All vertices from C belong to the same component H_1 of H by Lemmas 4.1 and 4.2. We can assume (with respect to Lemma 4.2) that $b_1, b_2 \in V(H_1)$. Since $e(b_1) = 5$ we may suppose (without loss of generality) that $d(b_1, c_1) = 5$. Therefore, $|V(H_1)| = 6$ and a shortest path between b_1 and c_1 (in H_1) contains every vertex from $B \cup C$. This yields $e(b_2) \leq 4$, a contradiction.

b) Suppose that $\deg s = 3$.

Then $|B| + |C| = 7$ and the eccentricity of at most one vertex from B is 4. By Lemma 4.2 the degree of at most one vertex from C is 1. We get (by Lemma 4.1) $|C| > 2$ and moreover if $|C| = 3$ then $|B| \geq 5$ (a contradiction).

Therefore, we may assume that $|C| \geq 4$. We distinguish two subcases.

1) Let $|C| = 4$, $|B| = 3$.

Consider the subgraph $H = \langle c_1, c_2, c_3, c_4, b_1, b_2, b_3 \rangle$ of G. There are two possibilities.

A) All vertices from C belong to the same component H_1 of H.

Firstly, if $|V(H_1)| = 7$ then the eccentricities of at least 3 vertices from $V(H_1)$ are at most 4 (in H_1 and so in G, too), a contradiction.

Secondly, if (for example) $b_3 \notin V(H_1)$ then $e(b_1) \leq 4$ and $e(b_2) \leq 4$, a contradiction.

B) In the opposite case to A) the vertices c_1, c_2, c_3, c_4 belong to two components H_1, H_2 of the graph H (by Lemma 4.2). We may assume, without loss of generality, that $b_1, c_1 \in V(H_1)$ and $b_2, b_3, c_4 \in V(H_2)$. This yields $e(b_1) = 4$ by Lemma 4.2. A shortest path between c_1 and c_4 (in G) contains the vertex b_2 or b_3 and so $e(b_2) \leq 4$ or $e(b_3) \leq 4$, a contradiction.

2) Let $|C| = 5$, $|B| = 2$.

By Lemmas 2.1 and 4.2 the graph $H = \langle c_1, \ldots, c_5, b_1, b_2 \rangle$ is connected and so $e(b_1) \leq 4$ and $e(b_2) \leq 4$.

The set B has at least two elements by Lemma 2.1 and so the proof is complete. □

5. The main results.

Theorem 5.1. *There are exactly 13 minimal eccentric sequences with least eccentricity three, namely*

$S_1 :$	3^6	$S_7 :$	$3, 4^2, 5^{12}$
$S_2 :$	$3^5, 4^2$	$S_8 :$	$3, 4^3, 5^9$
$S_3 :$	$3^4, 4^4$	$S_9 :$	$3, 4^4, 5^7$
$S_4 :$	$3^3, 4^6$	$S_{10} :$	$3, 4^5, 5^4$
$S_5 :$	$3^2, 4^8$	$S_{11} :$	$3, 4^7, 5^2$
$S_6 :$	$3, 4^{10}$	$S_{12} :$	$3^2, 4^2, 5^2$

$$S_{13} : \quad 3, 4^2, 5^2, 6^2$$

Proof. The sequence S_1 is the eccentric sequence of the graph in Fig 5.1. We will show that it is minimal. Let G be a connected graph with $|V(G)| \leq 5$ and let H be its spanning tree. Let a vertex v belong to the centre of H. It is obvious that $e_H(v) \leq 2$ and so $e_G(v) \leq 2$.

The sequences S_2, S_3, S_4, S_5 and S_6 are the eccentric sequences of the graphs in Figs. 5.2, 5.3, 5.4, 5.5 and 2.5, respectively.

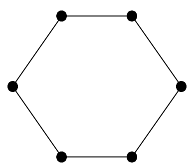


Fig. 5.1

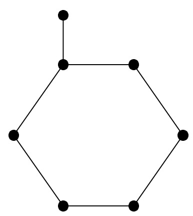


Fig. 5.2

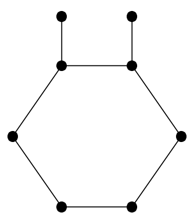


Fig. 5.3

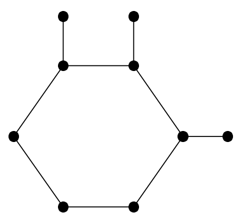


Fig. 5.4

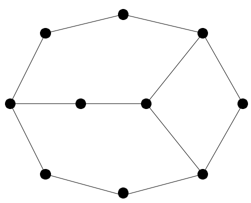


Fig. 5.5

By Theorems 1.1, 1.2, Lemmas 3.7, 3.6, 3.5 and Corollary 2.5 the eccentic sequences S_2, S_3, S_4, S_5 and S_6 are minimal and there are no other minimal eccentric sequences of type $3^k, 4^l$.

S_7, S_8, S_9, S_{10}, S_{11} and S_{12} are the eccentric sequences of the graphs in Figs. 4.1, 5.6, 5.7, 2.3, 2.4 and 5.8.

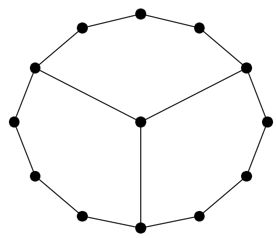


Fig. 5.6

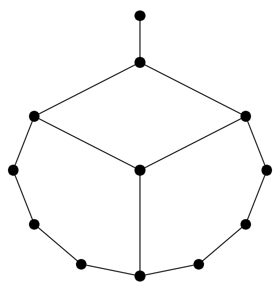


Fig. 5.7

Fig. 5.8

Fig. 5.9

By Theorems 1.1, 1.2, Lemmas 4.3, 4.4, 4.5 and Corollaries 2.6, 2.7 the eccentric sequences S_7, S_8, S_9, S_{10}, S_{11} and S_{12} are minimal and there are no other minimal eccentric sequences of type $3^k, 4^l, 5^r$.

S_{13} is the eccentric sequence of the graph in Fig 5.9. By Theorem 1.1 it is minimal $\square$

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