Acta Univ. M. Belii
Math. no. 5 (1997), pp. 63-72

DUALITY OF BOUNDED DISTRIBUTIVE q-LATTICES

Ivan Chajda and Miroslav Ploščica

Abstract. By a q-lattice is meant an algebra with two binary operations satisfying all normal lattice identities. We establish a duality in the sense of B. Davey and H. Werner for the quasivarieties of constant q-lattices and bounded distributive q-lattices.

1. Introduction

Let p, q be terms of the same similarity type. An identity $p = q$ is called *normal* (see [3], [7], [8], [9]) if it is either of the form $x = x$ (x is a variable) or none of p, q is equal identically to a single variable. So the lattice idempotence or absorption are not normal identities.

An algebra $\mathcal{A} = (A; \vee, \wedge)$ of type $(2,2)$ is called a *q-lattice* if it satisfies all normal identities of lattices. In fact, see [1], [2], [3], $\mathcal{A}$ is a q-lattice if it satisfies the following identities:

(commutativity)	$x \vee y = y \vee x$	$x \wedge y = y \wedge x$
(associativity)	$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(weak idempotence)	$x \vee (y \vee y) = x \vee y$	$x \wedge (y \wedge y) = x \wedge y$
(weak absorption)	$x \vee (x \wedge y) = x \vee x$	$x \wedge (x \vee y) = x \wedge x$
(equalization)	$x \vee x = x \wedge x\,.$	

A q-lattice $\mathcal{A}$ is *distributive* if it satisfies the distributive identity:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

(which is equal to its dual similarly as in the case of lattices). In every q-lattice $\mathcal{A} = (A; \vee, \wedge)$ we can introduce a binary relation Q as follows:

$$(a, b) \in Q \quad \text{if and only if} \quad a \wedge b = a \wedge a\,.$$

1991 *Mathematics Subject Classification.* 06D05, 06E15, 08C15.

Key words and phrases. Algebraic duality, normal identity, q-lattice

The first author was supported by GAČR - Grant Agency of Czech Republic, Grant No 201/98/0330.
The second author was supported by the Slovak VEGA Grant 1/4379/97.

It is easy to show that it is equivalent to $a \vee b = b \vee b$ and, moreover, Q is a *quasiorder* on A (i.e. a reflexive and transitive relation). A q-lattice $\mathcal{A}$ is a lattice if and only if Q is an order on A; in such a case, Q is the lattice order of $\mathcal{A}$.

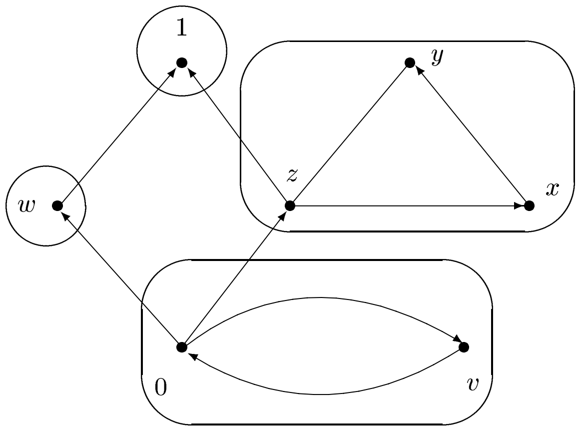

Fig. 1

Example. The following quasiordered set $\{0, x, y, z, v, w, 1\}$ is a distributive q-lattice which is not a lattice (the fact $(a, b) \in Q$ is visualized in Fig. 1 by a connected path of arrows from a to b):

Here e.g. $(0, v) \in Q$ and $(v, 0) \in Q$, $(w, 1) \in Q$, $(1, w) \notin Q$ etc. The operation tables for $\vee$ and $\wedge$ are as follows:

$\wedge$	0	v	w	x	y	z	1
0	0	0	0	0	0	0	0
v	0	0	0	0	0	0	0
w	0	0	w	0	0	0	w
x	0	0	0	z	z	z	z
y	0	0	0	z	z	z	z
z	0	0	0	z	z	z	z
1	0	0	w	z	z	z	1

	0	v	w	x	y	z	1
0	0	0	w	z	z	z	1
v	0	0	w	z	z	z	1
w	w	w	w	1	1	1	1
x	z	z	1	z	z	z	1
y	z	z	1	z	z	z	1
z	z	z	1	z	z	z	1
1	1	1	1	1	1	1	1

An element b of a q-lattice $\mathcal{A} = (A; \vee, \wedge)$ is called an *idempotent* if $b \vee b = b$ (or, equivalently, $b \wedge b = b$). The set of all idempotents of $\mathcal{A}$ is called the *skeleton* of $\mathcal{A}$ and it is denoted by $Sk\,\mathcal{A}$. It is easy to see that $Sk\,\mathcal{A}$ is the maximal sublattice of $\mathcal{A}$.

Evidently, the restriction of Q onto $Sk\,\mathcal{A}$ is the lattice order of $Sk\,\mathcal{A}$. Moreover, see [1], the q-lattice $\mathcal{A}$ is distributive if and only if $Sk\,\mathcal{A}$ is a distributive lattice.

Further, we introduce a binary relation ψ on A by $(a,b) \in \psi$ iff $(a,b) \in Q$ and $(b,a) \in Q$. Then (see [4] or [3]), ψ is a congruence on $\mathcal{A}$ and $\mathcal{A}/\psi \cong Sk\,\mathcal{A}$. The congruence classes of ψ are called *cells* of $\mathcal{A}$.

In the foregoing Example, the elements $0, w, z, 1$ are idempotents and the skeleton $Sk\,\mathcal{A}$ is the four element distributive lattice $\{0, z, w, 1\}$. $\mathcal{A}$ has four cells and the congruence ψ is shown in Fig. 1.

Hence, every cell of $\mathcal{A}$ contains exactly one idempotent (idempotents are the only results of operations). If d is the idempotent of a cell C of $\mathcal{A}$, then $x \vee y = d = x \wedge y$ for any $x, y \in C$. For more details of q-lattices, see e.g. [1], [2] and [4].

Let us recall some necessary concepts of duality theory given by B. Davey and H. Werner [6]. Let $\mathcal{V} = ISP(\underline{P})$ be a quasivariety generated by a non–trivial finite algebra $\underline{P} = (P; F)$. Let $\underset{\sim}{P} = (P; G, H, R, \tau)$ where τ is the discrete topology on P and G is the set of operations, H is a set of partial operations and R is a set of relations on P. Suppose that all those operations, partial operations and relations are subalgebras of appropriate powers of $\underline{P}$. In this case, $\underset{\sim}{P}$ is called *algebraic over* $\underline{P}$.

Let $\mathcal{W} = IS_cP(\underset{\sim}{P})$ be the class of all topological structures of the same type as $\underset{\sim}{P}$ which are isomorphic (i.e. simultaneously isomorphic and homeomorphic) to a closed substructure of a power of $\underset{\sim}{P}$. For every $\mathcal{A} \in \mathcal{V}$ the set $D(\mathcal{A})$ of all homomorphisms $\mathcal{A} \to \underline{P}$ is a closed substructure of $\underset{\sim}{P}^A$, hence $D(\mathcal{A}) \in \mathcal{W}$. Similarly, for each $X \in \mathcal{W}$ the set $E(X)$ of all morphisms $X \to \underset{\sim}{P}$ (i.e. continuous maps that preserve G, H and R) is a subalgebra of $\underline{P}^X$, hence $E(X) \in \mathcal{V}$. (See Lemmas 1.1 and 1.2 in [5].) We have thereby defined two contravariant hom-functors

$$D:\ \mathcal{V} \longrightarrow \mathcal{W}, \qquad E:\ \mathcal{W} \longrightarrow \mathcal{V},$$

which are adjoint to each other. Further, for every $a \in A$, the *evaluation mapping* $e_a : D(\mathcal{A}) \to \underset{\sim}{P}$ given by

$$e_a(x) = x(a) \quad \text{for each} \quad x \in D(A)$$

is a morphism. Similarly, the evaluation mapping $\varepsilon_x : E(X) \to \underline{P}$ given by $\varepsilon_x(\alpha) = \alpha(x)$ for each $\alpha \in E(X)$ is a homomorphism for each $x \in X$. The natural maps $e:\ \mathcal{A} \to ED(\mathcal{A})$ and $\varepsilon:\ X \to DE(X)$ given by evaluation (i.e. $e(a) = e_a$, $\varepsilon(x) = \varepsilon_x$) are embeddings for every $\mathcal{A} \in \mathcal{V}$, $X \in \mathcal{W}$.

Definition. If for all $\mathcal{A} \in \mathcal{V}$ the map e is an isomorphism (equivalently: if evaluation mappings are the only morphisms $D(\mathcal{A}) \to \underset{\sim}{P}$), we say that $\underset{\sim}{P}$ *yields a duality* on $\mathcal{V}$. If moreover, the map ε is an isomorphism for each $X \in \mathcal{W}$, the duality is called *full.*

Our aim is to establish a duality of this type for the quasivariety of all bounded distributive q-lattices. We shall use the well-known Priestley duality for bounded distributive lattices (see [11], [12], [13]).

2. Constant q-lattices

A q-lattice $\mathcal{C}$ is called a *constant q-lattice* if it contains exactly one cell, i.e. if $\mathcal{A}$ consists of one cell which equals to A. Hence, a constant q-lattice has the unique idempotent 0 and $x \vee y = 0 = x \wedge y$ for all elements x, y of $\mathcal{A}$.

We consider constant q-lattices as algebras with binary operations $\vee$, $\wedge$ and a nullary operation 0. It is easy to see that constant q-lattices form a quasivariety (in fact, a variety) $\mathcal{V}^* = ISP(\underline{B})$, where $\underline{B}$ is a two-element constant lattice defined on the set $B = \{0, c\}$.

Let us define $\underset{\sim}{B} = (B, \underline{\vee}, \underline{\wedge}, 0, \tau)$, where 0 is a nullary operation, τ is a discrete topology and $\underline{\vee}, \underline{\wedge}$ are the lattice operations derived from the ordering $0 < c$.

It is easy to see that the structure of $\underset{\sim}{B}$ is algebraic over $\underline{B}$. (Notice that $\rho \subseteq B^n$ is a subalgebra of $\underline{B}^n$ iff $(0, 0, ...0) \in \rho$.) Hence, for any $\mathcal{C} = (C; \vee, \wedge, 0) \in \mathcal{V}^*$, the dual space $D(\mathcal{C})$ is the set of all homomorphisms $\mathcal{C} \to \underline{B}$ with the structure inherited from $\underset{\sim}{B}$. This dual space is easy to describe. A map $\mathcal{C} \to \underline{B}$ is a homomorphism iff it preserves 0. Clearly, $(D(\mathcal{C}), \underline{\vee}, \underline{\wedge})$ is isomorphic to the Boolean lattice of all functions $(C \setminus \{0\} \to B)$. (Equivalently, the Boolean lattice of all subsets of $C \setminus \{0\}$.) The space $D(\mathcal{C})$ inherits its topology from the usual product topology of $\underset{\sim}{B}^C$. There is a close connection between this topology and the lattice operations $\underline{\vee}, \underline{\wedge}$. We need the following fact taken from [10].

Lemma 2.1. *Let a sublattice L of $\underset{\sim}{B}^X$ be topologically closed. Then $\bigwedge A \in L$, $\bigvee A \in L$ for every $\emptyset \neq A \subseteq L$.*

Proof. Let $a = \bigwedge A$. Clearly, for $x \in X$, $a(x) = 0$ iff $b(x) = 0$ for some $b \in A$. To prove that $a \in L$, it suffices to show that a belongs to the topological closure of L. The base of open sets of the topology of $\underset{\sim}{B}^X$ consists of all sets of the form

$$S = \{p \in \underset{\sim}{B}^X \mid p(x_1) = \cdots = p(x_m) = 0,\ p(y_1) = \cdots = p(y_n) = c\},$$

where $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$. Suppose that such a set S contains a, hence $a(x_1) = \cdots = a(x_m) = 0$, $a(y_1) = \cdots = a(y_n) = c$. We need to show that $S \cap L \neq \emptyset$. There exist $b_1, \ldots, b_m \in A$ such that $b_i(x_i) = 0$ and $b_i(y_1) = \cdots = b_i(y_n) = c$. If $m = 0$, then clearly $z \in S \cap L$ for arbitrary $z \in A$. If $m \geq 1$, we set $z = \bigwedge_{i=1}^m b_i$. Since the lattice operations in $\underset{\sim}{B}^X$ are pointwise, we have $z \in S$. Since L is a sublattice, we have $z \in L$. Hence, $S \cap L \neq \emptyset$.

Similarly we can prove that $\bigvee A \in L$. $\Box$

For any $a \in C$ we have $h_a \in D(\mathcal{C})$ defined by $h_a(a) = c$ and $h_a(x) = 0$ for every $x \neq a$. It is easy to see that this map is an atom in $D(\mathcal{C})$ and every atom has this form.

Theorem 2.2. *The structure $\underset{\sim}{B} = (B, \underline{\vee}, \underline{\wedge}, 0, \tau)$ yields a duality on $\mathcal{V}^*$.*

Proof. Let $\mathcal{C} \in \mathcal{V}$. Let $\delta : D(\mathcal{C}) \to \underset{\sim}{B}$ be a morphism. We need to show that δ is the evaluation map for some $a \in C$. If δ is constant 0 then $\delta = e_0$. Let δ be non-constant. Since δ is a lattice homomorphism and $D(\mathcal{C})$ is a Boolean algebra, the set $U = \delta^{-1}(c)$ must be an ultrafilter. Since δ is continuous, the set U is closed. By 2.1 used for $A = U = L$, the ultrafilter U must have a smallest element. This smallest element is some atom h_a of $D(\mathcal{C})$. Therefore, for $h \in D(\mathcal{C})$ we have $\delta(h) = c$ iff $h \in U$ iff $h_a \leq h$ iff $h(a) = c$, which shows that $\delta = e_a$. $\Box$

This duality is not full in the sense of [5] because every dual $D(\mathcal{C})$ is an atomic Boolean algebra but $IS_cP(\underset{\sim}{B})$ contains also non–Boolean lattices. It is worth mentioning that our duality is very similar to the duality for sets in [5].

3. Bounded q-lattices

A q-lattice $\mathcal{A}$ is called *bounded* if there exist elements 0 and 1 in $\mathcal{C}$ such that

$$x \wedge 0 = 0 \qquad \text{and} \qquad x \vee 1 = 1$$

are identities of $\mathcal{A}$ and either $0 \neq 1$ or $card\, A = 1$. Let us note that contrary to the case of lattices, it implies neither $x \vee 0 = x$ nor $x \wedge 1 = x$. Further, in a bounded q-lattice, $(0, a) \in Q$ and $(a, 1) \in Q$ for each $a \in A$. Let us note that it can happen also $(b, 0) \in Q$ or $(1, c) \in Q$ for some $b, c \in A$.

We consider bounded distributive q-lattices as algebras with binary operations $\wedge, \vee$ and nullary operations $0, 1$. They form the quasivariety $\mathcal{V} = ISP(\underline{P})$, where $\underline{P}$ is the four element bounded q-lattice visualized in Fig. 2. This follows from the fact that 2-element constant lattice and 2-element lattice are the only subdirectly irreducible distributive q-lattices. (See [4].)

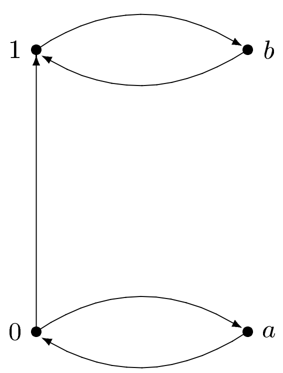

Fig. 2

Let us define the partial orderings $\leq, \leq_a, \leq_b, \leq_{ab}$ and the binary relation T on the set $P = \{0, 1, a, b\}$ by the following rules:
$x \leq y$ iff $x = y$ or $(x, y) = (0, 1)$ or $(x, y) = (a, b)$,
$x \leq_a y$ iff $x = y$ or $(x, y) = (0, a)$,
$x \leq_b y$ iff $x = y$ or $(x, y) = (1, b)$,
$x \leq_{ab} y$ iff $x \leq_a y$ or $x \leq_b y$,
$(x, y) \in T$ iff $\{x, y\} \subseteq \{0, 1\}$ or $\{x, y\} \subseteq \{a, b\}$.

Now we set $\underset{\sim}{P} = (P, \leq, \underline{\vee}, \underline{\wedge}, \leq_a, \leq_b, T, \{0, 1\}, \tau)$, where $\underline{\vee}, \underline{\wedge}$ are partial lattice operations determined by $\leq_{ab}$, $\{0, 1\}$ is a unary relation and τ is the discrete topology.

It is easy to verify that the structure of $\underset{\sim}{P}$ is algebraic over $\underline{P}$. Now we describe the dual space of a bounded distributive q-lattice $\mathcal{C} = (C; \vee, \wedge, 0, 1)$. Let N_C denote the set of non-idempotents of $\mathcal{C}$. Let $\mathcal{C}^*$ be the constant q-lattice defined on the set $\{0\} \cup N_C$ with a new idempotent $0 \notin N_C$. Every homomorphism $f : \mathcal{C} \to \underline{P}$ must map idempotents of $\mathcal{C}$ into $\{0, 1\}$. The restriction $f|_{Sk\,\mathcal{C}}$ is a lattice homomorphism.

Let us introduce an equivalence relation Φ on $D(\mathcal{C})$ by $(f,g) \in \Phi$ iff $f|_{Sk\,\mathcal{C}} = g|_{Sk\,\mathcal{C}}$. Let $[f]_\Phi$ denote the equivalence class containing f. It is easy to see that $f\underline{\vee}g$ and $f\underline{\triangle}g$ are defined iff $(f,g) \in \Phi$. In fact, every Φ-equivalence class with $\underline{\vee}$, $\underline{\triangle}$ is a Boolean lattice. Every such class contains a unique *skeletal* homomorphism, i. e. homomorphism that maps whole $\mathcal{C}$ into $\{0,1\}$. Every such skeletal homomorphism is the least element in its equivalence class (with respect to $\underline{\vee}$, $\underline{\triangle}$). Let $D_S(\mathcal{C})$ denote the set of all skeletal members of $D(\mathcal{C})$

Lemma 3.1. *$W = ([h]_\Phi; \underline{\vee}, \underline{\triangle}, h, \tau|_W)$ is the dual space of the constant q-lattice $\mathcal{C}^*$ for every skeletal $h \in D(\mathcal{C})$ (in the sense of the previous section).*

Proof. To every $f \in [h]_\Phi$ we assign $f^* : \mathcal{C}^* \to \underline{B}$ by $f^*(0) = 0$ and (for $x \in N_C$) $f^*(x) = 0$ if $f(x) \in \{0,1\}$ and $f^*(x) = c$ if $f(x) \in \{a,b\}$. This defines an isomorphism of W and the dual space of $\mathcal{C}^*$. $\square$

Similarly, the order relations $\leq_a$, $\leq_b$ compare only elements of the same Φ-equivalence class. On the other hand, the order relation $\leq$ compares only elements of different classes.

Lemma 3.2. *$V = (D_S(\mathcal{C}); \leq, \tau|_V)$ is isomorphic to the Priestley space of the lattice $Sk\,\mathcal{C}$.*

Proof. For every $f \in D_S(\mathcal{C})$ we define $f^* : Sk\,\mathcal{C} \to \{0,1\}$ by $f^* = f|_{Sk\,\mathcal{C}}$. This defines the required isomorphism. $\square$

Hence, we have the following picture of $D(\mathcal{C})$. In the Priestley space of $Sk\,\mathcal{C}$, every point is replaced by the Boolean lattice representing the constant q-lattice $\mathcal{C}^*$. Besides that, we have relations $\leq_a, \leq_b, T$ and $\{0,1\}$, whose role will be explained in the sequel.

Lemma 3.3. *Let $h,k \in D(\mathcal{C})$. Then there is exactly one $g \in D(\mathcal{C})$ such that $(g,h) \in \Phi$, $(g,k) \in T$.*

Proof. If $x \in Sk\,\mathcal{C}$ then we set $g(x) = h(x)$, to ensure that $(g,h) \in \Phi$. Let $x \in N_C$. We set

$$g(x) = \begin{cases} 0 & \text{if } k(x) \in \{0,1\} \text{ and } h(x) \in \{0,a\} \\ 1 & \text{if } k(x) \in \{0,1\} \text{ and } h(x) \in \{1,b\} \\ a & \text{if } k(x) \in \{a,b\} \text{ and } h(x) \in \{0,a\} \\ b & \text{if } k(x) \in \{a,b\} \text{ and } h(x) \in \{1,b\}. \end{cases}$$

It is clear that g has the required properties. $\square$

Theorem 3.4. *Let $\mathcal{V}$ be a quasivariety of all bounded distributive q-lattices. Then $\underset{\sim}{P}$ yields a duality on $\mathcal{V}$.*

Proof. Let $\delta \in ED(\mathcal{C})$. We need to show that δ is an evaluation map. The preservation of the unary relation $\{0,1\}$ means that δ must map skeletal members of $D(\mathcal{C})$ into $\{0,1\}$. By the Priestley duality and 3.2, there is an idempotent $z \in C$ such that $\delta(h) = (h|_{Sk\,\mathcal{C}})(z) = h(z)$ for every skeletal h.

Now, let $h \in D_S(\mathcal{C})$ and let $\mathcal{C}^*$ and W be as in 3.1. The morphism $\delta : D(\mathcal{C}) \to \underset{\sim}{P}$ induces the morphism $\delta' : W \to \underset{\sim}{B}$ by the rule $\delta'(k) = 0$ if $\delta(k) \in \{0,1\}$ and

$\delta'(k) = c$ otherwise. By 3.1, δ' must be an evaluation map, i.e. $\delta' = e_x$ for some $x \in \mathcal{C}^*$. Hence, $\delta'(k) = k^*(x)$ for every $k \in [h]_\Phi$. (See the proof of 3.1.)

If $x \in N_C$, we have

$$\delta'(k) = \begin{cases} 0 \text{ if } k(x) \in \{0,1\} \\ c \text{ otherwise.} \end{cases}$$

In other words,

$$\delta(k) \in \{0,1\} \text{ iff } k(x) \in \{0,1\}. \qquad (*)$$

Equivalently, $(\delta(k), k(x)) \in T$. We claim that this is true for arbitrary $k \in D(\mathcal{C})$, not only for $k \in [h]_\Phi$. Indeed, by 3.3 for every $k \in D(\mathcal{C})$ there is $g \in [h]_\Phi$ with $(g,k) \in T$. Then clearly $(g(x), k(x)) \in T$, $(g(x), \delta(g)) \in T$ and also, since δ preserves T, $(\delta(g), \delta(k)) \in T$. This implies that $(\delta(k), k(x)) \in T$.

Similarly, if $x = 0$, we have $\delta(k) \in \{0,1\}$, which also holds for every $k \in D(\mathcal{C})$.

Now we claim that $\delta = e_x$ if $x \in N_C$ and $\delta = e_z$ if $x = 0$. First we settle the case $x = 0$.

Let $x = 0$ and $k \in D(\mathcal{C})$. Then there is a skeletal $h \in D(\mathcal{C})$ with $(k,h) \in \Phi$. We have $\delta(h), \delta(k) \in \{0,1\}$. Since $h \underline{\vee} k = k$, also $\delta(h) \underline{\vee} \delta(k) = \delta(k)$ which is possible only if both $\delta(h), \delta(k)$ equal to 0 or both of them equal to 1. Since h and k coincide in idempotents, we have

$$\delta(k) = \delta(h) = h(z) = k(z),$$

which was to prove.

Suppose now that $x \in N_C$. Next we prove that x belongs to the cell containing z. For contradiction, suppose that this is not the case. Then there exists $k \in D_S(\mathcal{C})$ such that $k(x) \neq k(z)$. Without loss of generality, $k(x) = 0$, $k(z) = 1$. Let us define $k_1 \in D(\mathcal{C})$ by the rule that $k_1(x) = a$ and $k_1(y) = k(y)$ for all $y \neq x$. Then $k \leq_a k_1$, therefore $\delta(k) \leq_a \delta(k_1)$ which is a contradiction, because $\delta(k) = k(z) = 1$ and $\delta(k_1) \in \{a, b\}$ (since $k_1(x) = a$ and $(\delta(k_1), k_1(x)) \in T$).

Now let $k \in D(\mathcal{C})$ be arbitrary and let $h \in D_S(\mathcal{C})$ be such that $(k,h) \in \Phi$. Again we have $\delta(h) \underline{\vee} \delta(k) = \delta(k)$. We distinguish four cases.

If $\delta(k) = 0$ then $\delta(h) = 0$ (h is skeletal), hence $0 = h(z) = h(x \vee x) = h(x) \vee h(x) = h(x)$ and therefore $k(x) \in \{0, a\}$ (from $(h,k) \in \Phi$). On the other hand, from (*) we have $k(x) \in \{0,1\}$, hence $k(x) = 0 = \delta(k)$.

If $\delta(k) = a$ then $\delta(h) = 0$, which implies $k(x) \in \{0, a\}$. From (*) we have $k(x) \in \{a, b\}$, hence $k(x) = a = \delta(k)$.

If $\delta(k) = 1$ then $\delta(h) = 1$, which implies $h(x) = 1$ and $k(x) \in \{1, b\}$. From (*) we have $k(x) \in \{0,1\}$, hence $k(x) = 1 = \delta(k)$.

Finally, if $\delta(k) = b$ then we get $\delta(h) = 1$, $h(x) = 1$ and $k(x) \in \{1, b\}$. From (*) we have $k(x) \in \{a, b\}$, hence $k(x) = b = \delta(k)$.

Thus, $\delta(k) = k(x)$ holds in all cases, which means that $\delta = e_x$. The proof is complete. $\square$

Let us show how the duality works for the q-lattice $\mathcal{A}$ on Figure 1.

Evidently, $D(\mathcal{A})$ consists of exactly 16 homomorphisms given by the following table:

	0	v	w	z	x	y	1
h_0	0	0	0	1	1	1	1
h_1	0	0	0	1	1	b	1
h_2	0	0	0	1	b	1	1
h_3	0	0	0	1	b	b	1
h_4	0	a	0	1	1	1	1
h_5	0	a	0	1	1	b	1
h_6	0	a	0	1	b	1	1
h_7	0	a	0	1	b	b	1
h_8	0	0	1	0	0	0	1
h_9	0	0	1	0	0	a	1
h_{10}	0	0	1	0	a	0	1
h_{11}	0	0	1	0	a	a	1
h_{12}	0	a	1	0	0	0	1
h_{13}	0	a	1	0	0	a	1
h_{14}	0	a	1	0	a	0	1
h_{15}	0	a	1	0	a	a	1
	e_0	e_v	e_w	e_z	e_x	e_y	e_1

Clearly h_0, h_8 are skeletal homomorphisms, i.e. $D_S(\mathcal{A}) = \{h_0, h_8\}$. This corresponds to the fact that $Sk\,\mathcal{A}$ is the four element Boolean lattice and its Priestley space is the two–element antichain.

The dotted lines denote $\leq_b$ and the solid lines denote $\leq_a$. The equivalence relation T consists of all pairs (h_i, h_{i+8}), $i = 0, \ldots, 7$. The equivalence relation Φ has two equivalence classes, which are isomorphic to the dual of the constant q-lattice $\mathcal{A}^*$.

The dual space $D(\mathcal{A})$ looks as shown in Fig. 3.

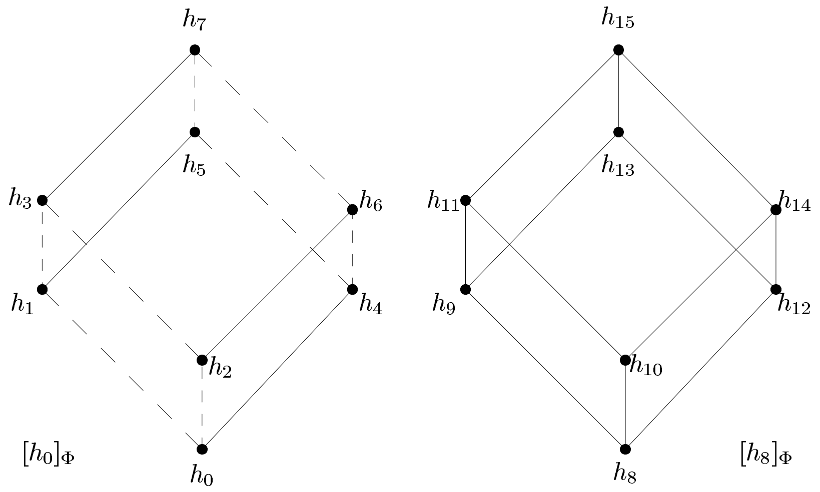

Fig. 3

References

[1] Chajda, I., *Lattices of quasiorder sets*, Acta Univ. Palacký Olomouc, Mathem. **31** (1992), 6-12.

[2] Chajda, I., *An algebra of quasiordered logic*, Math. Bohemica **119** (1994), 129-135.

[3] Chajda, I., *Congruence properties of algebras in nilpotent shifts of varieties*, in General Algebras and Discrete Mathem. (ed. K. Denecke and O. Lüders) Heldermann–Verlag Berlin (1995), 35-46.

[4] Chajda, I. and Kotrle, M., *Subdirectly irreducible and congruence distributive q-lattices*, Czech. Math. J. **43** (1993), 635-642.

[5] Davey, B., *Duality Theory on Ten Dollars a Day*, Algebra and Orders Vol. 389, NATO ASI series, Kluwer Academic Publ. (1993), 71-111.

[6] Davey, B and Werner, H., *Dualities and equivalences for varieties of algebras*, Coll. Math. Soc. J., Bolyai (Contributions to lattice theory) North–Holland **33** (1983), 101–275.

[7] Graczyńska, E., *On normal and regular identities*, Algebra Univ. **27** (1990), 387-397.

[8] Meľnik, I. I., *Nilpotent shifts of varieties*, Math. Notes **14** (1973), 692-696.

[9] Plonka, J., *On varieties of algebras defined by identities of some special forms*, Houston J. of Math. **14** (1988), 253-263.

[10] Ploščica, M. and Haviar, M., *Extension of Bikhoff's duality to the class of all partially ordered sets preprint.*

[11] Priestley, H. A., *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186-190.

[12] Priestley, H. A., *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. **24** (1972), 507-530.

[13] Priestley, H. A., *Ordered sets and duality for distributive lattices*, Annales of Discrete Math. **23** (1984), 39-60.

(Received October 20, 1997)

Department of Algebra and Geometry
Palacký University Olomouc
Tomkova 40
779 00 Olomouc
CZECH REPUBLIC

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
04001 Košice
SLOVAKIA

E-mail address: chajda@matnw.upol.cz
E-mail address: ploscica@Linux1.saske.sk