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HISTORY OF THE NUMBER OF FINITE POSETS

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Abstract. In this paper we introduce the survey of all main known results on the number of finite partially ordered sets. We also present the similar and connected problems. The historical review of related works is also included. In this context there are introduced author's works and results in this branch.

1. Introduction

In the first section we remind the main basic notions and some their relations. Let us denote by N the set of all positive integers and put $N_0 := N \cup \{0\}$. Further, let A be a finite n-element set, $n \in N_0$. By $|A|$ we shall denote the number of all elements of A. As usual, a binary relation ρ on A is a subset of $A \times A$. We define:

Definition 1. A binary relation ρ on A is called

(1) reflexive if $\forall x \in A : [x, x] \in \rho$,
(2) symmetric if $\forall x, y \in A : [x, y] \in \rho \Rightarrow [y, x] \in \rho$,
(3) antisymmetric if $\forall x, y \in A : [x, y] \in \rho \wedge [y, x] \in \rho \Rightarrow x = y$,
(4) transitive if $\forall x, y, z \in A : [x, y] \in \rho \wedge [y, z] \in \rho \Rightarrow [x, z] \in \rho$.

A binary relation ρ is called a quasi-order if it is reflexive and transitive. Furthermore, ρ is called an equivalence if it is reflexive, symmetric and transitive and finally ρ is called a partial order or an ordering if it is reflexive, antisymmetric and transitive. A partially ordered set (A, ρ) or poset, for short, is a set A together with a partial order ρ. We also call (A, ρ) a labelled poset.

Definition 2. A topology on A is a family τ of subsets of A such that

(1) $\emptyset \in \tau \wedge A \in \tau$,
(2) $\forall X, Y \in \tau : X \cup Y \in \tau$,
(3) $\forall X, Y \in \tau : X \cap Y \in \tau$.

The elements of τ are called open sets. A topology is said to be T_0 if for all a, b in A such that $a \neq b$ there exists an open set containing one of a, b but not the other.

At the very beginning we recall an important fact that there is a connection between binary relations on A and topologies on A. In 1937 P. S. Alexandrov [1] and also G. Birkhoff [2] observed that there is the one-to-one correspondence between topologies on A and quasi-orders on A, and furthermore, there is the one-to-one correspondence between partial orders on A and T_0-topologies on A (see [2], 3.ed, p.117).

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Definition 3. A partition of a finite n-set A is a collection $\mathscr{P} = \{A_1, \dots, A_k\}$ of subsets of A, where $1 \leqq k \leqq n$, such that

(1) $\forall i \in \{1, \dots, k\} : A_i \neq \emptyset$,
(2) $\forall i, j \in \{1, \dots, k\}, i \neq j : A_i \cap A_j = \emptyset$,
(3) $A_1 \cup \dots \cup A_k = A$.

We call A_i blocks of $\mathscr{P}$ and we say that $\mathscr{P}$ has k blocks. Then we define $S(n,k)$ to be the number of partitions of an n-set into k blocks. $S(n,k)$ is called a Stirling number of the second kind. By convention, we put $S(0,0) = 1$. Furthermore, the total number of partitions of an n-set A is called a Bell number and is denoted by $B(n)$. Thus we have the relation $B(n) = \sum_{k=1}^{n} S(n,k)$.

Now we remind further important and well-known result on the set-partitions and equivalence relations on a set A. We have the following assertion: There is a one-to-one correspondence between the set of all partitions of an n-set A and the set of all equivalence relations on A. Consequently, the Bell number $B(n)$ is the number of all equivalence relations on an n-set A. This correspondence is given in such a way that the elements which are equivalent lie in the same block.

Definition 4. Let ρ be an ordering on A and σ be an ordering on B. We say that two posets (A, ρ) and (B, σ) are isomorphic if there is an order-preserving bijection $f : A \to B$ whose inverse is order-preserving as well (i.e. $\forall x, y \in A : [x,y] \in \rho \Leftrightarrow [f(x), f(y)] \in \sigma$). This isomorphism decomposes the set of all posets on A into blocks, which we call non-isomorphic posets or also unlabelled posets.

Finally, it is necessary to recall the following notions of a partition and a composition of an integer n.

Definition 5. A partition of an integer $n \in N$ is a sequence $(x_1, \dots, x_k) \in N^k$, where $1 \leqq k \leqq n$, such that $x_1 + \dots + x_k = n$ and $x_1 \geqq \dots \geqq x_k$. A composition of n is a sequence $(x_1, \dots, x_k) \in N^k$, where $1 \leqq k \leqq n$, such that $x_1 + \dots + x_k = n$. If exactly k summands appear in a partition, we call it a k-partition. Analogously, a composition of n in which exactly k summands occur, is called a k-composition.

It is known that there is a bijection between all k–compositions of an integer n and $(k-1)$–subsets of $\{1, 2, \dots, n-1\}$. Hence there are $\binom{n-1}{k-1}$ k–compositions and 2^{n-1} compositions of n. On the other hand, it is not possible to count the number of partitions so easily. All the same, there are more ways to enumerate these numbers (see e.g. our paper [27]).

2. Counting the binary relations

One of the basic problems from the combinatorical analysis is to find the number of all configurations of the specific type. For example, to find the number of all binary relations, the number of set-partitions, the number of topologies and so on. It is well-known that the number of all binary relations on an n-set A is equal to 2^{n^2}. Quite easily we can count the numbers of reflexive, symmetric and antisymmetric relations. Let $\mathscr{R}(A)$ be the set of all reflexive relations on A, $\mathscr{S}(A)$ the set of all symmetric relations on A, $\mathscr{A}(A)$ the set of all antisymmetric relations on A and let

$\mathscr{T}(A)$ denote the set of all transitive relations on A. Thus we have:

$$|\mathscr{R}(A)| = 4^{\binom{n}{2}}, \tag{1}$$

$$|\mathscr{S}(A)| = 2^{\binom{n+1}{2}}, \tag{2}$$

$$|\mathscr{A}(A)| = 2^n \cdot 3^{\binom{n}{2}}, \tag{3}$$

$$|\mathscr{R}(A) \cap \mathscr{S}(A)| = 2^{\binom{n}{2}}, \tag{4}$$

$$|\mathscr{R}(A) \cap \mathscr{A}(A)| = 3^{\binom{n}{2}}, \tag{5}$$

$$|\mathscr{A}(A) \cap \mathscr{S}(A)| = 2^n. \tag{6}$$

These formulas can be deduced by means of elementary combinatorical techniques (i.e. by means of the rules of sum and product). The problem of finding these numbers is often submitted as an exercise. But the difficulties begin when we start to engage with counting binary relations which have the property of transitivity. For the number of equivalences and their classes we still have reasonable formulas. It is easy to verify that $S(n,k) = 0$ if $k > n$, $S(n,0) = 0$, $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$, $S(n,n-1) = \binom{n}{2}$, $S(n,n) = 1$. Now we introduce a short survey of possibilities how to count these numbers. We have the following formulas:

$$S(n,k) = kS(n-1,k) + S(n-1,k-1), \tag{7}$$

$$S(n,k) = \sum_{i=1}^{n-1} \binom{n-1}{i} S(i,k-1), \tag{8}$$

$$S(n,k) = (k!)^{-1} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n, \tag{9}$$

$$S(n,k) = \sum_{x_1+\cdots+x_k=n} 1^{x_1-1} 2^{x_2-1} \cdots k^{x_k-1}, \tag{10}$$

where the sum extends over all k-compositions of an integer n. Moreover, for the Bell numbers $B(n)$ we have the recursion

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k). \tag{11}$$

The Bell numbers can be also computed by means of the scheme (12), analogously to the Pascal triangle for counting binomial coefficients.

$$
\begin{array}{ccccccccc}
1 & & 1 & & 2 & & 5 & & 15 \\
 & 2 & & 3 & & 7 & & \ast & \\
 & & 5 & & 10 & & \ast & & \\
 & & & 15 & & \ast & & & \\
 & & & & \ast & & & &
\end{array}
\tag{12}
$$

In the following section we shall continue in our list of enumerative combinatorical results on binary relations with the property of transitivity. We shall also pay an attention to the numbers of such relations. We shall concentrate especially on our basic subject, which is the number of finite posets.

3. Counting finite posets

Our main combinatorical counting problem is the following: How many posets are there on n elements (enumeration of labelled posets) and what is the number of their isomorphism classes (enumeration of unlabelled posets)? These problems are unsolved up till now. No reasonable explicit or recursive formula for these numbers is still known. By reasonable we mean that the number of involved operations is of considerably smaller order than the numbers which we want to compute. The enumeration of all finite posets is a long-standing open problem. G. Birkhoff with his well-known book Lattice theory (see [2], third ed., pages 4 and 19) was in 1948 one of the first who formulated this problem. We quote: "Let $G(n)$ denote the number of nonisomorphic posets of n elements and $G^*(n)$ denote the number of different partial orderings of n elements. Compute for small n, and find asymptotic estimates and bounds for the rates of growth of the functions $G(n)$ and $G^*(n)$."

Now we make a short remark on the notation of the number of posets. This notation did still not stabilise. We already know the notation from Birkhoff's book. Several authors used this notation too, but it did not root. There was used the whole range of notations for the number of labelled posets up till now. For example $G^*(n)$ in [2], $H(n)$ in [46], d_n in [13], $A_0(n)$ in [22], $T_0(n)$ in [3], γ_n in [24], p_n in [26] and P_n in [23]. Next, in the case of numbers of unlabelled posets we have the similar situation. In this paper we shall use the following notation: p_n will denote the number of all partial orders of an n-element set A and P_n the number of non-isomorphic posets on A. Now we introduce the fundamental results on the number of posets.

For p_n we have the following explicit expression (see M. Erné, [23])

$$
p_n = \sum_{m=0}^{2^{n^2}-1} \prod_{i=0}^{n-1} x_{ni+i}^m \prod_{j=0}^{n-1} (1 - x_{ni+j}^m x_{nj+i}^m) \prod_{k=0}^{n-1} (1 - x_{ni+j}^m x_{nj+k}^m (1 - x_{ni+k}^m)),
\tag{13}
$$

where

$$
x_i^m = \lfloor 2^{-i} m \rfloor - 2\lfloor 2^{-i-1} m \rfloor
\tag{14}
$$

is the i-th digit in the binary expansion of m ($0 \leqq i \leqq n^2$). Each binary relation on the set $\{0, \ldots, n-1\}$ is represented by one summand, where m runs from 0 to

$2^{n^2}-1$. The value of x^m_{ni+j} is 1 if i and j are related, and 0 otherwise. The product over i codes reflexivity, that over j antisymmetry, and that over k transitivity. Thus the whole product is 1 iff the relation is a partial order, and it is 0 in all other cases. The formula (13) is evidently not practical for computing p_n.

In 1966 L. Comtet [13] introduced the important and often used and occured formula

$$p_n = \sum_{(x_1,\dots,x_m)} \frac{n!}{x_1!\dots x_m!} V(x_1,\dots,x_m), \tag{15}$$

where the sum is taken over all compositions of the number n and $V(x_1,\dots,x_m)$ is the number of certain posets with respect to the composition $x_1+\dots+x_m = n$. This formula was in 1979 rediscovered by Z. I. Borevich. More exactly, $V(x_1,\dots,x_m)$ is the number of all so called V–nets of the type $(x_1,\dots,x_m)$. The detailed definition and explanation of this concept can be founded in [3]. In [4] and [5] Borevich derived a special case of $V(x_1,\dots,x_m)$ and determined some values of p_n. He also proved that all values $V(x_1,\dots,x_m)$ are odd numbers.

The following enumerative results show that nearly all the problems on finding the number of binary relations, where transitivity is one of the properties, can be converted to finding the values of p_n. Above all, it is possible to show that it holds:

$$|\mathscr{A}(A) \cap \mathscr{T}(A)| = 2^n p_n, \tag{16}$$

$$|\mathscr{S}(A) \cap \mathscr{T}(A)| = B(n+1). \tag{17}$$

Now we introduce the following notation. Let t_n be the number of all transitive relations on an n-set A and q_n be the number of all quasiorders on A. In 1967 Evans, Harary and Lynn [24] derived a formula relating the number of all quasi-orders on a set of n elements and the number of all partial orders or equivalently the number of topologies on an n-set and T_0-topologies. In particular, they proved the following formula

$$q_n = \sum_{k=1}^{n} S(n,k) p_k. \tag{18}$$

This result contributes to the intensive interest in the number of posets. In spite of the fact that neither the explicit nor recursive formula is still known, there was discovered the asymptotic estimate for p_n. The significant results in this area were presented in 1970 by D. J. Kleitman and B. L. Rothschild. In [30] they deduced the formula

$$\log_2 p_n = \frac{1}{4} n^2 + o(n^2). \tag{19}$$

Furthermore, in 1975 the same authors proved in [31] that

$$p_n = (1 + O(\frac{1}{n})) \sum_{i=1}^{n} \sum_{j=1}^{n-i} \binom{n}{i} \binom{n-i}{j} (2^i-1)^j (2^j-1)^{n-i-j}. \tag{20}$$

This asymptotic formula was in 1981 simplified by K. H. Kim and F. W. Roush [26]. From further works which concern about the asymptotic behaviour of p_n we remark at least the papers of J. L. Davison [19] and D. Dhar [20].

In 1974 M. Erné, [22], showed that quasiorders are asymptotically posets, i.e.

$$\frac{q_n}{p_n} \to 1 \quad \text{for} \quad n \to \infty. \tag{21}$$

In 1992 we have derived in [29] the following formula (22) for the number of transitive relations, which enables to compute numbers t_n, if the values p_n are known:

$$t_n = \sum_{k=1}^{n} \alpha_k(n) p_k, \quad \text{where} \quad \alpha_k(n) = \sum_{s=0}^{k} \binom{n}{s} S(n-s, k-s). \tag{22}$$

In particular, using (22) we have computed the numbers t_n for $n \leqq 14$. Number t_{14} constitutes currently the greatest known value of the sequence t_n and exceeds 10^{28}. In [29] we have also proved the asymptotic formula

$$\frac{t_n}{2^n p_n} \to 1 \quad \text{for} \quad n \to \infty. \tag{23}$$

In 1987 H. J. Prömel [40] proved that the number of unlabelled structures is asymptotically $1/n!$ times the same labelled quantity. The paper [40] contains a short proof of this fact for all classes of structures whose logarithm approaches a quadratic in the size parameter n. In particular, for posets we have

$$\frac{p_n}{n! P_n} \to 1 \quad \text{for} \quad n \to \infty. \tag{24}$$

This problem was in 1981 introduced by K. H. Kim and F. W. Roush (see [26], Problem 3). At the end of this section we mention the solitary and very interesting result of Z. I. Borevich on the residual periodicity of the sequence p_n. In the period 1979-1982 Borevich published papers [6], [7] and [8], where he proved the following assertions. Let $m = \mathfrak{p}$ be an arbitrary prime number. Then the sequence $\{p_n \mod m\}_{n=1}^{\infty}$ is periodical and the length of its period is equal to $\mathfrak{p} - 1$. If $m = \mathfrak{p}^a$, where $a \in N$, then $\{p_n \mod m\}_{n=1}^{\infty}$ is periodical from $n \geqq \mathfrak{p}^{a-1}$ and the length of its period is equal to $\varphi(\mathfrak{p}^a) = \mathfrak{p}^a - \mathfrak{p}^{a-1}$. Furthermore, if $m = \mathfrak{p}_1 \dots \mathfrak{p}_k$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are the different primes, then the sequence $\{p_n \mod m\}_{n=1}^{\infty}$ is periodical and the length of its period is equal to the least common multiple of the numbers $\mathfrak{p}_1 - 1, \dots, \mathfrak{p}_k - 1$. Finally in the general case it holds: Let m be an arbitrary positive integer. Then there exists an index n_0 from which the sequence $\{p_n \mod m\}_{n=1}^{\infty}$ is periodical. Specially, the periodicity of the last figures of the sequence p_n follows from Borewich's results. This visible fact can be seen in Table 1 (see also the author's work [29]).

4. The history of the known values of p_n

As we remarked, the original stimulation for computing the values p_n for small n came from G. Birkhoff in 1948. First, finding the values p_2, p_3 and p_4 is not difficult. This problem was often submitted to the reader as an exercise. In 1966 L. Comtet has found in [13] the values p_5 and p_6. In 1967 J. W. Evans, F. Harary and M. S. Lynn have in [24] found the values p_n for $n \leqq 7$. Counting and verifying these values was left to the reader in the third edition of the Birkhoff's book [2].

The computer enumeration of the values p_n was based on the matrix representation of binary relations. Binary relations on a set of n elements can be represented as $n \times n$ matrices of zeros and ones. If ρ is a binary relation on a set of elements $x_1, \ldots, x_n$, then we associate to ρ the matrix $M = (m_{i,j})$ such that $m_{i,j} = 1$ if $[x_i, x_j] \in \rho$ and $m_{i,j} = 0$ if $[x_i, x_j] \notin \rho$. This gives a one-to-one correspondence between binary relations and $n \times n$ matrices of zeros and ones (see [24], [35] and [48]). The matrix representation of a partial order was given in 1972 by K. K. H. Butler [9]. However, the basic idea can be already found even in the paper [48] by H. Sharp from 1966. It holds that the $n \times n$ matrix of zeros and ones represents a partial order on some n-set iff it is nonsingular and idempotent.

In 1974 M. Erné has published the important paper [22], where he has computed the values p_n up to $n \leqq 9$. Further significant results were obtained in 1977 in the paper [18] of S. K. Das, where the values p_n up to $n \leqq 11$ are computed. At that time this work presented the full list of values p_n. In this historical review it is necessary to underline the works of the soviet mathematicians in the period 1978-1982. In the papers [4] and [5] from 1978 and 1979 Z. I. Borevich, V. I. Rodionov and their coauthors computed the values p_9 and p_{10}. But at that time these values were already known. Further, Rodionov in [44] and [45] independently resumed the common works with Borevich. In 1982 he computed the values p_{11} and p_{12}. Now we introduce the table of the numbers p_n by [23] up to $n = 14$.

Table 1. The numerical values p_n for $n \leq 14$.

$p_1 =$	1	(Folklore)	
$p_2 =$	3	(Folklore)	
$p_3 =$	19	(Folklore)	
$p_4 =$	219	(Folklore)	
$p_5 =$	4 231	(1966)	L. Comtet
$p_6 =$	130 023	(1966)	L. Comtet
$p_7 =$	6 129 859	(1967)	Evans, Harary and Lynn
$p_8 =$	431 723 379	(1967)	Evans, Harary and Lynn
$p_9 =$	44 511 042 511	(1974)	M. Erné
$p_{10} =$	6 611 065 248 783	(1977)	S. K. Das
$p_{11} =$	1 396 281 677 105 899	(1977)	S. K. Das
$p_{12} =$	414 864 951 055 853 499	(1982)	V. I. Rodionov
$p_{13} =$	171 850 728 381 587 059 351	(1991)	M. Erné and K. Stege
$p_{14} =$	98 484 324 257 128 207 032 183	(1991)	M. Erné and K. Stege

As late as 1991, after a long pause, M. Erné and K. Stege presented in [23] the values p_n up to $n \leqq 14$. Currently the number p_{14} constitutes the greatest known value of the sequence p_n. At the present time we can use computers for finding further values of p_n by means of contemporary known methods. But the necessary computing time constitutes the insuperable barrier. Interesting informations on the time-consuming computation of the values p_n can be found in [23].

5. The history of the known values of nonisomorphic posets P_n

The number of works which deal with the computation of values P_n for small n is much less than the number of works which deal with the computation of p_n. By G. Birkhoff, [2], the values P_n for $n \leqq 6$ were found by I. Rose and R. T. Sasaki. In 1981 N. P. Chaudhuri and A. A. J. Mohammed, [12], were concerned with finding a method for verifying the results of Rose and Sasaki. In their paper the verification for $n = 4$ is shown. The values P_n for $n \leqq 6$ can also be found in [46] by R. A. Rozenfeld from 1985. The value P_7 was discovered in 1972 by J. A. Wright in his PhD-thesis [50]. In 1977 S. K. Das found in [18] the value P_8. Seven years later in 1984 R. H. Möhring introduced the value P_9. Further progress came in 1990, when J. C. Culberson and G. J. E. Rawlins computed the numbers of non-isomorphic posets up to $n \leqq 11$. Further, in 1990 A. M. Kutin, [36], was also engaged in computing P_n. C. Chaunier and N. Lygerös found in 1991 the value P_{12} and finally the latest progress in the computation of P_n came in 1992 when the same authors computed in [14] the value P_{13}. Now we introduce the known values of P_n by C. Chaunier and N. Lygerös.

Table 2. The numerical values P_n for $n \leqq 13$.

$P_1 =$	1	(Folklore)	
$P_2 =$	2	(Folklore)	
$P_3 =$	5	(Folklore)	
$P_4 =$	16	(Folklore)	
$P_5 =$	63	(Folklore)	
$P_6 =$	318	(1967)	I. Rose and R. T. Sasaki
$P_7 =$	2 045	(1972)	J. Wright
$P_8 =$	16 999	(1977)	S. K. Das
$P_9 =$	183 231	(1984)	R. H. Möhring
$P_{10} =$	2 567 284	(1990)	J. C. Culberson and G. J. E. Rawlins
$P_{11} =$	46 749 427	(1990)	J. C. Culberson and G. J. E. Rawlins
$P_{12} =$	1 104 891 746	(1991)	C. Chaunier and N. Lygerös
$P_{13} =$	33 823 327 452	(1992)	C. Chaunier and N. Lygerös

In the following section we shall deal with the more special problem of finding the number of connected posets.

6. The number of connected posets

Let (A, ρ) be a partially ordered set, $x, y \in A$. We say that two elements x and y are comparable and we write $x \frown y$, if $[x, y] \in \rho$ or $[y, x] \in \rho$. For $x, y \in A$ we put $x \sim y$ iff there are $k \in N$ and k elements $x_1, \ldots, x_k \in A$ such that $x \frown x_1, \ldots, x_k \frown y$. The poset (A, ρ) is called connected, if for all $x, y \in A : x \sim y$. By c_n we shall denote the number of all connected posets on an n-set A. Furthermore, an isomorphism decomposes the set of all connected posets on A into blocks, which we call non-isomorphic connected posets. The number of all non-isomorphic connected posets on an n-set A will be denoted by C_n.

The first mention of the number of connected posets came probably in 1963 from R. A. Rankin [43]. In [43] there are introduced the values c_n for $n \leqq 4$. Next, 11 years later M. Erné [22] found the values c_n up to $n \leqq 9$. In 1991 the same author and K. Stege [23] presented these numbers for $n \leqq 14$ (see Table 3). Further, we have almost no references on the number C_n of non-isomorphic connected posets. Let us remark that G. Birkhoff [2] did not refer to the numbers c_n and C_n. In 1985 R. A. Rozenfeld [46] presented the numbers C_n for $n \leqq 6$. After this solitary paper we have computed in 1994 the values C_n up to $n \leqq 13$ (see Table 3), [28]. In [28] we have also derived the following formulas (25) and (26) (cf. also our paper [27]):

$$P_n = \frac{1}{n} \sum_{k=0}^{n-1} \alpha(n-k) P_k \quad \text{and} \quad \alpha(m) = \sum_{k|m} k C_k, \tag{25}$$

$$P_n = -\sum_{k=0}^{n-1} Q_{n-k} P_k \quad \text{and} \quad Q_n = \sum_{S} (-1)^{k_1+\cdots+k_n} \binom{C_1}{k_1} \cdots \binom{C_n}{k_n}, \tag{26}$$

where the sum extends over the set S of all solutions $[k_1, \ldots, k_n] \in \{0, 1, \ldots, n\}^n$ of the linear Diophantine equation $1k_1 + 2k_2 + \cdots + nk_n = n$.

Table 3. Initial values of the connected posets c_n and C_n.

$c_1 =$	1	$C_1 =$	1
$c_2 =$	2	$C_2 =$	1
$c_3 =$	12	$C_3 =$	3
$c_4 =$	146	$C_4 =$	10
$c_5 =$	3 060	$C_5 =$	44
$c_6 =$	101 642	$C_6 =$	238
$c_7 =$	5 106 612	$C_7 =$	1 650
$c_8 =$	377 403 266	$C_8 =$	14 512
$c_9 =$	40 299 722 580	$C_9 =$	163 341
$c_{10} =$	6 138 497 261 882	$C_{10} =$	2 360 719
$c_{11} =$	1 320 327 172 853 172	$C_{11} =$	43 944 974
$c_{12} =$	397 571 105 288 091 506	$C_{12} =$	1 055 019 099
$c_{13} =$	166 330 355 795 371 103 700	$C_{13} =$	32 664 484 238
$c_{14} =$	96 036 130 723 851 671 469 482		

7. The special classes of order structures

In the last section we make a short remark on enumeration in the special classes of order structures. In the scientific literature it was studied the whole range of ordered structures such as graded posets, interval orders, lattices, semiorders, series-parallel posets, tiered posets, two-dimensional posets and weak orders. We mention here only one particular. A lattice is a partially ordered set in which every pair of elements has the least upper bound (join) and the greatest lower bound (meet). G. Birkhoff already formulated in [2] the problem to find the number of all n-element lattices. For the number of lattices we have a similar situation as for the number of posets. No explicit or recursive formula is known. In 1979 S. Kyuno [37] described the algorithm for constructing Hasse diagrams of all n-element lattices and also found the number of lattices for $n \leqq 8$. As late as 1994 Y. Koda [34] computed these numbers up to $n \leqq 13$. In 1971 W. Klotz and L. Lucht [33] found the lower bound and in 1980 D. J. Kleitman and K. J. Winston [32] the upper bound for the number of lattices. The aim of this short section was to show that the enumeration problem of posets is not solitary and that there exists a whole family of similar problems. The survey of the enumeration problems in further classes of ordered structures together with the main results can be found e.g. in the paper [21] by M. El-Zahar.

At the end of this paper we present the survey of the main works related to our topic. Of course, this bibliography collection is not complete. The comprehensive resource of references can be found in the papers [6], [23] and [26].

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