Acta Univ. M. Belii
Math. no. 5 (1997), pp. 85-90

GREATEST COMMON SUBGROUP AND SMALLEST COMMON SUPERGROUP OF TWO FINITE GROUPS AND RELATED METRICS ON A SYSTEM OF FINITE GROUPS

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Abstract. The paper deals with metrics on a system of finite groups which are defined by the greatest common subgroup and the smallest common supergroup of two finite groups. An interesting result is obtained for groups S_4 and D_{12}.

INTRODUCTION

Metrics on systems of graphs and posets were investigated in papers [1], [7] and [4], [6] respectively. Paper [3] of A. Haviar investigated four metrics on a system of finite universal algebras. The present paper studies metrics on a system of finite groups which correspond to the substructure and superstructure metric of A. Haviar.

Two groups are considered to be near if they contain a large isomorphic subgroup. Alternatively, two groups are considered to be near if they are embedable into a small group.

1. GREATEST COMMON SUBGROUP AND SMALLEST COMMON SUPERGROUP OF TWO FINITE GROUPS

Definition 1.1: Let G_1 and G_2 be finite groups. The symbol $m(G_1, G_2)$ denotes the maximal order of a group G such that G_1 and G_2 contain subgroups K_1 and K_2 isomorphic to G. The symbol $M(G_1, G_2)$ denotes the minimal order of a group H containing subgroups H_1 and H_2 isomorphic to G_1 and G_2 respectively.

The product of two elements a and b of a group G will be denoted simply ab. If A and B are subset of a group, then the symbol AB donotes the set of all products ab, where $a \in A$ and $b \in B$. The unity of any group will be denoted by e. The symbol $|A|$ denotes the cardinality of a set A.

In the whole paper we shall use the following obvious lemma.

1991 *Mathematics Subject Classification.* 20E07, 54E35.

Key words and phrases. Group, metric

The author has been supported by Slovak grant agency, grant number 1/1466/1994.

Lemma 1.1: Let G be a group, G_1 and G_2 be finite subgroups of G.

Then $|G_1G_2| = \frac{|G_1|\cdot|G_2|}{|G_1\cap G_2|}$.

Proposition 1.2 For any two finite groups G_1 and G_2 the following inequalities hold.

(i) $1 \leq m(G_1, G_2) \leq$ g.c.d.$(|G_1|, |G_2|)$ and $m(G_1, G_2)$ is a common divisor of $|G_1|$ and $|G_2|$.

(ii) s.c.m. $(|G_1|, |G_2|) \leq M(G_1, G_2) \leq |G_1| \cdot |G_2|$ and $M(G_1, G_2)$ is a common multiple of $|G_1|$ and $|G_2|$.

(iii) $M(G_1, G_2) \geq \frac{|G_1|\cdot|G_2|}{m(G_1,G_2)}$

Proof: Parts (i) and (ii) are obvious. We shall prove (iii). Let H be a group of the minimal order containing subgroups H_1 and H_2 isomorphic to G_1 and G_2 respectively. Without loss of generality we may assume that $G_1 = H_1$ and $G_2 = H_2$. Then $G_1 \cap G_2$ is a subgroup of G_1 and G_2 which means $m(G_1, G_2) \geq |G_1 \cap G_2|$. Since G_1G_2 is a subset of H, we obtain

$M(G_1, G_2) = |H| \geq |G_1G_2| = \frac{|G_1|\cdot|G_2|}{|G_1\cap G_2|} \geq \frac{|G_1|\cdot|G_2|}{m(G_1,G_2)}$.

It completes the proof.

The symbol $D_n (n > 2)$ denotes the dihedral group, i.e. the symmetry group of a regular polygon with n edges. This group is generated by the elements r and t satisfying relations $r^n = t^2 = e$ and $trt = r^{-1}$. The element r is a rotation through angle $\frac{2\pi}{n}$ and t is an axial symmetry.

The symbol S_n denotes the group of all permutations of the set $\{1, ..., n\}$. It is easy to see that the cycle $\rho = (12...n)$ and the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & n & n-1 & \dots & 3 & 2 \end{pmatrix}$$

generate a subgroup of S_n isomorphic to D_n. For $n = 3$ the groups S_n and D_n are isomorphic.

Lemma 1.3: Let j and n be coprime integers, $1 \leq j \leq n-1$ and $0 \leq k \leq n-1$. There is a unique automorphism $\psi : D_n \to D_n$ such that $\psi(r) = r^j$ and $\psi(t) = r^k t$. Conversely, any automorphism $\psi : D_n \to D_n$ has such a form.

Proof: Under the above conditions about j and k the elements $\rho = r^j$ and $\tau = r^k t$ satisfy the same relations as r and t. Therefore, formulas $\psi(r) = r^j$ and $\psi(t) = r^k t$ define an automorphism. Let $\psi : D_n \to D_n$ be an automorphism. The order of the element $\psi(r)$ is n, so $\psi(r) = r^j$, where j and n are coprime integers and $1 \leq j \leq n-1$. The order of the element $\psi(t)$ is 2 and this element does not commute with $\psi(r) = r^j$. So, $\psi(t) = r^k t$, where $0 \leq k \leq n-1$.

If a natural k is a divisor of n, i.e. $n = jk$ for some natural j, then the elements $s = r^j$ and t satisfy relations $s^k = t^2 = e$ and $tst = s^{-1}$ and they generate a subgroup which may be identified with D_k. In this situation we shall assume $D_k \subset D_n$. The following lemma may be generalised, but we shall use only this special case.

Lemma 1.4: For any automorphism $\psi : D_4 \to D_4$ there is an automorphic extension $\varphi : D_{12} \to D_{12}$.

Proof: In this situation $n = 12, k = 4, j = 3$ and $s = r^3$. By Lemma 1.3., we have $\psi(s) = s$ or $\psi(s) = s^3$ and $\psi(t) = s^k t$, where $0 \leq k \leq 3$. For the definition of an automorphism $\varphi : D_{12} \to D_{12}$ it is sufficient to define $\varphi(t)$ and $\varphi(r)$. Put $\varphi(t) = \psi(t)$. If $\psi(s) = s$, then put $\varphi(r) = r$. In this case $\varphi(s) = \varphi(r^3) = (\varphi(r))^3 = r^3 = s = \psi(s)$. If $\psi(s) = s^3$, then put $\varphi(r) = r^{11}$. In this case $\varphi(s) = \varphi(r^3) = (\varphi(r))^3 = (r^{11})^3 = r^{33} = r^9 = s^3 = \psi(s)$. So, φ is an extension of ψ.

Lemma 1.5: There is a group H of order 96 which contains subgroups H_1 and H_2 isomorphic to S_4 and D_{12}.

Proof: Let C_4 be a subgroup of D_{12} generated by the element $s = r^3$ and D_3 be a subgroup generated by the elements $p = r^4$ and t. Then $|C_4 \cap D_3| = 1$ and $|C_4 D_3| = \frac{|C_4| \cdot |D_3|}{|C_4 \cap D_3|} = 24$ which means $C_4 D_3 = D_{12}$. Note that $xsx^{-1} = s$, when x is a rotation and $xsx^{-1} = s^{-1}$, when x is an axial symmetry. So, C_4 is a normal subgroup of D_{12} which is an internal semidirect product of C_4 and D_4, [5,p.27]. Using this fact and isomorphism of D_3 and S_3, it may be easily shown that D_{12} is isomorphic to the Cartesian product $C_4 \times S_3$ with the group operation defined by the formula $[x, \sigma][y, \tau] = [xy^{sgn\sigma}, \sigma\tau]$, where $x, y \in C_4, \sigma, \tau \in S_3$ and sgn σ denotes the sign of a permutation $\sigma \in S_3$. Replacing S_3 by S_4 in the above construction, we obtain the required group H.

The following theorem is the main result of this paper.

Theorem 1.6: Let $G_1 = S_4$ and $G_2 = D_{12}$. Then $m(G_1, G_2) = 8$ and $M(G_1, G_2) = 96$. It means that the inequality $M(G_1, G_2) \geq \frac{|G_1| \cdot |G_2|}{m(G_1, G_2)}$ can not be replaced by the equality $M(G_1, G_2) = \frac{|G_1| \cdot |G_2|}{m(G_1, G_2)}$.

Proof: We shall show the equality $m(G_1, G_2) = 8$. Clearly, both groups G_1 and G_2 contain a subgroup isomorphic to D_4 of order 8. It means $m(G_1, G_2) \geq 8$. The group D_{12} is generated by the elements r and t satisfying relations $r^{12} = e = t^2$ and $trt = r^{-1}$. The group S_4 contains only permutations of the form $(ijkl), (ijk), (ij), (kl)$ and (ij) the orders of which are $4, 3, 2$ and 2 respecively. On the other hand the group D_{12} contains the element r, the order of which is 12. The groups G_1 and G_2 are not isomorphic which means $m(G_1, G_2) \neq 24$. We shall show $m(G_1, G_2) \neq 12$. Let G be a subgroup of D_{12} with $|G| = 12$. Then the index of G is 2. So, G is a normal subgroup of D_{12} and the order of the factor group D_{12}/G is 2 which particulary means $r^2 \in G$. The order of r^2 is 6 and S_4 does not contain such elements. So, S_4 does not contain a subgroup isomorphic to G. It means $m(G_1, G_2) \neq 12$. It proves $m(G_1, G_2) = 8$. Proposition 1.2 implies $M(G_1, G_2) \geq 72$. We shall show $M(G_1, G_2) \neq 72$. Let H be a hypothetical group of order 72 which contains subgroups H_1 and H_2 isomorphic to S_4 and D_{12}. So, there are monomorphisms $\varphi_1 : S_4 \to H$ and $\varphi_2 : D_{12} \to H$ with

$\varphi_1(S_4) = H_1$ and $\varphi_2(D_{12}) = H_2$. Assume that S_4 is group of all permutation of the set $\{A, B, C, D\}$. The elements $s = r^3$ and t generate a subgroup of D_{12} which is identified with D_4. Let $\psi : D_4 \to S_4$ be a monomorphism defined by the formulas $\psi(s) = (ABCD)$ and $\psi(t) = (BD)$. Denote by $a = \varphi_1(ABC), x = \varphi_1(ABCD)$ and $y = \varphi_1(BD)$. Since $(ABC)^3 = e$ and $(ABC)(ABCD)(ABC) = (BD)$, we have $a^3 = e$ and $axa = y$ which means $ax = ya^2$. There is an element $b \in H$ such that $b \neq e, ab = ba, xb = bx, by = yb^2$ and $b^2y = yb$. This is a contradiction, because $ya^2b = axb = abx = bax = bya^2 = yb^2a^2$ which means $a^2b = b^2a^2 = a^2b^2$ and $e = b$. We shall show the existence of such an element $b \in H$. The images $\varphi_1(\psi(D_4))$ and $\varphi_2(D_4)$ are Sylow subgroups of order 8 in H. By Sylow theorem, they are conjugated by an inner automorphism in H, $[5, p.39]$. So, without loss of generality we may assume $\varphi_1(\psi(D_4)) = \varphi_2(D_4)$. Denote by $f = \varphi_2^{-1} \circ \varphi_1 \circ \psi$. The mapping f is an automorphism of D_4 and by Lemma 1.4., there is an automorphic extension $\varphi : D_{12} \to D_{12}$ of f. Now, the monomorphism $\varphi_2 \circ f : D_{12} \to H$ is an extension of $\varphi_1 \circ \psi : D_4 \to H$. Replacing φ_2 by $\varphi_2 \circ f$, we may assume that $\varphi_2(z) = \varphi_1(\psi(z))$ for any $z \in D_4$. The element $a = \varphi_1(ABC)$, generates a subgroup K_1 of H with $|K_1| = 3$. Since $|H| = 72$, the subgroup K_1 is contained in some Sylow group K of order 9, $[5, p.39]$. Denote by $G = \varphi_1(\psi(D_4)) = \varphi_2(D_4)$. Since $|G| = 8$, we have $|G \cap K| = 1$ and $|GK| = 72$ which means $GK = H$. Therefore, $H_2K = H$ and $|H_2 \cap K| = \frac{|H_2| \cdot |K|}{|H_2K|} = 3$. Put $K_2 = H_2 \cap K$. It is a subgroup of H_2 of order 3. The group D_{12} contains a unique group of order 3, it is a subgroup C_3 generated by the element $p = r^4$. It means $K_2 = \varphi_2(C_3)$. The element $b = \varphi_2(p) \neq e$ has the required properties. The group K is commutative, because any group of the order p^2 is commutative, $[5, p.39]$. It proves $ab = ba$. Since $ps = sp$ and $\varphi_2(s) = \varphi_1(\psi(s)) = \varphi_1(ABCD)) = x$, we have $xb = bx$. Finally, relations $by = yb^2$ and $b^2y = yb$ follow from relations $p^3 = t^2 = e, tpt = p^{-1}$ and $\varphi_2(t) = \varphi_1(\psi(t)) = \varphi_1(BD) = y$. The proof of $M(G_1, G_2) \neq 72$ is complete. The following multiple of 24 is 96. Now, $M(G_1, G_2) = 96$ by Lemma 1.5.

2.SUBGROUP METRICS

For two finite groups G_1 and G_2 put $d(G_1, G_2) = |G_1| + |G_2| - 2m(G_1, G_2)$.

Proposition 2.1: The function d is a metric, i.e. for any finite groups G_1, G_2 and G_3

(i) $d(G_1, G_2) \geq 0$ and $d(G_1, G_2) = 0$ if and only if G_1 and G_2 are isomorphic

(ii) $d(G_1, G_2) = d(G_2, G_1)$

(iii) $d(G_1, G_3) \leq d(G_1, G_2) + d(G_2, G_3)$ and the equality appears only in the case when G_2 is isomorphic to a subgroup of G_1 or G_3.

Proof: Parts (i) and (ii) are obvious. Let H_1 and H_2 be isomorphic subgroups of G_1 and G_2 respectively for which $|H_1| = |H_2| = m(G_1, G_2)$ and $\varphi : H_2 \to H_1$ be the corresponding isomorphism. Similarly, let K_2 and K_3 be isomorphic subgroups of G_2 and G_3 respectively for which $|K_2| = |K_3| = m(G_2, G_3)$ and $\psi : K_2 \to K_3$ be the corresponding isomorphism. Obviously, the groups $\varphi(H_2 \cap K_2)$ and $\psi(H_2 \cap K_2)$ are isomorphic which implies $m(G_1, G_3) \geq |H_2 \cap K_2| = |H_2| + |K_2| - |H_2 \cup K_2| \geq m(G_1, G_2) + m(G_2, G_3) - |G_2|$.

Therefore, $d(G_1, G_3) = |G_1| + |G_3| - 2m(G_1, G_3) \leq |G_1| + |G_3| + 2|G_2| -$

$2m(G_1,G_2)-2m(G_2,G_3) = d(G_1,G_2)+d(G_2,G_3)$. The equality appears if and only if $|H_2 \cup K_2| = |G_2|$ which is possible only in the case $H_2 = G_2$ or $K_2 = G_2$. In the opposite case we should have $|H_2 \cup K_2| = |H_2|+|K_2|-|H_2 \cap K_2| \leq \frac{1}{2}|G_2|+\frac{1}{2}|G_2|-1 < |G_2|$.

Proposition 2.2: For any two finite groups G_1 and G_2
(i) $d(G_1,G_2) = |G_1| + |G_2| - 2$ if the orders are coprime
(ii) $d(G_1,G_2) \leq |G_1| + |G_2| - 2p$ if the orders are not coprime and p is the greatest prime number dividing the orders $|G_1|$ and $|G_2|$.

Clearly, the metric d is unbouded. So, for any two finite groups G_1, G_2 put: $\delta(G_1,G_2) = 1 - \frac{m(G_1,G_2)}{max(|G_1|,|G_2|)}$.

Proposition 2.3: The function δ is a metric which attends values in the interval $< 0, 1)$. If $|G_1| = |G_2| = n$, then $d(G_1,G_2) = 2n\delta(G_1,G_2)$.
Proof: The triangle inequality is obvious if G_2 is isomorphic to G_1 or G_3. In the opposite case $m(G_1,G_2) \leq \frac{1}{2}max(|G_1|,|G_2|)$ and $m(G_2,G_3) \leq \frac{1}{2}max(|G_2|,|G_3|)$. Therefore $\delta(G_1,G_3) < 1 = \frac{1}{2} + \frac{1}{2} \leq \delta(G_1,G_2) + \delta(G_2,G_3)$. The other properties are obvious.

3.SUPERGROUP METRIC

Copying the superstructurre metric of [3], we define

$$\rho(G_1,G_2) = 2M(G_1,G_2) - |G_1| - |G_2|$$

.

Example 3.1: Let $|G_1| = 5, |G_2| = 2$ and $|G_3| = 3$. Then $M(G_1,G_3) = 15, M(G_1,G_2) = 10, M(G_2,G_3) = 6, \rho(G_1,G_3) = 22, \rho(G_1,G_2) = 13, \rho(G_2,G_3) = 7$ and $\rho(G_1,G_3) > \rho(G_1,G_2) + \rho(G_2,G_3)$. So, the function ρ is not a metric.

Example 3.2: Let $G_1 = C_8, G_2 = C_4 \times C_2$ and $G_3 = C_2 \times C_2 \times C_2$, where C_n denotes the cyclic group of order n. Then $m(G_1,G_3) = 2, m(G_1,G_2) = 4 = m(G_2,G_3)$. By Proposition 1.2., we have $M(G_1,G_3) \geq 32, M(G_1,G_2) \geq 16$ and $M(G_2,G_3) \geq 16$. Using the direct products $C_2 \times C_2 \times C_8, C_2 \times C_8$ and $C_2 \times C_2 \times C_4$, we obtain $M(G_1,G_3) = 32, M(G_1,G_2) = (G_2,G_3) = 16, \rho(G_1,G_3) = 48, \rho(G_1,G_2) = \rho(G_2,G_3) = 16$ and $\rho(G_1,G_3) > \rho(G_1,G_2) + \rho(G_2,G_3)$. Thus, the function ρ is not a metric on the system of all groups of order 8.

Part (iii) of Proposition 1.2. may be rewrite as $m(G_1,G_2) \geq \frac{|G_1|\cdot|G_2|}{M(G_1,G_2)}$. Now, the right side may be considered as an alternative of the left side and we define supergroup alternatives of subgroups metrics d and δ

$$d_1(G_1,G_2) = |G_1| + |G_2| - 2\frac{|G_1| \cdot |G_2|}{M(G_1,G_2)}$$

$$\delta_1(G_1,G_2) = 1 - \frac{min(|G_1|,|G_2|)}{M(G_1,G_2)}$$

The proof of the next proposition is similar to the proof of 2.3.

Proposition 3.3: The function δ_1 is a metric which attends values in the interval $< 0, 1)$. If $|G_1| = |G_2| = n$, then $d_1(G_1, G_2) = 2n\delta_1(G_1, G_2)$.

Collolary 3.4: The function d_1 is a metric on a system of all groups of order n.

Example 3.1: Let $G_1 = S_4, G_2 = D_4$ and $G_3 = D_{12}$. Then by Theorem 1.6., $M(G_1, G_3) = 96, M(G_1, G_2) = 24 = M(G_2, G_3), d_1(G_1, G_3) = 36, d_1(G_1, G_2) = 16 = d_1(G_2, G_3)$ and $d_1(G_1, G_3) > d_1(G_1, G_2) + d_1(G_2, G_3)$. So, the function d_1 is not a netric on the system of all groups.

Proposition 1.2. and Theorem 1.6. imply

Theorem3.5: For any finite groups G_1 and G_2

$$d_1(G_1, G_2) \leq d(G_1, G_2)$$

$$\delta_1(G_1, G_2) \leq \delta(G_1, G_2)$$

If $G_1 = S_4$ and $G_2 = D_{12}$ then the inequalities are strict.

All metrics considered in the present paper are not interesting from the topological point of view because they induce the discrete topology on any set of groups which does not contain isomorphic groups. These metrics are only number characteristics which express the degree of relationship of two groups. The same may be said about cited papers [1],[3],[4],[6] and[7].

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(Received October 2, 1997)

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