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LAGRANGEANS ON A MANIFOLD WITH A (1,1)-TENSOR FIELD

Anton Dekrét

Abstract. The main object of this paper is the Lagrange calculus of first order on a manifold with a given (1,1)-tensor field.

INTRODUCTION

In this paper we deal with Lagrangians of first order that are functions $L : TM \to R$ on the tangent bundle TM of a manifold M, when on M is given a (1,1)-tensor field $A : M \to T^*M \otimes TM$. Recall two canonical objects on TM: the Liouville field V the flow of which is determined by the homotheties on the fibres of $p_M : TM \to M$ and the endomorphism $v : TM \to T^*M \otimes VTM$ which is induced by the identity on TM and by the canonical identification $VTM = TMx_MTM$ of the subbundle VTM of vertical vectors on TM.

We use well known notions of the Lagrange formalism and the theory of lifting:

1. The Lagrange equation $i_X dd_v L = dL - VL$, see for example [1].
2. The Lagrange fields S_L that are the semisprays (vector fields S on TM with the property $v(S) = V$) satisfying the Lagrange equation.
3. The Lagrange forms $d_vL, \omega_L = dd_vL, dE = d(L - VL)$.
4. The connection Γ_S canonically determined by a semispray S on TM, see [2].
5. The natural lifts of a (1,1)-tensor field A on M in the tangent bundle TM first of all the vertical lift Av and the complete lift Ac, see [5].

In the first section we deal with the affine space of the connections on TM with the property $Ac(H\Gamma) \subset H\Gamma$, where $H\Gamma$ is the horizontal subbundle of a connection Γ. In general, a (1,1)tensor field α on TM is called Γ-parallel if $\alpha(H\Gamma) \subset H\Gamma$. We find conditions for A such that Ac is Γ_S-parallel and conditions under which there exists a unique connection Γ such that $L_S Av$ is Γ-parallel where L_S denotes the Lie derivative of Av with respect to a given semispray S.

The second section is devoted to the mutual relations between ω_L and A. Here some equalities of the first order Lagrangian calculus on a manifold M with a (1,1)-tensor field are introduced. Proposition 8 states the conditions under which there

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is a unique connection Γ such that $\omega_L(h_\Gamma, Ac\cdot h_\Gamma)=0$. In Proposition 9 it is proved that the Lagrange field S_L satisfies the equality

$$i_S dd_{Av}L = i_{Ac}dl - d(V_A L),\ V_A = Av(S).$$

It is proved (Proposition 11) that the equality

$$i_{AcX}dd_v L = i_X dd_{Av}L$$

is satisfied if the 2-form $di_{Ac}dL$ is semibasic. Proposition 12 states the conditions under which the equation

$$i_S i_{Ac}\omega_L = d(V_A L) - i_{Ac}(d(E-L))$$

has a unique solution.

In this paper we suppose that all manifolds and maps are smooth.

1. SEMISPRAYS AND CONNECTIONS ON A MANIFOLD WITH (1,1)-TENSOR FIELDS

Let M be a manifold, (x^i) be a local chart on M and (x^i, x^i_1) be the induced chart on TM. Denote by $V = x^i_1\partial/\partial x^i_1$ the Liouville field on TM, by $v = dx^i\otimes\partial/\partial x^i_1$ the canonical (1,1)-tensor field determined by the identity on TM and by the canonical identification $VTM = TMx_M TM$, where VTM is the vector bundle of all vertical vectors on TM.

Recall that a vector field $X : TM \to TTM$ on TM is called a differential equation of second order (shortly a semispray) if $v(X) = V$, i.e. if its coordinate form is

$$X = x^i_1\partial/\partial x^i + \eta^i(x, x_1)\partial/\partial x^i_1\ .$$

Let $\alpha = (a^i_j dx^j + b^i_j dx^j_i)\otimes\partial/\partial x^i + (c^i_j dx^j + h^i_j dx^j)\otimes\partial/\partial x^i_1$ be a (1,1)-tensor field on TM. We say that α is vertical if $\alpha(VTM)\subset VTM$, i.e. if $v\cdot\alpha\cdot v = 0$.

A connection Γ on the fibre manifold $p_m : TM \to M$ can be introduced as a (1,1)-tensor field h_Γ (the horizontal form of the connection) satisfying the conditions $h_\Gamma(VTM) = 0$, $Tp_m h_\Gamma = Tp_M$ where throughout this paper we use the denotation TF for the tangential prolongation of a map F. In coordinates $h_\Gamma = dx^i\otimes\partial/\partial c^i + \Gamma^i_j(x, x_1)dx^j\otimes\partial/\partial x^i_1$, where Γ^i_j are the local components of Γ. Then $H\Gamma := Imh_\Gamma \subset TTM$ is the horizontal subbundle of the connection Γ (satisfying the equation $dx^i_1 = \Gamma^i_j dx^j$ and the decomposition $TTM = VTM \oplus H\Gamma$) and $v_\Gamma = Id_{TTM} - h_\Gamma$ is the vertical form of the connection Γ. Recall that the set of all connections on TM is an afinne space associated with the vector space $C^\infty(T^*M\otimes VTM)$ of all semibasic 1-vector forms with values in VTM.

There is a unique semispray $S_\Gamma : x^i_1\partial/\partial x^i + \Gamma^i_j x^j_1\partial/\partial x^i$ which is Γ-horizontal.

Every semispray S determines the connection Γ_S the horizontal form of which is $h_\Gamma = \frac{1}{2}(Id_{TTM} - L_S v)$, where $L_S v$ is the Lie derivative of v with respect to S. Its components are $\Gamma^i_j = \frac{1}{2}\eta^i_{j_1}$, where the denotation $f_{i_1} : \partial f/\partial x^i_1$ together with $f_i := \partial f/\partial x^i$ will be used throughout our paper.

Definition 1. Let α be a (1,1)-tensor field and Γ be a connection on TM. The field α is called Γ-parallel if $\alpha(H\Gamma) \subset H\Gamma$.

We will introduce the coordinate condition for α to be Γ-parallel. If we use the above expression of α and if Γ^i_j are the components of Γ then

$$\alpha \cdot h_\Gamma = (a^i_j + b^i_t \Gamma^t_j) dx^j \otimes \partial/\partial x^i + (c^i_j + h^i_t \Gamma^t_j) dx^j \otimes \partial/\partial x^i_1 \ . \tag{1}$$

Then α is Γ-parallel iff

$$\Gamma^i_u (a^u_j + b^u_t \Gamma^t_j) = c^i_j + h^i_t \Gamma^t_j \ . \tag{2}$$

Let $A = a^i_j dx^j \otimes \partial/\partial x^i$ be a (1,1)-tensor field on a manifold M. We will prefer two natural lifts of A on TM, see [5]:

a) the vertical lift $Av = a^i_j dx^j \otimes \partial/\partial x^i_1$ that is a semi-basic vector 1-form on TM induced by the identification $VTM = TMx_MTM$,

b) the complete lift $Ac = a^i_j dx^j \otimes \partial/\partial x^i + (a^i_{jk} x^k_1 dx^j + a^i_j dx^j_1) \otimes \partial/\partial x^i_1$ which is determined by the map $i_2 \cdot TA \cdot i_2$, where $i_2 : (x^i, x^i_1, dx^i, dx^i_1) \to (x^i, dx^i, x^i_1, dx^i_1)$ is the canonical involution on TM.

Recall that the vector field Ac is vertical and

$$v \cdot Ac = Ac \cdot v = Av \ .$$

Definition 2. A (1,1)-tensor field A on M is called Γ-parallel if Ac is Γ-parallel.

We will need the coordinate form of the Lie derivatives $L_S Av$, $L_S Ac$ with respect to a semispray S. We get

$$\begin{aligned} L_S Av &= -a^i_j dx^j \otimes \partial/\partial x^i + [(a^i_{jk} x^k_1 - \eta^i_{k_1} a^k_j) dx^j + a^i_j dx^j_1] \otimes \partial/\partial x^i_1 \\ L_S Ac &= [(a^i_{jks} x^k_1 x^s_1 + a^i_{jk} \eta^k + a^i_k \eta^k_j - \eta^i_{t_1} a^t_{jk} x^k_1 - \eta^i_k a^k_j) dx^j + \\ &\quad + (2 a^i_{jk} x^k_1 + a^i_k \eta^k_{j_1} - \eta^i_{k_1} a^k_j) dx^j_1] \otimes \partial/\partial x^i_1 \ . \end{aligned}$$

So the field $L_S Ac$ is a vector 1-form with values in VTM. Therefore $v \cdot L_S Ac = 0$. The field $L_S Av$ is vertical and $v \cdot L_S Av = -L_S Av \cdot v = Av$.

Proposition 1. *Let S be a semispray and Γ_S be the canonical connection determined by S. Then $Ac - L_S Av = 2h_{\Gamma_S} \cdot Ac$.*

Proof. The equality $Av = v \cdot Ac$ gives $L_S Av = L_S v \cdot Ac + v \cdot L_S Ac = L_S v \cdot Ac$. Then the equality $L_S v = Id_{TTM} - 2h_{\Gamma_S}$ completes our proof.

Corollary. *The tensor field $Ac - L_S Av$ is a vector 1-form with values in $H\Gamma_S$.*

Remark. If A is a regular field then also $L_S Av$ is regular. It is easy to show that $(L_S Av)^{-1} \Gamma_S$ is just the connection on TM the horizontal subbundle of which is given by vectors Y such that $L_{\overline{S}}(L_S Av)(Y)$ is vertical and that the connection Γ does not depend on the choice of the semispray $\overline{S}$.

Let Γ be a connection on TM. If A is regular then also Ac is regular and then the subbundle $Im\ Ac \cdot h_\Gamma$ states the connection $A\Gamma$ the local components of which

$$\overline{\Gamma}^i_j = (a^i_{tk} x^k_1 + a^i_s \Gamma^s_t)\tilde{a}^t_j, \quad a^i_t \tilde{a}^t_j = \delta^i_j \ ,$$

immediately follow from the equality (1).

In the case of the tensor field $\alpha = Ac$ the equality (2) reads

$$\text{(2')} \qquad \Gamma^i_u a^u_j = a^i_{jk} x^k_1 + a^i_s \Gamma^s_j \ .$$

If Γ, $\overline{\Gamma}$ are two connections with respect to which is the tensor field A parallel then from (2') we get

$$(\Gamma^i_u - \overline{\Gamma}^i_u) a^u_j - a^i_s (\Gamma^s_j - \overline{\Gamma}^s_j) = 0 \ .$$

This equality together with the fact that Av and $\Gamma - \overline{\Gamma}$ are sections $TM \to T^*M \otimes_{TM} TM$ immediately give

Proposition 2. *Let Γ be a given connection on TM. The set of all (1,1)-tensor fields A on M which are Γ-parallel is a real vector space. Let A be a given (1,1)-tensor field on M. Then the set of all connections Γ on TM with respect to which A is Γ-parallel is an afinne space associated with the kern of the linear map*

$$\Phi_A : C^\infty(T^*M \otimes_{TM} TM) \to C^\infty(T^*M \otimes_{TM} TM), \quad \xi \to Av\xi - \xi Av \ .$$

Proposition 3. *Let S be a semispray on TM, Γ_S be the canonical connection determined by S and A be a (1,1)-tensor field on M. Then the following conditions are equivalent*

a) *A is Γ_S-parallel,*
b) *$L_S Ac$ is a semibasic vector 1-form with values in VTM,*
c) *$L_S Av$ is Γ_S-parallel.*

Proof. By the equality (2) the tensor field $L_S Av$ is Γ_S-parallel iff

$$-\frac{1}{2}\eta^i_{k_1} a^k_j = a^k_{jk} x^k_1 - \eta^i_{k_1} a^k_j + \frac{1}{2} a^i_k \eta^k_{j_1} \ .$$

This condition coincides with the equality (2) for the connection Γ_S, $\Gamma^i_j = \frac{1}{2}\eta^i_{j_1}$, and with the coordinate condition $2a^i_{jk} x^k_1 + a^i_k \eta^k_{j_1} - a^k_j \eta^i_{k_1} = 0$ for $L_S Ac$ to be semibasic. Proof is finished.

Proposition 4. *Let S be a semispray on TM and A be a (1,1)-tensor field on M. If the (2,2)-tensor field $AT := A \otimes Id_{TM} + Id_{TM} \otimes A$ is regular then there is a unique connection Γ on TM such that the tensor field $L_S Av$ is Γ-parallel.*

Proof. Let Γ^i_j be the components of a connection Γ. Then the condition (2) for the tensor field $L_S Av$ to be Γ-parallel reads

$$(a^i_u \delta^s_j + \delta^i_u a^s_j)\Gamma^u_s = \eta^i_{k_1} a^k_j - a^i_{jk} x^k_1 \ .$$

It completes our proof.

Remark. It is easy to prove that the tensor field $L_S L_S Av$ is not vertical and that it holds $v \cdot L_S L_S Av \cdot v = -2Av$. Therefore if A is regular then there are connections Γ_1, Γ_2 such that $H\Gamma_1 = L_S L_S Av(VTM)$ and $L_S L_S Av(H\Gamma_2) = VTM$.

2. GEOMETRY OF LAGRANGIANS ON MANIFOLDS WITH A (1,1)-TENSOR FIELD

First, recall some notions and properties.

Let α be a (1,1)-tensor field on TM, X be a vector field, ε be a k-form on TM. Then the symbols i_α and i_X denote derivatives

$$i_\alpha \varepsilon(Y_1, \dots, Y_k) = \sum_{i=1}^{k} \varepsilon(Y_1, \dots, \alpha(Y_i), \dots, Y_k),$$
$$i_X \varepsilon(Y_1, \dots, Y_{k-1}) = \varepsilon(X, Y_1, \dots, Y_{k-1}),\ d_\alpha = [i_\alpha, d] = i_\alpha d - d i_\alpha$$

where d denotes the exterior derivative.

It holds
$$d_\alpha d = -d d_\alpha, \quad L_X = i_X d + d i_X, \quad d L_X = L_X d\ ,$$

where L_X denotes the Lie derivative of exterior forms with respect to a vector field X.

When ε is a (0, 2)-tensor field on TM we will use the following denotations

$$\varepsilon^\alpha,\ \varepsilon^\alpha(X, Y) = \varepsilon(\alpha X, Y),$$
$$\varepsilon_\alpha,\ \varepsilon_\alpha(X, Y) = \varepsilon(X, \alpha Y),$$
$$\varepsilon\alpha,\ \varepsilon\alpha(X, Y) = \varepsilon(\alpha X, \alpha Y) = \alpha^* \varepsilon(X, Y)\ .$$

It is clair that if ε is a 2-form then $i_\alpha \varepsilon = \varepsilon^\alpha + \varepsilon_\alpha$.

Let L be a Lagrangian of first order on M, i.e. a function on TM. Then the forms

$$d_v L = L_{i_1} dx^i, \quad d_v L = i_v dL\ ,$$
$$\omega_L := dd_v L = L_{i_1 j} dx^j \wedge dx^i + L_{i_1 j_1} dx_1^j \wedge dx^i$$

are called the Lagrange 1- and 2-form of Lagrangian L. When the map $I_L : C^\infty(TM \to TTM) \to C^\infty(TM \to T^*TM), X \to i_X \omega_L$, is regular then the Lagrangian L is called regular. In this case, the Lagrange 2-form ω_L is symplectic. Locally L is regular iff det $L_{i_1 j_1} \neq 0$. The equation

$$i_X \omega_L = dE, \quad E = L - VL\ , \tag{3}$$

is a basic equation of the Lagrange formalism of first order. It is called the Lagrange equation. Every semispray, which is a solution of (3) is called the Lagrange field and denoted by S_L. Recall that when L is regular then there is a unique solution of the equation (3) and moreover it is a semispray, i.e. it is the Lagrange field.

We introduce some coordinate expressions we will need.

$$(4) \qquad L_{i_1t_1}\eta^t = L_i - L_{i_1k}x_1^k \ ,$$

that is the equation (3) for semisprays

$$\omega_L^\alpha = (L_{j_1s}a_i^s + L_{j_1s_1}c_i^s - L_{s_1j}a_i^s)dx^idy^j + (L_{j_1s}b_i^s + L_{j_1s_1}h_i^s - L_{s_1j}b_i^s)dx_1^idy^j -$$
$$- L_{s_1j_1}a_i^sdx^idy_1^j - L_{s_1j_1}b_i^sdx_1^idy_1^j \ ,$$
$$\omega_L\alpha = (L_{s_1u}a_i^sa_j^u + L_{s_1u_1}a_i^sc_j^u)dx^j \wedge dx^i + (L_{s_1u}a_i^sb_j^u + L_{s_1u_1}a_i^sh_j^u - L_{s_1u}b_j^sa_i^u -$$
$$-L_{s_1u}b_j^sc_i^u)dx_1^j \wedge dx^i + (L_{s_1u}b_i^sb_j^u + L_{s_1u_1}b_i^sc_j^u)dx_1^j \wedge dx_1^i \ ,$$
$$\omega_L^{Ac} = (L_{j_1s}a_i^s + L_{j_1s_1}a_{ik}^sx_1^k - L_{s_1j}a_i^s)dx^i \otimes dx^j + L_{j_1s_1}a_i^sdx_1^i \otimes dx^j - L_{s_1j_1}a_i^sdx^i \otimes$$
$$\otimes dx_1^j \ ,$$
$$\omega_L^{Av} = L_{j_1s_1}a_i^sdx^idy^j, \quad \omega_L^v = L_{i_1j_1}dx^idy^j \ .$$

The tensor fields ω_L^{Av}, ω_L^v can be interpreted as the sections $\tilde{\omega}_L^{Av}$, $\omega_L^v : TM \to V^*TM \otimes V^*TM$.

These expressions immediately give

Lemma 1. *If ω_L^{Ac} is symmetric or skew-symmetric then ω_L^{Av} is skew-symmetric or symmetric.*

Lemma 2. *The tensor field ω_L^α is symmetric or skew-symmetric iff $i_\alpha\omega_L = 0$ or $i_\alpha\omega_L = 2\omega_L$ respectively.*

Proof. Since ω_L is a 2-form therefore $\omega_{L\alpha}(X,Y) = \omega_L(X,\alpha Y) = -\omega_L(\alpha Y, Y) = -\omega_L^\alpha(Y,X) = -(\omega_L^\alpha)^t(X,Y)$. Then the equality $i_\alpha\omega_L = \omega_L^\alpha + \omega_{L\alpha}$ finishes our proof.

Definition 3. We say that vector fields X, Y on TM are ω_L-orthogonal if $\omega_L(X,Y) = 0$. Tensor (1,1)-fields α_1, α_2 on TM are called ω_L-orthogonal if $\omega_L(\alpha_1X, \alpha_2Y) = 0$ for any vector fields X, Y. A tensor (1,1)-field α on TM is said to be ω_L-isotropic if $\omega_L\alpha = 0$. A tensor (1,1)-field A on M is called ω_L-isotropic if its complete lift Ac is ω_L-isotropic.

Definition 4. Let $Z \subset TTM$ be a subbundle of the tangent bundle $p_{TM} : T(TM) \to TM$. The symbol $\mathrm{Orth}_L Z$ will denote the set of tangent vectors Y on TM such that $\omega_L(X,Y) = 0$ for any $X \in Z$ satisfying $p_{TM}X = p_{TM}Y$. We will say that Z is ω_L-Lagrange if $\mathrm{Orth}_L Z = Z$.

Definition 5. A connection Γ on TM is called ω_L-isotropic or ω_L-Lagrange if its horizontal form h_Γ is ω_L-isotropic or ω_L-Lagrange respectively. We will say that two connections Γ_1, Γ_2 are ω_L-orthogonal if h_{Γ_1}, h_{Γ_2} are ω_L-orthogonal. When a connection Γ is ω_L-isotropic we will say that ω_L is Γ-parallel as well.

Remark. Let (M,ε) be pseudo-Riemannian manifold and Γ be the Levi-Civita connection of (M,ε). Then $\nabla_\Gamma\varepsilon = 0$. Let $\bar{\varepsilon}$ be Sasaki metrics on TM which is the natural lift of ε. Then $\bar{\varepsilon}h_\Gamma = 0$, i.e. $\bar{\varepsilon}$ is Γ-parallel.

Lemma 3. *The tensor field ω_L^{Ac} is skew-symmetric if and only if the vector fields AcX and X are ω_L-orthogonal for every vector field X on TM.*

Proof. $\omega_L(AcX, X) = 0 \leftrightarrow \omega_L^{Ac}(X, X) = 0$. It finishes our proof.

Let Γ_j^i be the components of a connection Γ on TM. Then $\omega_L h_\Gamma = (L_{j_1 i} + L_{j_1 s_1}\Gamma_i^s)dx^j \wedge dx^i$. Therefore the equality

$$L_{j_1 i} - L_{i_1 j} + L_{j_1 s_1}\Gamma_i^s - L_{i_1 s_1}\Gamma_j^s = 0 \tag{5}$$

is the coordinate condition for ω_L to be Γ-parallel.

If Γ and $\overline{\Gamma}$ are two connections on TM such that ω_L is Γ- and $\overline{\Gamma}$-parallel then it holds from (5)

$$L_{j_1 s_1}(\Gamma_i^s - \overline{\Gamma}_i^s) - L_{i_1 s_1}(\Gamma_j^s - \overline{\Gamma}_j^s) = 0 \ . \tag{6}$$

We have proved

Proposition 5. *Let Γ be a given connection on TM. Then the set of all Lagrangians L such that Lagrange forms ω_L are Γ-parallel is a vector subspace of the vector space off all functions on TM. Let L be a given Lagrangian on TM. Then the set of all connections Γ on TM such that ω_L is Γ-parallel is an affine space associated to the kernel of the antisymmetrization of the map $\psi : V^*TM \otimes VTM \to V^*TM \otimes V^*TM$ determined by the rule $\beta \to (\tilde{\omega}_L^v)^\beta$, $\beta_j^i \to L_{i_1 t_1}\beta_j^t$.*

It is known, see for example [3], that ω_L is Γ_L parallel, i.e. we have

Proposition 6. *Let S_L be a Lagrange field. Then the connection Γ_L determined by the semispray S_L is ω_L-isotropic, i.e. ω_L is Γ_L- parallel.*

We will say that the tensor (2,2)-field $A \otimes Id_{TM} + Id_{TM} \otimes A = A_I$ is regular if the vector bundle morphism $\overline{A}_I : T^*M \otimes T^*M \to T^*M \otimes T^*M$ over Id_M, $(x_{km}) \to (a_i^t \delta_j^u + \delta_i^u a_j^t)x_{ut}$, is regular.

Proposition 7. *Let the tensor field A_I and the Lagrangian L be regular. Let the tensor (0,2)-field ω_L^{Av} be skew-symmetric. Then there is a unique connection Γ such that h_Γ and $Ac \cdot h$ are ω_L-orthogonal, i.e. $\omega_L(h_\Gamma X, \alpha \cdot h_\Gamma Y) = 0$.*

Proof. Recall that if ω_L^{Ac} is symmetric then ω_L^{Av} is skew-symmetric, i.e. $L_{j_1 t_1}a_i^t = -L_{i_1 t_1}a_j^t$. Let $X = \xi^i \partial/\partial x^i + \eta^i \partial/\partial x^i$, $Y = \overline{\xi}^i \partial/\partial x^i + \overline{\eta}^i \partial/\partial x_1^i$ be two vector fields on TM. Let Γ_j^i be the components of a connection Γ on TM. Then $\omega_L(h_\Gamma X, Ac \cdot h_\Gamma Y) = (L_{t_1 j}a_i^t - L_{j_1 t}a_i^t + L_{t_1 u_1}\Gamma_j^u a_i^t - L_{j_1 u_1}a_{ik}^u x_1^k - L_{j_1 u_1}a_s^u \Gamma_i^s)\xi^j \overline{\xi^i}$. Using the condition for ω_L^{Av} to be skew-symmetric the equality $\omega_L(h_\Gamma X, Ac \cdot h_\Gamma Y) = 0$ holds if and only if

$$(a_i^t \delta_j^u + \delta_i^u a_j^t)L_{s_1 t_1}\Gamma_u^s = L_{i_1 t}a_j^t - L_{t_1 i}a_j^t + L_{i_1 u_1}a_{jk}^u x_1^k \ .$$

It finishes our proof.

Remark. The equality $\omega_L^{Ac}(Ac \cdot h_\Gamma, h_\Gamma) = 0$ in the case when $A^2 = \pm Id_{TM}$ is equivalent to the ω_L-isotropy of Γ or of $Ac \cdot h_\Gamma$ in the case when ω_L^{Ac} is symmetric or skew-symmetric.

Inspiring by [4] we formulate

Lemma 4. *Let X, Y be vector fields, L be a Lagrangian, α be a (1,1)-tensor field and ε be a 1-form on TM. Then the conditions*

$$a, i_X dd_\alpha L = \varepsilon - dYL, \quad b, (L_Y - i_X d_\alpha)dL = \varepsilon$$

are equivalent.

Proof. $L_Y dL = (i_Y d + di_Y)dL = di_Y dL = d(YL), dd_\alpha L = -d_\alpha dL$. It completes our proof.

Let $\beta = \beta^i_j dx^j \otimes \partial/\partial x^i_1$ be a vector semibasic form on TM with values in VTM. Denote $V\beta := \beta(S)$, where S is an arbitrary semispray. In the case of $\beta = Av$ we will denote $Av(S) := VA$.

Lemma 5. *Let S be a semispray, L be a Lagrangian and β be a semibasic vector (1,1)-form with values in VTM. Then*

$$(L_{V\beta} - i_S d_\beta)L = 0 \ .$$

Proof. We get $d_\beta L = i_\beta dL = L_{t_1} \beta^t_k dx^k$. Then $i_S d_\beta L = d_\beta L(S)$. From the other side $L_{V\beta} L = V\beta(L) = L_{t_1} \beta^t_k x^k_1 = d_\beta L(S)$. Our proof is completed.

Corollary. *Under the conditions of Lemma 5 it holds*

$$d(V\beta L) = di_S d_\beta L \ . \tag{8}$$

We return to the case when $\alpha = Ac,\ \beta = Av$. We have $d_{Av}L = L_{t_1} a^t_i dx^i$

$$dd_{Av}L = (L_{t_1 j} a^t_i + L_{t_1} a^t_{ij}) dx^j \wedge dx^i + L_{t_1 j_1} a^t_i dx^j_1 \wedge dx^i \ .$$

It immediately gives

Proposition 8. *Let both the Lagrangian L and the (1,1)-tensor field A on M be regular. Then $dd_{Av}L$ is a symplectic form.*

Lemma 6. *Let S be a semispray. Then*

$$i_{Ac} L_S d_v L = L_S d_{Av} L \ .$$

Proof. $L_S d_v L = (L_{i_1 k} x^k_1 + L_{i_1 k_1} \eta^k) dx^i + L_{i_1} dx^i_1$,

$$L_S d_{Av} L = [(L_{t_1 k_1} a^t_i + L_{t_1} a^t_{ik}) x^k_1 + L_{t_1 k_1} a^t_i \eta^k] dx^i + L_{t_1} a^t_i dx^i_1 \ .$$

Now the equality of Lemma 6 follows from the expression of Ac.

Proposition 9. *Every Lagrange field S_L is also a solution of the equation*

$$(*) \qquad i_S dd_{Av} L = i_{Ac} dL - d(VA(L)) \ .$$

Proof. Being a Lagrange field S_L satisfies the equation $i_S dd_v L = dL - dVL$. Then

$$(9) \qquad i_{Ac}(d(VL) + i_S dd_v L) = i_{Ac} dL \ .$$

Using the equality (8) we get

$$d(VL) + i_S dd_v L = di_S d_v L + i_S dd_v L = (di_S + i_S d) d_v L = L_S d_v L \ .$$

Then by (9)

$$i_{Ac} dL = i_{Ac}[d(VL) + i_S dd_v L] = i_{Ac} L_S d_v L \ .$$

Analogously using (8) and Lemma 6 we get

$$d(VAL) + i_S dd_{Av} L = di_S d_{Av} L + i_S dd_{Av} L = L_S d_{Av} L = I_{Ac} L_S d_v L \ .$$

Therefore S_L satisfies the equation (*). Proof is finished.

Let $X = \xi^i \partial/\partial x^i + \eta^i \partial/\partial x_1^i$ be a vector field on TM. We calculate

$$i_X dd_{Av} L = [(L_{t_1 j} a_i^t - L_{t_1 i} a_j^t - L_{t_1} a_{ji}^t + L_{t_1} a_{ij}^t)\xi^j + L_{t_1 j_1} a_i^t \eta^j] dx^i - L_{t_1 i_1} a_j^t \xi^j dx_1^i$$

$$AcX = a_j^i \xi^j \partial/\partial x^i + (a_{jk}^i x_1^k \xi^j + a_j^i \eta^j)\partial/\partial x_1^i$$

$$i_{AcX} dd_v L = [(-L_{t_1 i} a_j^t + L_{i_1 t} a_j^t + L_{i_1 t_1} a_{jk}^t x_1^k)\xi^j + L_{i_1 t_1} a_j^t \eta^j] dx^i - L_{t_1 i_1} a_j^t \xi^j dx_1^i \ .$$

These expressions immediately give

Lemma 7. *For any vertical vector field X on TM the 1-forms $i_X dd_{Av} L$, $i_{AcX} dd_v L$ are semibasic. For every vector field X the 1-form $i_{AcX} dd_v L - i_X dd_{av} L$ is semibasic.*

Corollary. *Let Y be a vertical vector field on TM. Then it holds*

$$i_{AcX} dd_v L(Y) = i_X dd_{Av} L(Y)$$

for any vector field X on TM.

Definition 6. A (1,1)-tensor field A on M is called L-commutative if

$$i_{AcX} dd_v L = i_X dd_{Av} L$$

for any vector field X on TM.

Proposition 10. *If a (1,1)-tensor field A on M is L-commutative then ω_L^A is skew-symmetric.*

Proof. The (0,2)-form ω_L^A is skew-symmetric iff $dd_v L(AcY, X) = -dd_v L(AcX, Y)$. Let A be L-commutative. Then for any vector fields X, Y we get $dd_v L(AcX, Y) = i_{AcX} dd_v L(Y) = i_X dd_{Av} L(Y) = dd_{Av} L(X, Y)$. Analogously $dd_v L(AcY, X) = dd_{Av} L(Y, X) = -dd_{Av} L(X, Y)$. It completes our proof.

Proposition 11. *A (1,1)-tensor field A on M is L-commutative if and only if the 2-form $dd_{Ac}L$ is semibasic.*

Proof. By the direct computation we get

$$di_{Ac}dL = (L_{tj}a^t_i + L_t a^t_{ij} + L_{t_1 j}a^t_{ik}x^k_1 + L_{t_1}a^t_{ikj}x^k_1)dx^j \wedge dx^i +$$
$$+ (L_{tj_1}a^t_i + L_{t_1 j_1}a^t_{ik}x^k_1 + L_{t_1}a^t_{ij} - L_{t_1 i}a^t_j - L_{t_1}a^t_{ji})dx^j_1 \wedge dx^i +$$
$$+ L_{t_1 j_1}a^t_i dx^j_1 \wedge dx^i_1 \ .$$

Comparing it with the expression for $i_{AcX}dd_vL - i_X dd_{Av}L$ we finish our proof.

Remark. Recall the map $I_L : X \to i_X dd_v L$. If we denote by I_{LA} the map $X \to i_X dd_{Av}L$ we can say that A is L-commutative if and only if $I_{LA} = I_L \cdot Ac$.

Lemma 8. *ω^A_L is symmetric iff $i_{AcX}dd_vL = -i_{Ac}i_X dd_vL$. ω^A_L is skew-symmetric iff $i_{AcX}dd_vL = i_{Ac}i_X dd_vL$.*

Proof. $(i_{AcX}dd_vL)(X) = dd_vL(AcX, Y) = \omega^A_L(X,Y)$, $(i_{Ac}i_X dd_vL)(Y) = (i_X dd_vL)(AcY) = dd_vL(X, AcY) = \omega_{LA}(X,Y) = -(\omega^A_L)^t(X,Y)$. It completes our proof.

In the rest part of this paper we will deal with the 2-form $i_{Ac}\omega_L$. In general it is not closed. We introduce its expression $i_{Ac}\omega_L = (L_{t_1 j}a^t_i + L_{i_1 t}a^t_j + L_{i_1 t_1}a^t_{jk}x^k_1)dx^j \wedge dx^i + (L_{t_1 j_1}a^t_i + L_{i_1 t_1}a^t_j)dx^j_1 \wedge dx^i$.

Let $s(\omega^{Av}_L) = (L_{i_1 s_1}a^s_j + L_{j_1 s_1}a^s_i)dx^i dy^j$ denote the symmetrisation of ω^{Av}_L. It is clair that $i_{Ac}\omega_L$ is regular iff $s(\omega^{Av}_L)$ is regular.

Proposition 12. *If the (0,2)-tensor field $s(\omega^{Av}_L)$ is regular then there is a unique vector field X such that*

$$i_X i_{Ac}\omega_L = i_{Ac}(dE + dL) - dVAL, \quad E = L - VL \ .$$

This vector field is a semispray.

Proof. By direct computation we obtain that the form $i_X(i_{Ac}\omega_L) + d(VAL) - i_{Ac}(dE + dL)$ is semibasic iff X is a semispray. Then the assertion of Proposition (12) follows from the term $(L_{t_1 j_1}a^t_i + L_{i_1 t_1}a^t_j)dx^j_1 \wedge dx^i$ in the expression of $i_{Ac}\omega_L$.

Remarks.

1. If S is the Lagrange field then $i_S i_{Ac}\omega_L(Y) = i_{Ac}\omega_L(S,Y) = \omega_L(AcS, Y) + \omega_L(S, AcY) = i_{AcS}\omega_L(Y) + i_{Ac}dE$. Therefore if A is L-commutative then $i_{AcS}\omega_L = i_S dd_{Av}L = i_{Ac}dL - d(VAL)$, (see Proposition (9)), and then $i_S i_{Ac}\omega_L = i_{Ac}dL - d(VAL) + i_{Ac}dE$.
2. In the case when ω^A_L is skew-symmetric then $i_{Ac}\omega_L = 2\omega^A_L$. Then $i_{Ac}\omega_L$ is regular iff Lagrangian L is regular and A is regular. Then the Lagrange field satisfies the equation

$$i_S(i_{Ac}\omega_L) = 2i_{Ac}dE \ .$$

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Dept. of Computer Science
Matej Bel Univerzity
Tajovské ho 40
974 01 Banská Bystrica
SLOVAKIA

E-mail address: dekret@fpv.umb.sk

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KERNEL AND SOLUTION NUMBERS OF DIGRAPHS

Matúš Harminc

Abstract. In this paper we show that for given nonnegative integers k, s there exist infinitely many strongly connected digraphs with exactly k kernels and s solutions.

Kernels and solutions are certain vertex subsets of digraphs that are studied in many books and papers e.g., [2, 3, 4, 5]. The number of kernels (or solutions) was investigated in the papers [1, 6, 7]. The following provide a recapitulation of basic observations:

(i) A directed cycle of an odd length has no kernel and no solution.
(ii) A digraph with no directed cycles has exactly one kernel and one solution.
(iii) A cycle of an even length possesses exactly two kernels and two solutions.

In connection with this two natural questions arise:

1. How many kernels and solutions a digraph can have ?
2. Are these numbers mutually dependent or not ?

The answer to the first question is trivial when $k = s$ and the pairs of opposite arcs are allowed. In this case it is sufficient to take the complete digraph with k vertices (its kernels are vertex singletons). In this paper we show that for given nonnegative integers k, s there are infinitely many pairwise nonisomorphic strongly connected digraphs with no couple of opposite arcs that have exactly k kernels and s solutions.

1. Preliminaries

An ordered pair $D = (V, A)$ is said to be *a digraph* whenever V is a nonempty set (vertices of D) and A (arcs of D) is a subset of the set of the ordered pairs of V such that for each $v \in V$ it holds $\overrightarrow{vv} \notin A$, and if $u, v \in V$ then $\overrightarrow{uv} \in A$ implies $\overrightarrow{vu} \notin A$. A set of vertices $W \subseteq V$ is called *independent* if for every pair of vertices $u, v \in W$ neither of arcs $\overrightarrow{uv}, \overrightarrow{uv}$ is present in the digraph. $W \subseteq V$ is *absorbent* if for each $u \in V - W$ there exists $\overrightarrow{uv} \in A$ with $v \in W$ and *dominant* if for each $v \in V - W$ there exists $\overrightarrow{uv} \in A$ with $u \in W$. A set $W \subseteq V$ is *a kernel of* D if W is independent and absorbent and it is *a solution of* D if W is independent and dominant. *A conversion of* D is a digraph $c(D) = (V, B)$ with the same vertex set as D and with the arc set $B = \{\overrightarrow{uv} : \overrightarrow{vu} \in A\}$. It is easy to see that the following lemma is valid.

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1.1. Lemma. *Let $D = (V, A)$ be a digraph and let $W \subseteq V$. Then W is a kernel of D if and only if W is a solution of $c(D)$.*

As usual, a digraph is *strongly connected*, if for every $u, v \in V$ there exists a sequence $\overrightarrow{ua_1}, \overrightarrow{a_1a_2}, \overrightarrow{a_2a_3}, \ldots, \overrightarrow{a_kv}$ in A. Let $\mathcal{G}$ denote the class of all finite strongly connected digraphs. For a digraph D we immediately have:

1.2. Lemma. *$D \in \mathcal{G}$ if and only if $c(D) \in \mathcal{G}$.*

Let us denote by C_n the directed cycle with n vertices ($n \geq 3$). Evidently, $C_n \in \mathcal{G}$.

1.3. Remark. By (i) above, each of digraphs $C_3, C_5, \ldots, C_{2n+1}, \ldots$ has no kernel and no solution. By (iii), each of digraphs $C_4, C_6, \ldots, C_{2n}, \ldots$ has two kernels which are simultaneously the all solutions of the even cycle.

2. Results

The following lemma (c.f. [1]) provides a method for constructions of digraphs belonging to $\mathcal{G}$ with the same number of kernels.

2.1. Lemma. *Let $D = (V, A), \overrightarrow{ac} \in A$ and let $D(\overrightarrow{ac}) = (V', A')$ be a digraph such that $V' = V \cup \{v_1, v_2\}$, $v_1, v_2 \notin V$ and $A' = (A - \overrightarrow{ac}) \cup \{\overrightarrow{av_1}, \overrightarrow{v_1v_2}, \overrightarrow{v_2c}\}$. Then the number of kernels of $D(\overrightarrow{ac})$ equals to the number of kernels of D.*

Proof. Let $\mathcal{K}$ and $\mathcal{L}$ denote the systems of the all kernels of D and of $D(\overrightarrow{ac})$, respectively. Define a mapping f from $\mathcal{K}$ to $\mathcal{L}$ in the following way: for $K \in \mathcal{K}$ put

$$f(K) = \begin{cases} K \cup \{v_1\} & \text{if } c \in K, \\ K \cup \{v_2\} & \text{if } c \notin K. \end{cases}$$

It is easy to verify that the mapping f is a bijection between $\mathcal{K}$ and $\mathcal{L}$. Thus, $\operatorname{card} \mathcal{K} = \operatorname{card} \mathcal{L}$. $\square$

Let $\mathcal{G}_{(k,s)}$ denote the set of all strongly connected digraphs with k kernels and s solutions.

2.2. Corollary. *If $\mathcal{G}_{(k,s)}$ is not empty then it is infinite.*

Proof. Let D be a digraph from $\mathcal{G}$ and let k be the number of its kernels. Let $\overrightarrow{x_1y_1}$ be an arbitrary arc of D. Create $D' = D(\overrightarrow{x_1y_1})$ as above (see 2.1). Then choose an arbitrary arc $\overrightarrow{x_2y_2}$ of D' and create $D'' = D'(\overrightarrow{x_2y_2})$, etc.. Every member of the sequence $D, D', D'', \ldots$ belongs to $\mathcal{G}$ and by Lemma 2.1 each of them has exactly k kernels. Since the number of vertices in these digraphs increases, they are pairwise nonisomorphic.

Let s be the number of solutions of D. Then $c(D)$ belongs to $\mathcal{G}$ and it possesses s kernels as well as $c(D'), c(D''), \ldots$. In such way, $D, D', D'', \ldots$ have s solutions. $\square$

According to Corollary 2.2 it suffices to find one digraph of $\mathcal{G}$ with k kernels and s solutions for each pair of integers k, s. First, we shall present a digraph belonging to $\mathcal{G}$ with exactly one kernel. The smallest digraph with such property is $D_1 = (V_1, A_1)$ with $V_1 = \{a_1, b_1, c_1, d_1\}$ and with $A_1 = \{\overrightarrow{a_1b_1}, \overrightarrow{b_1c_1}, \overrightarrow{c_1d_1}, \overrightarrow{d_1a_1}, \overrightarrow{b_1d_1}\}$.

2.3. Corollary. *$\mathcal{G}$ contains infinitely many digraphs with exactly one kernel.*

2.4. Example. Let D_1 be defined as in the previous. Take the arc $\overrightarrow{b_1d_1}$ of D_1 and create $D_1' = D_1(\overrightarrow{b_1d_1})$. Then choose the arc $\overrightarrow{v_2d_1}$ of D_1' and construct $D_1'' = D_1'(\overrightarrow{v_2d_1})$, and continue in this way. Each member of the sequence $D_1, D_1', D_1'', \ldots$ belongs to $\mathcal{G}$ and by Lemma 2.1 they have exactly one kernel (and one solution).

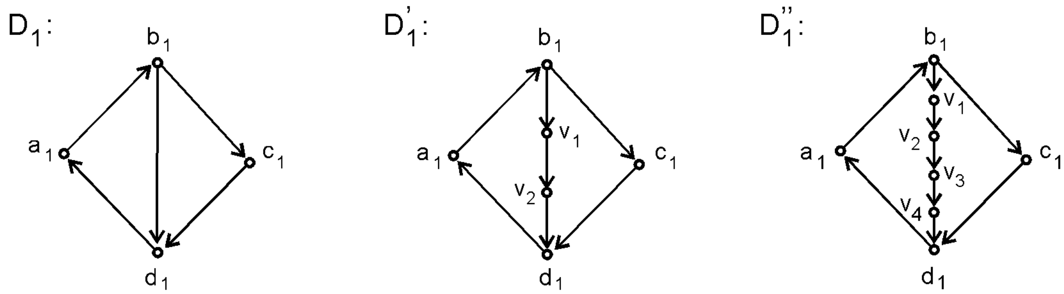

Fig.1: *The first three members of a sequence of digraphs $D_1, D_1', D_1'', \ldots$*

Let D_0 be the cycle with the vertices a_0, b_0, c_0 and with the arcs $\overrightarrow{a_0b_0}, \overrightarrow{b_0c_0}, \overrightarrow{c_0a_0}$. Let $D_2 = (V_2, A_2)$ where $V_2 = \{a_2, b_2, c_2, d_2\}$ and $A_2 = \{\overrightarrow{a_2b_2}, \overrightarrow{b_2c_2}, \overrightarrow{c_2d_2}, \overrightarrow{d_2a_2}\}$. Now we can construct digraphs with k kernels belonging to $\mathcal{G}$ for $k \geq 3$ in the following way: Let k be an integer, $k \geq 2$. Denote by $D_{k+1} = (V_{k+1}, A_{k+1})$ the digraph with

$$\begin{aligned} V_{k+1} &= V_k \cup \{a_{k+1}, b_{k+1}, c_{k+1}, d_{k+1}\} \qquad \text{and} \\ A_{k+1} &= A_k \cup \{\overrightarrow{a_{k+1}b_{k+1}}, \overrightarrow{b_{k+1}c_{k+1}}, \overrightarrow{c_{k+1}d_{k+1}}, \overrightarrow{d_{k+1}a_{k+1}}\} \cup \\ &\quad \cup \{\overrightarrow{c_{k+1}c_k}, \overrightarrow{a_2a_{k+1}}, \overrightarrow{a_3a_{k+1}}, \ldots, \overrightarrow{a_ka_{k+1}}\}. \end{aligned}$$

The following figure shows the digraph $D_3 \in \mathcal{G}$ with three kernels, namely $\{a_2, c_2, b_3, d_3\}, \{b_2, d_2, a_3, c_3\}, \{b_2, d_2, b_3, d_3\}$.

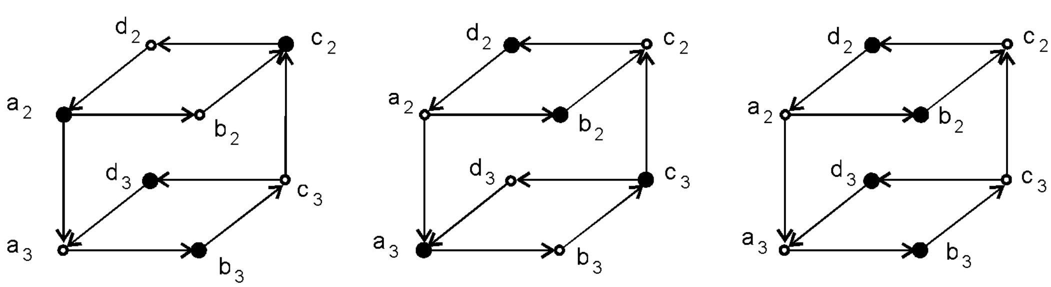

Fig.2: *The digraph D_3 is strongly connected and has three kernels.*

For every $k \geq 0$ let D_k be defined as above.

2.5. Proposition. *The digraph D_k has exactly k kernels for each nonnegative integer k.*

Proof. We have already stated that if $k \in \{0,1,2,3\}$ then D_k has k kernels. Now, we shall proceed by induction. Let k be an integer, $k \geq 3$. For every kernel K of D_{k+1} holds that either $b_{k+1} \notin K$ or $b_{k+1} \in K$. In the first case $c_{k+1} \in K$, $d_{k+1} \notin K$ and $a_{k+1} \in K$. Hence $a_i \notin K$ for $i \in \{2,3,\dots,k\}$ and $K = \{b_2, d_2, b_3, d_3, \dots, b_k, d_k, b_{k+1}, d_{k+1}\}$. In the second case $c_{k+1} \notin K, d_{k+1} \in K$ and there exists a bijective mapping ϕ from the set $\{K : K$ is a kernel of $D_k\}$ to the set $\{L : L$ is a kernel of $D_{k+1}\}$ given by $\phi(K) = K \cup \{b_{k+1}, d_{k+1}\}$. In this way we have shown that the digraph D_{k+1} has one kernel more than the digraph D_k. □

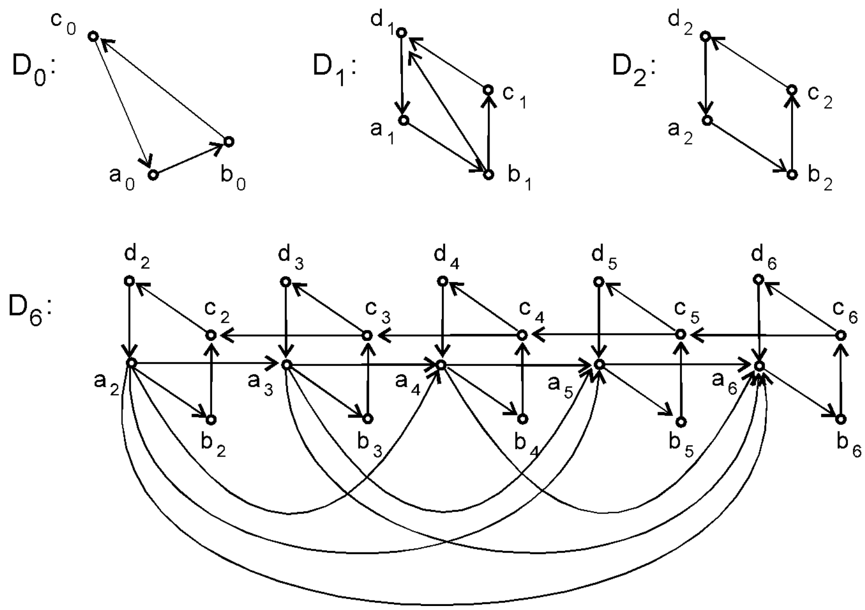

Fig.3: D_i *for* $i \in \{0,1,2,6\}$ *with* $0,1,2$ *and* 6 *kernels, respectively.*

2.6. Theorem. *Let k, s be nonnegative integers. Then $\mathcal{G}_{(k,s)}$ is infinite.*

Proof. By Corollary 2.2 it suffices to show that $\mathcal{G}_{(k,s)} \neq \emptyset$.

We take D_k and $c(D_s)$ with the disjoint vertex sets $\{a_i, b_i, \dots\}$ and $\{\overline{a_i}, \overline{b_i}, \dots\}$, respectively. First we add all the possible arcs beginning at a vertex of $c(D_s)$ and ending at a vertex of D_k. Then we add one new vertex v together with arcs from v to every a_i, and to every existing d_i of D_k and from v to every $\overline{b_i}$ and every $\overline{c_i}$ of $c(D_s)$. Also we add all the arcs from every b_i and every c_i of D_k to v and from every $\overline{a_i}$ and every existing $\overline{d_i}$ of $c(D_s)$ to v.

We shall show that the resulting digraph D is an element of $\mathcal{G}_{(k,s)}$. It is easy to see that the digraphs D_k and $c(D_s)$ are strongly connected and therefore the digraph D which we have constructed is strongly connected as well. Any kernel K of D must not contain vertex v because if v would be in some kernel K then the independence of K implies that there is no more vertices in K; it means $K = \{v\}$ what contradicts to the absorbency of K.

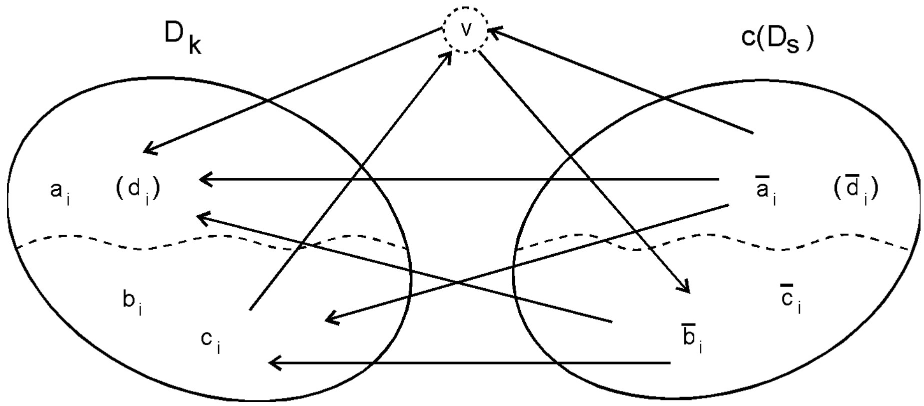

Fig.4: *The construction of a digraph belonging to* $\mathcal{G}_{(k,s)}$.

The independence of K implies that K cannot have nonempty intersections with D_k and with $c(D_s)$ simultaneously. It means that K is a subset of D_k or a subset of $c(D_s)$. But subsets of $c(D_s)$ are not absorbent in D. Therefore K is a kernel of D_k. Conversely, every kernel of D_k is a kernel of D because we did not add any new arc among the vertices of D_k (the independence) and each kernel of D_k contains either a_k or d_k, so it is absorbent not only in D_k but in D as well. Now we have verified that D has k kernels.

Analogously, the set of the all solutions of D is the same set as the set of the all solutions of $c(D_s)$, therefore D has s solutions, and the proof is complete. □

A digraph belonging to $\mathcal{G}_{(k,s)}$ with the minimal number of vertices is called *a minimal digraph* of $\mathcal{G}_{(k,s)}$. The number of the vertices of a minimal digraph of $\mathcal{G}_{(k,s)}$ will be denoted by $k \star s$.

2.7. Proposition. *Let* $k \star s = \min\{\text{card}\, V : (V, A) \in \mathcal{G}_{(k,s)}\}$. *Then*

(i) $0 \star 0 = 3, \qquad 0 \star 1 = 1 \star 0 = 5, \qquad 0 \star 2 = 2 \star 0 = 6,$

(ii) $1 \star 1 = 2 \star 2 = 4, \qquad 1 \star 2 = 2 \star 1 = 5,$

(iii) *if* $k > 1$ *then* $k \star 0 \leq 4k$ *and* $k \star 1 \leq 4k + 1$ *and*

(iv) $k \star s \leq 4(k + s) - 7$ *whenever* $k > 1$ *and* $s > 1$.

Proof. By Lemmas 1.1 and 1.2 the operation $\star$ is commutative. The only minimal digraph of $\mathcal{G}_{(0,0)}$ is D_0 thus $0 \star 0 = 3$. The Figure 5 shows minimal digraphs of the classes $\mathcal{G}_{(0,1)}$, $\mathcal{G}_{(0,2)}$ and $\mathcal{G}_{(1,2)}$. D_1 is the minimal digraph of $\mathcal{G}_{(1,1)}$ and D_2 is the minimal digraph of $\mathcal{G}_{(2,2)}$. In order to obtain the inequalities for $k > 1$ it suffices to enumerate the number of vertices of the digraphs constructed above. □

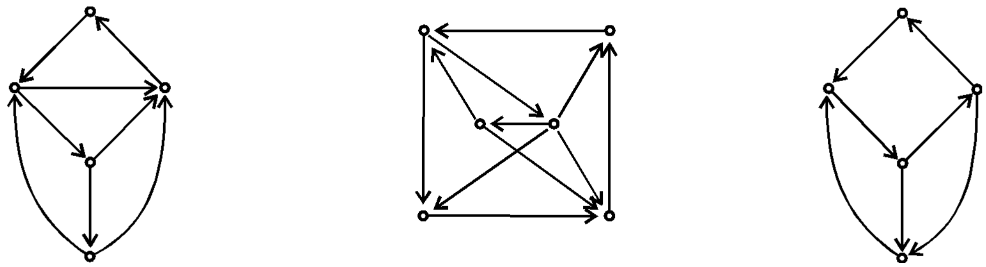

Fig.5: *Minimal digraphs of* $\mathcal{G}_{(0,1)}$, $\mathcal{G}_{(0,2)}$ *and* $\mathcal{G}_{(1,2)}$.

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Dept. of Geometry and Algebra
Faculty of Science, P.J. Šafárik University
Jesenná 5
041 54 Košice
SLOVAKIA

E-mail address: harminc@duro.upjs.sk

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ON $\vee$-IRREDUCIBLE ELEMENTS IN THE POSITIVE CONE OF AN ℓ-GROUP

Ján Jakubík and Gabriela Pringerová

Abstract. Let G be an ℓ-group. The relations between the structure of G and the conditions concerning $\vee$-irreducible elements in the lattice G^+ are investigated in this paper.

1. Introduction

The classical theorem of Birkhoff ([1], pp. 142-143) deals with the representation of elements of a distributive lattice as irredundant joins of $\vee$-irreducible elements.

For related results and for further references cf. the expository paper of Dilworth [3].

Let G^+ be the positive cone of an ℓ-group G. Then G^+ is a distributive lattice. In this note we are concerned with the following conditions for the lattice G^+:

(a) For each $x \in G^+$ there exists an irredundant representation

$$x = x_1 \vee x_2 \vee \cdots \vee x_n$$

such that each x_i $(i = 1, 2, \ldots, n)$ belongs to G^+ and is $\vee$-irreducible in G^+.

(b) For each $x \in G^+$ there exists an irredundant representation $x = \bigvee_{i \in I} x_i$ such that each x_i belongs to G^+ and is $\vee$-irreducible in G^+.

The notion of completely subdirect product of linearly ordered groups was introduced by Šik [6].

In the present note we prove

(A) The condition (a) holds if and only if G is a direct sum of linearly ordered groups.

(B) The condition (b) is valid if and only if G is a completely subdirect product of linearly ordered groups.

The main result of the recent paper [7] is the following theorem:

$(*)$ Let L be a lattice such that

(i) L satisfies the descending chain condition;

(ii) each element of L has one and only one representation as an irredundant join of a finite number of $\vee$-irreducible elements.

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Then the lattice L is distributive.

Unfortunately, in the proof of $(*)$ there was applied a lemma (given on p. 95 of [7]) which is false, and the assertion $(*)$ is false as well (cf. the remark at the end of Section 3 below).

2. Preliminaries

Let L be a lattice. An element $a \in L$ is called $\vee$-irreducible if, whenever $b, c \in L$ and $a = b \vee c$, then either $a = b$ or $a = c$.

Let $x \in L$ and let $(x_i)_{i \in I}$ be an indexed system of elements of L such that the relation

$$x = \bigvee_{i \in I} x_i \tag{1}$$

is valid in L. The representation (1) is said to be irredundant if either

(i) $\operatorname{card} I = 1$,

or

(ii) $\operatorname{card} I > 1$ and whenever $j \in I$, then the relation

$$x = \bigvee_{i \in I \setminus \{j\}} x_i$$

fails to be valid.

For lattice ordered groups (shortly ℓ-groups) we apply the notation as in Conrad [2]. In particular, the group operation is denoted by $+$, though we do not suppose this operation to be commutative.

The positive cone G^+ of an ℓ-group G is the set $\{x \in G : x \geqq 0\}$.

Let $(G_i)_{i \in I}$ be an indexed system of ℓ-groups . The direct product

$$\prod_{i \in I} G_i$$

is defined in the usual way.

Assume that G is an ℓ-group and that we have an isomorphism

$$\varphi : G \to \prod_{i \in I} G_i \tag{2}$$

of G into $\prod_{i \in I} G_i$. For $g \in G$ and $i \in I$ we denote by g_i the component of $\varphi(g)$ in G_i.

If for each $i \in I$ and each element $t^i \in G_i$ there exists $g \in G$ with $g_i = t^i$, then (2) is said to be a subdirect product representation of G.

If, moreover, for each $i \in I$ and each $t^i \in G_i$ there exists $g \in G$ such that $g_i = t^i$ and $g_j = 0$ whenever $j \in I \setminus \{i\}$, then (2) is called a completely subdirect product decomposition of G.

Let (2) be a completely subdirect product decomposition of G. For each $g \in G$ put

$$I(g) = \{i \in I : g_i \neq 0\}.$$

Assume that $I(g)$ is finite for each $g \in G$. Then (2) is called a direct sum representation of G.

3. Convex chains in G^+

Again, let G be an ℓ-group . For $a, b \in G$ with $a \leqq b$, the interval $[a, b]$ is the set $\{x \in G : a \leqq x \leqq b\}$. A nonempty subset H of G is *convex* in G if, whenever $h_1, h_2 \in H$ and $h_1 \leqq h_2$, then $[h_1, h_2] \subseteq H$. A subset of G which is linearly ordered under the induced partial order is called a *chain* in G.

3.1. Lemma. *Let $a \in G^+$. Then the following conditions are equivalent:*

(i) *The element a is $\vee$-irreducible in G^+.*
(ii) *The interval $[0, a]$ of G is a chain.*

Proof. The validity of the implication (ii)⇒(i) is obvious. Suppose that (i) holds. By way of contradiction, assume that the condition (ii) is not valid. Then there are $x_1, x_2 \in [0, a]$ such that x_1 and x_2 are incomparable. Put $v = x_1 \vee x_2$; hence $v \in [0, a]$. There is $t \in G^+$ with $v + t = a$. Denote

$$y_i = x_i + t \quad (i = 1, 2).$$

Then y_1 and y_2 are incomparable. Moreover, $y_1 \vee y_2 = a$ and $y_1 < a$, $y_2 < a$. Therefore the element a fails to be $\vee$-irreducible , which is a contradiction. □

We denote by $\mathcal{C}(G^+)$ the system of all convex chains in G^+ containing the element 0. This system is partially ordered by the set-theoretical inclusion. Further, let $\mathcal{C}_m(G^+)$ be the system of all maximal elements of $\mathcal{C}(G^+)$.

3.2. Lemma. (i) *Let $0 < z \in G$, $[0, z] \in \mathcal{C}(G^+)$. Then $[0, 2z] \in \mathcal{C}(G^+)$.*
(ii) *Let $X, Y \in \mathcal{C}(G^+)$, $X \cap Y \neq \{0\}$. Then either $X \subseteq Y$ or $Y \subseteq X$.*
(iii) *Let $X \in \mathcal{C}(G^+)$. Then there exists $\overline{X} \in \mathcal{C}_m(G^+)$ such that $X \subseteq \overline{X}$.*

Proof. (i) First we show that whenever $x \in [0, 2z]$, then either $x \in [0, z]$ or $x \in [z, 2z]$.

In fact, let $x \in [0, 2z]$. By way of contradiction, suppose that z and x are incomparable. Put

$$u = x \wedge z, \quad v = x \vee z,$$

$$p = z - u, \quad q = x - u.$$

Then p and q are incomparable as well. Moreover,

$$p \wedge q = (z - u) \wedge (x - u) = (z \wedge x) - u = 0.$$

Since $p, q \in [0, z]$, the interval $[0, z]$ fails to be a chain, which is a contradiction.

Now let x_i $(i = 1, 2)$ belong to the interval $[0, 2z]$. Let $i \in \{1, 2, \}$. Then either $x_i \in [0, z]$ or $x_i \in [z, 2z]$. The interval $[z, 2z]$ is isomorphic to $[0, z]$ hence it is a chain. Therefore x_1 and x_2 are comparable. Hence $[0, 2z]$ is a chain.

(ii) Let X and Y satisfy the assumptions of (ii). By way of contradiction, assume that neither $X \subseteq Y$ nor $Y \subseteq X$ is valid. Hence there exist x, y with

$$x \in X \setminus Y, \quad y \in Y \setminus X.$$

Then x and y must be incomparable. Put

$$x \wedge y = z, \quad x - z = x_1, \quad y - z = y_1.$$

Hence we have

$$x_1, z \in X, \quad y_1, z \in Y.$$

Therefore z is comparable with both x_1 and y_1. We distinguish the following cases:

a) $x_1 \leqq z$ and $y_1 \leqq z$. Then

$$0 \leqq x = x_1 + z \leqq 2z,$$

and similarly $0 \leqq y \leqq 2z$. Thus according to (i), x and y must be comparable, which is a contradction.

b) $x_1 \geqq z$ and $y_1 \geqq z$. Then $x_1 \wedge y_1 \geqq z$. Since

$$x_1 \wedge y_1 = (x - z) \wedge (y - z) = 0,$$

we get $z = 0$.

Let $x' \in X$; put $x' \wedge y = z_1$. Hence $z_1 \in X \cap Y$. If $z_1 \geqq x$, then $x \in Y$, which is impossible. If $z_1 < x$, then $z_1 \leqq x \wedge y$, whence $z_1 = 0$. Therefore $x' \wedge y = 0$ for each $x' \in X$.

Then by a similar argument we conclude that for each $x' \in X$ and each $y' \in Y$ we have $x' \wedge y' = 0$, whence $X \cap Y = \{0\}$, which is impossible.

c) If neither a) nor b) is valid, then without loss of generality we can suppose that

$$y_1 \leqq z < x_1.$$

Hence we have

$$y = y_1 + z \leqq x_1 + z = x,$$

yielding that $y \in X$, which is a contradiction.

(iii) By applying (ii), we can use the same method as in [5], Proof of 1.4. □

If $G = \{0\}$, then both the assertions (A) and (B) obviously hold. In what follows we suppose that $G \neq \{0\}$; hence $G^+ \neq \{0\}$.

In 3.3 - 3.10 we assume that the lattice G^+ satisfies the condition (b).

3.3. Lemma. *Let $Y \in \mathcal{C}_m(G^+)$. Then $Y \neq \{0\}$.*

Proof. In view of the assumption, there exists $0 < x \in G^+$. Hence in view of (b), there is an irredundant representation

$$x = \bigvee_{i \in I} x_i$$

such that all x_i are $\vee$-irreducible . In view of the irredundancy, $x_i > 0$ for each $i \in I$. Choose an arbitrary $i \in I$. Thus according to 3.1, $[0, x_i] \in \mathcal{C}(G^+)$. Then 3.2 yields that there is $Y_i \in \mathcal{C}_m(G^+)$ with $[0, x_i] \subseteq Y_i$, hence $Y_i \neq \{0\}$. If $Y = \{0\}$, then $Y \subset Y_i$, thus Y fails to be maximal in $\mathcal{C}(G^+)$, which is a contradiction. □

Let $0 < x \in G^+$ and suppose that x is $\vee$-irreducible in G^+. Then in view of 3.1 and 3.2 there exists a uniquely determined element $\overline{x}$ of $\mathcal{C}_m(G^+)$ such that $x \in \overline{x}$. Further, 3.2 immediately implies

3.4. Lemma. *Let* $0 < x \in G$, $0 < y \in G$. *Suppose that both* x *and* y *are* $\vee$*-irreducible . Then either* $\overline{x} = \overline{y}$ *or* $x \wedge y = 0$.

3.5. Lemma. *Let* $Y \in \mathcal{C}_M(G^+)$. *Then the set* Y *has no upper bound in* G^+.

Proof. a) In view of 3.3 there exists $0 < y \in Y$.

First we prove that $2y \in Y$. In fact, in view of 3.2 (i), $[0, 2y] \in \mathcal{C}(G^+)$. According to 3.2 (iii) there is $Y_1 \in \mathcal{C}_m(G^+)$ with $[0, 2y] \subseteq Y_1$. Hence $Y \cap Y_1 \neq \{0\}$. Then 3.2 (ii) yields that either $Y \subseteq Y_1$ or $Y_1 \subseteq Y$. Since both Y and Y_1 are maximal elements of $\mathcal{C}(G^+)$ we get $Y = Y_1$.

Thus for each $0 < t \in Y$ we have $t < 2t \in Y$. Therefore Y has no greatest element.

b) By way of contradiction, suppose that there is $x \in G^+$ such that x is an upper bound of the set Y in G^+. Clearly $x > 0$. Let $(x_i)_{i \in I}$ be as in the proof of 3.3.

It is well-known that the lattice G^+ is infinitely distributive, hence for each $0 < y \in Y$ we have

$$y = y \wedge x = y \wedge (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (y \wedge x_i).$$

According to 3.1, y is $\vee$-irreducible , hence there is $i \in I$ with $y = y \wedge x_i$. Thus $y \leqq x_i$. Hence in view of 3.4, $\overline{y} = \overline{x}_i$.

Let $i_1 \in I$, $i_1 \neq i$. Since the representation of x under consideration is irredundant, the elements x_i and x_{i_1} are incomparable. Thus 3.4 yields that $y_1 \wedge x_{i_1} = 0$ for each $y_1 \in Y$.

On the other hand, there exists $i_2 \in I$ such that $y_1 \leqq x_{i_2}$. Thus we must have $i_2 = i$. Then x_i is the greatest element of Y. In view of a), we arrived at a contradiction. □

For each $Y \in \mathcal{C}_m(G^+)$ we put

$$Y' = Y \cup \{-Y\},$$
$$Y^* = \{g \in G : |g| \wedge y = 0 \quad \text{for each } y \in Y\}.$$

3.6. Lemma. *Let* $Y \in \mathcal{C}_m(G^+)$. *Then*

(i) Y' *is a convex chain in* G*;*
(ii) Y' *is an* ℓ*-subgroup of* G*;*
(iii) Y' *fails to be bounded in* G.

Proof. (iii) is a consequence of 3.5. Then by applying [4] (Lemmas 3 and 5) we get that (i) and (ii) are valid. □

3.7. Lemma. *There exists a mapping* φ_Y *of* G *onto* $Y' \times Y^*$ *such that*

(i) φ_Y *is a direct product decomposition of* G*;*
(ii) *if* $g \in G^+$ *and* $\varphi_Y(g) = (y, y^*)$, *then*

$$y = \max\{y_1 \in Y : y_1 \leqq g\}.$$

In particular, if $g \in Y$, *then* $y = g$.

Proof. (i) is a consequence of 3.6 and of [4], Theorem 1. The assertion (ii) is well-known (moreover, for validity of (ii) the corresponding direct factor need not be linearly ordered). □

3.8. Lemma. *Let x and $(x_i)_{i\in I}$ be as in 3.3. Let $i_0 \in I$; put $\overline{x}_{i_0} = Y$. If $\varphi_Y(x) = (y, y^*)$, then $y = x_{i_0}$.*

Proof. We have $x_{i_0} \in Y$ and $x_{i_0} \leqq x$. Let $y_1 \in Y$, $y_1 \leqq x$. Then in view of 3.4, $y_1 \wedge x_i = 0$ for each $i \in I \setminus \{i_0\}$. Hence

$$y_1 = y_1 \wedge x = y_1 \wedge (\bigvee_{i\in I} x_i) = \bigvee_{i\in I} (y_1 \wedge x_i) = x_{i_0} \wedge y_1.$$

Thus $y_1 \leqq x_{i_0}$. According to 3.7 (ii), $y = x_{i_0}$. □

Let $\{Y_j\}_{j\in J}$ be the set $\mathcal{C}_m(G^+)$. For $g \in G$ and $j \in J$ we denote by g_j the element of Y_j' such that, under the direct product decomposition

$$\varphi_Y : G \to Y_j' \times Y_j^*,$$

the component of g in Y_j' is g_j.

Consider the mapping

$$\varphi : G \to \prod_{j\in J} Y_j'$$

defined by $\varphi(g) = (g_j)_{j\in J}$.

From the definition of φ we immediately obtain

3.9. Lemma. *φ is a homomorphism of the ℓ-group G into the ℓ-group $\prod_{j\in J} Y_j'$.*

Let $g \in G$ and assume that $\varphi(g) = 0$. Hence $\varphi(|g|) = 0$ and $|g| \geqq 0$. If $|g| > 0$, then there exists an irredundant representation

$$|g| = \bigvee_{k\in K} z_k$$

such that each z_k is $\vee$-irreducible . In particular, $z_k > 0$ for each $z_k \in Y_{j(k)}$. Then in view of 3.8,

$$|g|_{j(k)} = z_k,$$

whence $\varphi(|g|) \neq 0$, which is a contradiction.

From this and from 3.9 we infer

3.10. Lemma. *φ is an isomorphism of G into $\prod_{j\in J} Y_j'$.*

3.11. Lemma. *Let $j \in J$ and $t^j \in Y_j$. Then*

(i) *$(t^j)_j = t^j$;*

(ii) *if $j_1 \in J$ and $j_1 \neq j$, then $(t^j)_{j_1} = 0$.*

Proof. The case $t^j = 0$ is trivial. Let $t^j > 0$. Then in view of 3.7 (ii) we have $(t^j)_j = t^j$. If $j_1 \in J$, $j_1 \neq j$, then 3.4 and 3.7 yield that $(t^j)_{j_1} = 0$.

If $t^j < 0$, then it suffices to consider the element $-t^j$. □

3.12. Proposition. *Let G be an ℓ-group such that the lattice G^+ satisfies the condition* (b). *Let φ be as above. Then φ is a completely subdirect product decomposition of G.*

Proof. This is a consequence of 3.10 and 3.11. □

3.13. Lemma. *Let G be an ℓ-group which can be represented as a completely subdirect product of linearly ordered groups. Then the lattice G^+ satisfies the condition* (b).

Proof. Let

$$\varphi : G \to \prod_{t \in T} G_t$$

be a completely subdirect product decomposition of G. Without loss of generality we can suppose that $G_t \neq \{0\}$ for each $t \in T$. Let $0 < g \in G$ and $\varphi(g) = (g_t)_{t \in T}$. Then for each $t \in T$ there exists $\overline{g}_t \in G$ such that

$$(\overline{g_t})_{t \in T} = g_t \ ,$$

$$(\overline{g}_t)_{t_1} = 0 \quad \text{if} \quad t_1 \in T \setminus \{t\}.$$

Then $\overline{g}_t$ is $\vee$-irreducible for each $t \in T$.

Put $T_1 = \{t \in T : g_t \neq 0\}$. We have $T_1 \neq \emptyset$. Moreover,

$$g = \bigvee_{t \in T_1} \overline{g}_t$$

and this representation of g is irredundant. Therefore the condition (b) holds for the lattice G^+. □

From 3.12 and 3.13 we conclude that (B) is valid.

Now suppose that G is an ℓ-group such that the lattice G^+ satisfies the condition (a). Since (a) is stronger than (b), we can apply 3.12. For $g \in G$ and $j \in J$ we put $(\varphi(g))_j = g_j$.

3.14. Lemma. *Let $g \in G$. There exists a finite subset J_1 of J such that $g_j = 0$ whenever $j \in J \setminus J_1$.*

Proof. First suppose that $g > 0$. In view of (a) there exists an irredundant representation

$$g = x_1 \vee x_2 \vee \cdots \vee x_n \tag{1}$$

such that each x_i $(i = 1, 2, \ldots, n)$ is $\vee$-irreducible . Further, for each x_i there exists $j(i) \in J$ such that

$$\overline{x}_i = Y_{j(i)}.$$

Put $J_1 = \{j(1), j(2), \ldots, j(n)\}$. If $j \in J \setminus J_1$, then (1) and 3.11 yield that $g_j = 0$.

Thus the assertion of the lemma is valid in the case $g > 0$. Next, since each element g of G can be written in the form $g = u - v$ with $u, v \in G^+$, we conclude that the assertion is valid for an arbitrary element of G. □

3.15. Proposition. *Let G be an ℓ-group such that the lattice G^+ satisfies the condition* (a). *Let φ be as in 3.12. Then φ is a direct sum decomposition of G.*

Proof. This is a consequence of 3.12 and 3.14. □

3.16. Lemma. *Let G be an ℓ-group which can be represented as a direct sum of linearly ordered groups. Then the lattice G^+ satisfies to condition* (a).

Proof. We apply an analogous notation as in the proof of 3.13; the distinction is that φ is now a direct sum representation of G. Then the set T_1 is finite, whence (a) is valid. □

In view of 3.15 and 3.16 we infer that (A) holds.

We conclude with the following remark concerning the paper [7].

Let L be a lattice. For each $a \in L$ we put

$$L(a) = \{x \in L : x \leqq a\}.$$

Further, let $(*)$ be as in Section 1. The following lemma was presented in [7]:

$(**)$ If L is a lattice which satisfies the conditions (i) and (ii) from $(*)$, then for each $a \in L$, the set $L(a)$ is finite.

Let $\mathbb{N}$ be the set of all positive integers with the natural linear order and let ω be an infinite ordinal. We put $L = \mathbb{N} \cup \{\omega\}$ and for each $n \in \mathbb{N}$ we set $n < \omega$. Then L is a linearly ordered set, hence the condition (ii) is satisfied. Moreover, the descending chain condition is valid in L. But the set $L(\omega)$ fails to be finite. Hence $(**)$ does not hold.

Next let L be the lattice on Fig. 1. Then $\{u, a, b, c\}$ is the set of $\vee$-irreducible elements of L. The lattice L satisfies the conditions (i) and (ii) from $(*)$, but it fails to be distributive. Hence the assertion $(*)$ is not valid.

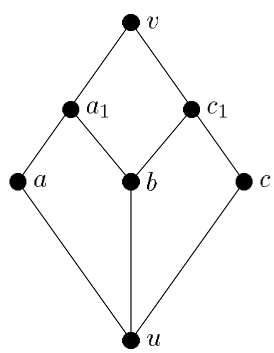

Fig. 1

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Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
040 01 Košice
SLOVAKIA

Dept. of Mathematics
Prešov University
17. novembra 1
080 00 Prešov
SLOVAKIA

E-mail address: pringer@unipo.sk

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FUZZY MAPPINGS AND FUZZY METHODS FOR CRISP MAPPINGS

Vladimír Janiš

Abstract. We deal with the notion of fuzziness from two different aspects. First we study the properties of fuzzy functions, mostly their derivatives, integrals and fixed point properties. The second aspect is the study of a classical real function, fuzzyfying the notions themselves. The last part of the paper is devoted to this aim, showing that introducing the methods of fuzzy mathematics can provide some interesting results for the real functions theory.

The work is a review of results already published or submitted.

1. Introduction

In a natural language we often use words describing the grade of some quality (very, somewhat, a little, ...) or the quality itself (young, heavy, dark, loud, etc.), quite often combined together. Usually a given object corresponds to the above mentioned linguistic construction only with some degree of membership. Therefore such constructions do not define sets, as we use the notion of a set in classical mathematics. We can define the set of all real numbers that are greater than ten, but the expression "numbers much greater than ten" does not define a set. On the other hand, in various situations we need to build e.g. a decision model based on "linguistic variables". The theory of fuzzy sets (or, more generally, fuzzy mathematics) provides us a tool to handle it.

The notion of a fuzzy set was introduced by Zadeh in [Za 1] in 1965. Since then the fuzzy mathematics has developed in a variety of directions. The most fruitful, also from the applications point of view, seems to be the fuzzy control. Moreover, the fuzzy theory appears to be a convenient tool for those applications, where the exact quantitative description of a particular model is either impossible or inappropriate. Hence we often find fuzzy methods used in various decision models designed for the real-time performance. Fuzzy objects and methods can be found also in other applications, like regulation, production control, household appliances, and many others, including music (see [Ha 1], [Ha 2]). The theoretical

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and methodological background of fuzzy mathematics is fuzzy logic and approximate reasoning closely connected to fuzzy logic. A review of the current state of fuzzy logic can be found in the work by Novák [No 1].

In this work we deal with two aspects of mappings analysis. The first one is the study of fuzzy mappings. These mappings are mainly understood as functions that assign a fuzzy set (or a fuzzy element) to a (crisp) element of a given domain. We study the derivatives of such mapping. In this area this work is a direct continuation of the study by Kalina [Ka 1]. The results concerning fixed points of fuzzy mappings, their derivatives and integrals have been published in Janiš [Ja 1], [Ja 2] and Janiš and Nedić [JN 1]. The second aspect of our study corresponds to the attitude proposed by Burgin and Šostak in [BS 1]. Here the authors deal with classical functions, but use methods of fuzzy mathematics. This way they obtain more compact and more general results for such functions. Namely, they study fuzzy continuity of classical functions. We continue in this direction and use fuzzy methods for the study of uniform continuity and properties of derivatives. It is shown that also here it is possible to achieve results that generalize results of classical mathematical analysis. This part of the work summarizes the results published in [Ja 3] and [Ja 4].

2. Basic notions and definitions

Let X be a nonempty set. Any of its subsets $A \subset X$ can be identified with its characteristic function $\chi_A(x)$. This function can be considered as a membership degree with which the element x belongs to the set A. Obviously in case of an ordinary (crisp) subset this value can be either one (the element x belongs to A) or zero (the element x does not belong to A). But if we deal with collections of object defined only vaguely, it is sometimes not possible to cope with only these two possibilities. Hence we generalize the concept of the characteristic function, allowing it to attain also values between zero and one. This attitude serves as a motivation for the definition of a fuzzy subset of X.

Definition 1. A *fuzzy subset* A of the set X is a function $A : X \to [0; 1]$. The set of all fuzzy subsets of X is denoted by $\mathcal{F}(X)$.

If the set X (the universe) is given, we speak briefly of fuzzy sets instead of fuzzy subsets of X.

A fuzzy set A is called *modal*, if there exists at least one element $x \in X$ for which $A(x) = 1$. The set of all those $x \in X$ for which $A(x) > 0$ is called the *support* of A and denoted by $supp(A)$.

Although fuzzy sets can be studied in more general frameworks, e.g. with values in a lattice (see Goguen in [Go 1]), for our purposes it will be sufficient to work only with fuzzy sets with values in the interval $[0; 1]$. To distinguish between fuzzy sets and classical ones we sometimes use the word *crisp* to denote usual non-fuzzy sets (or other objects).

A useful tool for the study of fuzzy sets are their *α-cuts*.

Definition 2. Let X be a metric space, $A \in \mathcal{F}(X), \alpha \in [0;1]$. The *$\alpha$-cut* of A is the set $A_\alpha = \{x \in X; A(x) \geq \alpha\}$ for $\alpha > 0$. For $\alpha = 0$ we put $A_0 = cl\{x \in X; A(x) > 0\}$, where cl is the closure operator.

In other words, if $A \in \mathcal{F}(X), \alpha \in (0;1]$, then

$$A_\alpha = A^{-1}([\alpha;1]), A_0 = cl\{\cup A_\alpha, \alpha > 0\}.$$

If we consider the strict inequality in the definition of A_α, then we speak of a *strict* α-cut.

Throughout this paper the symbol R will denote the set of all real numbers.

If $A \in \mathcal{F}(R)$ and there exists at least one $x \in R$ for which $A(x) = 1$, then A is called a *fuzzy number*. Sometimes this set is called just a *fuzzy quantity* and usually there are additional conditions on fuzzy numbers, very frequently we require that all the α-cuts should be convex sets (in that case we speak of *convex fuzzy numbers*). In some circumstances we assume that the value x for which $A(x) = 1$ is unique. If this is not the case, the term *fuzzy interval* is preferred. Very often, mainly in applications, we work with linear fuzzy numbers (we assume that the function A is partwise linear). For more information on the operations with linear fuzzy numbers see e.g. [Ko 1].

In some cases it is more convenient to define a fuzzy number as a non-decreasing left-continuous function $A : R \to [0;1]$, with the properties $\lim_{x\to-\infty} A(x) = 0$ and $\lim_{x\to\infty} A(x) = 1$. This representation corresponds to the linguistic construction "the number is much greater than A", while the representation with the unique modal value corresponds to the statement "the number is about equal to A". Because of the resemblance to the distribution function we sometimes address this representation as statistical one.

The complement of a fuzzy set is defined in the following way:

Definition 3. If A is a fuzzy set, then the function $1 - A$ is the *complement* of the fuzzy set A.

For the definition of the intersection of fuzzy sets the notion of a t-norm (a triangular norm, see Schweizer and Sklar in [SS 1]) is used. For more detailed study of t-norms see the monograph [KMP].

Definition 4. Let $T : [0;1]^2 \to [0;1]$ be a commutative, associative, non-decreasing function in both variables, fulfilling the boundary conditions $T(x,1) = x$. Then T is called a *t-norm* (a *triangular norm*).

Example 1. It is easy to show that the function

$$T : [0;1]^2 \to [0;1], T(x,y) = \min\{x,y\}$$

is a t-norm. We denote this t-norm by the symbol T_{min}. Note that if T is an arbitrary t-norm, then $T \leq T_{min}$.

Example 2. Let $T : [0;1]^2 \to [0;1]$,

$$T(x,y) = \begin{cases} 0 & \text{if } \max\{x,y\} < 1 \\ \min\{x,y\} & \text{if } \max\{x,y\} = 1 \end{cases}$$

Then T is a t-norm. It is called the weakest t-norm (or the drastic product) and denoted by T_W. Note that for an arbitrary t-norm T the inequality $T_W \leq T$ holds, which justifies the adjective "weakest".

In a lot of papers the minimum function is used to define the intersection of fuzzy sets, but it appears that this is not always the most convenient way to think of the intersection from practical applications point of view.

Definition 5. If A, B are fuzzy sets and a t-norm T is given, then the fuzzy set C for which $C(x) = T(A(x), B(x))$ for each $x \in X$ is the *intersection* of A and B based on the t-norm T.

Although we will not use the notion of a conorm, for the sake of completeness we mention that the union of fuzzy sets based on a given t-norm is given by the de Morgan rule, i.e.

$$S_T(A(x), B(x)) = 1 - T(1 - A(x), 1 - B(x)).$$

The function S_T is a *t-conorm* dual to T. The t-conorm is sometimes called also an S-norm, although this notation may lead to misunderstanding as the dependence on T cannot be seen here.

A t-norm is used also in addition of fuzzy numbers in the following way, which is the special case of the Zadeh extension principle (for more information on the extension principle see [Ng 1] and [BBT]):

Definition 6. If A, B are fuzzy numbers, then their *sum* based on the t-norm T is the fuzzy number $A +_T B$ such that

$$(A +_T B)(z) = \sup\{T(A(x), B(y)), z = x + y\}.$$

In the first part of this work we will focus on fuzzy mappings. There are several different definitions of this notion. We will present a review of the most frequent ones.

Perhaps the most general concept of a fuzzy function is the following, used in Negoita and Ralescu [NR 1]:

Definition 7. A *fuzzy function* from X to Y is a mapping $p : X \times Y \to [0;1]$.

The generality of the later definition (which is in fact a fuzzy relation) may not be convenient – we often require the values of a fuzzy mapping (i.e. the mappings $p(x,.) : Y \to [0;1]$ to be nonzero for at least one $y \in Y$. Such cases were studied e.g. in Dubois and Prade [DP 1]:

Definition 8. A mapping $p : X \times Y \to [0;1]$ is a *fuzzy function* if for all $x \in X$ there exists $y \in Y$ such that $p(x,y) > 0$.

If moreover modality of the values is required, then the following definition (see Ovchinnikov [Ov 1]) is convenient:

Definition 9. A mapping $p : X \times Y \to [0;1]$ is a *fuzzy function* if for all $x \in X$ there exists the unique $y \in Y$ such that $p(x,y) = 1$.

By omitting the uniqueness of the modal value in the later definition we obtain a fuzzy multifunction from X to Y. For more detailed study of this subject see e.g. Tsiporkova [Ts 1].

In the study of probabilistic and fuzzy metric spaces we sometimes represent fuzzy numbers as nondecreasing left-continuous functions from the real line into the unit interval. Another reason for using this representation is also the fact that it enables to introduce a topology on the fuzzy real line. In such situations the following attitude to fuzzy mappings used in Šešelja [Se 1] can be useful:

Definition 10. A mapping $p : X \times Y \to [0;1]$ is a *fuzzy function* if for all $x \in X$ there exists the unique $y \in Y$ such that $p(x,y) = 1$ and if each $\alpha \in (0;1)$ appears at most once as a membership value of p.

There are also other definitions of fuzzy functions used by various authors. A brief review of the most frequent concepts of this notion can be found in the work by Filep [Fi 1].

For our purposes the following convention will be most convenient: Speaking about a fuzzy function f from X to Y we mean a mapping that assigns a fuzzy number $f(x) \in \mathcal{F}(Y)$ to an element $x \in X$. Hence we deal with the case of crisp argument and fuzzy image, on the contrary to the paper [Ka 1], where the author studies also the cases of fuzzy argument and crisp image and both fuzzy argument and image. This does not apply for the section 6, where we study usual crisp functions but use fuzzy methods.

The particular representation of a fuzzy number (either "statistical" or "modal") and if necessary also other supplementary conditions required will be mentioned explicitly when necessary.

3. Derivatives of fuzzy functions

By a fuzzy real number we will understand a fuzzy set $\rho : [-\infty;\infty] \to [0;1]$ for which

(1) $\rho(-\infty) = 0$,
(2) $\rho(\infty) = 1$,
(3) $\rho(r) = \sup\{\rho(s), s < r\}$ for each $r \in R$.

Hence we define a fuzzy real number in the sense of the comment following the Definition 2 (the statistical representation). The set of all fuzzy real numbers will be denoted by $H(R)$.

The set of all (crisp) real numbers is embedded into $H(R)$ in the following way: a crisp real number t is represented by the function $\delta_t \in H(R)$ for which $\delta_t(r) = 0$ if $r \leq t$ and $\delta_t(r) = 1$ if $r > t$.

The partial ordering in $H(R)$ is given by the following way:

$$\rho \leq \sigma \text{ if and only if } \rho(r) \geq \sigma(r) \text{ for each } r \in R.$$

The addition of fuzzy numbers in $H(R)$ is based on the minimum t-norm T_{min} (see Definition 6), i.e.

$$(\rho + \sigma)(r) = \sup\{\min\{\rho(s), \sigma(t)\}; r = s + t\}.$$

The multiplication by a nonnegative real number is given by the formula

$$\begin{aligned}(c.\rho)(r) &= \rho(r/c) \quad \text{if } c > 0,\\ &= \delta_0(r) \quad\ \ \text{if } c = 0.\end{aligned}$$

A fuzzy mapping from R to R will be understood as a mapping that assigns a fuzzy real number in $H(R)$ to a real number. We will assume that a fuzzy mapping we are working with is defined for each $x \in R$.

Klement in [Kl 1] defines an extension of the Lebesgue integral for this type of fuzzy functions. Our aim will be to define a derivative of these functions that will be connected to the mentioned integral. A drawback of this attitude will be its validity only for the case of using the minimum t-norm.

First we introduce the notion of the pseudoinverse of a fuzzy real number. For more details see the work by Höhle [Ho 1].

Definition 11. Let $\rho \in H(R)$. Its *pseudoinverse* is the function $\rho^{(-1)} : [0;1] \to [-\infty;\infty]$ for which $\rho^{(-1)}(\alpha) = \sup\{r; \rho(r) < \alpha\}$.

In the last definition we use the convention $\sup \emptyset = -\infty$, hence the value of each pseudoinverse at zero is $-\infty$.

We will also need the pseudoinverse of a fuzzy function, which will be defined in the following way:

Definition 12. The pseudoinverse of a fuzzy function f is the mapping $f^{(-1)}$ for which $f^{(-1)}(x) = (f(x))^{(-1)}$.

We will denote by $H^{(-1)}(R)$ the set of all pseudoinverses of fuzzy real numbers. The author in [Kl 1] works only with the set of all positive fuzzy real number, but the following result holds also for our case: The mapping

$$p : H(R) \to H^{(-1)}(R), p(\rho) = \rho^{(-1)}$$

is an involutive order-preserving isomorphism, where the addition in $H^{(-1)}(R)$ is the usual addition of functions and the multiplication is the usual multiplication of a function by a nonnegative number.

We will use this isomorphism to define differentiability and the derivative of a fuzzy function in the following way:

Definition 13. Let $f : R \to R$ be a fuzzy function, let $x_0 \in R$. The fuzzy function f is *differentiable* at x_0, if there exists the mapping

$$h_{x_0} : a \mapsto \frac{df^{(-1)}(x_0)}{dx}(a)$$

as an element of $H^{(-1)}(R)$.

Definition 14. If f is differentiable at x_0, then the function $f'(x_0) = [h_{x_0}]^{(-1)}$ is the *derivative* of f at the point x_0.

Obviously if the derivative of a fuzzy function at a point exists, then it is a fuzzy number. Therefore it is possible to define the derivative of f, which will be a fuzzy function. Moreover, it is easy to verify that this notion is an extension of the classical derivative of a real function in the following way:

If f is a (crisp) function differentiable at the point a and if we identify $f(x)$ with $\delta_{f(x)}$ for each x in a neighborhood of a, then the derivative of f at a as indicated in the Definition 14 will be the fuzzy number δ_z, where z in the index is the usual (crisp) derivative of f at the point a.

The connection with the fuzzy integration from the Klement's work [Kl 1] is given by the theorem literally analogical to the classical theorem on the derivative of a function with the variable in the upper integral limit (the mean theorem of integral calculus).

4. Derivatives and fixed points of fuzzy functions

A great deal of this section is based on the paper by Kalina [Ka 1], where the derivative of a fuzzy valued mapping has been defined. We will briefly present basic notions and definitions.

Although a generalization into ordered Banach spaces would be possible, for the sake of simplicity the author demonstrates his apparatus on real fuzzy functions, that means on mappings that assign an LR-fuzzy number to a crisp real number. For more information on L-R-fuzzy numbers see e.g the paper by Mesiar [Me 1], and on their addition see papers by Marková [Ma 1] and [Ma 2].

Let f be a real fuzzy function. We introduce its *level functions* f_α and $f_{-\alpha}$ in the following way: Let $\alpha \in (0;1]$. Then

$$f_\alpha(x) = \sup\{z \in R; f(x)(z) \geq \alpha\}$$

and

$$f_{-\alpha}(x) = \inf\{z \in R; f(x)(z) \geq \alpha\}.$$

The level functions are used in [Ka 1] to define the derivative of f in the following way:

Definition 15. Let f_α and $f_{-\alpha}$ be level functions of a fuzzy function f. Suppose all the level functions are differentiable at a point x and $f_\alpha{}'(x)$ and $f_{-\alpha}{}'(x)$ be their derivatives at x. Denote

$$S = \sup\{z \in R; (\exists \alpha \in (0;1))(z \leq \max\{f_\alpha{}'(x), f_{-\alpha}{}'(x)\})\}$$
$$I = \inf\{z \in R; (\exists \alpha \in (0;1))(z \geq \min\{f_\alpha{}'(x), f_{-\alpha}{}'(x)\})\}.$$

By the *derivative of the fuzzy function* f at the point x we mean the fuzzy number $f'(x)$ defined in the following way:

$$f'(x)(y) = \begin{cases} 1 & \text{if } y = f_1{}'(x) \\ 0 & \text{if } y \notin (I; S) \\ \sup\{\alpha \in (0;1); \max\{f_\alpha{}'(x), f_{-\alpha}{}'(x)\} \geq y\} & \text{if } f_1{}'(x) < y \leq S \\ \sup\{\alpha \in (0;1); \min\{f_\alpha{}'(x), f_{-\alpha}{}'(x)\} \leq y\} & \text{if } f_1{}'(x) > y \geq I. \end{cases}$$

A derivative of a fuzzy function as a function with fuzzy values has been studied also in [FF 1], [PR 1] and some other works. The definition introduced in [Ka 1] is on one hand less general, but on the other hand provides a wider class of differentiable functions. Properties of this derivative are further studied in [Ka 2], [Ka 3], [Ja 1], [JN 1] and [Ja 2].

The basic question that arises with any type of differentiation is its linearity. In [Ja 2] we prove the following:

Proposition 1. *If the fuzzy functions f and g have fuzzy derivatives f' and g' at the point a, their sum $f +_T g$ has the fuzzy derivative $(f +_T g)'$ at a, where T is a t-norm, then $(f +_T g)'(a) \leq f'(a) +_T g'(a)$.*

The next example shows that the opposite inequality in general does not hold for any triangular norm T, not even if we use different t-norms for the additions on each of its sides:

Example 3. Let the fuzzy functions $f, g : [0; 2] \to R$ be given by the following formulas: For $x \in [0; 2]$ put

$$f(x)(t) = \max\left\{0; 1 - \frac{|t|}{x+1}\right\}, \quad t \in R,$$

$$g(x)(t) = \max\left\{0; 1 - \frac{|t|}{3-x}\right\}, \quad t \in R.$$

All the level functions of f and g are linear and hence differentiable on the interval $[0; 2]$. For an arbitrary $x \in [0; 2]$ the derivatives of f and g are equal (we take the one-side derivatives at the endpoints of the interval) and

$$f'(x)(t) = g'(x)(t) = \max\{0; 1 - |t|\}, \quad t \in R.$$

We see that both f' and g' are constant fuzzy functions on $[0; 2]$. Note that their sum with respect to an arbitrary t-norm is again a constant fuzzy function and its common value is not a crisp number.

On the other hand take an arbitrary t-norm T for which all the level functions of the fuzzy function $f +_T g$ are differentiable. Then using the fact that for any $x \in [0; 1]$ there is

$$f(1-x) +_T g(1-x) = f(1+x) +_T g(1+x)$$

as a consequence of the Rolle theorem we obtain that the derivative of all the level functions for the sum $f +_T g$ at the point $x = 1$ is zero. Therefore $(f +_T g)'(1)$ is the crisp number zero and so $(f +_T g)'(1) < f'(1) +_T g'(1)$ for an arbitrary t-norm T.

Moreover, this inequality holds for even an arbitrary pair of t-norms, as we have the crisp zero on the left-hand side for any t-norm and the fuzzy (not crisp) zero on the right-hand side again for any t-norm.

In the work [Ja 1] we deal with the existence of fuzzy fixed points for fuzzy functions and with their properties. Here the fuzzy function f is understood in the following way:

We take (X, d) a complete metric space. The corresponding Hausdorff metric in the space of all nonempty compact subsets of X will be denoted by h. Let for each $x \in X$ there exist an upper semicontinuous function $f(x) : X \to [0; 1]$. Moreover, we require that the inverse images $f_x^{-1}((\alpha; 1])$ are nonempty compact sets for any $\alpha \in (0; 1]$.

Let $\alpha \in (0; 1]$. A point $x \in X$ will be called an *α-fixed point* of f iff $x \in (f(x))_\alpha$. (This is a generalization of the classical case, when the crisp fixed point can be understood as 1-fixed point.)

The condition of contractivity is reformulated in the following way: Let $\alpha \in (0; 1]$. We will say that a fuzzy function f that maps X into itself is *α-contractive* iff there exists a real number q, $0 < q < 1$ such that for each $x_1, x_2 \in X$ there is

$$h((f(x_1))_\alpha, (f(x_2))_\alpha) < qd(x_1, x_2),$$

where h is the Hausdorff metric on X.

The main result of [Ja 1] is the following statement:

Proposition 2. *Let (X, d) be a complete metric space, $\alpha \in (0; 1]$ and let $f : X \to X$ be an α-contractive fuzzy function. Then there is an α-fixed point of f in X.*

Obviously, in contrary to the classical case, there can be more than one fuzzy fixed point of a fuzzy function. The set of all fuzzy fixed points can be characterized by the following propositions:

Proposition 3. *The set of all α-fixed points of an α-contractive function is closed in (X, d).*

Proposition 4. *The set of all α-fixed points of an α-contractive function f is bounded in (X, d). The upper bound for the diameter of this set is the number $\frac{1}{1-q} diam(f(x_0))_\alpha$, where x_0 is an arbitrary α-fixed point of f and q is the contraction coefficient.*

These propositions generalize results on fixed point of fuzzy mappings achieved in [He 1], [ST 1] and [Rh 1].

There is a relationship between the derivative of a differentiable fuzzy function and the existence of its fixed points similar to the crisp case. More details can be found in [Ka 1].

For the purpose of finding a fuzzy analogy to the Lebesgue (or Riemann) integral it is convenient to think of a fuzzy function in the sense of [Se 1] (see Definition 10). From practical reasons we restrict ourselves on nonnegative fuzzy functions. Therefore in this section we will use the concept of a nonnegative fuzzy number introduced by Höhle in [Ho 1]:

Definition 16. A *nonnegative fuzzy number* is a function $A : [0;\infty] \to [0;1]$ for which $A(0) = 0, A(\infty) = 1$ and $A(x) = \sup\{A(t); t \in [0;x)\}$.

Suppose f is a mapping that assigns a fuzzy number (in the sense of the previous definition) to each point of $X, X \subset R$. Evidently we can use the notions from the previous section to define level functions of the fuzzy function f. The only difference is that this time we will have only the functions $f_{-\alpha}$, not the functions f_α. Analogically we can define the derivative at a point for f.

As we have already mentioned in section 3, Klement in [Kl 1] defines the Lebesgue integral for f and states some properties of this integral. The core of this work lies in the isomorphism between the set of all nonnegative fuzzy numbers and the set of their pseudoinverses. Unfortunately the theory of integral based on this attitude works only under the assumption of minimum t-norm used in $\mathcal{F}(R)$. For other t-norm than T_{min} the integral even fails to be additive.

Using the concept of level functions (similar to that of Kalina in [Ka 1]) we can obtain the Lebesgue integral of a fuzzy function with properties resembling those of real functions.

This integral is defined in the following way: Let f be a fuzzy function defined on an interval J. Suppose the level functions $f_{-\alpha}$ are integrable on J for each $\alpha \in (0,1]$. Let

$$I = \inf\left\{\int_I f_{-\alpha}(x)dx; \alpha \in (0,1]\right\},$$

$$S = \sup\left\{\int_I f_{-\alpha}(x)dx; \alpha \in (0,1]\right\} = \int_I f_1(x)dx.$$

The integral of a fuzzy function f on the interval J can be defined as a fuzzy real number $i_J(f)$ in the following way:

$$i_J(f)(x) = \begin{cases} 0 & \text{if } x \leq I, \\ \alpha & \text{if } \alpha = \sup\left\{\gamma \in (0,1]; \int_J f_{-\gamma}(t)dt \leq x\right\}, \\ 1 & \text{if } x \geq S \end{cases}.$$

The integral defined as above has a close connection to the derivative from the previous section – this connection is given by the analogy of the mean integral calculus theorem, which holds for the derivative in the sense of Kalina's work [Ka 1] and above defined integral.

Proposition 5. *Let f be a fuzzy function with integrable level functions on the interval $J = [a, b]$. Let for $x \in J$ $F(x) = i_{[a,x]}(f)$. If f has all its level functions continuous at $x_0 \in J$, then $F'(x_0) = f(x_0)$.*

6. Fuzzy methods in crisp functions calculus

In [BS 1] the authors show how some classical results can obtain more compact form using terms of fuzzy set theory. We introduce the main notions and results of this work:

Let $f : X \to Y$ be a mapping, X, Y are sets of real numbers and let $x_0 \in X$. The *continuity defect* of f at the point x_0 is the value

$$\delta(f, x_0) = \sup\{|y - f(x_0)|; y = \lim_{n\to\infty} y_n, y_n = f(x_n), x_0 = \lim_{n\to\infty} x_n\}.$$

If the continuity defect is finite at $x \in X$ then we say that f is *fuzzy continuous* at the point x.

Next the authors in [BS 1] define the *local continuity measure* at x_0 denoted by $\lambda(f, x_0)$ by the equality

$$\lambda(f, x_0) = (1 + \delta(f, x_0))^{-1}.$$

(In case when $\delta(f, x_0) = \infty$ we put $\lambda(f, x_0) = 0$.)

The *continuity measure* $\lambda(f)$ on a set X is defined as

$$\lambda(f) = \inf\{\lambda(f, x); x \in X\}.$$

Finally, a function f is called *fuzzy continuous* on X if $\lambda(f) > 0$. Note that a function, that is fuzzy continuous at each point of a set need not be fuzzy continuous on that set.

The authors in [BS 1] claim that

> *There are classical results in mathematics which are incomplete. An example is given by the well-known result of the classical mathematical analysis stating that a continuous function defined on a closed interval is bounded. ... But if we ask whether the converse is true we reveal that the answer is negative. The criterion of boundedness may be found only in terms of fuzzy set theory.*

The above mentioned criterion is the proposition stating that a function f defined on a compact space X is bounded if and only if f is fuzzy continuous (see [BS 1], Theorem 2).

On the other hand, applying this approach, some other classical results are no more valid. Maybe the most obvious one is the intermediate value principle (known also as Bolzano lemma) which says that if f is continuous on a closed interval $[a; b]$ and y is an arbitrary number between $f(a)$ and $f(b)$, then there is $c \in [a; b]$ such that $f(c) = y$. (Shortly - a continuous function on a compact set has the Darboux property.) Clearly this is not true for fuzzy continuous functions, as they may have discontinuities.

But if we realise that the intermediate value principle is a tool that enables us e.g. to find approximate solutions of algebraic equations, then we see that in practical calculations we are often satisfied with the value $c \in [a; b]$ for which $f(c)$ is in some sense "not too far" from zero. This leads us again to terms from fuzzy set theory.

In [Ja 3] we define the fuzzy version of uniform continuity and show that for fuzzy uniformly continuous functions some version of intermediate value principle is fulfilled.

First we have to introduce the notion of nearness that was defined in [Ka 1]. Let us assume a fuzzy relation $N : R \times R \to [0; 1]$ is given satisfying the following properties:

(1) for each $x \in R$, $xNx = 1$,
(2) for each $x, y \in R$, $xNy = yNx$,
(3) for each $x, y, z \in R$ if $x < y < z$ then $xNy \geq xNz$,
(4) for each $x \in R$, $\lim_{y\to\infty} xNy = 0$,
(5) for each $x, y, z \in R$ there is $xNy = (x + z)N(y + z)$.

The last condition of nearness is not necessary for most results in [Ka 1] and here, but it simplifies the considerations a lot.

Here are some examples of nearness relations:

Example 4. If $xNy = \dfrac{1}{1 + |x - y|}$, then N is an example of nearness which has never the zero value.

Example 5. Let $k > 0$. The relation $xNy = \max\{1 - k\ |x - y|; 0\}$ is an example of a nearness that assigns nonzero values only to those pairs $(x; y)$ for which their distance does not exceed $\frac{1}{k}$.

Example 6. The relation $xNy = 1$ if $x = y, xNy = 0$ if $x \neq y$ is an example of a "crisp"nearness.

The nearness relation enables us to define the derivative also for various types of fuzzy mappings (see [Ka 1]). Here it will serve as a tool to introduce the α-fuzzy uniform continuity. In the rest of this work we assume a nearness N is given (hence the terms defined are dependent on that particular nearness).

Definition 17. A function $f : R \to R$ is an α-fuzzy uniformly continuous function on a set $M \subset R$ for the given $\alpha \in (0; 1)$ if there exists $\delta > 0$ such that for all $x, y \in M; |x - y| < \delta$ implies $f(x)Nf(y) \geq \alpha$.

Speaking about the α-fuzzy uniformly continuous function we assume the existence of such $\alpha \in (0; 1)$, for which the function fulfills the requirement of the previous definition.

Proposition 6. *If f is an α-fuzzy uniformly continuous function on the set $M \subset R$, then f is fuzzy continuous on M.*

A fuzzy continuous function need not be α-fuzzy uniformly continuous for any $\alpha \in (0;1)$. An easy example of such function is $f(x) = x^{-1}$ on the interval $(0;1)$.

An α-fuzzy uniformly continuous function satisfies the intermediate value principle in the following sense:

Proposition 7. *Let f be an α-fuzzy uniformly continuous function on the interval $[a;b]$, let c be any value between $f(a)$ and $f(b)$. Then there is a number $x \in [a;b]$ for which $f(x)Nc \geq \alpha$.*

In order to find the connection between the notion of an α-fuzzy uniformly continuous function and the uniformly continuous function in the classical sense we will assume to work with a "reasonable" nearness, i.e. we have to add the following conditions to the definition of nearness:

(6) for each $x, y \in R$ there is $xNy = 1$ if and only if $x = y$,

(7) the function $n(y) = xNy$ is continuous for any fixed $x \in R$.

Adding these conditions we can easily see that if f is an α-fuzzy uniformly continuous function for any $\alpha \in (0;1)$, then it is also uniformly continuous and vice versa.

The following proposition shows the connection between the fuzzy and classical uniform continuity.

Proposition 8. *If F_α denotes the set of all α-fuzzy uniformly continuous functions on a set $M \subset R$, then $\bigcap_{\alpha \in (0;1)} F_\alpha$ is the set of all uniformly continuous functions on M.*

In the classical mathematical analysis there is a well-known statement that a continuous function defined on a compact set is uniformly continuous on that set. A similar result holds for fuzzy continuity.

Proposition 9. *If f is fuzzy continuous on a compact set C, if N is a nearness with nonzero values, then there is $\alpha \in (0;1)$ such that f is an α-fuzzy uniformly continuous function (with respect to N).*

In [Ja 4] and [Ja 5] we study how introducing fuzzy methods into crisp functions analysis changes the basic theorems from classical mathematics. We show that a lot of them obtains more general and compact form. Here is a short summary of the results from the mentioned papers:

Definition 18. Let $f : R \to R$ be a function, $a \in R$, let N be a nearness on R and let $\alpha \in (0;1)$. Denote

$$D_\alpha(a) = \left\{ \frac{f(x) - f(a)}{x - a}; x \neq a, xNa \geq \alpha \right\}.$$

The function f is *fuzzy differentiable* at the point a on level α if the numbers $I = \inf D_\alpha(a), S = \sup D_\alpha(a)$ are both finite.

Note that $D_\alpha(a) \neq \emptyset$ for any $\alpha \in (0;1)$ because of the continuity of N.

Example 7. The function $f(x) = \sqrt{|x|}$ is not fuzzy differentiable at 0 at any level, as at this point $D_\alpha(0) = R$ for all $\alpha \in (0;1)$.

Example 8. The function $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$ (Dirac function) is not fuzzy differentiable at the point 0 on any level, as for any $\alpha \in (0;1)$ we have $\sup D_\alpha(0) = \infty$.

Definition 19. The interval $[I;S]$, where the numbers I and S are defined in Definition 18 is called an *α-nearness derivative* of f at a on the level α and denoted by $f'_\alpha(a)$.

In case when at some point $I = -\infty$ and $S = \infty$ we also call the interval $[I;S]$ an α-nearness derivative, although the function is not differentiable at that point. (This is a similar situation as with the integrability - a function that is not integrable may have the integral.)

We consider the arithmetical operations in the extended real line in the usual way: $a + \infty = \infty, a - \infty = -\infty, a.\infty = \infty, a(-\infty) = -\infty$ (the last two statements hold for positive a) and $0.\infty = 0$.

Example 9. The α-nearness derivative of the Dirac function (see Example 8) at the point 0 is the interval $[0;\infty]$. The α-nearness derivative of this function at some $a > 0$, for which $aN0 = \alpha$ is the interval $\left[0; \dfrac{1}{|a|}\right]$.

Here are some properties of fuzzy differentiable functions and their α-nearness derivatives:

Proposition 10. *If $\alpha, \beta \in (0;1)$, $\alpha < \beta$, then $f'_\beta(a) \subset f'_\alpha(a)$.*

The next propositions shows the connection between fuzzy and classical differentiability.

Proposition 11. *If f is differentiable at a (in classical sense), then there is an $\alpha_0 \in (0;1)$ such that f is fuzzy differentiable at a on level α_0 and*

$$\bigcap_{\alpha \in (0;1)} f'_\alpha(a) = \{f'(a)\}\,.$$

Instead of the linearity which holds for the classical derivative, we have only the following inclusion:

Proposition 12. *If f, g are real functions, $a \in R$, $\alpha \in (0;1)$, then*

$$(f+g)'_\alpha(a) \subset f'_\alpha(a) + g'_\alpha(a),$$

where the sum of the sets $f'_\alpha(a), g'_\alpha(a)$ is the set of all sums $x + y$, $x \in f'_\alpha(a), y \in g'_\alpha(a)$.

Proposition 13. *If $c \in R$, then $(cf)'_\alpha(a) = cf'_\alpha(a)$, where the product $cf'_\alpha(a)$ is the set of all products cx, $x \in f'_\alpha(a)$.*

From Proposition 10 we see that the α-nearness derivatives can be understood as level cuts for some fuzzy set. This fuzzy set seems to be a generalization of the crisp derivative (see Proposition 11). We are also able to formulate some classical statements of real variable calculus without assumptions of continuity or differentiability.

In all the remaining propositions in this work we restrict ourselves to functions defined only on a given interval. The reason is that we do not want the points outside this interval to influence the nearness derivative. Other possibility would be to consider the α-derivative of a function f at a point $x \in [m; n]$ as an α-derivative of f restricted on $[m; n]$ at x.

Proposition 14. *A function f is increasing on the interval $[m; n]$ if and only if at each $a \in [m; n]$ there is $f'_\alpha(a) \cap (0; \infty) \neq \emptyset$ and $f'_\alpha(a) \cap (-\infty; 0) = \emptyset$ for every $\alpha \in (0; 1)$.*

A dual necessary and sufficient condition can be formulated in similar way for decreasing functions. For non-decreasing functions we have:

Proposition 15. *A function f is non-decreasing on the interval $[m; n]$ if and only if at each $a \in [m; n]$ there is $f'_\alpha(a) \cap (-\infty; 0) = \emptyset$ for every $\alpha \in (0; 1)$.*

Again the dual condition can be stated for non-increasing functions. All the proofs of these statements are just modifications of the one in Proposition 14.

Finally we state generalized versions of Rolle, Lagrange and Darboux theorems.

Proposition 16 (Rolle Theorem). *If f is defined on the interval $[m; n]$, if $f(m) = f(n)$, then there exists $a \in (m; n)$ and $\alpha \in (0; 1)$ such that $0 \in f'_\alpha(a)$.*

Proposition 17 (Lagrange Mean Value Theorem). *If f is defined on $[m; n]$, then there is a number $a \in (m; n), \alpha \in (0; 1)$ such that $\dfrac{f(m) - f(n)}{m - n} \in f'_\alpha(a)$.*

Proposition 18 (Darboux Theorem). *If f is defined on $[m; n]$, if there is $z \in R$ such that for each $x \in supp(f'(m))$ and for each $y \in supp(f'(n))$ there is $x < z < y$, then there exists $c \in (m; n)$ such that $z \in supp(f'(c))$.*

7. Concluding remarks

This work shows two possible directions of fuzzy infinitesimal calculus. The first one deals with fuzzy object - fuzzy number and fuzzy functions. We show that using "reasonable" definitions we obtain similar results as in the classical case. On the other hand, the methods used to obtain these results sometimes differ a lot from the classical ones.

Another direction, less frequent, is applying fuzzy methods into classical mathematical analysis. In this work the section 6 is devoted to present results of this kind. It appears that using fuzzy methods enables us both to extend validity of some statements from the classical analysis and to formulate them in more general way.

The results of this work are just the basic principles of fuzzy differential calculus; there is a wide field for future research, both in theory and applications.

Finally it is worth to mention new ideas by Vojtáš and Kalina which show close links between fuzzy mathematics and non-standard analysis. This seems to be a fruitful topic both for the theory of fuzzy mathematics and for its applications, such as fuzzy measures or quantum computing.

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Dept. of Mathematics
Matej Bel University
Tajovského 40
974 01 Banská Bystrica
SLOVAKIA

E-mail address: janis@fpv.umb.sk

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INVARIANT MEASURES ON LOCALLY COMPACT SPACES

Peter Maličký

Abstract. The paper considers measures on a locally compact space which are invariant with respect to a given system of continuous maps.

Introduction

The present paper considers the following problem. There is given a locally compact space X and a system $\mathcal{F}$ of continuous maps of the space X into itself. We are interested in conditions under which there is a Borel $\mathcal{F}$-invariant measure on the space X. For example, let G be a locally compact topological group. A map T on the group G of the form $T(x) = ax$ ($T(x) = xb$, $T(x) = axb$) is said to be left (right, left-right) translation of G. If a system $\mathcal{F}$ consists of all left (right) translations, then a Borel $\mathcal{F}$-invariant measure exists and is called left (right) Haar measure of the group G, [2,p.246]. The group G is called unimodular if left Haar measure is also right invariant, [1,p.119]. It is well known that left (right) Haar measure need not be right (left) invariant, [2,p.248]. It means that the system of all left-right translations need not have an invariant Borel measure. For compact spaces the existence problem of an $\mathcal{F}$-invariant measure was fully solved in Roberts' paper [4]. The paper [3] of the author contains partial results about the locally compact case. The presented results are very similar to Roberts' results for the compact case. Without loss of generality we may assume that the system $\mathcal{F}$ contains the identity map and is closed with respect to the composition of maps, i.e. $\mathcal{F}$ is a monoid with respect to the composition. Moreover, we assume that $\mathcal{F}$ is a minimal monoid, which contains sufficiently many homeomorphisms, and we obtain a necessary and sufficient condition for the existence of an $\mathcal{F}$-invariant measure. Using this result we give a topological characterization of nonunimodular locally compact topological groups.

1. Preliminaries

For a locally compact space X the symbol $\mathcal{B}(X)$ denotes the minimal σ-ring containing all compact subsets of X. The members of $\mathcal{B}(X)$ are called Borel sets in X. A set A is called bounded if its closure $\overline{A}$ is a compact set in X. A Borel measure on the space X is a set function $\mu : \mathcal{B}(X) \to \langle 0, \infty)$ such that

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$\mu(\emptyset) = 0$, $\mu \bigcup_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} \mu(A_i)$ for any sequence $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint Borel sets and $\mu(K) < \infty$ for any compact set K in$X$. A map $T : X \to X$ is called measurable if $T^{-1}(A) \in \mathcal{B}(X)$ for any $A \in \mathcal{B}(X)$. A map $T : X \to X$ is called proper if it is continuous and $T^{-1}(K)$ is compact for any compact set K of X. A measure μ is said to be invariant with respect to a measurable map T if $\mu(T^{-1}(A)) = \mu(A)$ for any $A \in \mathcal{B}(X)$. If $\mathcal{F}$ is a system of measurable maps and a measure μ is T-invariant for any $T \in \mathcal{F}$, then μ is called $\mathcal{F}$-invariant. A system $\mathcal{F}$ of proper maps is called minimal if for any $x \in X$ the set $\{T(x) : T \in \mathcal{F}\}$ is dense in X. Clearly, the system $\mathcal{F}$ of proper maps is minimal if and if only for any nonempty open subset U of the space X the system $\{T^{-1}(U) : T \in \mathcal{F}\}$ is a covering of X. We say that a system $\mathcal{F}$ of proper maps contains sufficiently many homeomorphisms if there is a bounded open subset $U_0 \subset X$ such that for any $x \in X$ there is a homeomorphism $T \in \mathcal{F}$ for which $x \in T^{-1}(U_0)$, i.e. $T(x) \in U_0$. Equivalently, a system $\mathcal{F}$ of proper maps contains sufficiently many homeomorphisms if there is a bounded open subset $U_0 \subset X$ such that for any compact set K in X there are homeomorphisms $T_1, ..., T_n \in \mathcal{F}$ such that $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U_0)$. For example, if X is compact and $\mathcal{F}$ contains at least one homeomorphism (e. g. the identity map), then $\mathcal{F}$ contains sufficiently many homeomorphisms. So, the assumption of sufficiently many homeomorphisms in a system $\mathcal{F}$ is useless in the compact case, but not in the locally compact case. If f is a real function defined on a locally compact space X, then the symbol *supp* f denotes the support of the function f, i. e. the closure of the set $\{x : f(x) \neq 0\}$. The set of all continuous functions on X with a compact support is denoted by the symbol $\mathcal{C}_0(X)$. The subset of $\mathcal{C}_0(X)$ consisting of all nonnegative functions is denoted by $\mathcal{C}_0^+(X)$. The inequality $f < g$ means $f(x) \leq g(x)$ for all $x \in X$ and $f(x) < g(x)$ for some $x \in X$.

2. Construction of an invariant measure

To prove the main result we need two technical lemmas.

Lemma 2.1. *Let $\mathcal{F}$ be a minimal system of proper maps of a locally compact space X.*

(i) Let $g, \varphi \in \mathcal{C}_0^+(X)$ and $\varphi \neq 0$. Then there are $T_1, ..., T_n \in \mathcal{F}$, $\varphi_1, ..., \varphi_n \in \mathcal{C}_0^+(X)$ and a real number $a > 0$ such that $g \leq \alpha \sum_{i=1}^{n} \varphi_i \circ T_i$ and $\varphi = \sum_{i=1}^{n} \varphi_i$.

(ii) Moreover, if $\mathcal{F}$ contains sufficiently many homeomorphisms, then there is a function $\varphi_0 \in \mathcal{C}_0^+(X)$ such that any $g \in \mathcal{C}_0^+(X)$ may be represented in a form $g = \beta \sum_{i=1}^{n} f_i \circ T_i$, where $f_i \in \mathcal{C}_0^+(X)$, $0 \leq f_i \leq \varphi_0$, $T_i \in \mathcal{F}$ and β is a nonnegative real number.

Proof. Denote $K =$ *supp* g and $U = \{x : \varphi(x) \neq 0\}$. The set K is compact, U is nonempty open and $\mathcal{F}$ is a minimal system. Hence, there are $T_1, ..., T_n \in \mathcal{F}$ such

that $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U)$, which implies $\sum_{i=1}^{n} \varphi(T_i(x)) > 0$ for all $x \in K$. Put
$f = \sum_{i=1}^{n} \varphi \circ T_i$, $\alpha = \dfrac{n \cdot \sup_{x \in K} g(x)}{\inf_{x \in K} f(x)}$ and $\varphi_1 = ... = \varphi_n = \frac{\varphi}{n}$. Then $g \leq \alpha \sum_{i=1}^{n} \varphi_i \circ T_i$.
(ii) There is a bounded open set $U_0 \subset X$ such that for any compact set K in X there are homeomorphisms $T_1, ..., T_n \in \mathcal{F}$ for which $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U_0)$. There is another bounded open set W for which $\overline{U}_0 \subset W$. Take a function $\varphi_0 \in \mathcal{C}_0^+(X)$ such that $\varphi_0(x) = 1$ for $x \in \overline{U}_0$ and $\varphi_0(x) = 0$ for $x \notin W$, see [2,p.211]. Denote $U = \{x : \varphi_0(x) \neq 0\}$. Obviously, $U_0 \subset U$. Take an arbitrary function $g \in \mathcal{C}_0^+(X)$. Denote $K = supp\ g$. There are homeomorphisms $T_1, ..., T_n \in \mathcal{F}$ such that
$K \subset \bigcup_{i=1}^{n} T_i^{-1}(U_0) \subset \bigcup_{i=1}^{n} T_i^{-1}(U)$,which implies $\sum_{i=1}^{n} \varphi_0(T_i(x)) > 0$ for all $x \in K$. Put
$f = \sum_{i=1}^{n} \varphi_0 \circ T_i$. Define a function h by $h(x) = \frac{g(x)}{f(x)}$ if $f(x) \neq 0$ and $h(x) = 0$ if $x \notin supp\ g$. The function h is defined correctly, $h(x)f(x) = g(x)$ and $h \in \mathcal{C}_0^+(X)$. Put $\beta = \sup_{x \in X} h(x)$ and $f_i(x) = \frac{1}{\beta} h(T_i^{-1}(x)) \cdot \varphi_0(x)$. Then $f_i \in \mathcal{C}_0^+(X), 0 \leq f_i \leq \varphi_0$
and $\beta \sum_{i=1}^{n} f_i(T_i(x)) = \beta \sum_{i=1}^{n} \dfrac{1}{\beta} h(T_i^{-1}(T_i(x)))\varphi_0(T_i(x)) = \sum_{i=1}^{n} h(x)\varphi_0(T_i(x)) =$
$= h(x) \sum_{i=1}^{n} \varphi_0(T_i(x)) = h(x)f(x) = g(x)$. It proves (ii).

Lemma 2.2. *Let $\mathcal{F}$ be a minimal system of proper maps of a locally compact space X. If μ is a nonzero $\mathcal{F}$-invariant Borel measure, then $\mu(U) > 0$ for any nonempty open Borel subset U of X.*

Proof. Let μ be a nonzero $\mathcal{F}$-invariant Borel measure and $\mu(U) = 0$ for some nonempty open Borel subset U of X. Let K be a compact subset of X. Since the system $\{T^{-1}(U) : T \in \mathcal{F}\}$ is a covering of X, there are $T_1, ..., T_n \in \mathcal{F}$ such that $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U)$. $\mathcal{F}$-invariance of the measure μ implies $\mu(K) = 0$. Any Borel set in X may be covered by a sequence of compact sets, [2,p.214]. Therefore, the measure μ is zero.

The following theorem is the main result of the paper.

Theorem 2.3. *Let $\mathcal{F}$ be a minimal monoid of proper maps of a locally compact space X which contains sufficiently many homeomorphisms. The following properties are equivalent.*

(i) There exists a nonzero $\mathcal{F}$-invariant Borel measure on X.

(ii) For any open subsets $U_1, ..., U_n$, any compact subsets $K_1, ..., K_m$ and any maps $T_1, ...T_n, S_1, ..., S_m \in \mathcal{F}$
$\sum_{i=1}^{n} \chi_{U_i} > \sum_{j=1}^{m} \chi_{K_j}$ *implies* $\exists x \in X : \sum_{i=1}^{n} \chi_{\tilde{U}_i}(x) > \sum_{j=1}^{m} \chi_{\tilde{K}_j}(x)$,

where $\tilde{U}_i = T_i^{-1}(U_i)$ *and* $\tilde{K}_j = S_j^{-1}(K_j)$.

(iii) If $T_1, ..., T_n, S_1, ..., S_m \in \mathcal{F}$, $f_1, ..., f_n$ *and* $g_1, ..., g_m$ *are linear combinations of characteristic functions of open and compact sets respectively with positive rational coefficients, then*

$\sum_{i=1}^{n} f_i > \sum_{j=1}^{m} g_j$ *implies* $\exists x \in X : \sum_{i=1}^{n} f_i(T_i(x)) > \sum_{j=1}^{m} g_j(S_j(x))$.

(iv) For any functions $\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m \in \mathcal{C}_0^+(X)$ *and any maps* $T_1, ..., T_n, S_1, ..., S_m \in \mathcal{F}$

$\sum_{i=1}^{n} \varphi_i > \sum_{j=1}^{m} \psi_j$ *implies* $\exists x \in X : \sum_{i=1}^{n} \varphi_i(T_i(x)) > \sum_{j=1}^{m} \psi_j(S_j(x))$.

Proof. (i)⇒(ii) Let μ be a nonzero $\mathcal{F}$-invariant Borel measure on X. Take open subsets $U_1, ..., U_n$ and compact subsets $K_1, ...K_m$ such that $\sum_{i=1}^{n} \chi_{U_i} > \sum_{j=1}^{m} \chi_{K_j}$. Without loss of generality we may assume that the sets $U_1, ..., U_n$ are bounded to be Borel. (An open set is Borel if and only if it is σ-bounded). We have $\sum_{i=1}^{n} \chi_{U_i}(x) \geq \sum_{j=1}^{m} \chi_{K_j}(x)$ for all $x \in X$ and $\sum_{i=1}^{n} \chi_{U_i}(x) > \sum_{j=1}^{m} \chi_{K_j}(x)$ for some $x \in X$. The last inequality holds on a nonempty open subset, because the sets U_i are open and K_j are closed. Lemma 2.2. implies $\sum_{i=1}^{n} \mu(U_i) > \sum_{j=1}^{m} \mu(K_j)$. Take maps $T_1, ..., T_n, S_1, ..., S_m \in \mathcal{F}$. Suppose $\sum_{i=1}^{n} \chi_{\tilde{U}_i} \leq \sum_{i=1}^{m} \chi_{\tilde{K}_j}$, where $\tilde{U}_i = T_i^{-1}(U_i)$ and $\tilde{K}_j = S_j^{-1}(K_j)$. Then $\sum_{j=1}^{m} \mu(\tilde{K}_j) \geq \sum_{i=1}^{n} \mu(\tilde{U}_i) = \sum_{i=1}^{n} \mu(U_i) > \sum_{j=1}^{m} \mu(K_j)$, which is a contradiction.

(ii) ⇒ (iii) This is obvious.

(iii) ⇒ (iv) Take maps $T_1, ..., T_n, S_1, ..., S_m \in \mathcal{F}$ and functions $\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m \in \mathcal{C}_0^+(X)$ such that $\sum_{i=1}^{n} \varphi_i > \sum_{j=1}^{m} \psi_j$. Put

$$h = \frac{1}{2}\left(\sum_{i=1}^{n} \varphi_i - \sum_{j=1}^{m} \psi_j\right). \tag{1}$$

Obviously, $h \in \mathcal{C}_0^+(X)$ and $h \neq 0$. Denote $U = \{x : h(x) \neq 0\}$ and

$$K = \bigcup_{i=1}^{n} supp(\varphi_i \circ T_i). \tag{2}$$

Since $\mathcal{F}$ is minimal and K is compact, there are $R_1, ..., R_p \in \mathcal{F}$ such that

$$K \subset \bigcup_{j=1}^{p} R_j^{-1}(U). \tag{3}$$

Then $0 < \inf_{x\in K} \sum_{j=1}^{p} h(R_j(x))$. Put

$$\alpha = \inf_{x\in K} \sum_{j=1}^{p} h(R_j(x)), \tag{4}$$

$$\psi_{m+j} = \frac{1}{p} h \text{ and} \tag{5}$$

$$S_{m+j} = R_j \text{ for } j = 1, ..., p. \tag{6}$$

Then we have

$$\sum_{i=1}^{n} \varphi_i > \sum_{j=1}^{m+p} \psi_j. \tag{7}$$

Take a natural number

$$r > \frac{p(p+m+n)}{\alpha}. \tag{8}$$

Denote $U_{i,k} = \{x : \varphi_i(x) > \frac{k}{r}\}$ for integers i and k such that $1 \leq i \leq n$, $0 \leq k$ and $K_{j,k} = \{x : \psi_j(x) \geq \frac{k}{r}\}$ for integers j and k such that $1 \leq j \leq m$, $1 \leq k$. Obviously, the sets $U_{i,k}$ are open and the stets $K_{j,k}$ are compact for the corresponding integers. Put $f_i = \frac{1}{r} \sum_{k=0}^{\infty} \chi_{U_{i,k}}$ and $g_j = \frac{1}{r} \sum_{k=1}^{\infty} \chi_{K_{j,k}}$. In fact, both sums are finite. The functions f_i and g_j have properties:

$$0 \leq \varphi_i(x) \leq f_i(x) < \varphi_i(x) + \frac{1}{r}, \tag{9}$$

$$0 \leq g_j(x) \leq \psi_j(x) < g_j(x) + \frac{1}{r} \text{ and} \tag{10}$$

$$0 < f_i(x) \Leftrightarrow 0 < \varphi_i(x) \text{ for all } x \in X. \tag{11}$$

Now,

$$\sum_{i=1}^{n} f_i > \sum_{j=1}^{m+p} g_j \tag{12}$$

and by (iii)

$$\exists x \in X : \sum_{i=1}^{n} f_i(T_i(x)) > \sum_{j=1}^{m+p} g_j(S_j(x)). \tag{13}$$

This element x must belong to the compact K by (10), (11) and (2). Therefore, $\sum_{j=1}^{p} h(R_j(x)) \geq \alpha$ and

$$\sum_{j=m+1}^{m+p} \psi_j(S_j(x)) \geq \frac{\alpha}{p} \tag{14}$$

by (5) and (6). Using (9), (13), (10), (14), and (8) we obtain

$$\sum_{i=1}^{n}\varphi_i(T_i(x)) > \sum_{i=1}^{n}(f_i(T_i(x)) - \frac{1}{r}) > \frac{-n}{r} + \sum_{j=1}^{m+p} g_j(S_j(x)) >$$

$$> \tfrac{-n}{r} + \sum_{j=1}^{m+p}(\psi_j(S_j(x)) - \frac{1}{r}) = \frac{-(n+m+p)}{r} + \sum_{j=1}^{m}\psi_j(S_j(x)) + \sum_{j=m+1}^{m+p}\psi_j(S_j(x)) \geq$$

$$\geq \tfrac{-(n+m+p)}{r} + \sum_{j=1}^{m}\psi_j(S_j(x)) + \frac{\alpha}{p} > \sum_{j=1}^{m}\psi_j(S_j(x)).$$ It proves (iv).

(iv)$\Rightarrow$ (i) Take $\varphi_0 \in C_0^+(X)$ from (ii) of Lemma 2.1. Let A be the set of all functions φ of the form $\varphi = \sum_{i=1}^{n}\varphi_i \circ T_i$, where $\varphi_i \in \mathcal{C}_0^+(X)$, $T_i \in \mathcal{F}$ and $\varphi_0 = \sum_{i=1}^{n}\varphi_i$. Since $\mathcal{F}$ is a monoid, we have

(15) $\qquad \varphi_0 \in A,$

(16) $\qquad A$ is a convex subset of $\mathcal{C}_0(X)$ and

(17) $\qquad \forall\varphi \in A\ \forall T \in \mathcal{F} : \varphi \circ T \in A.$

By (i) of Lemma 2.1., we obtain

(18) $\qquad \forall f \in \mathcal{C}_0(X)\ \exists\varphi \in A\ \exists\alpha > 0 :\ |f| \leq \alpha|\varphi|.$

Property (iv) implies

(19) $\qquad \forall\varphi,\psi \in A\ \forall\alpha > 0 :\ \varphi \leq \alpha\psi \Rightarrow \alpha \geq 1.$

Let $p : \mathcal{C}_0(X) \to \langle 0,\infty)$ be defined as follows $p(f) = \inf\{\alpha : \exists\varphi \in A\ |f| \leq \alpha\varphi\}$. Then

(20) $\qquad p$ is a seminorm on $\mathcal{C}_0(X)$,

(21) $\qquad 0 \leq g \leq f \Rightarrow p(g) \leq p(f)$ for all $f, g \in \mathcal{C}_0(X)$,

(22) $\qquad p(\varphi) = 1$ for all $\varphi \in A$,

(23) $\qquad p(f \circ T) \leq p(f)$ for all $f \in \mathcal{C}_0(X)$ and $T \in \mathcal{F}$,

(24) $\qquad p(|f|) = p(f)$ for all $f \in \mathcal{C}_0(X)$ and

(25) $\qquad p$ is a norm.

Homogenity of p is obvious, subadditivity of p follows from (16). Relations (22) and (23) follow from (19) and (17) respectively. Relations (21) and (24) are obvious. We shall prove (25). Let $f \neq 0$. We may assume $f \in \mathcal{C}_0^+(X)$.

From (i) of Lemma 2.1. it follows that $\varphi_0 \leq a \cdot \sum_{i=1}^{n} f \circ T_i$ for some $a > 0$ and $T_1, ..., T_n \in \mathcal{F}$. Then we have $1 = p(\varphi_0) \leq p(a \cdot \sum_{i=1}^{n} f \circ T_i) \leq a \sum_{i=1}^{n} p(f \circ T_i) \leq a\, n\, p(f)$. Therefore, $p(f) \geq \frac{1}{n\ a}$.

Put $B = \{f : p(f) < 1\}$. Then we have two disjoint convex sets A and B such that B is open (with respect to the topology induced by the norm p). By Hahn-Banach theorem, there is a nontrivial linear functional $\Phi : C(X) \to \mathbb{R}$ such that $\Phi(f) \leq \Phi(\varphi)$ for all $f \in B$ and $\varphi \in A$. We may assume that

$$\sup_{f \in B} \Phi(f) = 1. \tag{26}$$

Then $\Phi(\varphi) \geq 1$ for all $\varphi \in A$. On the other hand $p((1-\varepsilon)\varphi) = 1-\varepsilon$ for $\varphi \in A$ and $\varepsilon \in (0,1)$ by (22). Therefore, $(1-\varepsilon)\varphi \in B$ and $(1-\varepsilon)\Phi(\varphi) = \Phi((1-\varepsilon)\varphi) \leq 1$. It means

$$\Phi(\varphi) = 1 \text{ for all } \varphi \in A. \tag{27}$$

Let $0 \leq f \leq \varphi_0$. Then $0 \leq \varphi_0 - f \leq \varphi_0$. Relations (15), (21) and (22) imply $p(\varphi_0 - f) \leq 1$. Then (26) implies $\Phi(\varphi_0 - f) \leq 1$. By (15) and (27), we have $1 = \Phi(\varphi_0) = \Phi(f) + \Phi(\varphi_0 - f)$. Therefore, $\Phi(f) = 1 - \Phi(\varphi_0 - f) \geq 1 - 1 = 0$, i.e.

$$\Phi(f) \geq 0.$$

Moreover, $f \circ T + (\varphi_0 - f) \in A$ for all $T \in \mathcal{F}$. By (27) we have $1 = \Phi(f \circ T + (\varphi_0 - f)) = \Phi(f \circ T) + \Phi(\varphi_0) - \Phi(f) = \Phi(f \circ T) + 1 - \Phi(f)$. Hence,

$$\Phi(f \circ T) = \Phi(f).$$

Let $g \in C_0^+(X)$ be arbitrary. Lemma 2.1. implies

$$g = \beta \sum_{i=1}^{n} f_i \circ T_i,$$

where $\beta > 0$, $0 \leq f_i \leq \varphi_0$ and $T_i \in \mathcal{F}$. Therefore,

$$\Phi(g) = \beta \sum_{i=1}^{n} \Phi(f_i \circ T_i) = \beta \sum_{i=1}^{n} \Phi(f_i) \geq 0$$

and

$$\Phi(g \circ T) = \beta \sum_{i=1}^{n} \Phi(f_i \circ T_i \circ T) = \beta \sum_{i=1}^{n} \Phi(f_i) = \Phi(g)$$

whenever $T \in \mathcal{F}$. Let $g \in \mathcal{C}_0(X)$ be arbitrary and $T \in \mathcal{F}$. Then

$$\Phi(g \circ T) = \Phi((g^+ - g^-) \circ T) = \Phi(g^+ \circ T) - \Phi((g^- \circ T) = \Phi(g^+) - \Phi(g^-) = \Phi(g).$$

So, Φ is a positive $\mathcal{F}$-invariant linear functional on $\mathcal{C}_0(X)$. There is a unique regular Borel measure μ on the space X such that

$$\Phi(g) = \int_X g d\mu \text{ for any } g \in \mathcal{C}_0(X),$$

see [2,p.240]. Obviously, the measure μ must be $\mathcal{F}$-invariant.

Now, we can give a topological characterization of nonunimodular locally compact topological groups.

Corollary 2.4. *A locally compact group G is nonunimodular if and only if there are open subset $U_1, ..., U_n$, compact subsets $K_1, ..., K_m$ of G such that*

$$\sum_{i=1}^{n} \chi_{U_i} > \sum_{j=1}^{m} \chi_{K_j} \text{ and } \sum_{i=1}^{n} \chi_{\tilde{U}_i} \leq \sum_{j=1}^{m} \chi_{\tilde{K}_j},$$

where $\tilde{U}_i = a_i U_i b_i$ and $\tilde{K}_j = c_j K_j d_j$ for some $a_i, b_i, c_j, d_j \in G$.

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Dept. of Mathematics
Matej Bel University
Tajovského 40
974 01 Banská Bystrica
SLOVAKIA

E-mail address: malicky@fpv.umb.sk

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DOMINATION IN PRODUCTS OF CIRCUITS

Bohdan Zelinka

Abstract. Three numerical invariants of graphs concerning the domination are considered, namely the domatic number, the doubly domatic number and the total domatic number of a graph. These invariants are investigated for Cartesian products of circuits. Such graphs are treated algebraically as Cayley graphs of direct products of finite cyclic groups.

In this paper we shall study the domatic number, the doubly domatic number and the total domatic numbers of graphs which are Cartesian products of circuits.

We shall consider finite undirected graphs without loops and multiple edges. By $V(G)$ we denote the vertex set of a graph G, by $N_G[v]$ the set consisting of v and of all vertices which are adjacent to v in G. By C_n we denote the circuit of length n. If $G_1, G_2, ..., G_n$ are graphs, then their Cartesian product $G_1 \times G_2 \times ... \times G_n$ is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2) \times ... \times V(G_n)$ and in which two vertices $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)$ are adjacent if and only if there exists an integer i such that $1 \leqq i \leqq n, x_i$ and y_i are adjacent in G_i and $x_j = y_j$ for all $j \in \{1, ..., n\} - \{i\}$.

A subset D of the vertex set $V(G)$ of a graph G is called dominating in G (or total dominating in G), if for each $x \in V(G) - D$ (or for each $x \in V(G)$ respectively) there exists a vertex $y \in D$ adjacent to x. The set S is called doubly dominating in G, if for each $x \in V(G) - D$ there exist two vertices y_1, y_2 in D which are adjacent to x.A domatic (or total domatic, or doubly domatic) partition of G is a partition of $V(G)$, all of whose classes are dominating (or total dominating, or doubly dominating respectively) sets of G. The minimum number of vertices of a dominating set in G is its domination number $\gamma(G)$, the maximum number of classes of a domatic partition of G is its domatic number $d(G)$. Analogously the total domination number $\gamma_t(G)$, the total domatic number $d_t(G)$, the doubly domination number $\gamma^2(G)$ and the doubly domatic number $d^2(G)$ are defined.

The domatic number was introduced by E. J. Cockayne and S. T. Hedetniemi in [1], the total domatic number by the same authors and R. M. Dawes in [2]. The doubly domatic number is a particular case of the k-ply domatic number introduced in [3].

We shall study Cartesian products of circuits. Let $G = H_1 \times H_2 \times ... \times H_n$, where $H_1, H_2, ..., H_n$ are circuits. The lengths of $H_1, H_2, ..., H_n$ will be $h_1, h_2, ..., h_n$

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respectively. The graph G may be considered as the Cayley graph of a direct product of finite cyclic groups of orders $h_1, h_2, ..., h_n$. (Among such direct products of groups there are all finite Abelian groups.)

We shall treat finite Abelian groups and thus we shall use the additive notation as it is usual in this case. The group operation is denoted by $+$ as an addition, the neutral element is denoted by 0 and called the zero element, the inverse element to x is denoted by $-x$.

Let A be a subset of a group $\mathcal{G}$ such that $0 \notin A$ and $x \in A$ implies $-x \in A$ for each $x \in \mathcal{G}$. The Cayley graph $G(\mathcal{G}, A)$ is the graph whose vertex set is $\mathcal{G}$ and in which two vertices x, y are adjacent if and only if $x - y \in A$.

By $\mathcal{H}$ we shall denote the Abelian group which is a direct product of finite cyclic subgroups $\mathcal{H}_1, ..., \mathcal{H}_n$. For $i = 1, ..., n$ let a_i be a generator of $\mathcal{H}_i$ and let h_i be its order. Each element of $\mathcal{H}$ can be expressed as $\sum_{i=1}^{n} \alpha_1 a_i$, where $\alpha_1, ..., \alpha_n$ are integers. The expressions $\sum_{i=1}^{n} \alpha_i a_i, \sum_{i=1}^{n} \beta_i a_i$ denote the same element of $\mathcal{H}$ if and only if $\alpha_i \equiv \beta_i \pmod{h_i}$ for $i = 1, ..., n$.

For each element of $\mathcal{H}$ there exists a unique expression $\sum_{i=1}^{n} \alpha_i a_i$ with $0 \leqq \alpha_i \leq h_i$ for $i = 1, ..., n$.

Let p be a positive integer. By $\mathcal{H}_0(p)$ we denote the subset of H consisting of the elements $\sum_{i=1}^{n} \alpha_i a_i$ such that $\sum_{i=1}^{n} i\alpha_i \equiv 0 \pmod{p}$. We shall prove a lemma.

Lemma 1. *The set $\mathcal{H}_0(p)$ is a subgroup of $\mathcal{H}$. If $h_i \equiv 0$ (mod p) for $i = 1, ..., n$, then the index of $\mathcal{H}_0(p)$ in $\mathcal{H}$ is p. In the case when p is a prime number and $n < p$, also the inverse implication holds.*

Proof. Evidently $\mathcal{H}_0(p)$ contains the zero element o of $\mathcal{H}$, for any two elements of $\mathcal{H}_0(p)$ their sum is in $\mathcal{H}_0(p)$ and for any element of $\mathcal{H}_0(p)$ its inverse is in $\mathcal{H}_0(p)$; therefore $\mathcal{H}_0(p)$ is a subgroup of $\mathcal{H}$. Suppose $h_i \equiv 0 \pmod{p}$ for $i = 1, ..., n$. If $\sum_{i=1}^{n} \alpha_i a_i = \sum_{i=1}^{n} \beta_i a_i$ then $\sum_{i=1}^{n} i\alpha_i \equiv \sum_{i=1}^{n} i\beta_i \pmod{p}$. For each integer j such that $0 \leqq j \leqq p-1$ the set of all elements $\sum_{i=1}^{n} \alpha_i a_i$ with $\sum_{i=1}^{n} ix_i \equiv j \pmod{p}$ is evidently a class of $\mathcal{H}$ by $\mathcal{H}_0(p)$ and the index of $\mathcal{H}_0(p)$ in $\mathcal{H}$ is p. Now let there exist $k \in \{1, ..., n\}$ such that h_k is not divisible by p. Suppose that p is a prime number and $n < p$. Then there exists a solution x of the congruence $px \equiv 1 \pmod{h_k}$. We have $pa_k \in \mathcal{H}_0(p)$ by the definition and also $xpa_k = a_k \in \mathcal{H}_0(p)$. Let $b = \sum_{i=1}^{n} \beta_i a_i$ be an arbitrary element of $\mathcal{H}$ and let $\sigma = \sum_{i}^{n} = 1i\beta_i$. As p is prime, there exists τ such that $k\tau \equiv \sigma \pmod p$. Let $c = b - \tau a_k$. Then $c = \sum_{i=1}^{n} \gamma_i a_i$, where $\gamma_k = \beta_k - \tau$ and $\gamma_i = \beta_i$ for $i \neq k$. We have $\sum_{i=1}^{n} i\gamma_i = \sum_{i=1}^{n} i\beta_i - k\tau \equiv 0 \pmod p$ and $c \in \mathcal{H}_0(p)$. As $a_k \in \mathcal{H}_0(p)$, also $\tau a_k \in \mathcal{H}_0(p)$ and $b = c + \tau a_k \in \mathcal{H}_0(p)$. As b was chosen arbitrarily, we have $\mathcal{H}_0(p) = \mathcal{H}$.

Now we prove a theorem.

Theorem 1.. *Let $G = H_1 \times ... \times H_n$, where $H_1, ..., H_2$ are circuits, let h_i be the length of H_i for $i = 1, ..., n$. If $h_i \equiv 0$ (mod $(2n+1)$) for $i = 1, ..., n$, then*

$$d(G) = 2n + 1,$$

$$\gamma(G) = (\Pi_{i=1}^{n} h_i)/(2n+1).$$

Proof. The graph G is a regular graph of degree $2n$, therefore by a result from [1] we have $d(G) \leqq 2n + 1$. Therefore it suffices to show a domatic partition of G having $2n+1$ classes. We may consider G as the Cayley graph $G(\mathcal{H}, A)$, where $\mathcal{H}$ is the above mentioned group and $A = \{a_1, ..., a_n, -a_1, ..., -a_n\}$ and its vertices may be considered as elements of $\mathcal{H}$. For $k = 0, ..., 2n$ put $D_k = \{\sum_{i=1}^{n} \alpha_i a_i \mid \sum_{i=1}^{n} i\alpha_i \equiv k$ (mod $(2n+1))\}$.

Denote $\mathcal{D} = \{D_{,} ..., D_{2n}\}$. The classes of $\mathcal{D}$ are classes of $\mathcal{H}$ by $\mathcal{H}_0(2n+1)$. We shall prove that D_0 is a dominating set in G. For each vertex $x = \sum_{i=1}^{n} \alpha_i a_i$ let $k(x)$ be the integer such that $0 \leqq k(x) \leqq 2n$ and $\sum_{i=1}^{n} i\alpha_i \equiv k(x)$ (mod $(2n+1)$): this number $k(x)$ is determined uniquely. If $k(x) = 0$, then $x \in D_0$. If $1 \leqq k(x) \leqq n$, then let $y = \sum_{i=1}^{n} \beta_i a_i$, where $\beta_{k(x)} = \alpha_{k(x)} - 1$ and $\beta_i = \alpha_i$ for $i \neq k(x)$. If $n+1 \leqq k(x) \leqq 2n$, then let $y = \sum_{i=1}^{n} \gamma_i a_i$, where $\gamma_{2n-k(x)+1} = \alpha_{2n-k(x)+1} + 1, \gamma_i = \alpha_j$ for $j \neq 2n - k(x) + 1$. In both the cases $y \in D_0$ and is adjacent to x. Analogously as for D_0 the proof can be done for any other class of $\mathcal{D}$. Therefore $\mathcal{D}$ is a domatic partition of G and $d(G) = 2n + 1$.

As G is regular of degree $2n$, each vertex of G is adjacent only to vertices of other classes of $\mathcal{D}$ than its own one and is not adjacent to two vertices of the same class. Therefore the system of sets $\{N[x] \mid x \in D_0\}$ is a partition of $V(G)$. As the number of vertices of G is $\Pi_{i=1}^{n} h_i$, we have $|D_0| = (\Pi_{i=1}^{n})/(2n+1)$ and evidently this is $\gamma(G)$. □

In the following theorem we shall consider only $n = 2$.

Theorem 2. *Let $G = H_1 \times H_2$, where H_1, H_2 are circuits of lengths h_1, h_2 respectively. Then the following two assertions are equivalent:*

(i) $h_1 \equiv 0 \pmod 5$ and $h_2 \equiv 0 \pmod 5$;

(ii) $d(G) = 5$ and $\gamma(G) = h_1h_2/5$.

Proof. The implication (i)$\Rightarrow$ (ii) follows from Theorem 1. Consider the groups $\mathcal{H}, \mathcal{H}_0(5)$ in this case. Let $\mathcal{D}$ be a domatic partition of G with 5 classes, let D_0 be the class of $\mathcal{D}$ which contains the vertex o (the zero of $\mathcal{H}$). As $d(G) = 5$, the closed neighbourhoods of any two distinct vertices of D_0 are disjoint and therefore any two distinct vertices of D_0 have distance at least 3 in G. Therefore the vertex $a_1 + a_2$ is not in D_0 and must be adjacent to a vertex of D_0. Such a vertex is neither a_1, or a_2 and therefore it is $a_1 + 2a_2$ or $2a_1 + a_2$. Suppos that $a_1 + 2a_2 \in D_0$. Also $-a_1 + a_2 \notin D_0$ and must be adjacent to a vertex of D_0. The vertices $-a_1, +a_2$ are adjacent to o and the vertex $2a, -a_2$ has the distance 2 from $a_1 + 2a_2$. Therefore $-2a_1 + a_2 \in D_0$. Silmilarly we prove that $-a_1 - 2a_2 \in D_0$ and $2a_1 - a_2 \in D_0$. Denote $b = a_1 + 2a_2, c = -2a_1 + a_2$. Therefore the assumption $o \in D_0$ and $b \in D_0$ implies $c \in D_0, -b \in D_0, -c \in D_0$. Analogously $b \in D$ and $o \in D$ implies $c \in D_0, -b \in D_0, -c \in D_0$. Now we may proceed further in such a way and prove that all elements of the subgroup $\mathcal{H}$ of generated by the elements b, c are in D_0. Here we use the symmetry of the graph G. We have $b \in \mathcal{H}_0(5), c \in \mathcal{H}_0(5)$. Any element of $\mathcal{H}_0(5)$ has the expression $\alpha_1 a_1 + \alpha_2 a_2$, where $\alpha_1 + 2\alpha_2 \equiv 0 \pmod 5$ and therefore it may be expressed as $\beta b + \gamma c$, where $\beta = (\alpha_1 + 2\alpha_2)/5, \gamma = \alpha_2 - 2(\alpha_1 + 2\alpha_2)/5$ and β, γ are integers, Therefore the subgroup of $\mathcal{H}$ generated by b, c is $\mathcal{H}_0(5)$. As $\mathcal{D}$ has to be a domatic partition, the index of $\mathcal{H}_0(5)$ in G must be 5 and (i) holds by Lemma 1. If $2a_1 + a_2 \in D_0$ instead of $a_1 + 2a_2 \in D_0$, then the proof is the same, only with interchanging a_1 and a_2. $\square$

Theorem 3. *Let G be the same graph as in Theorem 1. If $h_i \equiv 0$ (mod $(n+1)$) for $i = 1, ..., n$, then $d^2(G) = n + 1$.*

Proof. A result from [3] implies that $d^2(G) \leqq n + 1$. Therefore it suffices to show a doubly domatic partition $\mathcal{D}$ of G with $n + 1$ classes. For $k = 0, ..., n$ put $D_k = \{\sum_{i=1}^{n} \alpha_i a | \sum_{i=1}^{n} i\alpha_i \equiv k \pmod{(n+1)}\}$

Denote $\mathcal{D} = \{D_0, ..., D_n\}$. The classes of D are classes of $\mathcal{H}$ by $\mathcal{H}_0(n+1)$. We shall prove that D_0 is a doubly dominating set in G. For each vertex $x = \sum_{i=1}^{n} \alpha_i a_i$ let $k(x)$ be the integer such that $0 \leqq k(x) \leqq n$ and $\sum_{i=1}^{n} i\alpha_i \equiv k(x) \pmod{(n+1)}$: this number $k(x)$ is determined uniquely. If $k(x) = 0$, then $x \in D_0$. Otherwise let $y = \sum_{i=n}^{n} \beta_i a_i, z = \sum_{i=1}^{n} \gamma_i a_i$, where $\beta_{k(x)} - 1, \beta_j = \alpha_j$ for $j \neq k(x), \gamma_{n-k(x)+1} = \alpha_{n-k(x)+1} + 1, \gamma_j = \alpha_j$, for $j \neq n - k(x) + 1$. Evidently $y \in D_0, z \in D_0$ and both y, z are adjacent to x. Therefore D_0 is a doubly dominating set. Analogously as for D_0, the proof can be done for any other class of $\mathcal{D}$. Therefore $\mathcal{D}$ is a doubly domatic partition and $d^2(G) = n + 1$.$\square$

Lemma 2. *If $h_i \equiv 0$ (mod 2 $(n+1)$) for $i = 1, ..., n$, then there exists a doubly domatic partition $\mathcal{D} = \{D_0, ..., D_n\}$ in G such that for each $k = 0, ..., n$ the set D_k*

is the union of two disjoint non-empty sets D'_k, D''_k with the property that for each $x \in V(G) - D_k$ there exist vertices $y \in D'_k, z \in D''_k$ adjacent to x.

Proof. In this case we may take
$D'_k = \{\sum_{i=1}^{n} i\alpha_i \equiv k \pmod{2(n+1)}\}$ $D''_k = \{\sum_{i=1}^{n} \alpha_i a_i | \sum_{i=1}^{n} i\alpha_i \equiv n+1+k$ (mod $2(n+1)$)$\}$ for $k = 0, ..., n$. If we put $D_k = D'_k \cup D''_k$, then $\{D_0, ..., D_n\}$ is the doubly domatic partition from Theorem 3. □

With help of this lemma we prove the following theorem.

Theorem 4. *Let G be the same graph as in Theorem 1. If $h_n \equiv 0$ (mod 4) and $h_i \equiv 0$ (mod $2n$) for $i < n$, then $d_t(G) = 2n$.*

Poof. As G is regular of degree $2n$, by a result from [2] we have $d_t(G) \leqq 2n$. Therefore it suffices to show a total domatic partition of G having $2n$ classes. If $n = 1$, then this is $\mathcal{D} = \{D_), D_1\}$, where $D_0 = \{\alpha a_1 | \alpha \equiv 0 \pmod 4\} \cup \{\alpha a_1 | \alpha \equiv 1$ (mod 4)$\}, D_1 = \{\alpha a_1 | \alpha \equiv 2 \pmod 4\} \cup \{\alpha a_1 | \alpha \equiv 3 \pmod 4\}$. If $n \geqq 2$, then let $G_o = H_1 \times ... \times H_{n-1}$. By Lemma 2 there exists a doubly domatic partition $\tilde{\mathcal{D}} = \{\tilde{D}_0, ..., \tilde{D}_{n-1}\}$ of G with n classes such that for each $k = 0, ..., n-1$ the set $\tilde{D}_k$ is the union of two subsets $\tilde{D}'_k, \tilde{D}''_k$ such that for each vertex $x \in V(G_0) - \tilde{D}_k$ there ewist vertices $y \in \tilde{D}_k, z \in \tilde{D}''_k$ adjacent to x. Now for $k = 0, ..., n-1$ put

$$\begin{aligned} D_k = &\{b + \alpha a_n | b \in \tilde{D}'_k \ \& \ \alpha \equiv 0 (mod 4)\} \cup \{b + \alpha a_n | b \in \tilde{D}'_k \ \& \ \alpha \equiv 1 (mod 4)\} \cup \\ &\cup \{b + \alpha a_n | b \in \tilde{D}''_k \ \& \ \alpha \equiv 2 (mod 4)\} \cup \{b + \alpha a_n | b \in \tilde{D}''_k \ \& \ \alpha \equiv 3 (mod 4)\}, \\ D_{n+k} = &\{b + \alpha a_n | b \in \tilde{D}'_k \ \& \ \alpha \equiv 2 (mod 4)\} \cup \{b + \alpha a_n | b \in \tilde{D}'_k \ \& \ \alpha \equiv 3 (mod 4)\} \cup \\ &\cup \{b + \alpha a_n | b \in \tilde{D}''_k \ \& \ \alpha \equiv 0 (mod 4)\} \cup \{b + \alpha a_n | b \in \tilde{D}''_k \ \& \ \alpha \equiv 1 (mod 4)\}. \end{aligned}$$

We prove that D_0 is a total dominating set. Let $x = b + \alpha a_n$ be a vertex of G. If $b \in \tilde{D}'_0$ and $\alpha \equiv 0 \pmod 4$ or $\alpha \equiv 3 \pmod 4$, then x is adjacent to $b + (\alpha + 1)a_n \in D_0$. If $b \in \tilde{D}'_0$ and $\alpha \equiv 1 \pmod 4$, or $\alpha \equiv 2 \pmod 4$, then x is adjacent to $b + (\alpha - 1)a_n \in D_0$. Analogously for $b \in D''_0$. If $b \notin \tilde{D}_0$, then there exist $y \in \tilde{D}'_0$ and $z \in \tilde{D}''_0$ adjacent to x. If $\alpha \equiv 0 \pmod 4$ or $\alpha \equiv 1 \pmod 4$ then x is adjacent to $y + \alpha a_n \in D_0$. For other classes of $\mathcal{D} = \{D_0, ..., D_{2n-1}\}$ other than D_0 the proof is analogous. Therefore $\mathcal{D}$ is a total domatic partition of G.

At the end we prove again a theorem concerning only $n = 2$.

Theorem 5. *Let $G = H_1 \times H_2$, where H_1, H_2 are circuits of lengths h_1, h_2 respectively. If at least one of the numbers h_1, h_2 is divisible by 4, then $d(G) \geqq 4$.*

Remark. In such a case, if G satisfies the conditions of Theorem 1, then $d(G) = 5$, otherwise $d(G) = 4$.

Proof. Without loss of generality let h_1 be divisible by 4. We shall consider again the vertices of G as elements of a group and express them in the form $\alpha_1 a_1 + \alpha_2 a_z$, where $0 \leqq \alpha_1 \leqq h_1 - 1, 0 \leqq \alpha_2 \leqq h_2 - 1$. For each $k \in \{0, 1, 2, 3\}$ let $D_k = \{\alpha_1 a_1 + \alpha_2 a_2 | \alpha_1 + 2\alpha_2 \equiv k \pmod 4\}$. The reader may verify himself that $\mathcal{D} = \{D_0, D_1, D_2, D_3\}$ is a domatic partition of G and this $d(G) \geqq 4$.□

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Dept. of Applied Mathematics
Technical University of Liberec
Voroněžská 13
461 17 Liberec
CZECH REPUBLIC

E-mail adress: bohdan.zelinka@vslib.cz

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EXISTENCE OF INVARIANT TORI OF CRITICAL DIFFERENTIAL-EQUATION SYSTEMS DEPENDING ON MORE-DIMENSIONAL PARAMETER. PART II

Rudolf Zimka

Abstract. In the paper a system of differential equations depending on more-dimensional parameter is studied. It is supposed that the matrix of the first linear approximation P has m pairs of pure imaginary eigenvalues while the others do not lie on the imaginary axis. Conditions under which such a system in the cases when $m = 3,4$ has invariant tori are presented (in Part I the cases when $m = 1,2$ were analysed).

Introduction

Consider the system of differential equations

$$\dot{x} = X(x,\mu) + X^*(x,\mu), \tag{1}$$

where $x \in R^n, \mu \in R^d, \dot{x} = \frac{dx}{dt}, X(x,\mu)$ - a vector polynomial with respect to $x, \mu, X(0,0) = 0, X^*(x,\mu)$ - a continuous function in $\mathbb{M} = \{(x,\mu) : ||x|| < K, ||\mu|| < < L\}$ with the property:

$$X^*(\sqrt{\varepsilon}x, \varepsilon\mu_0) = (\sqrt{\varepsilon})^{3p+2}\tilde{X}(x,\varepsilon,\mu_0), \tag{2}$$

$\tilde{X}(x,\varepsilon,\mu_0)$ - a continuous function with respect to x, ε, μ_0 of the class $C^1_x(\mathbb{M}), \mu_0 = = \frac{\mu}{||\mu||}, 0 \le \varepsilon < L, p$ - a natural number.

It is supposed that:

1. the matrix $P = \frac{\partial X(0,0)}{\partial x}$ has m pairs of pure imaginary eigenvalues $\pm i\lambda_1, ..., \pm i\lambda_m$ and the others $\lambda_{2m+1}, ..., \lambda_n$ have non-zero real parts
2. det $P \neq 0$
3. $q_1\lambda_1 + ... + q_m\lambda_m \neq 0, 0 < |q| \le 3p+2, |q| = |q_1| + ... + |q_m|, q_i$ - integer numbers, $i = 1, ..., m$.

The bifurcation equation of the system (1) is (see [6]):

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(3) $$B\rho^2 + C\mu = 0,$$

$$\rho^2 = col(\rho_1^2, ..., \rho_m^2), \mu = col(\mu_1, ..., \mu_d),$$

$$B = \begin{pmatrix} B_{11} & ... & B_{1m} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ B_{m1} & ... & B_{mm} \end{pmatrix}, C = \begin{pmatrix} C_{11} & ... & C_{1d} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ C_{m1} & ... & C_{md} \end{pmatrix}.$$

Suppose that $\det B \neq 0$. From (3) we have on the beams $\delta(\mu_0) = \{\varepsilon\mu_0 : \mu = \frac{\mu}{||\mu||}, \mu \in \mathbb{M}, 0 \leq \varepsilon < L\}$:

$$\rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0),$$

$$\alpha^2(\mu_0) = \Lambda\mu_0, \Lambda = \begin{pmatrix} \alpha_{11} & ... & \alpha_{1d} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \alpha_{m1} & ... & \alpha_{md} \end{pmatrix}, \mu_0 = \frac{1}{||\mu||} col(\mu_1, ..., \mu_d)$$

(the notions and the notations in this article have the same meaning as in [6]).

It was shown in [6] that on the beams $\delta(\mu_0), \mu \in \mathcal{DP}$, the system (1) can be reduced to the system

$$\dot{x}_1 = \varepsilon X_1(x_1, \varepsilon, \mu_0) + X_1^0(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0) + (\sqrt{\varepsilon})^{3p+1}\tilde{X}_1(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0)$$

(4) $$\dot{\varphi}_1 = \lambda_1(\varepsilon) + \varepsilon\Phi_1(x_1, \varepsilon, \mu_0) + \Phi_1^0(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0) + (\sqrt{\varepsilon})^{3p+1}\tilde{\Phi}_1(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0)$$

$$\dot{\nu}_1 = J\nu_1 + V_1^0(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0) + (\sqrt{\varepsilon})^{3p+1}\tilde{V}_1(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0),$$

where X_1, Φ_1 - vector polynomials with respect to $x_1, \varepsilon, X_1(0, 0, \mu_0) = 0$, $\Phi_1(0, \varepsilon, \mu_0) = 0, \lambda_1(0) = \lambda = col(\lambda_1, ..., \lambda_m), X_1^0, \Phi_1^0, V_1^0, \tilde{X}_1, \tilde{\Phi}_1, \tilde{V}_1$ - continuous 2π - periodic with respect to φ_1 functions in the domain $\mathbb{M}_1 \{(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0) : x_1 \in R^m, ||x_1|| < K_1, \nu_1 \in R^{n-2m}, ||\nu_1|| < K_1, \varphi_1 \in R^m, 0 \leq \varepsilon < L, \mu \in \mathcal{DP}\}$ of the class $C^1_{x_1,\varphi_1,\nu_1}, X_1^0, \Phi_1^0, V_1^0$ - vanishing at $\nu_1 = 0, J$ - a Jordan canonical lower matrix.

It holds (see [1]):

(5) $$P_1(\mu_0) = \frac{\partial X_1(0, 0, \mu_0)}{\partial x_1} = 2[diag\ \alpha(\mu_0)]B[diag\ \alpha(\mu_0)].$$

Suppose that the domain of criticalness $\mathcal{DC}$ of the bifurcation equation (3) is non-empty set. Take $\mu \in \mathcal{DC}$. On the beam $\delta(\mu_0)$ the system (4) is the system with one dimensional positive parameter ε which was investigated in [1]. We can perform on the system (4) on the beam $\delta(\mu_0)$ the transformation procedure that was described in [1]. This procedure consists of p steps if the following conditions are satisfied:

1. $q_1\lambda_1^k + ... + q_{m_k}\lambda_{m_k}^k \neq 0, 0 < |q| \leq 3(p-k)+2$, where $\pm i\lambda_1^k, ..., \pm i\lambda_{m_k}^k$ are the pure imaginary eigenvalues of $P_k(\mu_0), k = 1, ..., p-1$.
2. $\det B_k \neq 0, \beta_k^2(\mu_0) = -B_k^{-1}C_k(\mu_0) > 0$, where $B_k, C_k(\mu_0)$ are the matrices

of the bifurcation equation $B_k\rho_k^2 + \varepsilon C_k(\mu_0) = 0$ arising at the $(k+1)^{st}$ step, $k = 1, ..., p-2$.

3. $P_{k+1}(\mu_0) = 2[diag\ \beta_k(\mu_0)]B_k[diag\ \beta_k(\mu_0)]$ is critical, $k = 1, ..., p-2$.

Performing this transformation procedure consisting of p steps on the beam $\delta(\mu_0)$ (the transformation of the system (1) to the system (4) is the 1^{st} step) the system (4) is reduced to the system

$$\dot{x}_p = \varepsilon^p X_p(x_p, \varepsilon, \mu_0) + X_p^0(x_p, \varphi_1, ..., \varphi_p, \nu_1, ..., \nu_p, \varepsilon, \mu_0) + \\ +\varepsilon^{p+1}\tilde{X}_p(x_p, \varphi_1, ..., \varphi_p, \nu_1, ..., \nu_p, \varepsilon, \mu_0)$$

$$(6) \quad \dot{\varphi}_k = \varepsilon^{k-1}\lambda_k(\varepsilon) + \varepsilon^p \Phi_k(x_k, \varepsilon, \mu_0) + \Phi_k^0(x_p, \varphi_1, ..., \varphi_p, \nu_1, ..., \nu_p, \varepsilon, \mu_0) +$$

$$+\varepsilon^{p+1}\tilde{\Phi}_k(x_p, \varphi_1, ..., \varphi_p, \nu_1, ..., \nu_p, \varepsilon, \mu_0)$$

$$\dot{\nu}_k = \varepsilon^{k-1}J_{k-1}\nu_k + V_k^0(x_p, \varphi_1, ..., \varphi_p, \nu_1, ..., \nu_p, \varepsilon, \mu_0) + \\ +(\sqrt{\varepsilon})^{3p+2-k}\tilde{V}_k(x_p, \varphi_1, ..., \varphi_p, \nu_1, ..., \nu_p, \varepsilon, \mu_0), k = 1, ..., p,$$

where X_p, Φ_k - polynomials with respect to $x_p, \varepsilon, X_p(0,0,\mu_0) = 0, \Phi_k(0,\varepsilon,\mu_0) = = 0, \lambda_k(0) = \lambda^{k-1} = col(\lambda_1^{k-1}, ..., \lambda_{m_{k-1}}^{k-1}), \pm i\lambda_1^{k-1}, ..., \pm i\lambda_{m_{k-1}}^{k-1}$ - the eigenvalues of the matrix $P_{k-1}, \lambda^0 = \lambda, m_0 = m, P_0 = P, X_p^0, \Phi_k^0, V_k^0, \tilde{X}_p, \tilde{\Phi}_k, \tilde{V}_k$ - continuous functions 2π - periodic with respect to $\varphi_1, ..., \varphi_p$ in the domain $\mathbb{M}_p = \{(x_p, \varphi_1, ..., \varphi_p, \nu_1, ..., \nu_p, \varepsilon) : ||x_p|| < K_p, ||\nu_k|| < K_p, \varphi_k \in R^{m_{k-1}}, k = 1, ..., p, 0 \leq \varepsilon < L\}$ of the class C^1 with respect to all variables with the exception of ε, X_p^0, Φ_k^0, V_k^0 - vanishing at $\nu_1 = ... = \nu_p = 0, P_p = \frac{\partial X_p(0,0,\mu_0)}{\partial x_p}$ - regular matrice, J_{k-1} - non-critical Jordan matrices, $J_0 = J$.

In this article the existence of invariant tori of the system (1) is studied in the cases when the matrix P has three and four pairs of pure imaginary eigenvalues.

1. Three pairs of pure imaginary eigenvalues

Suppose that the matrix P of the system (1) has three pairs of pure imaginary eigenvalues $\pm i\lambda_1, \pm i\lambda_2, \pm i\lambda_3$ and the others $\lambda_7, ..., \lambda_n$ have non-zero real parts.

The bifurcation equation (3) is:

$$(1.1) \qquad B\rho^2 + C\mu = 0,$$

where $\rho^2 = col(\rho_1^2, \rho_2^2, \rho_3^2), \mu = col(\mu_1, ..., \mu_d)$,

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}, C = \begin{pmatrix} C_{11} & ... & C_{1d} \\ C_{21} & ... & C_{2d} \\ C_{31} & ... & C_{3d} \end{pmatrix}.$$

Suppose that $\det B \neq 0$. Take $\mu \in \mathbb{M}$ and consider the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 \leq \leq \varepsilon < L\}$. The solution of (1.1) with respect to ρ^2 on the beam $\delta(\mu_0)$ is:

$$(1.2) \qquad \rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0),$$

$$\alpha^2(\mu_0) = \begin{pmatrix} \alpha_1^2(\mu_0) \\ \alpha_2^2(\mu_0) \\ \alpha_3^2(\mu_0) \end{pmatrix} = \Lambda\mu_0, \Lambda = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & ... & \alpha_{1d} \\ \alpha_{21} & ... & \alpha_{2d} \\ \alpha_{31} & ... & \alpha_{3d} \end{pmatrix}.$$

The matrix $P_1(\mu_0)$ which is defined by (5) has the form:

$$P_1(\mu_0) = 2\begin{pmatrix} \alpha_1^2(\mu_0)B_{11} & \alpha_1(\mu_0)\alpha_2(\mu_0)B_{12} & \alpha_1(\mu_0)\alpha_3(\mu_0)B_{13} \\ \alpha_1(\mu_0)\alpha_2(\mu_0)B_{21} & \alpha_2^2(\mu_0)B_{22} & \alpha_2(\mu_0)\alpha_3(\mu_0)B_{23} \\ \alpha_1(\mu_0)\alpha_3(\mu_0)B_{31} & \alpha_2(\mu_0)\alpha_3(\mu_0)B_{32} & \alpha_3^2(\mu_0)B_{33} \end{pmatrix},$$

where $\alpha_i(\mu_0) = \sqrt{\frac{1}{||\mu||}(\alpha_{i1}\mu_1 + ... + \alpha_{id}\mu_d)}$, $i = 1,2,3$.

Denote the rank of the matrix Λ in (1.2) by the symbol $h(\Lambda)$ and the domain of positiveness and the domain of criticalness of the bifurcation equation (1.1) by the symbols $\mathcal{D}P$ and $\mathcal{D}C$.

Lemma 1.1. *Let be $h(\Lambda) = 1$. Then $\mathcal{D}P \neq \emptyset$ if and only if $\alpha_1 \neq 0, \alpha_i = k_i\alpha_1, k_i >$ $> 0,\ i = 2,3$.*

Proof. $\mathcal{D}P$ of (1.1) is determined by the inequalities:

$$\alpha_1^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d) > 0$$

(1.3)
$$\alpha_2^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{21}\mu_1 + ... + \alpha_{2d}\mu_d) > 0$$

$$\alpha_3^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{31}\mu_1 + ... + \alpha_{3d}\mu_d) > 0.$$

The first inequality in (1.3) is satisfied at all parameters $\mu \in \mathbb{M}$ which belong to that half-sphere of the sphere $O = \{\mu = (\mu_1, ..., \mu_d) : 0 < ||\mu|| < L\}$ that is determined by the hyperplane $\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d = 0$ and by a point $\mu^* \in O$ at which $\alpha_1^2(\mu^*) > 0$. As $h(\Lambda) = 1$ and $\alpha_1 \neq 0$ so there exist $k_2 \in R, k_3 \in R$ such that $\alpha_2 = k_2\alpha_1, \alpha_3 = k_3\alpha_1$. Using this we can express the second and the third inequality in (1.3) in the form: $\frac{k_2}{||\mu||}(\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d) > 0, \frac{k_3}{||\mu||}(\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d) > 0$. From these inequalities it follows that $\mathcal{D}P \neq \emptyset$ only if $k_2 > 0, k_3 > 0$. If $\alpha_1 = 0$ then $\mathcal{D}P = \emptyset$. The proof is over.

Lemma 1.2. *Let be $h(\Lambda) = 2$. Let $\alpha_{i_1}, \alpha_{i_2}$ be the linear independent pair from the triad $\{\alpha_1, \alpha_2, \alpha_3\}$. If for the third member α_{i_3} from this triad it holds: $\alpha_{i_3} =$ $= k_1\alpha_{i_1} + k_2\alpha_{i_2}, k_1 \geq 0, k_2 \geq 0, k_1 + k_2 > 0$, then $\mathcal{D}P \neq \emptyset$.*

The proof of Lemma 1.2. is similar to the proof of Lemma 1.1.

Lemma 1.3. *Let be $h(\Lambda) = 3$. Then $\mathcal{D}P \neq \emptyset$.*

Proof. The set $\mathcal{D}P$ consists of those parameters $\mu \in \mathbb{M}$ which satisfy the inequalities (1.3). Solving them we get:

$$\begin{aligned}
&\alpha_{11}\mu_1+...+\alpha_{1d}\mu_d-t_1 &&=0\\
(1.4)\qquad &\alpha_{21}\mu_1+...+\alpha_{2d}\mu_d \quad -t_2 &&=0\\
&\alpha_{31}\mu_1+...+\alpha_{3d}\mu_d \qquad -t_3 &&=0,\ t_i>0, i=1,2,3.
\end{aligned}$$

As $h(\Lambda)=3$ the system (1.4) has solutions with d parameters, $d\geq 3$. Among these parameters the variables t_1,t_2,t_3 can always be. Those parameters $\mu\in\mathbb{M}$ corresponding to positive numbers t_1,t_2,t_3 create $\mathcal{D}P$. The proof is over.

Denote

$$\begin{aligned}
&a_1(\mu_0)=\alpha_1^2(\mu_0)B_{11}+\alpha_2^2(\mu_0)B_{22}+\alpha_3^2(\mu_0)B_{33}\\
(1.5)\quad &a_2(\mu_0)=\alpha_1^2(\mu_0)\alpha_2^2(\mu_0)|M_{33}|+\alpha_1^2(\mu_0)\alpha_3^2(\mu_0)|M_{22}|+\alpha_2^2(\mu_0)\alpha_3^2(\mu_0)|M_{11}|\\
&a_3(\mu_0)=\alpha_1^2(\mu_0)\alpha_2^2(\mu_0)\alpha_3^2(\mu_0)\quad \det B,
\end{aligned}$$

where $|M_{ii}|$ is the minor of the element B_{ii} of $\det B$, $i=1,2,3, \mu\in\mathcal{D}P$.

Lemma 1.4. *The matrix $P_1(\mu_0)$ is critical at $\mu\in\mathcal{D}P$ if and only if the following two conditions are satisfied:*

1. $a_1(\mu_0)a_2(\mu_0)=a_3(\mu_0)$

2. $a_2(\mu_0)>0$.

The eigenvalues $\pm i\lambda_1^1,\lambda_3^1$ of the matrix $P_1(\mu_0)$ are defined by the formulae:

$$\lambda_1^1=2\sqrt{a_2(\mu_0)}, \lambda_3^1=2a_1(\mu_0).$$

Proof. If λ is the eigenvalue of $P_1(\mu_0)$ then $\tilde{\lambda}=\frac{\lambda}{2}$ is the eigenvalue of $\frac{P_1(\mu_0)}{2}$. The characteristic equation of the matrix $\frac{P_1(\mu_0)}{2}$ is:

$$(1.6)\qquad \lambda^3-a_1(\mu_0)\lambda^2+a_2(\mu_0)\lambda-a_3(\mu_0)=0,$$

where $a_1(\mu_0),a_2(\mu_0),a_3(\mu_0)$ have the form (1.5). Comparing (1.6) with its expression by means of the roots $\pm i\tilde{\lambda}_1^1,\tilde{\lambda}_3$ of $\frac{P_1(\mu_0)}{2}$ what is

$$\lambda^3-\tilde{\lambda}_3^1\lambda^2+(\tilde{\lambda}_1^1)^2\lambda-(\tilde{\lambda}_1^1)^2\tilde{\lambda}_3^1=0,$$

we have:

$$a_1(\mu_0)=\tilde{\lambda}_3^1, a_2(\mu_0)=(\tilde{\lambda}_1^1)^2, a_3(\mu_0)=(\tilde{\lambda}_1^1)^2\tilde{\lambda}_3^1.$$

From this we get the assertion of lemma. The proof is over.

Lemma 1.5. *Let be $h(\Lambda)=1$ and $\mathcal{D}P\neq\emptyset$. Then $\mathcal{D}C=\emptyset$ or $\mathcal{D}C\equiv\mathcal{D}P$.*

Proof. When $\mathcal{D}P\neq\emptyset$ then according to Lemma 1.1 $\alpha_2=k_2\alpha_1,\alpha_3=k_3\alpha_1$, $k_2>0,k_3>0$. The expressions $a_1(\mu_0),a_2(\mu_0),a_3(\mu_0)$ from (1.5) can be expressed in the following way:

$$
\begin{aligned}
a_1(\mu_0) &= \alpha_1^2(\mu_0)(B_{11} + k_2 B_{22} + k_3 B_{33}) \\
a_2(\mu_0) &= \alpha_1^4(\mu_0)(k_2|M_{33}| + k_3|M_{22}| + k_2 k_3|M_{11}|) \qquad (1.7)\\
a_3(\mu_0) &= \alpha_1^6(\mu_0) k_2 k_3 \det B.
\end{aligned}
$$

According to Lemma 1.4 the conditions for criticalness are:

1. $a_1(\mu_0)a_2(\mu_0) = a_3(\mu_0)$.
2. $a_2(\mu_0) > 0$.

Putting the expressions (1.7) into these conditions we get the conditions for criticalness which do not depend on $\mu \in \mathcal{D}P$:

(1.8)

1. $(B_{11} + k_2 B_{22} + k_3 B_{33})(k_2|M_{33}| + k_3|M_{22}| + k_2 k_3|M_{11}|) = k_2 k_3 \det B$.
2. $k_2|M_{33}| + k_3|M_{22}| + k_2 k_3|M_{11}| > 0$.

Suppose that $\mathcal{D}C \neq \emptyset$ and take $\mu^* \in \mathcal{D}C$. This means that the conditions (1.8) are satisfied at $\mu^* \in \mathcal{D}C$ and as they do not depend on $\mu \in \mathcal{D}P$ they are satisfied at every $\mu \in \mathcal{D}P$. The proof is over.

Consider now $\mathcal{D}P$ and $\mathcal{D}C$ of the bifurcation equation (1.1) and suppose that $\mathcal{D}P \neq \emptyset$. Then on $\mathcal{D}P$ the system (1) can be reduced to the system (4) with $x_1 \in R^3, \varphi_1 \in R^3, \nu_1 \in R^{n-6}$.

Theorem 1.1. *Let be $\mathcal{D}P \neq \emptyset$. Then to every small enough $\mu \in \mathcal{D}P \backslash \mathcal{D}C$ there exists the invariant manifold*

$$
\begin{aligned}
x_1 &= ||\mu||\eta(\varphi_1, ||\mu||, \mu_0) \\
\nu_1 &= ||\mu||^2\Theta_1(\varphi_1, ||\mu||, \mu_0),
\end{aligned}
\qquad \textit{(1.9)}
$$

where η, Θ_1 are continuous functions 2π - periodic in all components of $\varphi_1, \varphi_1 \in$ $\in R^3, x_1 \in R^3, \nu_1 \in R^{n-6}$. The natural number p can be taken $p = 1$.

Proof. Consider an arbitrary $\mu \in \mathcal{D}P \backslash \mathcal{D}C$. This parameter μ lies on the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 < \varepsilon < L\}$. On this beam the system (1) can be reduced to the system (4) what is the system with one positive parameter ε. According to Theorem from Section 3 of Chapter 1 in [1] the invariant manifold (1.9) exists. The proof is over.

Suppose that $\mu \in \mathcal{D}C$ of the bifurcation equation (1.1). On the beam $\delta(\mu_0)$ we can perform the second step of the transformation procedure. The bifurcation equation of the system (4) on the beam $\delta(\mu_0)$is

$$
B_1(\mu_0)\rho_1^2 + \varepsilon C_1(\mu_0) = 0,
$$

where $B_1(\mu_0) \in R$.

Assume that $B_1(\mu_0) \neq 0$ and $\beta_1^2(\mu_0) = -\frac{1}{B_1(\mu_0)}C_1(\mu_0) > 0$. Then the system (4) can be reduced to the system (6) with $p = 2, x_1 \in R, \varphi_1 \in R^3, \varphi_2 \in R, \nu_1 \in$ $\in R^{n-6}, \nu_2 \in R$ and $P_2(\mu_0) = 2\beta_1^2(\mu_0)B_1(\mu_0) \neq 0$.

Utilizing Theorem from Section 3 of Chapter 1 in [1] the following theorem can be formulated.

Theorem 1.2. *Let be $\mu \in \mathcal{D}C$. If $B_1(\mu_0) \neq 0$ and $\beta_1^2(\mu_0) > 0$ then to every small enough $\mu \in \delta(\mu_0)$ there exists the invariant manifold*

$$\begin{aligned} x_2 &= ||\mu||\eta(\varphi_1, \varphi_2, ||\mu||, \mu_0) \\ \nu_1 &= ||\mu||^3\Theta_1(\varphi_1, \varphi_2, ||\mu||, \mu_0) \\ \nu_2 &= ||\mu||^2\Theta_2(\varphi_1, \varphi_2, ||\mu||, \mu_0), \end{aligned}$$

where η, Θ_1, Θ_2 are continuous functions 2π - periodic in all components of φ_1, φ_2, $\varphi_1 \in R^3, \varphi_2 \in R, x_2 \in R, \nu_1 \in R^{n-6}, \nu_2 \in R$. The natural number p has the value $p = 2$.

2. Four pairs of pure imaginary eigenvalues

Suppose that the matrix P of the system (1) has four pairs of pure imaginary eigenvalues $\pm i\lambda_1, \pm i\lambda_2, \pm i\lambda_3, \pm i\lambda_4$ and the others $\lambda_9, ..., \lambda_n$ have non-zero real parts.

The bifurcation equation (3) of the system (1) is:

$$B\rho^2 + C\mu = 0, \tag{2.1}$$

where $\rho^2 = col(\rho_1^2, \rho_2^2, \rho_3^2, \rho_4^2), \mu = col(\mu_1, ..., \mu_d)$,

$$B = \begin{pmatrix} B_{11} & ... & B_{14} \\ \dots & \dots & \dots \\ B_{41} & ... & B_{44} \end{pmatrix}, C = \begin{pmatrix} C_{11} & ... & C_{1d} \\ \dots & \dots & \dots \\ C_{41} & ... & C_{1d} \end{pmatrix}.$$

Suppose that $\det B \neq 0$. Take $\mu \in \mathbb{M}$ and consider the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 \leq \leq \varepsilon < L\}$. The solution of (2.1) with respect to ρ^2 on the beam $\delta(\mu_0)$ is:

$$\rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0), \tag{2.2}$$

where $\alpha^2(\mu_0) = col(\alpha_1^2(\mu_0), ..., \alpha_4^2(\mu_0)) = \Lambda\mu_0, \Lambda = col(\alpha_1, ..., \alpha_4) =$

$$= \begin{pmatrix} \alpha_{11} & ... & \alpha_{1d} \\ \dots & \dots & \dots \\ \alpha_{41} & ... & \alpha_{4d} \end{pmatrix}.$$

The matrix $P_1(\mu_0)$ which is defined by (5) has the form:

$$P_1(\mu_0) = 2\begin{pmatrix} \alpha_1^2(\mu_0)B_{11} & \alpha_1(\mu_0)\alpha_2(\mu_0)B_{12} & \alpha_1(\mu_0)\alpha_3(\mu_0)B_{13} & \alpha_1(\mu_0)\alpha_4(\mu_0)B_{14} \\ \alpha_1(\mu_0)\alpha_2(\mu_0)B_{21} & \alpha_2^2(\mu_0)B_{22} & \alpha_2(\mu_0)\alpha_3(\mu_0)B_{23} & \alpha_2(\mu_0)\alpha_4(\mu_0)B_{24} \\ \alpha_1(\mu_0)\alpha_3(\mu_0)B_{31} & \alpha_2(\mu_0)\alpha_3(\mu_0)B_{32} & \alpha_3^2(\mu_0)B_{33} & \alpha_3(\mu_0)\alpha_4(\mu_0)B_{34} \\ \alpha_1(\mu_0)\alpha_4(\mu_0)B_{41} & \alpha_2(\mu_0)\alpha_4(\mu_0)B_{42} & \alpha_3(\mu_0)\alpha_4(\mu_0)B_{43} & \alpha_4^2(\mu_0)B_{44} \end{pmatrix},$$

where $\alpha_i(\mu_0) = \sqrt{\frac{1}{||\mu||}(\alpha_{i1}\mu_1 + ... + \alpha_{id}\mu_d)}, \quad i = 1, 2, 3, 4.$

The following 4 lemmas say how the existence of $\mathcal{D}P$ of the bifurcation equation (2.1) depends on the rank of the matrix Λ from (2.2). The proofs of these lemmas can be performed in the same way as they were done in the lemmas 1.1 - 1.3 of the section 1.

Lemma 2.1. *Let be $h(\Lambda)=1$. Then $\mathcal{D}P\neq\emptyset$ if and only if $\alpha_1\neq 0, \alpha_i=k_i\alpha_1$, $k_i>0, i=2,3,4$.*

Lemma 2.2. *Let be $h(\Lambda)=2$. Let $\alpha_{i_1},\alpha_{i_2}$ be a linearly independent pair from the tetrad $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$. If for the third and the fourth members of this tetrad is holds:*

$$\alpha_{i_3}=k_1\alpha_{i_1}+k_2\alpha_{i_2}, \alpha_{i_4}=k_3\alpha_{i_1}+k_4\alpha_{i_2},$$

$$k_i\geq 0, i=1,2,3,4, \ \ k_1+k_2>0, k_3+k_4>0,$$

then $\mathcal{D}P\neq\emptyset$.

Lemma 2.3. *Let be $h(\Lambda)=3$. Let $\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}$ be a linearly independent triad from the tetrad $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$. If for the fourth member α_{i_4} of this tetrad it holds:*

$$\alpha_{i_4}=k_1\alpha_{i_1}+k_2\alpha_{i_2}+k_3\alpha_{i_3}, k_i\geq 0, i=1,2,3, k_1+k_2, k_3>0,$$

then $\mathcal{D}P\neq\emptyset$.

Lemma 2.4. *Let be $h(\Lambda)=4$. Then $\mathcal{D}P\neq\emptyset$.*

Now we shall deal with the question when the matrix $P_1(\mu_0)$ is critical. As at every $\mu\in\mathcal{D}P$ det $P_1(\mu_0)=\det\{2[diag\ \ \alpha(\mu_0)]B[diag\ \ \alpha(\mu_0)]\}\neq 0$ the eigenvalues of $P_1(\mu_0)$ are different from zero. Therefore $P_1(\mu_0)$ is critical at $\mu\in\mathcal{D}P$ only when its eigenvalues are one of the following kinds:

A. $\pm i\lambda_1^1, \pm i\lambda_2^1$
B. $\pm i\lambda_1^1, \lambda_3^1, \lambda_4^1=-\lambda_3^1$
C. $\pm i\lambda_1^1, \lambda_3^1, Re\lambda_3^1\neq 0, \lambda_4^1\neq -\lambda_3^1$.

Consider the characteristic equation of the matrix $\frac{P_1(\mu_0)}{2}, \mu\in\mathcal{D}P$:

$$\lambda^4-a_1(\mu_0)\lambda^3+a_2(\mu_0)\lambda^2-a_3(\mu_0)\lambda+a_4(\mu_0)=0, \tag{2.3}$$

where $a_1(\mu_0)=Tr\frac{P_1(\mu_0)}{2}, a_2(\mu_0)$ - the sum of all principal minors of order 2 of $\frac{P_1(\mu_0)}{2}, a_3(\mu_0)$ - the sum of all principal minors of order 3 of $\frac{P_1(\mu_0)}{2}, a_4(\mu_0)=$ $=\det\frac{P_1(\mu_0)}{2}$.

Lemma 2.5. *The matrix $P_1(\mu_0)$ has at $\mu\in\mathcal{D}P$ the eigenvalues $\pm i\lambda_1^1, \pm i\lambda_2^1$ if and only if*

$$a_1(\mu_0)=0, a_2(\mu_0)>0, a_3(\mu_0)=0, a_4(\mu_0)>0, a_2^2(\mu_0)\geq 4a_4(\mu_0). \tag{2.4}$$

The values λ_1^1,λ_2^1 are determined by the formulae:

$$\begin{aligned}\lambda_1^1&=\sqrt{2}\sqrt{|-a_2(\mu_0)+\sqrt{a_2^2(\mu_0)-4a_4(\mu_0)}|},\\ \lambda_2^1&=\sqrt{2}\sqrt{|-a_2(\mu_0)-\sqrt{a_2^2(\mu_0)-4a_4(\mu_0)}|}.\end{aligned} \tag{2.5}$$

Proof. Comparing (2.3) with its expression by means of its roots $\pm i\tilde{\lambda}_1^1, \pm i\tilde{\lambda}_2^1$ what is

$$\lambda^4 + [(\tilde{\lambda}_1^1)^2 + (\tilde{\lambda}_2^1)^2]\lambda^2 + (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_2^1)^2 = 0, \tag{2.6}$$

we get the assertion (2.4). The roots of the equation (2.6) using the notation $a_2(\mu_0) = (\tilde{\lambda}_1^1)^2 + (\tilde{\lambda}_2^1)^2, a_4(\mu_0) = (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_2^1)^2$ are determined by the equation $\lambda^4 + +a_2(\mu_0)\lambda^2 + a_4(\mu_0) = 0$. Putting $u = \lambda^2$ we have: $u^2 + a_2(\mu_0)u + a_4(\mu_0) = 0$. The roots of this equation are given by the formula: $u_{12} = \frac{-a_2(\mu_0)\pm\sqrt{a_2^2(\mu_0)-4a_4(\mu_0)}}{2}$.

From this we have:

$$\lambda_1 = \pm i\frac{\sqrt{2}}{2}\sqrt{|-a_2(\mu_0) + \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}|},$$

$$\lambda_2 = \pm i\frac{\sqrt{2}}{2}\sqrt{|-a_2(\mu_0) - \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}|}.$$

Taking into account that $\lambda_i^1 = 2\tilde{\lambda}_i^1, i = 1, 2$, we get the assertion (2.5). The proof is over.

Note 2.1. If follows from (2.5) that when $a_2^2(\mu_0) = 4a_4(\mu_0)$ then $\lambda_1^1 = \sqrt{2a_2(\mu_0)}, \lambda_2^1 = \sqrt{2a_2(\mu_0)}$. This means that the eigenvalues $\pm i\lambda_1^1$ have the multiplicity two.

Lemma 2.6. *The matrix $P_1(\mu_0)$ has at $\mu \in \mathcal{D}P$ the eigenvalues $\pm i\lambda_1^1, \lambda_3^1, \lambda_4^1 = -\lambda_3^1$ if and only if*

$$a_1(\mu_0) = 0, a_3(\mu_0) = 0, a_4(\mu_0) < 0. \tag{2.7}$$

The values λ_1^1, λ_3^1 are determined by the formulae:

$$\lambda_1^1 = \sqrt{2}\sqrt{a_2(\mu_0) + \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}},$$

(2.8)

$$\lambda_2^1 = \sqrt{2}\sqrt{-a_2(\mu_0) + \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}}.$$

Proof. Comparing (2.3) with its expression by means of its roots $\pm i\tilde{\lambda}_1^1, \tilde{\lambda}_3^1, \tilde{\lambda}_4^1 = -\tilde{\lambda}_3^1$ what is

$$\lambda^4 + [(\tilde{\lambda}_1^1)^2 - (\tilde{\lambda}_3^1)^2]\lambda^2 - (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_3^1)^2 = 0, \tag{2.9}$$

we get the assertion (2.7). Putting $u = \lambda^2$ we have from (2.9): $u^2 + a_2(\mu_0)u + +a_4(\mu_0) = 0, a_2(\mu_0) = (\tilde{\lambda}_1^1)^2 - (\tilde{\lambda}_3^1)^2, a_4(\mu_0) = -(\tilde{\lambda}_1^1)^2(\tilde{\lambda}_3^1)^2$. The roots of this equation are given by the formula: $u_{12} = \frac{-a_2(\mu_0)\pm\sqrt{a_2^2(\mu_0)-4a_4(\mu_0)}}{2}$.From this we have:

$$\lambda_1 = \pm i\frac{\sqrt{2}}{2}\sqrt{|-a_2(\mu_0) - \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}|},$$

$$\lambda_2 = \frac{\sqrt{2}}{2}\sqrt{-a_2(\mu_0) + \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}}.$$

This gives (2.8). The proof is over.

Lemma 2.7. *The matrix $P_1(\mu_0)$ has at $\mu \in \mathcal{D}P$ the eigenvalues $\pm i\lambda_1^1, \lambda_3^1, \lambda_4^1$, $Re\lambda_3^1 \neq 0, \lambda_4^1 \neq -\lambda_3^1$, if and only if the following conditions are satisfied:*

(2.10)
$$a_1(\mu_0) \neq 0, a_3(\mu_0) \neq 0, a_1(\mu_0)a_3(\mu_0) > 0,$$

$$a_4(\mu_0) = \frac{a_2(\mu_0)a_3(\mu_0)}{a_1(\mu_0)} - \frac{a_3(\mu_0)}{a_1(\mu_0)}^2.$$

The values of $\lambda_1^1, \lambda_3^1, \lambda_4^1$ are given by the formulae:

$$\lambda_1^1 = 2\sqrt{\frac{a_3(\mu_0)}{a_1(\mu_0)}},$$

(2.11)
$$\lambda_3^1 = a_1(\mu_0) + \sqrt{a_1^2(\mu_0) - 4[a_2(\mu_0) - \frac{a_3(\mu_0)}{a_1(\mu_0)}]},$$

$$\lambda_4^1 = a_1(\mu_0) - \sqrt{a_1^2(\mu_0) - 4[a_2(\mu_0) - \frac{a_3(\mu_0)}{a_1(\mu_0)}]}.$$

Proof. Comparing (2.3) with its expression by means of its roots $\pm i\tilde{\lambda}_1^1, \tilde{\lambda}_3^1, \tilde{\lambda}_4^1$ what is $\lambda^4 - (\tilde{\lambda}_3^1 + \tilde{\lambda}_4^1)\lambda^3 + [(\tilde{\lambda}_1^1)^2 + \tilde{\lambda}_3^1\tilde{\lambda}_4^1]\lambda^2 - (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_3^1 + \tilde{\lambda}_4^1)\lambda + (\tilde{\lambda}^1)^2\tilde{\lambda}_3^1\tilde{\lambda}_4^1 = 0$, we have:

(2.12)
$$a_1(\mu_0) = \tilde{\lambda}_3^1 + \tilde{\lambda}_4^1, a_2(\mu_0) = (\tilde{\lambda}_1^1)^2 + \tilde{\lambda}_3^1\tilde{\lambda}_4^1,$$

$$a_3(\mu_0) = (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_3^1 + \tilde{\lambda}_4^1), a_4(\mu_0) = (\tilde{\lambda}_1^1)^2\tilde{\lambda}_3^1\tilde{\lambda}_4^1.$$

From (2.12) we get the assertion (2.10). Solving (2.12) with respect to $\tilde{\lambda}_1^1, \tilde{\lambda}_3^1, \tilde{\lambda}_4^1$ and taking into account the relation between the eigenvalues of the matrices $P_1(\mu_0)$ and $\frac{P_1(\mu_0)}{2}$ we get the assertion (2.11). The proof is over.

Lemma 2.8. *Let be $h(\Lambda) = 1$ and $\mathcal{D}P \neq \emptyset$. Then $\mathcal{D}C = \emptyset$ or $\mathcal{D}C \equiv \mathcal{D}P$.*

The proof of this lemma is similar to the proof of Lemma 1.5.

Consider $\mathcal{D}P$ and $\mathcal{D}C$ of the bifurcation equation (2.1). Suppose that $\mathcal{D}P$ is non-empty set. Then on $\mathcal{D}P$ the system (1) can be reduced to the system (4) with $x_1 \in R^4, \varphi_1 \in R^4, \nu_1 \in R^{n-8}, p = 1$.

Theorem 2.1. *Let be $\mathcal{D}P \neq \emptyset$. Then to every small enough $\mu \in \mathcal{D}P \backslash \mathcal{D}C$ there exists the invariant manifold*

$$x_1 = ||\mu||\eta(\varphi_1, ||\mu||, \mu_0)$$
$$\nu_1 = ||\mu||^2\Theta_1(\varphi_1, ||\mu||, \mu_0),$$

where η, Θ_1 are continuous functions 2π - periodic in all components of φ_1, $\varphi_1 \in R^4, x_1 \in R^4, \nu_1 \in R^{n-8}$. The natural number p can be taken $p = 1$.

The proof of this theorem is similar to the proof of Theorem 1.1.

Suppose that $\mathcal{D}C \neq \emptyset$. Take $\mu \in \mathcal{D}C$. We can perform on the beam $\delta(\mu_0)$ the second step of the transformation procedure. The bifurcation equation of the system (4) on the beam $\delta(\mu_0)$ is:

$$B_1(\mu_0)\rho_1^2 + \varepsilon C_1(\mu_0) = 0, \tag{2.13}$$

where:

1. $B_1(\mu_0)$ is the matrix of the order 2 when the eigenvalues of $P_1(\mu_0)$ are of the kind A
2. $B_1(\mu_0) \in R$ when the eigenvalues of $P_1(\mu_0)$ are of the kind B, C.

Consider firstly the cases when the eigenvalues of $P_1(\mu_0)$ are of the type B, C. Suppose that $B_1(\mu_0) \neq 0$ and $\beta_1^2(\mu_0) = -\frac{1}{B_1(\mu_0)}C_1(\mu_0) > 0$. Then the system (4) can be reduced to the system (6) with $x_2 \in R, \varphi_1 \in R^4, \varphi_2 \in R, \nu_1 \in R^{n-8}, \nu_2 \in$ $\in R^2, p = 2$ and $P_2(\mu_0) = 2\beta_1^2(\mu_0)B_1(\mu_0) \neq 0$. Utilizing Theorem from Section 3 of Chapter 1 in [1] we can formulate the following theorem.

Theorem 2.2. *Let be $\mu \in \mathcal{D}C$ and the eigenvalues of $P_1(\mu_0)$ of the kind B or C. If $B_1(\mu_0) \neq 0$ and $\beta_1^2(\mu_0) > 0$ then to every small enough $\mu \in \delta(\mu_0)$ there exists the invariant manifold*

$$x_2 = ||\mu||\eta(\varphi_1, \varphi_2, ||\mu||, \mu_0)$$
$$\nu_1 = ||\mu||^3\Theta_1(\varphi_1, \varphi_2, ||\mu||, \mu_0)$$
$$\nu_2 = ||\mu||^2\Theta_2(\varphi_1, \varphi_2, ||\mu||, \mu_0),$$

where η, Θ_1, Θ_2 are continuous functions 2π - periodic in all components of φ_1, φ_2, $\varphi_1 \in R^4, \varphi_2 \in R, x_2 \in R, \nu_1 \in R^{n-8}, \nu_2 \in R^2$. The natural number p has the value $p = 2$.

Suppose now that the eigenvalues of $P_1(\mu_0)$ at $\mu \in \mathcal{D}C$ are $\pm i\lambda_1^1, \pm i\lambda_2^1$. Let the following conditions be satisfied:

$$\begin{aligned} &1.\ q_1\lambda_1^1 + q_2\lambda_2^1 \neq 0, 0 < |q| \leq 5 \\ &2.\ \det\ B_1(\mu_0) \neq 0 \\ &3.\ \beta_1^2(\mu_0) = -B_1^{-1}(\mu_0)C_1(\mu_0) > 0. \end{aligned} \tag{2.14}$$

Then on the beam $\delta(\mu_0)$ the system (4) can be reduced to the system (6) with $p = 2, x_2 \in R^2, \varphi_1 \in R^4, \varphi_2 \in R^2, \nu_1 \in R^{n-8}$ and

$$P_2(\mu_0) = \frac{\partial X_2(0, 0, \mu_0)}{\partial x_2} = 2[diag\ \ \beta_1(\mu_0)]B_1(\mu_0)[diag\ \ \beta_1(\mu_0)].$$

On the base of Theorem from Section 3 of Chapter 1 in [1] the following theorem is valid.

Theorem 2.3. *Let be $\mu \in \mathcal{D}C$, the eigenvalues of $P_1(\mu_0)$ of the kind A and the conditions (2.14) statisfied. If the matrix $P_2(\mu_0)$ in non-critical then to every small enough $\mu \in \delta(\mu_0)$ there exists the invariant manifold*

$$x_2 = ||\mu||\eta(\varphi_1, \varphi_2, ||\mu||, \mu_0)$$

$$\nu_1 = ||\mu||^2\Theta_1(\varphi_1, \varphi_2, ||\mu||, \mu_0),$$

where η, Θ_1 are continuous functions 2π - periodic in all components of $\varphi_1, \varphi_2, \varphi_1 \in \in R^4, \varphi_2 \in R^2, x_2 \in R^2, \nu_1 \in R^{n-8}$. The natural number p has the value $p = 2$.

Suppose that the matrix $P_2(\mu_0)$ is critical. Then the third step of the transformation procedure can be performed on the beam $\delta(\mu_0)$. The bifurcation equation of the system (6) is

$$B_2(\mu_0)\rho_2^2 + \varepsilon C_2(\mu_0) = 0, \tag{2.15}$$

where $B_2(\mu_0) \in R$.

Assume that $B_2(\mu_0) \neq 0$ and $\beta_2^2(\mu_0) = -\frac{1}{B_2(\mu_0)}C_2(\mu_0) > 0$. Then the system (6) with $p = 2$ can be reduced to the system (6) with $p = 3, x_3 \in R, \varphi_1 \in R^4, \varphi_2 \in \in R^2, \varphi_3 \in R, \nu_1 \in R^{n-8}$ and

$$P_3(\mu_0) = \frac{\partial X_3(0, 0, \mu_0)}{\partial x_3} = 2\beta_2^2(\mu_0)B_2(\mu_0) \neq 0.$$

On the base of Theorem from Section 3 of Chapter 1 in [1] the following theorem is valid.

Theorem 2.4. *If $B_2(\mu_0) \neq 0$ and $\beta_2^2(\mu_0) > 0$ then to every small enough $\mu \in \delta(\mu_0)$ there exists the invariant manifold*

$$x_3 = ||\mu||\eta(\varphi_1, \varphi_2, \varphi_3, ||\mu||, \mu_0)$$
$$\nu_1 = ||\mu||^4\Theta_1(\varphi_1, \varphi_2, \varphi_3, ||\mu||, \mu_0),$$

where η, Θ_1 are continuous functions 2π - periodic in all components of $\varphi_1, \varphi_2, \varphi_3$, $\varphi_1 \in R^4, \varphi_2 \in R^2, \varphi_3 \in R, x_3 \in R, \nu_1 \in R^{n-8}$. The natural number p has the value $p = 3$.

Note Many significant results in the bifurcation theory of dynamical systems were achieved during last three decades. A nice survey of them can be found in the books [4], [5] in which also the relations among reached results are discussed. The question of the existence of bifurcations in the case of two pairs of pure imaginary eigenvalues is for example studied in the articles [2], [3], [7], [8].

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Dept. of Applied Informatics
Matej Bel University
Tajovského 10
974 01 Banská Bystrica
SLOVAKIA

E-mail address: zimka@econ.umb.sk