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LAGRANGEANS ON A MANIFOLD WITH A (1,1)-TENSOR FIELD

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Abstract. The main object of this paper is the Lagrange calculus of first order on a manifold with a given (1,1)-tensor field.

INTRODUCTION

In this paper we deal with Lagrangians of first order that are functions $L : TM \to R$ on the tangent bundle TM of a manifold M, when on M is given a (1,1)-tensor field $A : M \to T^*M \otimes TM$. Recall two canonical objects on TM: the Liouville field V the flow of which is determined by the homotheties on the fibres of $p_M : TM \to M$ and the endomorphism $v : TM \to T^*M \otimes VTM$ which is induced by the identity on TM and by the canonical identification $VTM = TMx_MTM$ of the subbundle VTM of vertical vectors on TM.

We use well known notions of the Lagrange formalism and the theory of lifting:

1. The Lagrange equation $i_X dd_v L = dL - VL$, see for example [1].
2. The Lagrange fields S_L that are the semisprays (vector fields S on TM with the property $v(S) = V$) satisfying the Lagrange equation.
3. The Lagrange forms $d_v L, \omega_L = dd_v L, dE = d(L - VL)$.
4. The connection Γ_S canonically determined by a semispray S on TM, see [2].
5. The natural lifts of a (1,1)-tensor field A on M in the tangent bundle TM first of all the vertical lift Av and the complete lift Ac, see [5].

In the first section we deal with the affine space of the connections on TM with the property $Ac(H\Gamma) \subset H\Gamma$, where $H\Gamma$ is the horizontal subbundle of a connection Γ. In general, a (1,1)tensor field α on TM is called Γ-parallel if $\alpha(H\Gamma) \subset H\Gamma$. We find conditions for A such that Ac is Γ_S-parallel and conditions under which there exists a unique connection Γ such that $L_S Av$ is Γ-parallel where L_S denotes the Lie derivative of Av with respect to a given semispray S.

The second section is devoted to the mutual relations between ω_L and A. Here some equalities of the first order Lagrangian calculus on a manifold M with a (1,1)-tensor field are introduced. Proposition 8 states the conditions under which there

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is a unique connection Γ such that $\omega_L(h_\Gamma, Ac\cdot h_\Gamma)=0$. In Proposition 9 it is proved that the Lagrange field S_L satisfies the equality

$$i_S dd_{Av}L = i_{Ac}dl - d(V_A L),\ V_A = Av(S).$$

It is proved (Proposition 11) that the equality

$$i_{AcX}dd_v L = i_X dd_{Av}L$$

is satisfied if the 2-form $di_{Ac}dL$ is semibasic. Proposition 12 states the conditions under which the equation

$$i_S i_{Ac}\omega_L = d(V_A L) - i_{Ac}(d(E-L))$$

has a unique solution.

In this paper we suppose that all manifolds and maps are smooth.

1. SEMISPRAYS AND CONNECTIONS ON A MANIFOLD WITH (1,1)-TENSOR FIELDS

Let M be a manifold, (x^i) be a local chart on M and (x^i, x_1^i) be the induced chart on TM. Denote by $V = x_1^i\partial/\partial x_1^i$ the Liouville field on TM, by $v = dx^i\otimes\partial/\partial x_1^i$ the canonical (1,1)-tensor field determined by the identity on TM and by the canonical identification $VTM = TMx_MTM$, where VTM is the vector bundle of all vertical vectors on TM.

Recall that a vector field $X : TM \to TTM$ on TM is called a differential equation of second order (shortly a semispray) if $v(X) = V$, i.e. if its coordinate form is

$$X = x_1^i\partial/\partial x^i + \eta^i(x, x_1)\partial/\partial x_1^i\ .$$

Let $\alpha = (a_j^i dx^j + b_j^i dx_i^j)\otimes\partial/\partial x^i + (c_j^i dx^j + h_j^i dx^j)\otimes\partial/\partial x_1^i$ be a (1,1)-tensor field on TM. We say that α is vertical if $\alpha(VTM)\subset VTM$, i.e. if $v\cdot\alpha\cdot v = 0$.

A connection Γ on the fibre manifold $p_m : TM \to M$ can be introduced as a (1,1)-tensor field h_Γ (the horizontal form of the connection) satisfying the conditions $h_\Gamma(VTM) = 0$, $Tp_m h_\Gamma = Tp_M$ where throughout this paper we use the denotation TF for the tangential prolongation of a map F. In coordinates $h_\Gamma = dx^i\otimes\partial/\partial c^i + \Gamma_j^i(x, x_1)dx^j\otimes\partial/\partial x_1^i$, where Γ_j^i are the local components of Γ. Then $H\Gamma := Imh_\Gamma\subset TTM$ is the horizontal subbundle of the connection Γ (satisfying the equation $dx_1^i = \Gamma_j^i dx^j$ and the decomposition $TTM = VTM\oplus H\Gamma$) and $v_\Gamma = Id_{TTM} - h_\Gamma$ is the vertical form of the connection Γ. Recall that the set of all connections on TM is an afinne space associated with the vector space $C^\infty(T^*M\otimes VTM)$ of all semibasic 1-vector forms with values in VTM.

There is a unique semispray $S_\Gamma : x_1^i\partial/\partial x^i + \Gamma_j^i x_1^j\partial/\partial x^i$ which is Γ-horizontal.

Every semispray S determines the connection Γ_S the horizontal form of which is $h_\Gamma = \frac{1}{2}(Id_{TTM} - L_S v)$, where $L_S v$ is the Lie derivative of v with respect to S. Its components are $\Gamma_j^i = \frac{1}{2}\eta_{j1}^i$, where the denotation $f_{i_1} : \partial f/\partial x_1^i$ together with $f_i := \partial f/\partial x^i$ will be used throughout our paper.

Definition 1. Let α be a (1,1)-tensor field and Γ be a connection on TM. The field α is called Γ-parallel if $\alpha(H\Gamma) \subset H\Gamma$.

We will introduce the coordinate condition for α to be Γ-parallel. If we use the above expression of α and if Γ^i_j are the components of Γ then

$$\alpha \cdot h_\Gamma = (a^i_j + b^i_t \Gamma^t_j) dx^j \otimes \partial/\partial x^i + (c^i_j + h^i_t \Gamma^t_j) dx^j \otimes \partial/\partial x^i_1 \ . \tag{1}$$

Then α is Γ-parallel iff

$$\Gamma^i_u (a^u_j + b^u_t \Gamma^t_j) = c^i_j + h^i_t \Gamma^t_j \ . \tag{2}$$

Let $A = a^i_j dx^j \otimes \partial/\partial x^i$ be a (1,1)-tensor field on a manifold M. We will prefer two natural lifts of A on TM, see [5]:

a) the vertical lift $Av = a^i_j dx^j \otimes \partial/\partial x^i_1$ that is a semi-basic vector 1-form on TM induced by the identification $VTM = TMx_M TM$,

b) the complete lift $Ac = a^i_j dx^j \otimes \partial/\partial x^i + (a^i_{jk} x^k_1 dx^j + a^i_j dx^j_1) \otimes \partial/\partial x^i_1$ which is determined by the map $i_2 \cdot TA \cdot i_2$, where $i_2 : (x^i, x^i_1, dx^i, dx^i_1) \to (x^i, dx^i, x^i_1, dx^i_1)$ is the canonical involution on TM.

Recall that the vector field Ac is vertical and

$$v \cdot Ac = Ac \cdot v = Av \ .$$

Definition 2. A (1,1)-tensor field A on M is called Γ-parallel if Ac is Γ-parallel.

We will need the coordinate form of the Lie derivatives $L_S Av, L_S Ac$ with respect to a semispray S. We get

$$\begin{aligned} L_S Av &= -a^i_j dx^j \otimes \partial/\partial x^i + [(a^i_{jk} x^k_1 - \eta^i_{k_1} a^k_j) dx^j + a^i_j dx^j_1] \otimes \partial/\partial x^i_1 \\ L_S Ac &= [(a^i_{jks} x^k_1 x^s_1 + a^i_{jk} \eta^k + a^i_k \eta^k_j - \eta^i_{t_1} a^t_{jk} x^k_1 - \eta^i_k a^k_j) dx^j + \\ &\quad + (2a^i_{jk} x^k_1 + a^i_k \eta^k_{j_1} - \eta^i_{k_1} a^k_j) dx^j_1] \otimes \partial/\partial x^i_1 \ . \end{aligned}$$

So the field $L_S Ac$ is a vector 1-form with values in VTM. Therefore $v \cdot L_S Ac = 0$. The field $L_S Av$ is vertical and $v \cdot L_S Av = -L_S Av \cdot v = Av$.

Proposition 1. *Let S be a semispray and Γ_S be the canonical connection determined by S. Then $Ac - L_S Av = 2h_{\Gamma_S} \cdot Ac$.*

Proof. The equality $Av = v \cdot Ac$ gives $L_S Av = L_S v \cdot Ac + v \cdot L_S Ac = L_S v \cdot Ac$. Then the equality $L_S v = Id_{TTM} - 2h_{\Gamma_S}$ completes our proof.

Corollary. *The tensor field $Ac - L_S Av$ is a vector 1-form with values in $H\Gamma_S$.*

Remark. If A is a regular field then also $L_S Av$ is regular. It is easy to show that $(L_S Av)^{-1}\Gamma_S$ is just the connection on TM the horizontal subbundle of which is given by vectors Y such that $L_{\overline{S}}(L_S Av)(Y)$ is vertical and that the connection Γ does not depend on the choice of the semispray $\overline{S}$.

Let Γ be a connection on TM. If A is regular then also Ac is regular and then the subbundle $Im\ Ac \cdot h_\Gamma$ states the connection $A\Gamma$ the local components of which

$$\overline{\Gamma}^i_j = (a^i_{tk}x^k_1 + a^i_s\Gamma^s_t)\tilde{a}^t_j, \quad a^i_t\tilde{a}^t_j = \delta^i_j \ ,$$

immediately follow from the equality (1).

In the case of the tensor field $\alpha = Ac$ the equality (2) reads

(2')
$$\Gamma^i_u a^u_j = a^i_{jk}x^k_1 + a^i_s\Gamma^s_j \ .$$

If Γ, $\overline{\Gamma}$ are two connections with respect to which is the tensor field A parallel then from (2') we get

$$(\Gamma^i_u - \overline{\Gamma}^i_u)a^u_j - a^i_s(\Gamma^s_j - \overline{\Gamma}^s_j) = 0 \ .$$

This equality together with the fact that Av and $\Gamma - \overline{\Gamma}$ are sections $TM \to T^*M \otimes_{TM} TM$ immediately give

Proposition 2. *Let Γ be a given connection on TM. The set of all (1,1)-tensor fields A on M which are Γ-parallel is a real vector space. Let A be a given (1,1)-tensor field on M. Then the set of all connections Γ on TM with respect to which A is Γ-parallel is an afinne space associated with the kern of the linear map*

$$\Phi_A : C^\infty(T^*M \otimes_{TM} TM) \to C^\infty(T^*M \otimes_{TM} TM), \quad \xi \to Av\xi - \xi Av \ .$$

Proposition 3. *Let S be a semispray on TM, Γ_S be the canonical connection determined by S and A be a (1,1)-tensor field on M. Then the following conditions are equivalent*

a) *A is Γ_S-parallel,*
b) *L_SAc is a semibasic vector 1-form with values in VTM,*
c) *L_SAv is Γ_S-parallel.*

Proof. By the equality (2) the tensor field L_SAv is Γ_S-parallel iff

$$-\frac{1}{2}\eta^i_{k_1}a^k_j = a^k_{jk}x^k_1 - \eta^i_{k_1}a^k_j + \frac{1}{2}a^i_k\eta^k_{j_1} \ .$$

This condition coincides with the equality (2) for the connection Γ_S, $\Gamma^i_j = \frac{1}{2}\eta^i_{j_1}$, and with the coordinate condition $2a^i_{jk}x^k_1 + a^i_k\eta^k_{j_1} - a^k_j\eta^i_{k_1} = 0$ for L_SAc to be semibasic. Proof is finished.

Proposition 4. *Let S be a semispray on TM and A be a (1,1)-tensor field on M. If the (2,2)-tensor field $AT := A \otimes Id_{TM} + Id_{TM} \otimes A$ is regular then there is a unique connection Γ on TM such that the tensor field L_SAv is Γ-parallel.*

Proof. Let Γ^i_j be the components of a connection Γ. Then the condition (2) for the tensor field L_SAv to be Γ-parallel reads

$$(a^i_u\delta^s_j + \delta^i_u a^s_j)\Gamma^u_s = \eta^i_{k_1}a^k_j - a^i_{jk}x^k_1 \ .$$

It completes our proof.

Remark. It is easy to prove that the tensor field $L_S L_S Av$ is not vertical and that it holds $v \cdot L_S L_S Av \cdot v = -2Av$. Therefore if A is regular then there are connections Γ_1, Γ_2 such that $H\Gamma_1 = L_S L_S Av(VTM)$ and $L_S L_S Av(H\Gamma_2) = VTM$.

2. GEOMETRY OF LAGRANGIANS ON MANIFOLDS WITH A (1,1)-TENSOR FIELD

First, recall some notions and properties.

Let α be a (1,1)-tensor field on TM, X be a vector field, ε be a k-form on TM. Then the symbols i_α and i_X denote derivatives

$$i_\alpha \varepsilon(Y_1, \ldots, Y_k) = \sum_{i=1}^{k} \varepsilon(Y_1, \ldots, \alpha(Y_i), \ldots, Y_k),$$
$$i_X \varepsilon(Y_1, \ldots, Y_{k-1}) = \varepsilon(X, Y_1, \ldots, Y_{k-1}), \ d_\alpha = [i_\alpha, d] = i_\alpha d - d i_\alpha$$

where d denotes the exterior derivative.

It holds

$$d_\alpha d = -d d_\alpha, \quad L_X = i_X d + d i_X, \quad d L_X = L_X d \ ,$$

where L_X denotes the Lie derivative of exterior forms with respect to a vector field X.

When ε is a (0, 2)-tensor field on TM we will use the following denotations

$$\varepsilon^\alpha, \ \varepsilon^\alpha(X, Y) = \varepsilon(\alpha X, Y),$$
$$\varepsilon_\alpha, \ \varepsilon_\alpha(X, Y) = \varepsilon(X, \alpha Y),$$
$$\varepsilon\alpha, \ \varepsilon\alpha(X, Y) = \varepsilon(\alpha X, \alpha Y) = \alpha^* \varepsilon(X, Y) \ .$$

It is clair that if ε is a 2-form then $i_\alpha \varepsilon = \varepsilon^\alpha + \varepsilon_\alpha$.

Let L be a Lagrangian of first order on M, i.e. a function on TM. Then the forms

$$d_v L = L_{i_1} dx^i, \quad d_v L = i_v dL \ ,$$
$$\omega_L := d d_v L = L_{i_1 j} dx^j \wedge dx^i + L_{i_1 j_1} dx_1^j \wedge dx^i$$

are called the Lagrange 1- and 2-form of Lagrangian L. When the map $I_L : C^\infty(TM \to TTM) \to C^\infty(TM \to T^*TM), X \to i_X \omega_L$, is regular then the Lagrangian L is called regular. In this case, the Lagrange 2-form ω_L is symplectic. Locally L is regular iff $\det L_{i_1 j_1} \neq 0$. The equation

$$i_X \omega_L = dE, \quad E = L - VL \ , \tag{3}$$

is a basic equation of the Lagrange formalism of first order. It is called the Lagrange equation. Every semispray, which is a solution of (3) is called the Lagrange field and denoted by S_L. Recall that when L is regular then there is a unique solution of the equation (3) and moreover it is a semispray, i.e. it is the Lagrange field.

We introduce some coordinate expressions we will need.

$$L_{i_1t_1}\eta^t = L_i - L_{i_1k}x_1^k \ , \tag{4}$$

that is the equation (3) for semisprays

$$\begin{aligned}
\omega_L^\alpha &= (L_{j_1s}a_i^s + L_{j_1s_1}c_i^s - L_{s_1j}a_i^s)dx^i dy^j + (L_{j_1s}b_i^s + L_{j_1s_1}h_i^s - L_{s_1j}b_i^s)dx_1^i dy^j - \\
&\quad - L_{s_1j_1}a_i^s dx^i dy_1^j - L_{s_1j_1}b_i^s dx_1^i dy_1^j \ , \\
\omega_L\alpha &= (L_{s_1u}a_i^s a_j^u + L_{s_1u_1}a_i^s c_j^u)dx^j \wedge dx^i + (L_{s_1u}a_i^s b_j^u + L_{s_1u_1}a_i^s h_j^u - L_{s_1u}b_j^s a_i^u - \\
&\quad - L_{s_1u}b_j^s c_i^u)dx_1^j \wedge dx^i + (L_{s_1u}b_i^s b_j^u + L_{s_1u_1}b_i^s c_j^u)dx_1^j \wedge dx_1^i \ , \\
\omega_L^{Ac} &= (L_{j_1s}a_i^s + L_{j_1s_1}a_{ik}^s x_1^k - L_{s_1j}a_i^s)dx^i \otimes dx^j + L_{j_1s_1}a_i^s dx_1^i \otimes dx^j - L_{s_1j_1}a_i^s dx^i \otimes \\
&\quad \otimes dx_1^j \ , \\
\omega_L^{Av} &= L_{j_1s_1}a_i^s dx^i dy^j, \quad \omega_L^v = L_{i_1j_1}dx^i dy^j \ .
\end{aligned}$$

The tensor fields ω_L^{Av}, ω_L^v can be interpreted as the sections $\tilde{\omega}_L^{Av}$, $\omega_L^v : TM \to V^*TM \otimes V^*TM$.

These expressions immediately give

Lemma 1. *If ω_L^{Ac} is symmetric or skew-symmetric then ω_L^{Av} is skew-symmetric or symmetric.*

Lemma 2. *The tensor field ω_L^α is symmetric or skew-symmetric iff $i_\alpha\omega_L = 0$ or $i_\alpha\omega_L = 2\omega_L$ respectively.*

Proof. Since ω_L is a 2-form therefore $\omega_{L\alpha}(X,Y) = \omega_L(X,\alpha Y) = -\omega_L(\alpha Y, Y) = -\omega_L^\alpha(Y,X) = -(\omega_L^\alpha)^t(X,Y)$. Then the equality $i_\alpha\omega_L = \omega_L^\alpha + \omega_{L\alpha}$ finishes our proof.

Definition 3. We say that vector fields X, Y on TM are ω_L-orthogonal if $\omega_L(X,Y) = 0$. Tensor (1,1)-fields α_1, α_2 on TM are called ω_L-orthogonal if $\omega_L(\alpha_1 X, \alpha_2 Y) = 0$ for any vector fields X, Y. A tensor (1,1)-field α on TM is said to be ω_L-isotropic if $\omega_L\alpha = 0$. A tensor (1,1)-field A on M is called ω_L-isotropic if its complete lift Ac is ω_L-isotropic.

Definition 4. Let $Z \subset TTM$ be a subbundle of the tangent bundle $p_{TM} : T(TM) \to TM$. The symbol $\mathrm{Orth}_L Z$ will denote the set of tangent vectors Y on TM such that $\omega_L(X,Y) = 0$ for any $X \in Z$ satisfying $p_{TM}X = p_{TM}Y$. We will say that Z is ω_L-Lagrange if $\mathrm{Orth}_L Z = Z$.

Definition 5. A connection Γ on TM is called ω_L-isotropic or ω_L-Lagrange if its horizontal form h_Γ is ω_L-isotropic or ω_L-Lagrange respectively. We will say that two connections $\Gamma_1, \ \Gamma_2$ are ω_L-orthogonal if $h_{\Gamma_1}, \ h_{\Gamma_2}$ are ω_L-orthogonal. When a connection Γ is ω_L-isotropic we will say that ω_L is Γ-parallel as well.

Remark. Let (M,ε) be pseudo-Riemannian manifold and Γ be the Levi-Civita connection of (M,ε). Then $\nabla_\Gamma\varepsilon = 0$. Let $\overline{\varepsilon}$ be Sasaki metrics on TM which is the natural lift of ε. Then $\overline{\varepsilon}h_\Gamma = 0$, i.e. $\overline{\varepsilon}$ is Γ-parallel.

Lemma 3. *The tensor field ω_L^{Ac} is skew-symmetric if and only if the vector fields AcX and X are ω_L-orthogonal for every vector field X on TM.*

Proof. $\omega_L(AcX, X) = 0 \leftrightarrow \omega_L^{Ac}(X, X) = 0$. It finishes our proof.

Let Γ_j^i be the components of a connection Γ on TM. Then $\omega_L h_\Gamma = (L_{j_1 i} + L_{j_1 s_1}\Gamma_i^s) dx^j \wedge dx^i$. Therefore the equality

$$L_{j_1 i} - L_{i_1 j} + L_{j_1 s_1}\Gamma_i^s - L_{i_1 s_1}\Gamma_j^s = 0 \tag{5}$$

is the coordinate condition for ω_L to be Γ-parallel.

If Γ and $\overline{\Gamma}$ are two connections on TM such that ω_L is Γ- and $\overline{\Gamma}$-parallel then it holds from (5)

$$L_{j_1 s_1}(\Gamma_i^s - \overline{\Gamma}_i^s) - L_{i_1 s_1}(\Gamma_j^s - \overline{\Gamma}_j^s) = 0\ . \tag{6}$$

We have proved

Proposition 5. *Let Γ be a given connection on TM. Then the set of all Lagrangians L such that Lagrange forms ω_L are Γ-parallel is a vector subspace of the vector space off all functions on TM. Let L be a given Lagrangian on TM. Then the set of all connections Γ on TM such that ω_L is Γ-parallel is an affine space associated to the kernel of the antisymmetrization of the map $\psi : V^*TM \otimes VTM \to V^*TM \otimes V^*TM$ determined by the rule $\beta \to (\tilde{\omega}_L^v)^\beta$, $\beta_j^i \to L_{i_1 t_1}\beta_j^t$.*

It is known, see for example [3], that ω_L is Γ_L parallel, i.e. we have

Proposition 6. *Let S_L be a Lagrange field. Then the connection Γ_L determined by the semispray S_L is ω_L-isotropic, i.e. ω_L is Γ_L- parallel.*

We will say that the tensor (2,2)-field $A \otimes Id_{TM} + Id_{TM} \otimes A = A_I$ is regular if the vector bundle morphism $\overline{A}_I : T^*M \otimes T^*M \to T^*M \otimes T^*M$ over Id_M, $(x_{km}) \to (a_i^t \delta_j^u + \delta_i^u a_j^t) x_{ut}$, is regular.

Proposition 7. *Let the tensor field A_I and the Lagrangian L be regular. Let the tensor (0,2)-field ω_L^{Av} be skew-symmetric. Then there is a unique connection Γ such that h_Γ and $Ac \cdot h$ are ω_L-orthogonal, i.e. $\omega_L(h_\Gamma X, \alpha \cdot h_\Gamma Y) = 0$.*

Proof. Recall that if ω_L^{Ac} is symmetric then ω_L^{Av} is skew-symmetric, i.e. $L_{j_1 t_1} a_i^t = -L_{i_1 t_1} a_j^t$. Let $X = \xi^i \partial/\partial x^i + \eta^i \partial/\partial x^i$, $Y = \overline{\xi}^i \partial/\partial x^i + \overline{\eta}^i \partial/\partial x_1^i$ be two vector fields on TM. Let Γ_j^i be the components of a connection Γ on TM. Then $\omega_L(h_\Gamma X, Ac \cdot h_\Gamma Y) = (L_{t_1 j} a_i^t - L_{j_1 t} a_i^t + L_{t_1 u_1}\Gamma_j^u a_i^t - L_{j_1 u_1} a_{ik}^u x_1^k - L_{j_1 u_1} a_s^u \Gamma_i^s)\xi^j \overline{\xi^i}$. Using the condition for ω_L^{Av} to be skew-symmetric the equality $\omega_L(h_\Gamma X, Ac \cdot h_\Gamma Y) = 0$ holds if and only if

$$(a_i^t \delta_j^u + \delta_i^u a_j^t) L_{s_1 t_1}\Gamma_u^s = L_{i_1 t} a_j^t - L_{t_1 i} a_j^t + L_{i_1 u_1} a_{jk}^u x_1^k\ .$$

It finishes our proof.

Remark. The equality $\omega_L^{Ac}(Ac \cdot h_\Gamma, h_\Gamma) = 0$ in the case when $A^2 = \pm Id_{TM}$ is equivalent to the ω_L-isotropy of Γ or of $Ac \cdot h_\Gamma$ in the case when ω_L^{Ac} is symmetric or skew-symmetric.

Inspiring by [4] we formulate

Lemma 4. *Let X, Y be vector fields, L be a Lagrangian, α be a (1,1)-tensor field and ε be a 1-form on TM. Then the conditions*

$$a, i_X dd_\alpha L = \varepsilon - dYL, \quad b, (L_Y - i_X d_\alpha)dL = \varepsilon$$

are equivalent.

Proof. $L_Y dL = (i_Y d + di_Y)dL = di_Y dL = d(YL), dd_\alpha L = -d_\alpha dL$. It completes our proof.

Let $\beta = \beta^i_j dx^j \otimes \partial/\partial x^i_1$ be a vector semibasic form on TM with values in VTM. Denote $V\beta := \beta(S)$, where S is an arbitrary semispray. In the case of $\beta = Av$ we will denote $Av(S) := VA$.

Lemma 5. *Let S be a semispray, L be a Lagrangian and β be a semibasic vector (1,1)-form with values in VTM. Then*

$$(L_{V\beta} - i_S d_\beta)L = 0 \ .$$

Proof. We get $d_\beta L = i_\beta dL = L_{t_1}\beta^t_k dx^k$. Then $i_S d_\beta L = d_\beta L(S)$. From the other side $L_{V\beta}L = V\beta(L) = L_{t_1}\beta^t_k x^k_1 = d_\beta L(S)$. Our proof is completed.

Corollary. *Under the conditions of Lemma 5 it holds*

$$d(V\beta L) = di_S d_\beta L \ . \tag{8}$$

We return to the case when $\alpha = Ac$, $\beta = Av$. We have $d_{Av}L = L_{t_1}a^t_i dx^i$

$$dd_{Av}L = (L_{t_1 j}a^t_i + L_{t_1}a^t_{ij})dx^j \wedge dx^i + L_{t_1 j_1}a^t_i dx^j_1 \wedge dx^i \ .$$

It immediately gives

Proposition 8. *Let both the Lagrangian L and the (1,1)-tensor field A on M be regular. Then $dd_{Av}L$ is a symplectic form.*

Lemma 6. *Let S be a semispray. Then*

$$i_{Ac}L_S d_v L = L_S d_{Av} L \ .$$

Proof. $L_S d_v L = (L_{i_1 k}x^k_1 + L_{i_1 k_1}\eta^k)dx^i + L_{i_1}dx^i_1$,

$$L_S d_{Av}L = [(L_{t_1 k_1}a^t_i + L_{t_1}a^t_{ik})x^k_1 + L_{t_1 k_1}a^t_i \eta^k]dx^i + L_{t_1}a^t_i dx^i_1 \ .$$

Now the equality of Lemma 6 follows from the expression of Ac.

Proposition 9. *Every Lagrange field S_L is also a solution of the equation*

$$(*)\qquad i_S dd_{Av}L = i_{Ac}dL - d(VA(L)) \ .$$

Proof. Being a Lagrange field S_L satisfies the equation $i_S dd_v L = dL - dVL$. Then

$$(9)\qquad i_{Ac}(d(VL) + i_S dd_v L) = i_{Ac}dL \ .$$

Using the equality (8) we get

$$d(VL) + i_S dd_v L = di_S d_v L + i_S dd_v L = (di_S + i_S d)d_v L = L_S d_v L \ .$$

Then by (9)

$$i_{Ac}dL = i_{Ac}[d(VL) + i_S dd_v L] = i_{Ac}L_S d_v L \ .$$

Analogously using (8) and Lemma 6 we get

$$d(VAL) + i_S dd_{Av}L = di_S d_{Av}L + i_S dd_{Av}L = L_S d_{Av}L = I_{Ac}L_S d_v L \ .$$

Therefore S_L satisfies the equation (*). Proof is finished.

Let $X = \xi^i \partial/\partial x^i + \eta^i \partial/\partial x_1^i$ be a vector field on TM. We calculate

$$i_X dd_{Av}L = [(L_{t_1 j}a_i^t - L_{t_1 i}a_j^t - L_{t_1}a_{ji}^t + L_{t_1}a_{ij}^t)\xi^j + L_{t_1 j_1}a_i^t \eta^j]dx^i - L_{t_1 i_1}a_j^t \xi^j dx_1^i$$
$$AcX = a_j^i \xi^j \partial/\partial x^i + (a_{jk}^i x_1^k \xi^j + a_j^i \eta^j)\partial/\partial x_1^i$$
$$i_{AcX}dd_v L = [(-L_{t_1 i}a_j^t + L_{i_1 t}a_j^t + L_{i_1 t_1}a_{jk}^t x_1^k)\xi^j + L_{i_1 t_1}a_j^t \eta^j]dx^i - L_{t_1 i_1}a_j^t \xi^j dx_1^i \ .$$

These expressions immediately give

Lemma 7. *For any vertical vector field X on TM the 1-forms $i_X dd_{Av}L$, $i_{AcX}dd_v L$ are semibasic. For every vector field X the 1-form $i_{AcX}dd_v L - i_X dd_{av}L$ is semibasic.*

Corollary. *Let Y be a vertical vector field on TM. Then it holds*

$$i_{AcX}dd_v L(Y) = i_X dd_{Av}L(Y)$$

for any vector field X on TM.

Definition 6. A (1,1)-tensor field A on M is called L-commutative if

$$i_{AcX}dd_v L = i_X dd_{Av}L$$

for any vector field X on TM.

Proposition 10. *If a (1,1)-tensor field A on M is L-commutative then ω_L^A is skew-symmetric.*

Proof. The (0,2)-form ω_L^A is skew-symmetric iff $dd_v L(AcY, X) = -dd_v L(AcX, Y)$. Let A be L-commutative. Then for any vector fields X, Y we get $dd_v L(AcX, Y) = i_{AcX}dd_v L(Y) = i_X dd_{Av}L(Y) = dd_{Av}L(X, Y)$. Analogously $dd_v L(AcY, X) = dd_{Av}L(Y, X) = -dd_{Av}L(X, Y)$. It completes our proof.

Proposition 11. *A (1,1)-tensor field A on M is L-commutative if and only if the 2-form $dd_{Ac}L$ is semibasic.*

Proof. By the direct computation we get

$$di_{Ac}dL = (L_{tj}a_i^t + L_t a_{ij}^t + L_{t_1 j} a_{ik}^t x_1^k + L_{t_1} a_{ikj}^t x_1^k) dx^j \wedge dx^i + \\ + (L_{tj_1} a_i^t + L_{t_1 j_1} a_{ik}^t x_1^k + L_{t_1} a_{ij}^t - L_{t_1 i} a_j^t - L_{t_1} a_{ji}^t) dx_1^j \wedge dx^i + \\ + L_{t_1 j_1} a_i^t dx_1^j \wedge dx_1^i \ .$$

Comparing it with the expression for $i_{AcX} dd_v L - i_X dd_{Av} L$ we finish our proof.

Remark. Recall the map $I_L : X \to i_X dd_v L$. If we denote by I_{LA} the map $X \to i_X dd_{Av} L$ we can say that A is L-commutative if and only if $I_{LA} = I_L \cdot Ac$.

Lemma 8. *ω_L^A is symmetric iff $i_{AcX} dd_v L = -i_{Ac} i_X dd_v L$. ω_L^A is skew-symmetric iff $i_{AcX} dd_v L = i_{Ac} i_X dd_v L$.*

Proof. $(i_{AcX} dd_v L)(X) = dd_v L(AcX, Y) = \omega_L^A(X, Y)$, $(i_{Ac} i_X dd_v L)(Y) = (i_X dd_v L)(AcY) = dd_v L(X, AcY) = \omega_{LA}(X, Y) = -(\omega_L^A)^t(X, Y)$. It completes our proof.

In the rest part of this paper we will deal with the 2-form $i_{Ac}\omega_L$. In general it is not closed. We introduce its expression $i_{Ac}\omega_L = (L_{t_1 j} a_i^t + L_{i_1 t} a_j^t + L_{i_1 t_1} a_{jk}^t x_1^k) dx^j \wedge dx^i + (L_{t_1 j_1} a_i^t + L_{i_1 t_1} a_j^t) dx_1^j \wedge dx^i$.

Let $s(\omega_L^{Av}) = (L_{i_1 s_1} a_j^s + L_{j_1 s_1} a_i^s) dx^i dy^j$ denote the symmetrisation of ω_L^{Av}. It is clair that $i_{Ac}\omega_L$ is regular iff $s(\omega_L^{Av})$ is regular.

Proposition 12. *If the (0,2)-tensor field $s(\omega_L^{Av})$ is regular then there is a unique vector field X such that*

$$i_X i_{Ac} \omega_L = i_{Ac}(dE + dL) - dVAL, \quad E = L - VL \ .$$

This vector field is a semispray.

Proof. By direct computation we obtain that the form $i_X(i_{Ac}\omega_L) + d(VAL) - i_{Ac}(dE + dL)$ is semibasic iff X is a semispray. Then the assertion of Proposition (12) follows from the term $(L_{t_1 j_1} a_i^t + L_{i_1 t_1} a_j^t) dx_1^j \wedge dx^i$ in the expression of $i_{Ac}\omega_L$.

Remarks.

1. If S is the Lagrange field then $i_S i_{Ac} \omega_L(Y) = i_{Ac}\omega_L(S, Y) = \omega_L(AcS, Y) + \omega_L(S, AcY) = i_{AcS}\omega_L(Y) + i_{Ac} dE$. Therefore if A is L-commutative then $i_{AcS}\omega_L = i_S dd_{Av} L = i_{Ac} dL - d(VAL)$, (see Proposition (9)), and then $i_S i_{Ac}\omega_L = i_{Ac} dL - d(VAL) + i_{Ac} dE$.
2. In the case when ω_L^A is skew-symmetric then $i_{Ac}\omega_L = 2\omega_L^A$. Then $i_{Ac}\omega_L$ is regular iff Lagrangian L is regular and A is regular. Then the Lagrange field satisfies the equation

$$i_S(i_{Ac}\omega_L) = 2 i_{Ac} dE \ .$$

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