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KERNEL AND SOLUTION NUMBERS OF DIGRAPHS

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Abstract. In this paper we show that for given nonnegative integers k, s there exist infinitely many strongly connected digraphs with exactly k kernels and s solutions.

Kernels and solutions are certain vertex subsets of digraphs that are studied in many books and papers e.g., [2, 3, 4, 5]. The number of kernels (or solutions) was investigated in the papers [1, 6, 7]. The following provide a recapitulation of basic observations:

(i) A directed cycle of an odd length has no kernel and no solution.
(ii) A digraph with no directed cycles has exactly one kernel and one solution.
(iii) A cycle of an even length possesses exactly two kernels and two solutions.

In connection with this two natural questions arise:

1. How many kernels and solutions a digraph can have ?
2. Are these numbers mutually dependent or not ?

The answer to the first question is trivial when $k = s$ and the pairs of opposite arcs are allowed. In this case it is sufficient to take the complete digraph with k vertices (its kernels are vertex singletons). In this paper we show that for given nonnegative integers k, s there are infinitely many pairwise nonisomorphic strongly connected digraphs with no couple of opposite arcs that have exactly k kernels and s solutions.

1. Preliminaries

An ordered pair $D = (V, A)$ is said to be *a digraph* whenever V is a nonempty set (vertices of D) and A (arcs of D) is a subset of the set of the ordered pairs of V such that for each $v \in V$ it holds $\overrightarrow{vv} \notin A$, and if $u, v \in V$ then $\overrightarrow{uv} \in A$ implies $\overrightarrow{vu} \notin A$. A set of vertices $W \subseteq V$ is called *independent* if for every pair of vertices $u, v \in W$ neither of arcs $\overrightarrow{uv}, \overrightarrow{uv}$ is present in the digraph. $W \subseteq V$ is *absorbent* if for each $u \in V - W$ there exists $\overrightarrow{uv} \in A$ with $v \in W$ and *dominant* if for each $v \in V - W$ there exists $\overrightarrow{uv} \in A$ with $u \in W$. A set $W \subseteq V$ is *a kernel of* D if W is independent and absorbent and it is *a solution of* D if W is independent and dominant. *A conversion of* D is a digraph $c(D) = (V, B)$ with the same vertex set as D and with the arc set $B = \{\overrightarrow{uv} : \overrightarrow{vu} \in A\}$. It is easy to see that the following lemma is valid.

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1.1. Lemma. *Let $D = (V, A)$ be a digraph and let $W \subseteq V$. Then W is a kernel of D if and only if W is a solution of $c(D)$.*

As usual, a digraph is *strongly connected*, if for every $u, v \in V$ there exists a sequence $\overrightarrow{ua_1}, \overrightarrow{a_1a_2}, \overrightarrow{a_2a_3}, \ldots, \overrightarrow{a_kv}$ in A. Let $\mathcal{G}$ denote the class of all finite strongly connected digraphs. For a digraph D we immediately have:

1.2. Lemma. *$D \in \mathcal{G}$ if and only if $c(D) \in \mathcal{G}$.*

Let us denote by C_n the directed cycle with n vertices ($n \geq 3$). Evidently, $C_n \in \mathcal{G}$.

1.3. Remark. By (i) above, each of digraphs $C_3, C_5, \ldots, C_{2n+1}, \ldots$ has no kernel and no solution. By (iii), each of digraphs $C_4, C_6, \ldots, C_{2n}, \ldots$ has two kernels which are simultaneously the all solutions of the even cycle.

2. Results

The following lemma (c.f. [1]) provides a method for constructions of digraphs belonging to $\mathcal{G}$ with the same number of kernels.

2.1. Lemma. *Let $D = (V, A)$, $\overrightarrow{ac} \in A$ and let $D(\overrightarrow{ac}) = (V', A')$ be a digraph such that $V' = V \cup \{v_1, v_2\}$, $v_1, v_2 \notin V$ and $A' = (A - \overrightarrow{ac}) \cup \{\overrightarrow{av_1}, \overrightarrow{v_1v_2}, \overrightarrow{v_2c}\}$. Then the number of kernels of $D(\overrightarrow{ac})$ equals to the number of kernels of D.*

Proof. Let $\mathcal{K}$ and $\mathcal{L}$ denote the systems of the all kernels of D and of $D(\overrightarrow{ac})$, respectively. Define a mapping f from $\mathcal{K}$ to $\mathcal{L}$ in the following way: for $K \in \mathcal{K}$ put

$$f(K) = \begin{cases} K \cup \{v_1\} & \text{if } c \in K, \\ K \cup \{v_2\} & \text{if } c \notin K. \end{cases}$$

It is easy to verify that the mapping f is a bijection between $\mathcal{K}$ and $\mathcal{L}$. Thus, $\operatorname{card} \mathcal{K} = \operatorname{card} \mathcal{L}$. $\square$

Let $\mathcal{G}_{(k,s)}$ denote the set of all strongly connected digraphs with k kernels and s solutions.

2.2. Corollary. *If $\mathcal{G}_{(k,s)}$ is not empty then it is infinite.*

Proof. Let D be a digraph from $\mathcal{G}$ and let k be the number of its kernels. Let $\overrightarrow{x_1y_1}$ be an arbitrary arc of D. Create $D' = D(\overrightarrow{x_1y_1})$ as above (see 2.1). Then choose an arbitrary arc $\overrightarrow{x_2y_2}$ of D' and create $D'' = D'(\overrightarrow{x_2y_2})$, etc.. Every member of the sequence $D, D', D'', \ldots$ belongs to $\mathcal{G}$ and by Lemma 2.1 each of them has exactly k kernels. Since the number of vertices in these digraphs increases, they are pairwise nonisomorphic.

Let s be the number of solutions of D. Then $c(D)$ belongs to $\mathcal{G}$ and it possesses s kernels as well as $c(D'), c(D''), \ldots$. In such way, $D, D', D'', \ldots$ have s solutions. $\square$

According to Corollary 2.2 it suffices to find one digraph of $\mathcal{G}$ with k kernels and s solutions for each pair of integers k, s. First, we shall present a digraph belonging to $\mathcal{G}$ with exactly one kernel. The smallest digraph with such property is $D_1 = (V_1, A_1)$ with $V_1 = \{a_1, b_1, c_1, d_1\}$ and with $A_1 = \{\overrightarrow{a_1b_1}, \overrightarrow{b_1c_1}, \overrightarrow{c_1d_1}, \overrightarrow{d_1a_1}, \overrightarrow{b_1d_1}\}$.

2.3. Corollary. *$\mathcal{G}$ contains infinitely many digraphs with exactly one kernel.*

2.4. Example. Let D_1 be defined as in the previous. Take the arc $\overrightarrow{b_1d_1}$ of D_1 and create $D_1' = D_1(\overrightarrow{b_1d_1})$. Then choose the arc $\overrightarrow{v_2d_1}$ of D_1' and construct $D_1'' = D_1'(\overrightarrow{v_2d_1})$, and continue in this way. Each member of the sequence $D_1, D_1', D_1'', \ldots$ belongs to $\mathcal{G}$ and by Lemma 2.1 they have exactly one kernel (and one solution).

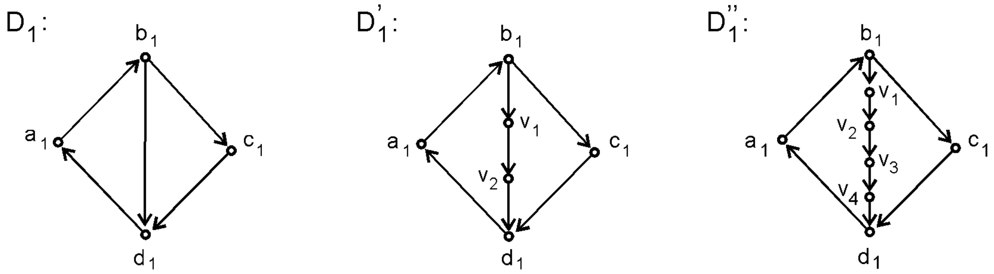

Fig.1: *The first three members of a sequence of digraphs $D_1, D_1', D_1'', \ldots$*

Let D_0 be the cycle with the vertices a_0, b_0, c_0 and with the arcs $\overrightarrow{a_0b_0}, \overrightarrow{b_0c_0}, \overrightarrow{c_0a_0}$. Let $D_2 = (V_2, A_2)$ where $V_2 = \{a_2, b_2, c_2, d_2\}$ and $A_2 = \{\overrightarrow{a_2b_2}, \overrightarrow{b_2c_2}, \overrightarrow{c_2d_2}, \overrightarrow{d_2a_2}\}$. Now we can construct digraphs with k kernels belonging to $\mathcal{G}$ for $k \geq 3$ in the following way: Let k be an integer, $k \geq 2$. Denote by $D_{k+1} = (V_{k+1}, A_{k+1})$ the digraph with

$$V_{k+1} = V_k \cup \{a_{k+1}, b_{k+1}, c_{k+1}, d_{k+1}\} \qquad \text{and}$$
$$A_{k+1} = A_k \cup \{\overrightarrow{a_{k+1}b_{k+1}}, \overrightarrow{b_{k+1}c_{k+1}}, \overrightarrow{c_{k+1}d_{k+1}}, \overrightarrow{d_{k+1}a_{k+1}}\} \cup$$
$$\cup \{\overrightarrow{c_{k+1}c_k}, \overrightarrow{a_2a_{k+1}}, \overrightarrow{a_3a_{k+1}}, \ldots, \overrightarrow{a_ka_{k+1}}\}.$$

The following figure shows the digraph $D_3 \in \mathcal{G}$ with three kernels, namely $\{a_2, c_2, b_3, d_3\}, \{b_2, d_2, a_3, c_3\}, \{b_2, d_2, b_3, d_3\}$.

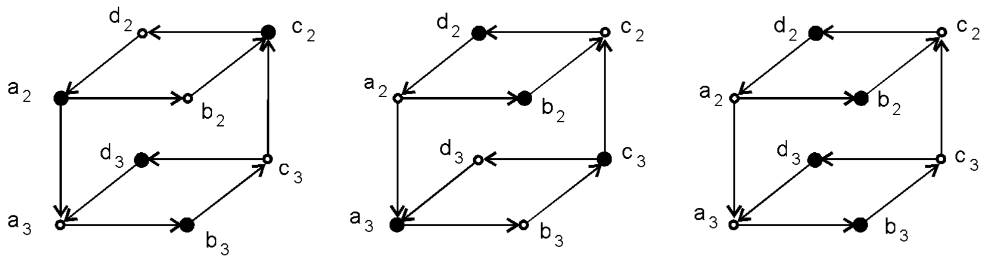

Fig.2: *The digraph D_3 is strongly connected and has three kernels.*

For every $k \geq 0$ let D_k be defined as above.

2.5. Proposition. *The digraph D_k has exactly k kernels for each nonnegative integer k.*

Proof. We have already stated that if $k \in \{0, 1, 2, 3\}$ then D_k has k kernels. Now, we shall proceed by induction. Let k be an integer, $k \geq 3$. For every kernel K of D_{k+1} holds that either $b_{k+1} \notin K$ or $b_{k+1} \in K$. In the first case $c_{k+1} \in K$, $d_{k+1} \notin K$ and $a_{k+1} \in K$. Hence $a_i \notin K$ for $i \in \{2, 3, \ldots, k\}$ and $K = \{b_2, d_2, b_3, d_3, \ldots, b_k, d_k, b_{k+1}, d_{k+1}\}$. In the second case $c_{k+1} \notin K, d_{k+1} \in K$ and there exists a bijective mapping ϕ from the set $\{K : K \text{ is a kernel of } D_k\}$ to the set $\{L : L \text{ is a kernel of } D_{k+1}\}$ given by $\phi(K) = K \cup \{b_{k+1}, d_{k+1}\}$. In this way we have shown that the digraph D_{k+1} has one kernel more than the digraph D_k. □

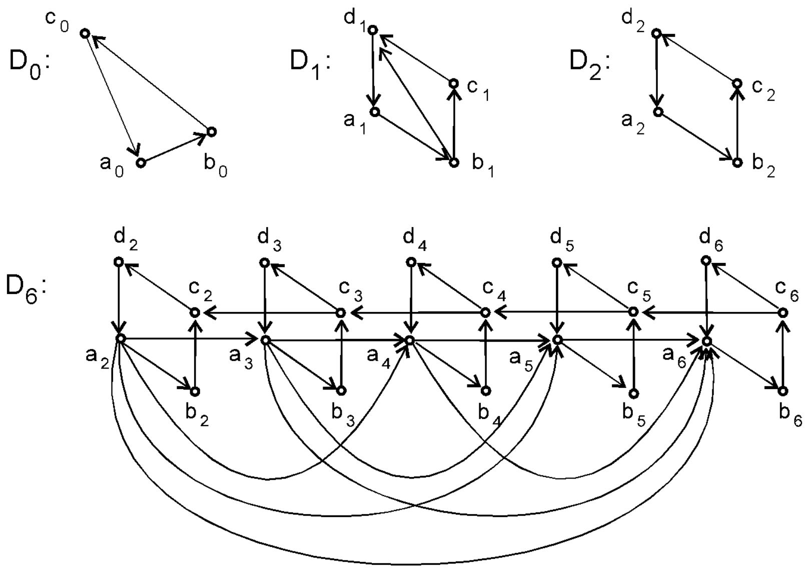

Fig.3: D_i *for* $i \in \{0, 1, 2, 6\}$ *with* $0, 1, 2$ *and* 6 *kernels, respectively.*

2.6. Theorem. *Let k, s be nonnegative integers. Then $\mathcal{G}_{(k,s)}$ is infinite.*

Proof. By Corollary 2.2 it suffices to show that $\mathcal{G}_{(k,s)} \neq \emptyset$.

We take D_k and $c(D_s)$ with the disjoint vertex sets $\{a_i, b_i, \ldots\}$ and $\{\overline{a_i}, \overline{b_i}, \ldots\}$, respectively. First we add all the possible arcs beginning at a vertex of $c(D_s)$ and ending at a vertex of D_k. Then we add one new vertex v together with arcs from v to every a_i, and to every existing d_i of D_k and from v to every $\overline{b_i}$ and every $\overline{c_i}$ of $c(D_s)$. Also we add all the arcs from every b_i and every c_i of D_k to v and from every $\overline{a_i}$ and every existing $\overline{d_i}$ of $c(D_s)$ to v.

We shall show that the resulting digraph D is an element of $\mathcal{G}_{(k,s)}$. It is easy to see that the digraphs D_k and $c(D_s)$ are strongly connected and therefore the digraph D which we have constructed is strongly connected as well. Any kernel K of D must not contain vertex v because if v would be in some kernel K then the independence of K implies that there is no more vertices in K; it means $K = \{v\}$ what contradicts to the absorbency of K.

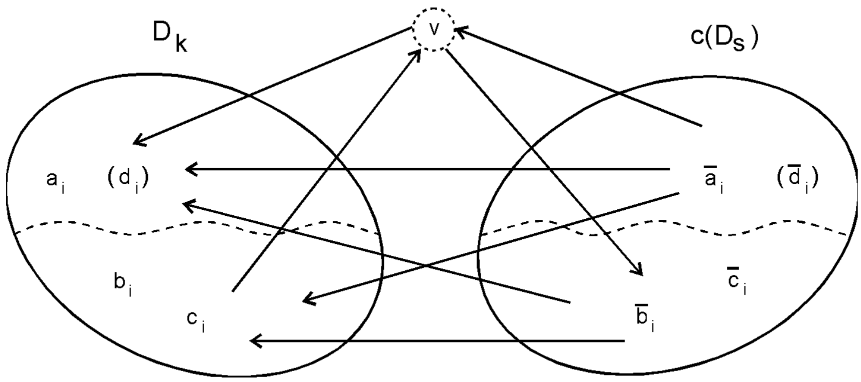

Fig.4: *The construction of a digraph belonging to* $\mathcal{G}_{(k,s)}$.

The independence of K implies that K cannot have nonempty intersections with D_k and with $c(D_s)$ simultaneously. It means that K is a subset of D_k or a subset of $c(D_s)$. But subsets of $c(D_s)$ are not absorbent in D. Therefore K is a kernel of D_k. Conversely, every kernel of D_k is a kernel of D because we did not add any new arc among the vertices of D_k (the independence) and each kernel of D_k contains either a_k or d_k, so it is absorbent not only in D_k but in D as well. Now we have verified that D has k kernels.

Analogously, the set of the all solutions of D is the same set as the set of the all solutions of $c(D_s)$, therefore D has s solutions, and the proof is complete. □

A digraph belonging to $\mathcal{G}_{(k,s)}$ with the minimal number of vertices is called *a minimal digraph* of $\mathcal{G}_{(k,s)}$. The number of the vertices of a minimal digraph of $\mathcal{G}_{(k,s)}$ will be denoted by $k \star s$.

2.7. Proposition. *Let* $k \star s = \min\{\operatorname{card} V : (V,A) \in \mathcal{G}_{(k,s)}\}$. *Then*

(i) $0 \star 0 = 3, \qquad 0 \star 1 = 1 \star 0 = 5, \qquad 0 \star 2 = 2 \star 0 = 6,$

(ii) $1 \star 1 = 2 \star 2 = 4, \qquad 1 \star 2 = 2 \star 1 = 5,$

(iii) *if* $k > 1$ *then* $k \star 0 \leq 4k$ *and* $k \star 1 \leq 4k + 1$ *and*

(iv) $k \star s \leq 4(k+s) - 7$ *whenever* $k > 1$ *and* $s > 1$.

Proof. By Lemmas 1.1 and 1.2 the operation $\star$ is commutative. The only minimal digraph of $\mathcal{G}_{(0,0)}$ is D_0 thus $0 \star 0 = 3$. The Figure 5 shows minimal digraphs of the classes $\mathcal{G}_{(0,1)}$, $\mathcal{G}_{(0,2)}$ and $\mathcal{G}_{(1,2)}$. D_1 is the minimal digraph of $\mathcal{G}_{(1,1)}$ and D_2 is the minimal digraph of $\mathcal{G}_{(2,2)}$. In order to obtain the inequalities for $k > 1$ it suffices to enumerate the number of vertices of the digraphs constructed above. □

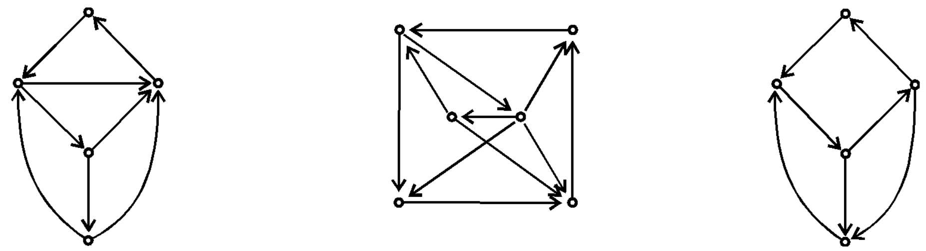

Fig.5: *Minimal digraphs of* $\mathcal{G}_{(0,1)}$, $\mathcal{G}_{(0,2)}$ *and* $\mathcal{G}_{(1,2)}$.

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