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ON $\vee$-IRREDUCIBLE ELEMENTS IN THE POSITIVE CONE OF AN ℓ-GROUP

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Abstract. Let G be an ℓ-group. The relations between the structure of G and the conditions concerning $\vee$-irreducible elements in the lattice G^+ are investigated in this paper.

1. Introduction

The classical theorem of Birkhoff ([1], pp. 142-143) deals with the representation of elements of a distributive lattice as irredundant joins of $\vee$-irreducible elements.

For related results and for further references cf. the expository paper of Dilworth [3].

Let G^+ be the positive cone of an ℓ-group G. Then G^+ is a distributive lattice. In this note we are concerned with the following conditions for the lattice G^+:

(a) For each $x \in G^+$ there exists an irredundant representation

$$x = x_1 \vee x_2 \vee \cdots \vee x_n$$

such that each x_i $(i = 1, 2, \ldots, n)$ belongs to G^+ and is $\vee$-irreducible in G^+.

(b) For each $x \in G^+$ there exists an irredundant representation $x = \bigvee_{i\in I} x_i$ such that each x_i belongs to G^+ and is $\vee$-irreducible in G^+.

The notion of completely subdirect product of linearly ordered groups was introduced by Šik [6].

In the present note we prove

(A) The condition (a) holds if and only if G is a direct sum of linearly ordered groups.

(B) The condition (b) is valid if and only if G is a completely subdirect product of linearly ordered groups.

The main result of the recent paper [7] is the following theorem:

$(*)$ Let L be a lattice such that

(i) L satisfies the descending chain condition;

(ii) each element of L has one and only one representation as an irredundant join of a finite number of $\vee$-irreducible elements.

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Then the lattice L is distributive.

Unfortunately, in the proof of $(*)$ there was applied a lemma (given on p. 95 of [7]) which is false, and the assertion $(*)$ is false as well (cf. the remark at the end of Section 3 below).

2. Preliminaries

Let L be a lattice. An element $a \in L$ is called $\vee$-irreducible if, whenever $b, c \in L$ and $a = b \vee c$, then either $a = b$ or $a = c$.

Let $x \in L$ and let $(x_i)_{i \in I}$ be an indexed system of elements of L such that the relation

$$x = \bigvee_{i \in I} x_i \tag{1}$$

is valid in L. The representation (1) is said to be irredundant if either

(i) $\operatorname{card} I = 1$,

or

(ii) $\operatorname{card} I > 1$ and whenever $j \in I$, then the relation

$$x = \bigvee_{i \in I \setminus \{j\}} x_i$$

fails to be valid.

For lattice ordered groups (shortly ℓ-groups) we apply the notation as in Conrad [2]. In particular, the group operation is denoted by $+$, though we do not suppose this operation to be commutative.

The positive cone G^+ of an ℓ-group G is the set $\{x \in G : x \geqq 0\}$.

Let $(G_i)_{i \in I}$ be an indexed system of ℓ-groups . The direct product

$$\prod_{i \in I} G_i$$

is defined in the usual way.

Assume that G is an ℓ-group and that we have an isomorphism

$$\varphi : G \to \prod_{i \in I} G_i \tag{2}$$

of G into $\prod_{i \in I} G_i$. For $g \in G$ and $i \in I$ we denote by g_i the component of $\varphi(g)$ in G_i.

If for each $i \in I$ and each element $t^i \in G_i$ there exists $g \in G$ with $g_i = t^i$, then (2) is said to be a subdirect product representation of G.

If, moreover, for each $i \in I$ and each $t^i \in G_i$ there exists $g \in G$ such that $g_i = t^i$ and $g_j = 0$ whenever $j \in I \setminus \{i\}$, then (2) is called a completely subdirect product decomposition of G.

Let (2) be a completely subdirect product decomposition of G. For each $g \in G$ put

$$I(g) = \{i \in I : g_i \neq 0\}.$$

Assume that $I(g)$ is finite for each $g \in G$. Then (2) is called a direct sum representation of G.

3. Convex chains in G^+

Again, let G be an ℓ-group . For $a, b \in G$ with $a \leqq b$, the interval $[a, b]$ is the set $\{x \in G : a \leqq x \leqq b\}$. A nonempty subset H of G is *convex* in G if, whenever $h_1, h_2 \in H$ and $h_1 \leqq h_2$, then $[h_1, h_2] \subseteq H$. A subset of G which is linearly ordered under the induced partial order is called a *chain* in G.

3.1. Lemma. *Let $a \in G^+$. Then the following conditions are equivalent:*

(i) *The element a is $\vee$-irreducible in G^+.*
(ii) *The interval $[0, a]$ of G is a chain.*

Proof. The validity of the implication (ii)⇒(i) is obvious. Suppose that (i) holds. By way of contradiction, assume that the condition (ii) is not valid. Then there are $x_1, x_2 \in [0, a]$ such that x_1 and x_2 are incomparable. Put $v = x_1 \vee x_2$; hence $v \in [0, a]$. There is $t \in G^+$ with $v + t = a$. Denote

$$y_i = x_i + t \quad (i = 1, 2).$$

Then y_1 and y_2 are incomparable. Moreover, $y_1 \vee y_2 = a$ and $y_1 < a$, $y_2 < a$. Therefore the element a fails to be $\vee$-irreducible , which is a contradiction. □

We denote by $\mathcal{C}(G^+)$ the system of all convex chains in G^+ containing the element 0. This system is partially ordered by the set-theoretical inclusion. Further, let $\mathcal{C}_m(G^+)$ be the system of all maximal elements of $\mathcal{C}(G^+)$.

3.2. Lemma. (i) *Let $0 < z \in G$, $[0, z] \in \mathcal{C}(G^+)$. Then $[0, 2z] \in \mathcal{C}(G^+)$.*
(ii) *Let $X, Y \in \mathcal{C}(G^+)$, $X \cap Y \neq \{0\}$. Then either $X \subseteq Y$ or $Y \subseteq X$.*
(iii) *Let $X \in \mathcal{C}(G^+)$. Then there exists $\overline{X} \in \mathcal{C}_m(G^+)$ such that $X \subseteq \overline{X}$.*

Proof. (i) First we show that whenever $x \in [0, 2z]$, then either $x \in [0, z]$ or $x \in [z, 2z]$.

In fact, let $x \in [0, 2z]$. By way of contradiction, suppose that z and x are incomparable. Put

$$u = x \wedge z, \quad v = x \vee z,$$

$$p = z - u, \quad q = x - u.$$

Then p and q are incomparable as well. Moreover,

$$p \wedge q = (z - u) \wedge (x - u) = (z \wedge x) - u = 0.$$

Since $p, q \in [0, z]$, the interval $[0, z]$ fails to be a chain, which is a contradiction.

Now let x_i $(i = 1, 2)$ belong to the interval $[0, 2z]$. Let $i \in \{1, 2, \}$. Then either $x_i \in [0, z]$ or $x_i \in [z, 2z]$. The interval $[z, 2z]$ is isomorphic to $[0, z]$ hence it is a chain. Therefore x_1 and x_2 are comparable. Hence $[0, 2z]$ is a chain.

(ii) Let X and Y satisfy the assumptions of (ii). By way of contradiction, assume that neither $X \subseteq Y$ nor $Y \subseteq X$ is valid. Hence there exist x, y with

$$x \in X \setminus Y, \quad y \in Y \setminus X.$$

Then x and y must be incomparable. Put

$$x \wedge y = z, \quad x - z = x_1, \quad y - z = y_1.$$

Hence we have

$$x_1, z \in X, \quad y_1, z \in Y.$$

Therefore z is comparable with both x_1 and y_1. We distinguish the following cases:

a) $x_1 \leqq z$ and $y_1 \leqq z$. Then

$$0 \leqq x = x_1 + z \leqq 2z,$$

and similarly $0 \leqq y \leqq 2z$. Thus according to (i), x and y must be comparable, which is a contradction.

b) $x_1 \geqq z$ and $y_1 \geqq z$. Then $x_1 \wedge y_1 \geqq z$. Since

$$x_1 \wedge y_1 = (x - z) \wedge (y - z) = 0,$$

we get $z = 0$.

Let $x' \in X$; put $x' \wedge y = z_1$. Hence $z_1 \in X \cap Y$. If $z_1 \geqq x$, then $x \in Y$, which is impossible. If $z_1 < x$, then $z_1 \leqq x \wedge y$, whence $z_1 = 0$. Therefore $x' \wedge y = 0$ for each $x' \in X$.

Then by a similar argument we conclude that for each $x' \in X$ and each $y' \in Y$ we have $x' \wedge y' = 0$, whence $X \cap Y = \{0\}$, which is impossible.

c) If neither a) nor b) is valid, then without loss of generality we can suppose that

$$y_1 \leqq z < x_1.$$

Hence we have

$$y = y_1 + z \leqq x_1 + z = x,$$

yielding that $y \in X$, which is a contradiction.

(iii) By applying (ii), we can use the same method as in [5], Proof of 1.4. □

If $G = \{0\}$, then both the assertions (A) and (B) obviously hold. In what follows we suppose that $G \neq \{0\}$; hence $G^+ \neq \{0\}$.

In 3.3 - 3.10 we assume that the lattice G^+ satisfies the condition (b).

3.3. Lemma. *Let $Y \in \mathcal{C}_m(G^+)$. Then $Y \neq \{0\}$.*

Proof. In view of the assumption, there exists $0 < x \in G^+$. Hence in view of (b), there is an irredundant representation

$$x = \bigvee_{i \in I} x_i$$

such that all x_i are $\vee$-irreducible . In view of the irredundancy, $x_i > 0$ for each $i \in I$. Choose an arbitrary $i \in I$. Thus according to 3.1, $[0, x_i] \in \mathcal{C}(G^+)$. Then 3.2 yields that there is $Y_i \in \mathcal{C}_m(G^+)$ with $[0, x_i] \subseteq Y_i$, hence $Y_i \neq \{0\}$. If $Y = \{0\}$, then $Y \subset Y_i$, thus Y fails to be maximal in $\mathcal{C}(G^+)$, which is a contradiction. □

Let $0 < x \in G^+$ and suppose that x is $\vee$-irreducible in G^+. Then in view of 3.1 and 3.2 there exists a uniquely determined element $\overline{x}$ of $\mathcal{C}_m(G^+)$ such that $x \in \overline{x}$. Further, 3.2 immediately implies

3.4. Lemma. *Let $0 < x \in G$, $0 < y \in G$. Suppose that both x and y are $\vee$-irreducible . Then either $\overline{x} = \overline{y}$ or $x \wedge y = 0$.*

3.5. Lemma. *Let $Y \in \mathcal{C}_M(G^+)$. Then the set Y has no upper bound in G^+.*

Proof. a) In view of 3.3 there exists $0 < y \in Y$.

First we prove that $2y \in Y$. In fact, in view of 3.2 (i), $[0, 2y] \in \mathcal{C}(G^+)$. According to 3.2 (iii) there is $Y_1 \in \mathcal{C}_m(G^+)$ with $[0, 2y] \subseteq Y_1$. Hence $Y \cap Y_1 \neq \{0\}$. Then 3.2 (ii) yields that either $Y \subseteq Y_1$ or $Y_1 \subseteq Y$. Since both Y and Y_1 are maximal elements of $\mathcal{C}(G^+)$ we get $Y = Y_1$.

Thus for each $0 < t \in Y$ we have $t < 2t \in Y$. Therefore Y has no greatest element.

b) By way of contradiction, suppose that there is $x \in G^+$ such that x is an upper bound of the set Y in G^+. Clearly $x > 0$. Let $(x_i)_{i \in I}$ be as in the proof of 3.3.

It is well-known that the lattice G^+ is infinitely distributive, hence for each $0 < y \in Y$ we have

$$y = y \wedge x = y \wedge (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (y \wedge x_i).$$

According to 3.1, y is $\vee$-irreducible , hence there is $i \in I$ with $y = y \wedge x_i$. Thus $y \leqq x_i$. Hence in view of 3.4, $\overline{y} = \overline{x}_i$.

Let $i_1 \in I$, $i_1 \neq i$. Since the representation of x under consideration is irredundant, the elements x_i and x_{i_1} are incomparable. Thus 3.4 yields that $y_1 \wedge x_{i_1} = 0$ for each $y_1 \in Y$.

On the other hand, there exists $i_2 \in I$ such that $y_1 \leqq x_{i_2}$. Thus we must have $i_2 = i$. Then x_i is the greatest element of Y. In view of a), we arrived at a contradiction. □

For each $Y \in \mathcal{C}_m(G^+)$ we put

$$Y' = Y \cup \{-Y\},$$

$$Y^* = \{g \in G : |g| \wedge y = 0 \quad \text{for each } y \in Y\}.$$

3.6. Lemma. *Let $Y \in \mathcal{C}_m(G^+)$. Then*

(i) *Y' is a convex chain in G;*
(ii) *Y' is an ℓ-subgroup of G;*
(iii) *Y' fails to be bounded in G.*

Proof. (iii) is a consequence of 3.5. Then by applying [4] (Lemmas 3 and 5) we get that (i) and (ii) are valid. □

3.7. Lemma. *There exists a mapping φ_Y of G onto $Y' \times Y^*$ such that*

(i) *φ_Y is a direct product decomposition of G;*
(ii) *if $g \in G^+$ and $\varphi_Y(g) = (y, y^*)$, then*

$$y = \max\{y_1 \in Y : y_1 \leqq g\}.$$

In particular, if $g \in Y$, then $y = g$.

Proof. (i) is a consequence of 3.6 and of [4], Theorem 1. The assertion (ii) is well-known (moreover, for validity of (ii) the corresponding direct factor need not be linearly ordered). □

3.8. Lemma. *Let x and $(x_i)_{i\in I}$ be as in 3.3. Let $i_0 \in I$; put $\overline{x}_{i_0} = Y$. If $\varphi_Y(x) = (y, y^*)$, then $y = x_{i_0}$.*

Proof. We have $x_{i_0} \in Y$ and $x_{i_0} \leqq x$. Let $y_1 \in Y$, $y_1 \leqq x$. Then in view of 3.4, $y_1 \wedge x_i = 0$ for each $i \in I \setminus \{i_0\}$. Hence

$$y_1 = y_1 \wedge x = y_1 \wedge (\bigvee_{i\in I} x_i) = \bigvee_{i\in I}(y_1 \wedge x_i) = x_{i_0} \wedge y_1.$$

Thus $y_1 \leqq x_{i_0}$. According to 3.7 (ii), $y = x_{i_0}$. □

Let $\{Y_j\}_{j\in J}$ be the set $\mathcal{C}_m(G^+)$. For $g \in G$ and $j \in J$ we denote by g_j the element of Y'_j such that, under the direct product decomposition

$$\varphi_Y : G \to Y'_j \times Y^*_j,$$

the component of g in Y'_j is g_j.

Consider the mapping

$$\varphi : G \to \prod_{j\in J} Y'_j$$

defined by $\varphi(g) = (g_j)_{j\in J}$.

From the definition of φ we immediately obtain

3.9. Lemma. *φ is a homomorphism of the ℓ-group G into the ℓ-group $\prod_{j\in J} Y'_j$.*

Let $g \in G$ and assume that $\varphi(g) = 0$. Hence $\varphi(|g|) = 0$ and $|g| \geqq 0$. If $|g| > 0$, then there exists an irredundant representation

$$|g| = \bigvee_{k\in K} z_k$$

such that each z_k is $\vee$-irreducible . In particular, $z_k > 0$ for each $z_k \in Y_{j(k)}$. Then in view of 3.8,

$$|g|_{j(k)} = z_k,$$

whence $\varphi(|g|) \neq 0$, which is a contradiction.

From this and from 3.9 we infer

3.10. Lemma. *φ is an isomorphism of G into $\prod_{j\in J} Y'_j$.*

3.11. Lemma. *Let $j \in J$ and $t^j \in Y_j$. Then*

(i) $(t^j)_j = t^j$;

(ii) *if $j_1 \in J$ and $j_1 \neq j$, then $(t^j)_{j_1} = 0$.*

Proof. The case $t^j = 0$ is trivial. Let $t^j > 0$. Then in view of 3.7 (ii) we have $(t^j)_j = t^j$. If $j_1 \in J$, $j_1 \neq j$, then 3.4 and 3.7 yield that $(t^j)_{j_1} = 0$.

If $t^j < 0$, then it suffices to consider the element $-t^j$. □

3.12. Proposition. *Let G be an ℓ-group such that the lattice G^+ satisfies the condition* (b). *Let φ be as above. Then φ is a completely subdirect product decomposition of G.*

Proof. This is a consequence of 3.10 and 3.11. □

3.13. Lemma. *Let G be an ℓ-group which can be represented as a completely subdirect product of linearly ordered groups. Then the lattice G^+ satisfies the condition* (b).

Proof. Let

$$\varphi : G \to \prod_{t\in T} G_t$$

be a completely subdirect product decomposition of G. Without loss of generality we can suppose that $G_t \neq \{0\}$ for each $t \in T$. Let $0 < g \in G$ and $\varphi(g) = (g_t)_{t\in T}$. Then for each $t \in T$ there exists $\overline{g}_t \in G$ such that

$$(\overline{g_t})_{t\in T} = g_t \ ,$$

$$(\overline{g}_t)_{t_1} = 0 \quad \text{if} \quad t_1 \in T \setminus \{t\}.$$

Then $\overline{g}_t$ is $\vee$-irreducible for each $t \in T$.

Put $T_1 = \{t \in T : g_t \neq 0\}$. We have $T_1 \neq \emptyset$. Moreover,

$$g = \bigvee_{t\in T_1} \overline{g}_t$$

and this representation of g is irredundant. Therefore the condition (b) holds for the lattice G^+. □

From 3.12 and 3.13 we conclude that (B) is valid.

Now suppose that G is an ℓ-group such that the lattice G^+ satisfies the condition (a). Since (a) is stronger than (b), we can apply 3.12. For $g \in G$ and $j \in J$ we put $(\varphi(g))_j = g_j$.

3.14. Lemma. *Let $g \in G$. There exists a finite subset J_1 of J such that $g_j = 0$ whenever $j \in J \setminus J_1$.*

Proof. First suppose that $g > 0$. In view of (a) there exists an irredundant representation

$$g = x_1 \vee x_2 \vee \cdots \vee x_n \tag{1}$$

such that each x_i $(i = 1, 2, \dots, n)$ is $\vee$-irreducible . Further, for each x_i there exists $j(i) \in J$ such that

$$\overline{x}_i = Y_{j(i)}.$$

Put $J_1 = \{j(1), j(2), \dots, j(n)\}$. If $j \in J \setminus J_1$, then (1) and 3.11 yield that $g_j = 0$.

Thus the assertion of the lemma is valid in the case $g > 0$. Next, since each element g of G can be written in the form $g = u - v$ with $u, v \in G^+$, we conclude that the assertion is valid for an arbitrary element of G. □

3.15. Proposition. *Let G be an ℓ-group such that the lattice G^+ satisfies the condition* (a). *Let φ be as in* 3.12. *Then φ is a direct sum decomposition of G.*

Proof. This is a consequence of 3.12 and 3.14. □

3.16. Lemma. *Let G be an ℓ-group which can be represented as a direct sum of linearly ordered groups. Then the lattice G^+ satisfies to condition* (a).

Proof. We apply an analogous notation as in the proof of 3.13; the distinction is that φ is now a direct sum representation of G. Then the set T_1 is finite, whence (a) is valid. □

In view of 3.15 and 3.16 we infer that (A) holds.

We conclude with the following remark concerning the paper [7].

Let L be a lattice. For each $a \in L$ we put

$$L(a) = \{x \in L : x \leqq a\}.$$

Further, let $(*)$ be as in Section 1. The following lemma was presented in [7]:

$(**)$ If L is a lattice which satisfies the conditions (i) and (ii) from $(*)$, then for each $a \in L$, the set $L(a)$ is finite.

Let $\mathbb{N}$ be the set of all positive integers with the natural linear order and let ω be an infinite ordinal. We put $L = \mathbb{N} \cup \{\omega\}$ and for each $n \in \mathbb{N}$ we set $n < \omega$. Then L is a linearly ordered set, hence the condition (ii) is satisfied. Moreover, the descending chain condition is valid in L. But the set $L(\omega)$ fails to be finite. Hence $(**)$ does not hold.

Next let L be the lattice on Fig. 1. Then $\{u, a, b, c\}$ is the set of $\vee$-irreducible elements of L. The lattice L satisfies the conditions (i) and (ii) from $(*)$, but it fails to be distributive. Hence the assertion $(*)$ is not valid.

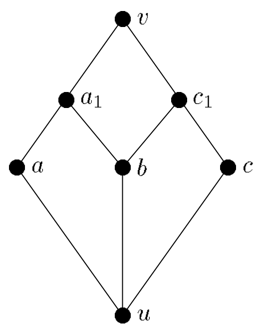

Fig. 1

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