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FUZZY MAPPINGS AND FUZZY METHODS FOR CRISP MAPPINGS

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Abstract. We deal with the notion of fuzziness from two different aspects. First we study the properties of fuzzy functions, mostly their derivatives, integrals and fixed point properties. The second aspect is the study of a classical real function, fuzzyfying the notions themselves. The last part of the paper is devoted to this aim, showing that introducing the methods of fuzzy mathematics can provide some interesting results for the real functions theory.

The work is a review of results already published or submitted.

1. Introduction

In a natural language we often use words describing the grade of some quality (very, somewhat, a little, ...) or the quality itself (young, heavy, dark, loud, etc.), quite often combined together. Usually a given object corresponds to the above mentioned linguistic construction only with some degree of membership. Therefore such constructions do not define sets, as we use the notion of a set in classical mathematics. We can define the set of all real numbers that are greater than ten, but the expression "numbers much greater than ten" does not define a set. On the other hand, in various situations we need to build e.g. a decision model based on "linguistic variables". The theory of fuzzy sets (or, more generally, fuzzy mathematics) provides us a tool to handle it.

The notion of a fuzzy set was introduced by Zadeh in [Za 1] in 1965. Since then the fuzzy mathematics has developed in a variety of directions. The most fruitful, also from the applications point of view, seems to be the fuzzy control. Moreover, the fuzzy theory appears to be a convenient tool for those applications, where the exact quantitative description of a particular model is either impossible or inappropriate. Hence we often find fuzzy methods used in various decision models designed for the real-time performance. Fuzzy objects and methods can be found also in other applications, like regulation, production control, household appliances, and many others, including music (see [Ha 1], [Ha 2]). The theoretical

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and methodological background of fuzzy mathematics is fuzzy logic and approximate reasoning closely connected to fuzzy logic. A review of the current state of fuzzy logic can be found in the work by Novák [No 1].

In this work we deal with two aspects of mappings analysis. The first one is the study of fuzzy mappings. These mappings are mainly understood as functions that assign a fuzzy set (or a fuzzy element) to a (crisp) element of a given domain. We study the derivatives of such mapping. In this area this work is a direct continuation of the study by Kalina [Ka 1]. The results concerning fixed points of fuzzy mappings, their derivatives and integrals have been published in Janiš [Ja 1], [Ja 2] and Janiš and Nedić [JN 1]. The second aspect of our study corresponds to the attitude proposed by Burgin and Šostak in [BS 1]. Here the authors deal with classical functions, but use methods of fuzzy mathematics. This way they obtain more compact and more general results for such functions. Namely, they study fuzzy continuity of classical functions. We continue in this direction and use fuzzy methods for the study of uniform continuity and properties of derivatives. It is shown that also here it is possible to achieve results that generalize results of classical mathematical analysis. This part of the work summarizes the results published in [Ja 3] and [Ja 4].

2. Basic notions and definitions

Let X be a nonempty set. Any of its subsets $A \subset X$ can be identified with its characteristic function $\chi_A(x)$. This function can be considered as a membership degree with which the element x belongs to the set A. Obviously in case of an ordinary (crisp) subset this value can be either one (the element x belongs to A) or zero (the element x does not belong to A). But if we deal with collections of object defined only vaguely, it is sometimes not possible to cope with only these two possibilities. Hence we generalize the concept of the characteristic function, allowing it to attain also values between zero and one. This attitude serves as a motivation for the definition of a fuzzy subset of X.

Definition 1. A *fuzzy subset* A of the set X is a function $A : X \to [0;1]$. The set of all fuzzy subsets of X is denoted by $\mathcal{F}(X)$.

If the set X (the universe) is given, we speak briefly of fuzzy sets instead of fuzzy subsets of X.

A fuzzy set A is called *modal*, if there exists at least one element $x \in X$ for which $A(x) = 1$. The set of all those $x \in X$ for which $A(x) > 0$ is called the *support* of A and denoted by $supp(A)$.

Although fuzzy sets can be studied in more general frameworks, e.g. with values in a lattice (see Goguen in [Go 1]), for our purposes it will be sufficient to work only with fuzzy sets with values in the interval $[0;1]$. To distinguish between fuzzy sets and classical ones we sometimes use the word *crisp* to denote usual non-fuzzy sets (or other objects).

A useful tool for the study of fuzzy sets are their *α-cuts*.

Definition 2. Let X be a metric space, $A \in \mathcal{F}(X), \alpha \in [0;1]$. The *$\alpha$-cut* of A is the set $A_\alpha = \{x \in X; A(x) \geq \alpha\}$ for $\alpha > 0$. For $\alpha = 0$ we put $A_0 = cl\{x \in X; A(x) > 0\}$, where *cl* is the closure operator.

In other words, if $A \in \mathcal{F}(X), \alpha \in (0;1]$, then

$$A_\alpha = A^{-1}([\alpha;1]), A_0 = cl\{\cup A_\alpha, \alpha > 0\}.$$

If we consider the strict inequality in the definition of A_α, then we speak of a *strict* α-cut.

Throughout this paper the symbol R will denote the set of all real numbers.

If $A \in \mathcal{F}(R)$ and there exists at least one $x \in R$ for which $A(x) = 1$, then A is called a *fuzzy number*. Sometimes this set is called just a *fuzzy quantity* and usually there are additional conditions on fuzzy numbers, very frequently we require that all the α-cuts should be convex sets (in that case we speak of *convex fuzzy numbers*). In some circumstances we assume that the value x for which $A(x) = 1$ is unique. If this is not the case, the term *fuzzy interval* is preferred. Very often, mainly in applications, we work with linear fuzzy numbers (we assume that the function A is partwise linear). For more information on the operations with linear fuzzy numbers see e.g. [Ko 1].

In some cases it is more convenient to define a fuzzy number as a non-decreasing left-continuous function $A : R \to [0;1]$, with the properties $\lim_{x\to-\infty} A(x) = 0$ and $\lim_{x\to\infty} A(x) = 1$. This representation corresponds to the linguistic construction "the number is much greater than A", while the representation with the unique modal value corresponds to the statement "the number is about equal to A". Because of the resemblance to the distribution function we sometimes address this representation as statistical one.

The complement of a fuzzy set is defined in the following way:

Definition 3. If A is a fuzzy set, then the function $1 - A$ is the *complement* of the fuzzy set A.

For the definition of the intersection of fuzzy sets the notion of a t-norm (a triangular norm, see Schweizer and Sklar in [SS 1]) is used. For more detailed study of t-norms see the monograph [KMP].

Definition 4. Let $T : [0;1]^2 \to [0;1]$ be a commutative, associative, non-decreasing function in both variables, fulfilling the boundary conditions $T(x,1) = x$. Then T is called a *t-norm* (a *triangular norm*).

Example 1. It is easy to show that the function

$$T : [0;1]^2 \to [0;1], T(x,y) = \min\{x,y\}$$

is a t-norm. We denote this t-norm by the symbol T_{min}. Note that if T is an arbitrary t-norm, then $T \leq T_{min}$.

Example 2. Let $T : [0;1]^2 \to [0;1]$,

$$T(x,y) = \begin{cases} 0 & \text{if } \max\{x,y\} < 1 \\ \min\{x,y\} & \text{if } \max\{x,y\} = 1 \end{cases}$$

Then T is a t-norm. It is called the weakest t-norm (or the drastic product) and denoted by T_W. Note that for an arbitrary t-norm T the inequality $T_W \leq T$ holds, which justifies the adjective "weakest".

In a lot of papers the minimum function is used to define the intersection of fuzzy sets, but it appears that this is not always the most convenient way to think of the intersection from practical applications point of view.

Definition 5. If A, B are fuzzy sets and a t-norm T is given, then the fuzzy set C for which $C(x) = T(A(x), B(x))$ for each $x \in X$ is the *intersection* of A and B based on the t-norm T.

Although we will not use the notion of a conorm, for the sake of completeness we mention that the union of fuzzy sets based on a given t-norm is given by the de Morgan rule, i.e.

$$S_T(A(x), B(x)) = 1 - T(1 - A(x), 1 - B(x)).$$

The function S_T is a *t-conorm* dual to T. The t-conorm is sometimes called also an S-norm, although this notation may lead to misunderstanding as the dependence on T cannot be seen here.

A t-norm is used also in addition of fuzzy numbers in the following way, which is the special case of the Zadeh extension principle (for more information on the extension principle see [Ng 1] and [BBT]):

Definition 6. If A, B are fuzzy numbers, then their *sum* based on the t-norm T is the fuzzy number $A +_T B$ such that

$$(A +_T B)(z) = \sup\{T(A(x), B(y)), z = x + y\}.$$

In the first part of this work we will focus on fuzzy mappings. There are several different definitions of this notion. We will present a review of the most frequent ones.

Perhaps the most general concept of a fuzzy function is the following, used in Negoita and Ralescu [NR 1]:

Definition 7. A *fuzzy function* from X to Y is a mapping $p : X \times Y \to [0;1]$.

The generality of the later definition (which is in fact a fuzzy relation) may not be convenient – we often require the values of a fuzzy mapping (i.e. the mappings $p(x,.) : Y \to [0;1]$ to be nonzero for at least one $y \in Y$. Such cases were studied e.g. in Dubois and Prade [DP 1]:

Definition 8. A mapping $p : X \times Y \to [0;1]$ is a *fuzzy function* if for all $x \in X$ there exists $y \in Y$ such that $p(x,y) > 0$.

If moreover modality of the values is required, then the following definition (see Ovchinnikov [Ov 1]) is convenient:

Definition 9. A mapping $p : X \times Y \to [0;1]$ is a *fuzzy function* if for all $x \in X$ there exists the unique $y \in Y$ such that $p(x, y) = 1$.

By omitting the uniqueness of the modal value in the later definition we obtain a fuzzy multifunction from X to Y. For more detailed study of this subject see e.g. Tsiporkova [Ts 1].

In the study of probabilistic and fuzzy metric spaces we sometimes represent fuzzy numbers as nondecreasing left-continuous functions from the real line into the unit interval. Another reason for using this representation is also the fact that it enables to introduce a topology on the fuzzy real line. In such situations the following attitude to fuzzy mappings used in Šešelja [Se 1] can be useful:

Definition 10. A mapping $p : X \times Y \to [0;1]$ is a *fuzzy function* if for all $x \in X$ there exists the unique $y \in Y$ such that $p(x, y) = 1$ and if each $\alpha \in (0;1)$ appears at most once as a membership value of p.

There are also other definitions of fuzzy functions used by various authors. A brief review of the most frequent concepts of this notion can be found in the work by Filep [Fi 1].

For our purposes the following convention will be most convenient: Speaking about a fuzzy function f from X to Y we mean a mapping that assigns a fuzzy number $f(x) \in \mathcal{F}(Y)$ to an element $x \in X$. Hence we deal with the case of crisp argument and fuzzy image, on the contrary to the paper [Ka 1], where the author studies also the cases of fuzzy argument and crisp image and both fuzzy argument and image. This does not apply for the section 6, where we study usual crisp functions but use fuzzy methods.

The particular representation of a fuzzy number (either "statistical" or "modal") and if necessary also other supplementary conditions required will be mentioned explicitly when necessary.

3. Derivatives of fuzzy functions

By a fuzzy real number we will understand a fuzzy set $\rho : [-\infty; \infty] \to [0;1]$ for which

(1) $\rho(-\infty) = 0$,
(2) $\rho(\infty) = 1$,
(3) $\rho(r) = \sup\{\rho(s), s < r\}$ for each $r \in R$.

Hence we define a fuzzy real number in the sense of the comment following the Definition 2 (the statistical representation). The set of all fuzzy real numbers will be denoted by $H(R)$.

The set of all (crisp) real numbers is embedded into $H(R)$ in the following way: a crisp real number t is represented by the function $\delta_t \in H(R)$ for which $\delta_t(r) = 0$ if $r \leq t$ and $\delta_t(r) = 1$ if $r > t$.

The partial ordering in $H(R)$ is given by the following way:

$$\rho \leq \sigma \text{ if and only if } \rho(r) \geq \sigma(r) \text{ for each } r \in R.$$

The addition of fuzzy numbers in $H(R)$ is based on the minimum t-norm T_{min} (see Definition 6), i.e.

$$(\rho + \sigma)(r) = \sup\{\min\{\rho(s), \sigma(t)\}; r = s + t\}.$$

The multiplication by a nonnegative real number is given by the formula

$$\begin{aligned}(c.\rho)(r) &= \rho(r/c) \quad \text{if } c > 0,\\ &= \delta_0(r) \quad \text{if } c = 0.\end{aligned}$$

A fuzzy mapping from R to R will be understood as a mapping that assigns a fuzzy real number in $H(R)$ to a real number. We will assume that a fuzzy mapping we are working with is defined for each $x \in R$.

Klement in [Kl 1] defines an extension of the Lebesgue integral for this type of fuzzy functions. Our aim will be to define a derivative of these functions that will be connected to the mentioned integral. A drawback of this attitude will be its validity only for the case of using the minimum t-norm.

First we introduce the notion of the pseudoinverse of a fuzzy real number. For more details see the work by Höhle [Ho 1].

Definition 11. Let $\rho \in H(R)$. Its *pseudoinverse* is the function $\rho^{(-1)} : [0;1] \to [-\infty;\infty]$ for which $\rho^{(-1)}(\alpha) = \sup\{r; \rho(r) < \alpha\}$.

In the last definition we use the convention $\sup \emptyset = -\infty$, hence the value of each pseudoinverse at zero is $-\infty$.

We will also need the pseudoinverse of a fuzzy function, which will be defined in the following way:

Definition 12. The pseudoinverse of a fuzzy function f is the mapping $f^{(-1)}$ for which $f^{(-1)}(x) = (f(x))^{(-1)}$.

We will denote by $H^{(-1)}(R)$ the set of all pseudoinverses of fuzzy real numbers. The author in [Kl 1] works only with the set of all positive fuzzy real number, but the following result holds also for our case: The mapping

$$p : H(R) \to H^{(-1)}(R), p(\rho) = \rho^{(-1)}$$

is an involutive order-preserving isomorphism, where the addition in $H^{(-1)}(R)$ is the usual addition of functions and the multiplication is the usual multiplication of a function by a nonnegative number.

We will use this isomorphism to define differentiability and the derivative of a fuzzy function in the following way:

Definition 13. Let $f : R \to R$ be a fuzzy function, let $x_0 \in R$. The fuzzy function f is *differentiable* at x_0, if there exists the mapping

$$h_{x_0} : a \mapsto \frac{df^{(-1)}(x_0)}{dx}(a)$$

as an element of $H^{(-1)}(R)$.

Definition 14. If f is differentiable at x_0, then the function $f'(x_0) = [h_{x_0}]^{(-1)}$ is the *derivative* of f at the point x_0.

Obviously if the derivative of a fuzzy function at a point exists, then it is a fuzzy number. Therefore it is possible to define the derivative of f, which will be a fuzzy function. Moreover, it is easy to verify that this notion is an extension of the classical derivative of a real function in the following way:

If f is a (crisp) function differentiable at the point a and if we identify $f(x)$ with $\delta_{f(x)}$ for each x in a neighborhood of a, then the derivative of f at a as indicated in the Definition 14 will be the fuzzy number δ_z, where z in the index is the usual (crisp) derivative of f at the point a.

The connection with the fuzzy integration from the Klement's work [Kl 1] is given by the theorem literally analogical to the classical theorem on the derivative of a function with the variable in the upper integral limit (the mean theorem of integral calculus).

4. Derivatives and fixed points of fuzzy functions

A great deal of this section is based on the paper by Kalina [Ka 1], where the derivative of a fuzzy valued mapping has been defined. We will briefly present basic notions and definitions.

Although a generalization into ordered Banach spaces would be possible, for the sake of simplicity the author demonstrates his apparatus on real fuzzy functions, that means on mappings that assign an LR-fuzzy number to a crisp real number. For more information on L-R-fuzzy numbers see e.g the paper by Mesiar [Me 1], and on their addition see papers by Marková [Ma 1] and [Ma 2].

Let f be a real fuzzy function. We introduce its *level functions* f_α and $f_{-\alpha}$ in the following way: Let $\alpha \in (0;1]$. Then

$$f_\alpha(x) = \sup\{z \in R; f(x)(z) \geq \alpha\}$$

and

$$f_{-\alpha}(x) = \inf\{z \in R; f(x)(z) \geq \alpha\}.$$

The level functions are used in [Ka 1] to define the derivative of f in the following way:

Definition 15. Let f_α and $f_{-\alpha}$ be level functions of a fuzzy function f. Suppose all the level functions are differentiable at a point x and $f_\alpha{}'(x)$ and $f_{-\alpha}{}'(x)$ be their derivatives at x. Denote

$$S = \sup\{z \in R; (\exists\alpha \in (0;1))(z \leq \max\{f_\alpha{}'(x), f_{-\alpha}{}'(x)\})\}$$
$$I = \inf\{z \in R; (\exists\alpha \in (0;1))(z \geq \min\{f_\alpha{}'(x), f_{-\alpha}{}'(x)\})\}.$$

By the *derivative of the fuzzy function* f at the point x we mean the fuzzy number $f'(x)$ defined in the following way:

$$f'(x)(y) = \begin{cases} 1 & \text{if } y = f_1{}'(x) \\ 0 & \text{if } y \notin (I;S) \\ \sup\{\alpha \in (0;1); \max\{f_\alpha{}'(x), f_{-\alpha}{}'(x)\} \geq y\} & \text{if } f_1{}'(x) < y \leq S \\ \sup\{\alpha \in (0;1); \min\{f_\alpha{}'(x), f_{-\alpha}{}'(x)\} \leq y\} & \text{if } f_1{}'(x) > y \geq I. \end{cases}$$

A derivative of a fuzzy function as a function with fuzzy values has been studied also in [FF 1], [PR 1] and some other works. The definition introduced in [Ka 1] is on one hand less general, but on the other hand provides a wider class of differentiable functions. Properties of this derivative are further studied in [Ka 2], [Ka 3], [Ja 1], [JN 1] and [Ja 2].

The basic question that arises with any type of differentiation is its linearity. In [Ja 2] we prove the following:

Proposition 1. *If the fuzzy functions f and g have fuzzy derivatives f' and g' at the point a, their sum $f +_T g$ has the fuzzy derivative $(f +_T g)'$ at a, where T is a t-norm, then $(f +_T g)'(a) \leq f'(a) +_T g'(a)$.*

The next example shows that the opposite inequality in general does not hold for any triangular norm T, not even if we use different t-norms for the additions on each of its sides:

Example 3. Let the fuzzy functions $f, g : [0;2] \to R$ be given by the following formulas: For $x \in [0;2]$ put

$$f(x)(t) = \max\left\{0; 1 - \frac{|t|}{x+1}\right\}, \quad t \in R,$$

$$g(x)(t) = \max\left\{0; 1 - \frac{|t|}{3-x}\right\}, \quad t \in R.$$

All the level functions of f and g are linear and hence differentiable on the interval $[0;2]$. For an arbitrary $x \in [0;2]$ the derivatives of f and g are equal (we take the one-side derivatives at the endpoints of the interval) and

$$f'(x)(t) = g'(x)(t) = \max\{0; 1 - |t|\}, \quad t \in R.$$

We see that both f' and g' are constant fuzzy functions on $[0;2]$. Note that their sum with respect to an arbitrary t-norm is again a constant fuzzy function and its common value is not a crisp number.

On the other hand take an arbitrary t-norm T for which all the level functions of the fuzzy function $f +_T g$ are differentiable. Then using the fact that for any $x \in [0;1]$ there is

$$f(1-x) +_T g(1-x) = f(1+x) +_T g(1+x)$$

as a consequence of the Rolle theorem we obtain that the derivative of all the level functions for the sum $f +_T g$ at the point $x = 1$ is zero. Therefore $(f +_T g)'(1)$ is the crisp number zero and so $(f +_T g)'(1) < f'(1) +_T g'(1)$ for an arbitrary t-norm T.

Moreover, this inequality holds for even an arbitrary pair of t-norms, as we have the crisp zero on the left-hand side for any t-norm and the fuzzy (not crisp) zero on the right-hand side again for any t-norm.

In the work [Ja 1] we deal with the existence of fuzzy fixed points for fuzzy functions and with their properties. Here the fuzzy function f is understood in the following way:

We take (X, d) a complete metric space. The corresponding Hausdorff metric in the space of all nonempty compact subsets of X will be denoted by h. Let for each $x \in X$ there exist an upper semicontinuous function $f(x) : X \to [0; 1]$. Moreover, we require that the inverse images $f_x^{-1}((\alpha; 1])$ are nonempty compact sets for any $\alpha \in (0; 1]$.

Let $\alpha \in (0; 1]$. A point $x \in X$ will be called an *α-fixed point* of f iff $x \in (f(x))_\alpha$. (This is a generalization of the classical case, when the crisp fixed point can be understood as 1-fixed point.)

The condition of contractivity is reformulated in the following way: Let $\alpha \in (0; 1]$. We will say that a fuzzy function f that maps X into itself is *α-contractive* iff there exists a real number q, $0 < q < 1$ such that for each $x_1, x_2 \in X$ there is

$$h((f(x_1))_\alpha, (f(x_2))_\alpha) < q d(x_1, x_2),$$

where h is the Hausdorff metric on X.

The main result of [Ja 1] is the following statement:

Proposition 2. *Let (X, d) be a complete metric space, $\alpha \in (0; 1]$ and let $f : X \to X$ be an α-contractive fuzzy function. Then there is an α-fixed point of f in X.*

Obviously, in contrary to the classical case, there can be more than one fuzzy fixed point of a fuzzy function. The set of all fuzzy fixed points can be characterized by the following propositions:

Proposition 3. *The set of all α-fixed points of an α-contractive function is closed in (X, d).*

Proposition 4. *The set of all α-fixed points of an α-contractive function f is bounded in (X, d). The upper bound for the diameter of this set is the number $\frac{1}{1-q} diam(f(x_0))_\alpha$, where x_0 is an arbitrary α-fixed point of f and q is the contraction coefficient.*

These propositions generalize results on fixed point of fuzzy mappings achieved in [He 1], [ST 1] and [Rh 1].

There is a relationship between the derivative of a differentiable fuzzy function and the existence of its fixed points similar to the crisp case. More details can be found in [Ka 1].

For the purpose of finding a fuzzy analogy to the Lebesgue (or Riemann) integral it is convenient to think of a fuzzy function in the sense of [Se 1] (see Definition 10). From practical reasons we restrict ourselves on nonnegative fuzzy functions. Therefore in this section we will use the concept of a nonnegative fuzzy number introduced by Höhle in [Ho 1]:

Definition 16. A *nonnegative fuzzy number* is a function $A : [0;\infty] \to [0;1]$ for which $A(0) = 0, A(\infty) = 1$ and $A(x) = \sup\{A(t); t \in [0;x)\}$.

Suppose f is a mapping that assigns a fuzzy number (in the sense of the previous definition) to each point of $X, X \subset R$. Evidently we can use the notions from the previous section to define level functions of the fuzzy function f. The only difference is that this time we will have only the functions $f_{-\alpha}$, not the functions f_α. Analogically we can define the derivative at a point for f.

As we have already mentioned in section 3, Klement in [Kl 1] defines the Lebesgue integral for f and states some properties of this integral. The core of this work lies in the isomorphism between the set of all nonnegative fuzzy numbers and the set of their pseudoinverses. Unfortunately the theory of integral based on this attitude works only under the assumption of minimum t-norm used in $\mathcal{F}(R)$. For other t-norm than T_{min} the integral even fails to be additive.

Using the concept of level functions (similar to that of Kalina in [Ka 1]) we can obtain the Lebesgue integral of a fuzzy function with properties resembling those of real functions.

This integral is defined in the following way: Let f be a fuzzy function defined on an interval J. Suppose the level functions $f_{-\alpha}$ are integrable on J for each $\alpha \in (0,1]$. Let

$$I = \inf\left\{\int_I f_{-\alpha}(x)dx; \alpha \in (0,1]\right\},$$

$$S = \sup\left\{\int_I f_{-\alpha}(x)dx; \alpha \in (0,1]\right\} = \int_I f_1(x)dx.$$

The integral of a fuzzy function f on the interval J can be defined as a fuzzy real number $i_J(f)$ in the following way:

$$i_J(f)(x) = \begin{cases} 0 & \text{if } x \le I, \\ \alpha & \text{if } \alpha = \sup\left\{\gamma \in (0,1]; \int_J f_{-\gamma}(t)dt \le x\right\}, \\ 1 & \text{if } x \ge S \end{cases}.$$

The integral defined as above has a close connection to the derivative from the previous section – this connection is given by the analogy of the mean integral calculus theorem, which holds for the derivative in the sense of Kalina's work [Ka 1] and above defined integral.

Proposition 5. *Let f be a fuzzy function with integrable level functions on the interval $J = [a, b]$. Let for $x \in J$ $F(x) = i_{[a,x]}(f)$. If f has all its level functions continuous at $x_0 \in J$, then $F'(x_0) = f(x_0)$.*

6. Fuzzy methods in crisp functions calculus

In [BS 1] the authors show how some classical results can obtain more compact form using terms of fuzzy set theory. We introduce the main notions and results of this work:

Let $f : X \to Y$ be a mapping, X, Y are sets of real numbers and let $x_0 \in X$. The *continuity defect* of f at the point x_0 is the value

$$\delta(f, x_0) = \sup\{|y - f(x_0)|; y = \lim_{n\to\infty} y_n, y_n = f(x_n), x_0 = \lim_{n\to\infty} x_n\}.$$

If the continuity defect is finite at $x \in X$ then we say that f is *fuzzy continuous* at the point x.

Next the authors in [BS 1] define the *local continuity measure* at x_0 denoted by $\lambda(f, x_0)$ by the equality

$$\lambda(f, x_0) = (1 + \delta(f, x_0))^{-1}.$$

(In case when $\delta(f, x_0) = \infty$ we put $\lambda(f, x_0) = 0$.)

The *continuity measure* $\lambda(f)$ on a set X is defined as

$$\lambda(f) = \inf\{\lambda(f, x); x \in X\}.$$

Finally, a function f is called *fuzzy continuous* on X if $\lambda(f) > 0$. Note that a function, that is fuzzy continuous at each point of a set need not be fuzzy continuous on that set.

The authors in [BS 1] claim that

> *There are classical results in mathematics which are incomplete. An example is given by the well-known result of the classical mathematical analysis stating that a continuous function defined on a closed interval is bounded. ... But if we ask whether the converse is true we reveal that the answer is negative. The criterion of boundedness may be found only in terms of fuzzy set theory.*

The above mentioned criterion is the proposition stating that a function f defined on a compact space X is bounded if and only if f is fuzzy continuous (see [BS 1], Theorem 2).

On the other hand, applying this approach, some other classical results are no more valid. Maybe the most obvious one is the intermediate value principle (known also as Bolzano lemma) which says that if f is continuous on a closed interval $[a; b]$ and y is an arbitrary number between $f(a)$ and $f(b)$, then there is $c \in [a; b]$ such that $f(c) = y$. (Shortly - a continuous function on a compact set has the Darboux property.) Clearly this is not true for fuzzy continuous functions, as they may have discontinuities.

But if we realise that the intermediate value principle is a tool that enables us e.g. to find approximate solutions of algebraic equations, then we see that in practical calculations we are often satisfied with the value $c \in [a;b]$ for which $f(c)$ is in some sense "not too far" from zero. This leads us again to terms from fuzzy set theory.

In [Ja 3] we define the fuzzy version of uniform continuity and show that for fuzzy uniformly continuous functions some version of intermediate value principle is fulfilled.

First we have to introduce the notion of nearness that was defined in [Ka 1]. Let us assume a fuzzy relation $N : R \times R \to [0;1]$ is given satisfying the following properties:

(1) for each $x \in R$, $xNx = 1$,
(2) for each $x, y \in R$, $xNy = yNx$,
(3) for each $x, y, z \in R$ if $x < y < z$ then $xNy \geq xNz$,
(4) for each $x \in R$, $\lim_{y\to\infty} xNy = 0$,
(5) for each $x, y, z \in R$ there is $xNy = (x+z)N(y+z)$.

The last condition of nearness is not necessary for most results in [Ka 1] and here, but it simplifies the considerations a lot.

Here are some examples of nearness relations:

Example 4. If $xNy = \dfrac{1}{1+|x-y|}$, then N is an example of nearness which has never the zero value.

Example 5. Let $k > 0$. The relation $xNy = \max\{1 - k\ |x-y|; 0\}$ is an example of a nearness that assigns nonzero values only to those pairs $(x;y)$ for which their distance does not exceed $\frac{1}{k}$.

Example 6. The relation $xNy = 1$ if $x = y, xNy = 0$ if $x \neq y$ is an example of a "crisp" nearness.

The nearness relation enables us to define the derivative also for various types of fuzzy mappings (see [Ka 1]). Here it will serve as a tool to introduce the α-fuzzy uniform continuity. In the rest of this work we assume a nearness N is given (hence the terms defined are dependent on that particular nearness).

Definition 17. A function $f : R \to R$ is an α-fuzzy uniformly continuous function on a set $M \subset R$ for the given $\alpha \in (0;1)$ if there exists $\delta > 0$ such that for all $x, y \in M; |x-y| < \delta$ implies $f(x)Nf(y) \geq \alpha$.

Speaking about the α-fuzzy uniformly continuous function we assume the existence of such $\alpha \in (0;1)$, for which the function fulfills the requirement of the previous definition.

Proposition 6. *If f is an α-fuzzy uniformly continuous function on the set $M \subset R$, then f is fuzzy continuous on M.*

A fuzzy continuous function need not be α-fuzzy uniformly continuous for any $\alpha \in (0;1)$. An easy example of such function is $f(x) = x^{-1}$ on the interval $(0;1)$.

An α-fuzzy uniformly continuous function satisfies the intermediate value principle in the following sense:

Proposition 7. *Let f be an α-fuzzy uniformly continuous function on the interval $[a;b]$, let c be any value between $f(a)$ and $f(b)$. Then there is a number $x \in [a;b]$ for which $f(x)Nc \geq \alpha$.*

In order to find the connection between the notion of an α-fuzzy uniformly continuous function and the uniformly continuous function in the classical sense we will assume to work with a "reasonable" nearness, i.e. we have to add the following conditions to the definition of nearness:

(6) for each $x, y \in R$ there is $xNy = 1$ if and only if $x = y$,

(7) the function $n(y) = xNy$ is continuous for any fixed $x \in R$.

Adding these conditions we can easily see that if f is an α-fuzzy uniformly continuous function for any $\alpha \in (0;1)$, then it is also uniformly continuous and vice versa.

The following proposition shows the connection between the fuzzy and classical uniform continuity.

Proposition 8. *If F_α denotes the set of all α-fuzzy uniformly continuous functions on a set $M \subset R$, then $\bigcap_{\alpha \in (0;1)} F_\alpha$ is the set of all uniformly continuous functions on M.*

In the classical mathematical analysis there is a well-known statement that a continuous function defined on a compact set is uniformly continuous on that set. A similar result holds for fuzzy continuity.

Proposition 9. *If f is fuzzy continuous on a compact set C, if N is a nearness with nonzero values, then there is $\alpha \in (0;1)$ such that f is an α-fuzzy uniformly continuous function (with respect to N).*

In [Ja 4] and [Ja 5] we study how introducing fuzzy methods into crisp functions analysis changes the basic theorems from classical mathematics. We show that a lot of them obtains more general and compact form. Here is a short summary of the results from the mentioned papers:

Definition 18. Let $f : R \to R$ be a function, $a \in R$, let N be a nearness on R and let $\alpha \in (0;1)$. Denote

$$D_\alpha(a) = \left\{ \frac{f(x) - f(a)}{x - a}; x \neq a, xNa \geq \alpha \right\}.$$

The function f is *fuzzy differentiable* at the point a on level α if the numbers $I = \inf D_\alpha(a), S = \sup D_\alpha(a)$ are both finite.

Note that $D_\alpha(a) \neq \emptyset$ for any $\alpha \in (0;1)$ because of the continuity of N.

Example 7. The function $f(x) = \sqrt{|x|}$ is not fuzzy differentiable at 0 at any level, as at this point $D_\alpha(0) = R$ for all $\alpha \in (0;1)$.

Example 8. The function $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$ (Dirac function) is not fuzzy differentiable at the point 0 on any level, as for any $\alpha \in (0;1)$ we have $\sup D_\alpha(0) = \infty$.

Definition 19. The interval $[I;S]$, where the numbers I and S are defined in Definition 18 is called an *α-nearness derivative* of f at a on the level α and denoted by $f'_\alpha(a)$.

In case when at some point $I = -\infty$ and $S = \infty$ we also call the interval $[I;S]$ an α-nearness derivative, although the function is not differentiable at that point. (This is a similar situation as with the integrability - a function that is not integrable may have the integral.)

We consider the arithmetical operations in the extended real line in the usual way: $a + \infty = \infty, a - \infty = -\infty, a.\infty = \infty, a(-\infty) = -\infty$ (the last two statements hold for positive a) and $0.\infty = 0$.

Example 9. The α-nearness derivative of the Dirac function (see Example 8) at the point 0 is the interval $[0;\infty]$. The α-nearness derivative of this function at some $a > 0$, for which $aN0 = \alpha$ is the interval $\left[0; \frac{1}{|a|}\right]$.

Here are some properties of fuzzy differentiable functions and their α-nearness derivatives:

Proposition 10. *If $\alpha, \beta \in (0;1)$, $\alpha < \beta$, then $f'_\beta(a) \subset f'_\alpha(a)$.*

The next propositions shows the connection between fuzzy and classical differentiability.

Proposition 11. *If f is differentiable at a (in classical sense), then there is an $\alpha_0 \in (0;1)$ such that f is fuzzy differentiable at a on level α_0 and*

$$\bigcap_{\alpha \in (0;1)} f'_\alpha(a) = \{f'(a)\}.$$

Instead of the linearity which holds for the classical derivative, we have only the following inclusion:

Proposition 12. *If f, g are real functions, $a \in R$, $\alpha \in (0;1)$, then*

$$(f+g)'_\alpha(a) \subset f'_\alpha(a) + g'_\alpha(a),$$

where the sum of the sets $f'_\alpha(a), g'_\alpha(a)$ is the set of all sums $x + y$, $x \in f'_\alpha(a), y \in g'_\alpha(a)$.

Proposition 13. *If $c \in R$, then $(cf)'_\alpha(a) = cf'_\alpha(a)$, where the product $cf'_\alpha(a)$ is the set of all products cx, $x \in f'_\alpha(a)$.*

From Proposition 10 we see that the α-nearness derivatives can be understood as level cuts for some fuzzy set. This fuzzy set seems to be a generalization of the crisp derivative (see Proposition 11). We are also able to formulate some classical statements of real variable calculus without assumptions of continuity or differentiability.

In all the remaining propositions in this work we restrict ourselves to functions defined only on a given interval. The reason is that we do not want the points outside this interval to influence the nearness derivative. Other possibility would be to consider the α-derivative of a function f at a point $x \in [m;n]$ as an α-derivative of f restricted on $[m;n]$ at x.

Proposition 14. *A function f is increasing on the interval $[m;n]$ if and only if at each $a \in [m;n]$ there is $f'_\alpha(a) \cap (0;\infty) \neq \emptyset$ and $f'_\alpha(a) \cap (-\infty;0) = \emptyset$ for every $\alpha \in (0;1)$.*

A dual necessary and sufficient condition can be formulated in similar way for decreasing functions. For non-decreasing functions we have:

Proposition 15. *A function f is non-decreasing on the interval $[m;n]$ if and only if at each $a \in [m;n]$ there is $f'_\alpha(a) \cap (-\infty;0) = \emptyset$ for every $\alpha \in (0;1)$.*

Again the dual condition can be stated for non-increasing functions. All the proofs of these statements are just modifications of the one in Proposition 14.

Finally we state generalized versions of Rolle, Lagrange and Darboux theorems.

Proposition 16 (Rolle Theorem). *If f is defined on the interval $[m;n]$, if $f(m) = f(n)$, then there exists $a \in (m;n)$ and $\alpha \in (0;1)$ such that $0 \in f'_\alpha(a)$.*

Proposition 17 (Lagrange Mean Value Theorem). *If f is defined on $[m;n]$, then there is a number $a \in (m;n), \alpha \in (0;1)$ such that $\dfrac{f(m) - f(n)}{m - n} \in f'_\alpha(a)$.*

Proposition 18 (Darboux Theorem). *If f is defined on $[m;n]$, if there is $z \in R$ such that for each $x \in supp(f'(m))$ and for each $y \in supp(f'(n))$ there is $x < z < y$, then there exists $c \in (m;n)$ such that $z \in supp(f'(c))$.*

7. Concluding remarks

This work shows two possible directions of fuzzy infinitesimal calculus. The first one deals with fuzzy object - fuzzy number and fuzzy functions. We show that using "reasonable" definitions we obtain similar results as in the classical case. On the other hand, the methods used to obtain these results sometimes differ a lot from the classical ones.

Another direction, less frequent, is applying fuzzy methods into classical mathematical analysis. In this work the section 6 is devoted to present results of this kind. It appears that using fuzzy methods enables us both to extend validity of some statements from the classical analysis and to formulate them in more general way.

The results of this work are just the basic principles of fuzzy differential calculus; there is a wide field for future research, both in theory and applications.

Finally it is worth to mention new ideas by Vojtáš and Kalina which show close links between fuzzy mathematics and non-standard analysis. This seems to be a fruitful topic both for the theory of fuzzy mathematics and for its applications, such as fuzzy measures or quantum computing.

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