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INVARIANT MEASURES ON LOCALLY COMPACT SPACES

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Abstract. The paper considers measures on a locally compact space which are invariant with respect to a given system of continuous maps.

Introduction

The present paper considers the following problem. There is given a locally compact space X and a system $\mathcal{F}$ of continuous maps of the space X into itself. We are interested in conditions under which there is a Borel $\mathcal{F}$-invariant measure on the space X. For example, let G be a locally compact topological group. A map T on the group G of the form $T(x) = ax$ ($T(x) = xb$, $T(x) = axb$) is said to be left (right, left-right) translation of G. If a system $\mathcal{F}$ consists of all left (right) translations, then a Borel $\mathcal{F}$-invariant measure exists and is called left (right) Haar measure of the group G, [2,p.246]. The group G is called unimodular if left Haar measure is also right invariant, [1,p.119]. It is well known that left (right) Haar measure need not be right (left) invariant, [2,p.248]. It means that the system of all left-right translations need not have an invariant Borel measure. For compact spaces the existence problem of an $\mathcal{F}$-invariant measure was fully solved in Roberts' paper [4]. The paper [3] of the author contains partial results about the locally compact case. The presented results are very similar to Roberts' results for the compact case. Without loss of generality we may assume that the system $\mathcal{F}$ contains the identity map and is closed with respect to the composition of maps, i.e. $\mathcal{F}$ is a monoid with respect to the composition. Moreover, we assume that $\mathcal{F}$ is a minimal monoid, which contains sufficiently many homeomorphisms, and we obtain a necessary and sufficient condition for the existence of an $\mathcal{F}$-invariant measure. Using this result we give a topological characterization of nonunimodular locally compact topological groups.

1. Preliminaries

For a locally compact space X the symbol $\mathcal{B}(X)$ denotes the minimal σ-ring containing all compact subsets of X. The members of $\mathcal{B}(X)$ are called Borel sets in X. A set A is called bounded if its closure $\overline{A}$ is a compact set in X. A Borel measure on the space X is a set function $\mu : \mathcal{B}(X) \to \langle 0, \infty)$ such that

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$\mu(\emptyset) = 0$, $\mu \bigcup_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} \mu(A_i)$ for any sequence $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint Borel sets and $\mu(K) < \infty$ for any compact set K in$X$. A map $T : X \to X$ is called measurable if $T^{-1}(A) \in \mathcal{B}(X)$ for any $A \in \mathcal{B}(X)$. A map $T : X \to X$ is called proper if it is continuous and $T^{-1}(K)$ is compact for any compact set K of X. A measure μ is said to be invariant with respect to a measurable map T if $\mu(T^{-1}(A)) = \mu(A)$ for any $A \in \mathcal{B}(X)$. If $\mathcal{F}$ is a system of measurable maps and a measure μ is T-invariant for any $T \in \mathcal{F}$, then μ is called $\mathcal{F}$-invariant. A system $\mathcal{F}$ of proper maps is called minimal if for any $x \in X$ the set $\{T(x) : T \in \mathcal{F}\}$ is dense in X. Clearly, the system $\mathcal{F}$ of proper maps is minimal if and if only for any nonempty open subset U of the space X the system $\{T^{-1}(U) : T \in \mathcal{F}\}$ is a covering of X. We say that a system $\mathcal{F}$ of proper maps contains sufficiently many homeomorphisms if there is a bounded open subset $U_0 \subset X$ such that for any $x \in X$ there is a homeomorphism $T \in \mathcal{F}$ for which $x \in T^{-1}(U_0)$, i.e. $T(x) \in U_0$. Equivalently, a system $\mathcal{F}$ of proper maps contains sufficiently many homeomorphisms if there is a bounded open subset $U_0 \subset X$ such that for any compact set K in X there are homeomorphisms $T_1, ..., T_n \in \mathcal{F}$ such that $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U_0)$. For example, if X is compact and $\mathcal{F}$ contains at least one homeomorphism (e. g. the identity map), then $\mathcal{F}$ contains sufficiently many homeomorphisms. So, the assumption of sufficiently many homeomorphisms in a system $\mathcal{F}$ is useless in the compact case, but not in the locally compact case. If f is a real function defined on a locally compact space X, then the symbol *supp* f denotes the support of the function f, i. e. the closure of the set $\{x : f(x) \neq 0\}$. The set of all continuous functions on X with a compact support is denoted by the symbol $\mathcal{C}_0(X)$. The subset of $\mathcal{C}_0(X)$ consisting of all nonnegative functions is denoted by $\mathcal{C}_0^+(X)$. The inequality $f < g$ means $f(x) \leq g(x)$ for all $x \in X$ and $f(x) < g(x)$ for some $x \in X$.

2. Construction of an invariant measure

To prove the main result we need two technical lemmas.

Lemma 2.1. *Let $\mathcal{F}$ be a minimal system of proper maps of a locally compact space X.*

(i) Let $g, \varphi \in \mathcal{C}_0^+(X)$ and $\varphi \neq 0$. Then there are $T_1, ..., T_n \in \mathcal{F}$, $\varphi_1, ..., \varphi_n \in \mathcal{C}_0^+(X)$ and a real number $a > 0$ such that $g \leq \alpha \sum_{i=1}^{n} \varphi_i \circ T_i$ and $\varphi = \sum_{i=1}^{n} \varphi_i$.

(ii) Moreover, if $\mathcal{F}$ contains sufficiently many homeomorphisms, then there is a function $\varphi_0 \in \mathcal{C}_0^+(X)$ such that any $g \in \mathcal{C}_0^+(X)$ may be represented in a form $g = \beta \sum_{i=1}^{n} f_i \circ T_i$, where $f_i \in \mathcal{C}_0^+(X)$, $0 \leq f_i \leq \varphi_0$, $T_i \in \mathcal{F}$ and β is a nonnegative real number.

Proof. Denote $K =$ *supp* g and $U = \{x : \varphi(x) \neq 0\}$. The set K is compact, U is nonempty open and $\mathcal{F}$ is a minimal system. Hence, there are $T_1, ..., T_n \in \mathcal{F}$ such that $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U)$, which implies $\sum_{i=1}^{n} \varphi(T_i(x)) > 0$ for all $x \in K$. Put $f = \sum_{i=1}^{n} \varphi \circ T_i$, $\alpha = \frac{n \cdot \sup_{x \in K} g(x)}{\inf_{x \in K} f(x)}$ and $\varphi_1 = ... = \varphi_n = \frac{\varphi}{n}$. Then $g \le \alpha \sum_{i=1}^{n} \varphi_i \circ T_i$.

(ii) There is a bounded open set $U_0 \subset X$ such that for any compact set K in X there are homeomorphisms $T_1, ..., T_n \in \mathcal{F}$ for which $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U_0)$. There is another bounded open set W for which $\overline{U}_0 \subset W$. Take a function $\varphi_0 \in \mathcal{C}_0^+(X)$ such that $\varphi_0(x) = 1$ for $x \in \overline{U}_0$ and $\varphi_0(x) = 0$ for $x \notin W$, see [2,p.211]. Denote $U = \{x : \varphi_0(x) \neq 0\}$. Obviously, $U_0 \subset U$. Take an arbitrary function $g \in \mathcal{C}_0^+(X)$. Denote $K = \mathit{supp}\ g$. There are homeomorphisms $T_1, ..., T_n \in \mathcal{F}$ such that $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U_0) \subset \bigcup_{i=1}^{n} T_i^{-1}(U)$,which implies $\sum_{i=1}^{n} \varphi_0(T_i(x)) > 0$ for all $x \in K$. Put $f = \sum_{i=1}^{n} \varphi_0 \circ T_i$. Define a function h by $h(x) = \frac{g(x)}{f(x)}$ if $f(x) \neq 0$ and $h(x) = 0$ if $x \notin \mathit{supp}\ g$. The function h is defined correctly, $h(x)f(x) = g(x)$ and $h \in \mathcal{C}_0^+(X)$. Put $\beta = \sup_{x \in X} h(x)$ and $f_i(x) = \frac{1}{\beta} h(T_i^{-1}(x)) \cdot \varphi_0(x)$. Then $f_i \in \mathcal{C}_0^+(X), 0 \le f_i \le \varphi_0$ and $\beta \sum_{i=1}^{n} f_i(T_i(x)) = \beta \sum_{i=1}^{n} \frac{1}{\beta} h(T_i^{-1}(T_i(x))) \varphi_0(T_i(x)) = \sum_{i=1}^{n} h(x) \varphi_0(T_i(x)) =$
$= h(x) \sum_{i=1}^{n} \varphi_0(T_i(x)) = h(x) f(x) = g(x)$. It proves (ii).

Lemma 2.2. *Let $\mathcal{F}$ be a minimal system of proper maps of a locally compact space X. If μ is a nonzero $\mathcal{F}$-invariant Borel measure, then $\mu(U) > 0$ for any nonempty open Borel subset U of X.*

Proof. Let μ be a nonzero $\mathcal{F}$-invariant Borel measure and $\mu(U) = 0$ for some nonempty open Borel subset U of X. Let K be a compact subset of X. Since the system $\{T^{-1}(U) : T \in \mathcal{F}\}$ is a covering of X, there are $T_1, ..., T_n \in \mathcal{F}$ such that $K \subset \bigcup_{i=1}^{n} T_i^{-1}(U)$. $\mathcal{F}$-invariance of the measure μ implies $\mu(K) = 0$. Any Borel set in X may be covered by a sequence of compact sets, [2,p.214]. Therefore, the measure μ is zero.

The following theorem is the main result of the paper.

Theorem 2.3. *Let $\mathcal{F}$ be a minimal monoid of proper maps of a locally compact space X which contains sufficiently many homeomorphisms. The following properties are equivalent.*

(i) There exists a nonzero $\mathcal{F}$-invariant Borel measure on X.

(ii) For any open subsets $U_1, ..., U_n$, any compact subsets $K_1, ..., K_m$ and any maps $T_1, ...T_n, S_1, ..., S_m \in \mathcal{F}$

$\sum_{i=1}^{n} \chi_{U_i} > \sum_{j=1}^{m} \chi_{K_j}$ *implies* $\exists x \in X : \sum_{i=1}^{n} \chi_{\tilde{U}_i}(x) > \sum_{j=1}^{m} \chi_{\tilde{K}_j}(x)$, where $\tilde{U}_i = T_i^{-1}(U_i)$ and $\tilde{K}_j = S_j^{-1}(K_j)$.

(iii) *If* $T_1, ..., T_n, S_1, ..., S_m \in \mathcal{F}$, $f_1, ..., f_n$ *and* $g_1, ..., g_m$ *are linear combinations of characteristic functions of open and compact sets respectively with positive rational coefficients, then*
$\sum_{i=1}^{n} f_i > \sum_{j=1}^{m} g_j$ *implies* $\exists x \in X : \sum_{i=1}^{n} f_i(T_i(x)) > \sum_{j=1}^{m} g_j(S_j(x))$.

(iv) *For any functions* $\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m \in \mathcal{C}_0^+(X)$ *and any maps* $T_1, ..., T_n, S_1, ..., S_m \in \mathcal{F}$
$\sum_{i=1}^{n} \varphi_i > \sum_{j=1}^{m} \psi_j$ *implies* $\exists x \in X : \sum_{i=1}^{n} \varphi_i(T_i(x)) > \sum_{j=1}^{m} \psi_j(S_j(x))$.

Proof. (i)⇒(ii) Let μ be a nonzero $\mathcal{F}$-invariant Borel measure on X. Take open subsets $U_1, ..., U_n$ and compact subsets $K_1, ...K_m$ such that $\sum_{i=1}^{n} \chi_{U_i} > \sum_{j=1}^{m} \chi_{K_j}$. Without loss of generality we may assume that the sets $U_1, ..., U_n$ are bounded to be Borel. (An open set is Borel if and only if it is σ-bounded). We have $\sum_{i=1}^{n} \chi_{U_i}(x) \geq \sum_{j=1}^{m} \chi_{K_j}(x)$ for all $x \in X$ and $\sum_{i=1}^{n} \chi_{U_i}(x) > \sum_{j=1}^{m} \chi_{K_j}(x)$ for some $x \in X$. The last inequality holds on a nonempty open subset, because the sets U_i are open and K_j are closed. Lemma 2.2. implies $\sum_{i=1}^{n} \mu(U_i) > \sum_{j=1}^{m} \mu(K_j)$. Take maps $T_1, ..., T_n, S_1, ..., S_m \in \mathcal{F}$. Suppose $\sum_{i=1}^{n} \chi_{\tilde{U}_i} \leq \sum_{i=1}^{m} \chi_{\tilde{K}_j}$, where $\tilde{U}_i = T_i^{-1}(U_i)$ and $\tilde{K}_j = S_j^{-1}(K_j)$. Then $\sum_{j=1}^{m} \mu(\tilde{K}_j) \geq \sum_{i=1}^{n} \mu(\tilde{U}_i) = \sum_{i=1}^{n} \mu(U_i) > \sum_{j=1}^{m} \mu(K_j)$, which is a contradiction.

(ii) ⇒ (iii) This is obvious.

(iii) ⇒ (iv) Take maps $T_1, ..., T_n, S_1, ..., S_m \in \mathcal{F}$ and functions $\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m \in \mathcal{C}_0^+(X)$ such that $\sum_{i=1}^{n} \varphi_i > \sum_{j=1}^{m} \psi_j$. Put

$$h = \frac{1}{2}\left(\sum_{i=1}^{n} \varphi_i - \sum_{j=1}^{m} \psi_j\right). \tag{1}$$

Obviously, $h \in \mathcal{C}_0^+(X)$ and $h \neq 0$. Denote $U = \{x : h(x) \neq 0\}$ and

$$K = \bigcup_{i=1}^{n} supp(\varphi_i \circ T_i). \tag{2}$$

Since $\mathcal{F}$ is minimal and K is compact, there are $R_1, ..., R_p \in \mathcal{F}$ such that

$$K \subset \bigcup_{j=1}^{p} R_j^{-1}(U). \tag{3}$$

Then $0 < \inf_{x\in K} \sum_{j=1}^{p} h(R_j(x))$. Put

$$\alpha = \inf_{x\in K} \sum_{j=1}^{p} h(R_j(x)), \tag{4}$$

$$\psi_{m+j} = \frac{1}{p} h \text{ and} \tag{5}$$

$$S_{m+j} = R_j \text{ for } j = 1, ..., p. \tag{6}$$

Then we have

$$\sum_{i=1}^{n} \varphi_i > \sum_{j=1}^{m+p} \psi_j. \tag{7}$$

Take a natural number

$$r > \frac{p(p+m+n)}{\alpha}. \tag{8}$$

Denote $U_{i,k} = \{x : \varphi_i(x) > \frac{k}{r}\}$ for integers i and k such that $1 \le i \le n$, $0 \le k$ and $K_{j,k} = \{x : \psi_j(x) \ge \frac{k}{r}\}$ for integers j and k such that $1 \le j \le m$, $1 \le k$. Obviously, the sets $U_{i,k}$ are open and the stets $K_{j,k}$ are compact for the corresponding integers. Put $f_i = \frac{1}{r}\sum_{k=0}^{\infty} \chi_{U_{i,k}}$ and $g_j = \frac{1}{r}\sum_{k=1}^{\infty} \chi_{K_{j,k}}$. In fact, both sums are finite. The functions f_i and g_j have properties:

$$0 \le \varphi_i(x) \le f_i(x) < \varphi_i(x) + \frac{1}{r}\ , \tag{9}$$

$$0 \le g_j(x) \le \psi_j(x) < g_j(x) + \frac{1}{r} \text{ and} \tag{10}$$

$$0 < f_i(x) \Leftrightarrow 0 < \varphi_i(x) \text{ for all } x \in X. \tag{11}$$

Now,

$$\sum_{i=1}^{n} f_i > \sum_{j=1}^{m+p} g_j \tag{12}$$

and by (iii)

$$\exists x \in X : \sum_{i=1}^{n} f_i(T_i(x)) > \sum_{j=1}^{m+p} g_j(S_j(x)). \tag{13}$$

This element x must belong to the compact K by (10), (11) and (2). Therefore, $\sum_{j=1}^{p} h(R_j(x)) \ge \alpha$ and

$$\sum_{j=m+1}^{m+p} \psi_j(S_j(x)) \ge \frac{\alpha}{p} \tag{14}$$

by (5) and (6). Using (9), (13), (10), (14), and (8) we obtain

$$\sum_{i=1}^{n}\varphi_i(T_i(x)) > \sum_{i=1}^{n}(f_i(T_i(x)) - \frac{1}{r}) > \frac{-n}{r} + \sum_{j=1}^{m+p} g_j(S_j(x)) >$$

$$> \tfrac{-n}{r} + \sum_{j=1}^{m+p}(\psi_j(S_j(x)) - \frac{1}{r}) = \frac{-(n+m+p)}{r} + \sum_{j=1}^{m}\psi_j(S_j(x)) + \sum_{j=m+1}^{m+p}\psi_j(S_j(x)) \geq$$

$$\geq \tfrac{-(n+m+p)}{r} + \sum_{j=1}^{m}\psi_j(S_j(x)) + \frac{\alpha}{p} > \sum_{j=1}^{m}\psi_j(S_j(x)).$$ It proves (iv).

(iv)$\Rightarrow$ (i) Take $\varphi_0 \in C_0^+(X)$ from (ii) of Lemma 2.1. Let A be the set of all functions φ of the form $\varphi = \sum_{i=1}^{n}\varphi_i \circ T_i$, where $\varphi_i \in \mathcal{C}_0^+(X)$, $T_i \in \mathcal{F}$ and $\varphi_0 = \sum_{i=1}^{n}\varphi_i$. Since $\mathcal{F}$ is a monoid, we have

(15) $\varphi_0 \in A$,

(16) A is a convex subset of $\mathcal{C}_0(X)$ and

(17) $\forall \varphi \in A\ \forall T \in \mathcal{F} : \varphi \circ T \in A.$

By (i) of Lemma 2.1., we obtain

(18) $\forall f \in \mathcal{C}_0(X)\ \exists \varphi \in A\ \exists \alpha > 0 :\ |f| \leq \alpha|\varphi|.$

Property (iv) implies

(19) $\forall \varphi, \psi \in A\ \forall \alpha > 0 :\ \varphi \leq \alpha\psi \Rightarrow \alpha \geq 1.$

Let $p : \mathcal{C}_0(X) \to \langle 0, \infty)$ be defined as follows $p(f) = \inf\{\alpha : \exists \varphi \in A\ |f| \leq \alpha\varphi\}$. Then

(20) p is a seminorm on $\mathcal{C}_0(X)$,

(21) $0 \leq g \leq f \Rightarrow p(g) \leq p(f)$ for all $f, g \in \mathcal{C}_0(X)$,

(22) $p(\varphi) = 1$ for all $\varphi \in A$,

(23) $p(f \circ T) \leq p(f)$ for all $f \in \mathcal{C}_0(X)$ and $T \in \mathcal{F}$,

(24) $p(|f|) = p(f)$ for all $f \in \mathcal{C}_0(X)$ and

(25) p is a norm.

Homogenity of p is obvious, subadditivity of p follows from (16). Relations (22) and (23) follow from (19) and (17) respectively. Relations (21) and (24) are obvious. We shall prove (25). Let $f \neq 0$. We may assume $f \in \mathcal{C}_0^+(X)$.

From (i) of Lemma 2.1. it follows that $\varphi_0 \leq a \cdot \sum_{i=1}^{n} f \circ T_i$ for some $a > 0$ and $T_1, ..., T_n \in \mathcal{F}$. Then we have $1 = p(\varphi_0) \leq p(a \cdot \sum_{i=1}^{n} f \circ T_i) \leq a \sum_{i=1}^{n} p(f \circ T_i) \leq a\, n\, p(f)$. Therefore, $p(f) \geq \frac{1}{n\ a}$.

Put $B = \{f : p(f) < 1\}$. Then we have two disjoint convex sets A and B such that B is open (with respect to the topology induced by the norm p). By Hahn-Banach theorem, there is a nontrivial linear functional $\Phi : C(X) \to \mathbb{R}$ such that $\Phi(f) \leq \Phi(\varphi)$ for all $f \in B$ and $\varphi \in A$. We may assume that

(26) $$\sup_{f \in B} \Phi(f) = 1.$$

Then $\Phi(\varphi) \geq 1$ for all $\varphi \in A$. On the other hand $p((1-\varepsilon)\varphi) = 1-\varepsilon$ for $\varphi \in A$ and $\varepsilon \in (0,1)$ by (22). Therefore, $(1-\varepsilon)\varphi \in B$ and $(1-\varepsilon)\Phi(\varphi) = \Phi((1-\varepsilon)\varphi) \leq 1$. It means

(27) $$\Phi(\varphi) = 1 \text{ for all } \varphi \in A.$$

Let $0 \leq f \leq \varphi_0$. Then $0 \leq \varphi_0 - f \leq \varphi_0$. Relations (15), (21) and (22) imply $p(\varphi_0 - f) \leq 1$. Then (26) implies $\Phi(\varphi_0 - f) \leq 1$. By (15) and (27), we have $1 = \Phi(\varphi_0) = \Phi(f) + \Phi(\varphi_0 - f)$. Therefore, $\Phi(f) = 1 - \Phi(\varphi_0 - f) \geq 1 - 1 = 0$, i.e.

$$\Phi(f) \geq 0.$$

Moreover, $f \circ T + (\varphi_0 - f) \in A$ for all $T \in \mathcal{F}$. By (27) we have $1 = \Phi(f \circ T + (\varphi_0 - f)) = \Phi(f \circ T) + \Phi(\varphi_0) - \Phi(f) = \Phi(f \circ T) + 1 - \Phi(f)$. Hence,

$$\Phi(f \circ T) = \Phi(f).$$

Let $g \in C_0^+(X)$ be arbitrary. Lemma 2.1. implies

$$g = \beta \sum_{i=1}^{n} f_i \circ T_i,$$

where $\beta > 0$, $0 \leq f_i \leq \varphi_0$ and $T_i \in \mathcal{F}$. Therefore,

$$\Phi(g) = \beta \sum_{i=1}^{n} \Phi(f_i \circ T_i) = \beta \sum_{i=1}^{n} \Phi(f_i) \geq 0$$

and

$$\Phi(g \circ T) = \beta \sum_{i=1}^{n} \Phi(f_i \circ T_i \circ T) = \beta \sum_{i=1}^{n} \Phi(f_i) = \Phi(g)$$

whenever $T \in \mathcal{F}$. Let $g \in \mathcal{C}_0(X)$ be arbitrary and $T \in \mathcal{F}$. Then

$$\Phi(g \circ T) = \Phi((g^+ - g^-) \circ T) = \Phi(g^+ \circ T) - \Phi((g^- \circ T) = \Phi(g^+) - \Phi(g^-) = \Phi(g).$$

So, Φ is a positive $\mathcal{F}$-invariant linear functional on $\mathcal{C}_0(X)$. There is a unique regular Borel measure μ on the space X such that

$$\Phi(g) = \int_X g d\mu \text{ for any } g \in \mathcal{C}_0(X),$$

see [2,p.240]. Obviously, the measure μ must be $\mathcal{F}$-invariant.

Now, we can give a topological characterization of nonunimodular locally compact topological groups.

Corollary 2.4. *A locally compact group G is nonunimodular if and only if there are open subset $U_1, ..., U_n$, compact subsets $K_1, ..., K_m$ of G such that*

$$\sum_{i=1}^{n} \chi_{U_i} > \sum_{j=1}^{m} \chi_{K_j} \text{ and } \sum_{i=1}^{n} \chi_{\tilde{U}_i} \leq \sum_{j=1}^{m} \chi_{\tilde{K}_j},$$

where $\tilde{U}_i = a_i U_i b_i$ and $\tilde{K}_j = c_j K_j d_j$ for some $a_i, b_i, c_j, d_j \in G$.

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