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DOMINATION IN PRODUCTS OF CIRCUITS

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Abstract. Three numerical invariants of graphs concerning the domination are considered, namely the domatic number, the doubly domatic number and the total domatic number of a graph. These invariants are investigated for Cartesian products of circuits. Such graphs are treated algebraically as Cayley graphs of direct products of finite cyclic groups.

In this paper we shall study the domatic number, the doubly domatic number and the total domatic numbers of graphs which are Cartesian products of circuits.

We shall consider finite undirected graphs without loops and multiple edges. By $V(G)$ we denote the vertex set of a graph G, by $N_G[v]$ the set consisting of v and of all vertices which are adjacent to v in G. By C_n we denote the circuit of length n. If $G_1, G_2, ..., G_n$ are graphs, then their Cartesian product $G_1 \times G_2 \times ... \times G_n$ is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2) \times ... \times V(G_n)$ and in which two vertices $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)$ are adjacent if and only if there exists an integer i such that $1 \leqq i \leqq n, x_i$ and y_i are adjacent in G_i and $x_j = y_j$ for all $j \in \{1, ..., n\} - \{i\}$.

A subset D of the vertex set $V(G)$ of a graph G is called dominating in G (or total dominating in G), if for each $x \in V(G) - D$ (or for each $x \in V(G)$ respectively) there exists a vertex $y \in D$ adjacent to x. The set S is called doubly dominating in G, if for each $x \in V(G) - D$ there exist two vertices y_1, y_2 in D which are adjacent to x.A domatic (or total domatic, or doubly domatic) partition of G is a partition of $V(G)$, all of whose classes are dominating (or total dominating, or doubly dominating respectively) sets of G. The minimum number of vertices of a dominating set in G is its domination number $\gamma(G)$, the maximum number of classes of a domatic partition of G is its domatic number $d(G)$. Analogously the total domination number $\gamma_t(G)$, the total domatic number $d_t(G)$, the doubly domination number $\gamma^2(G)$ and the doubly domatic number $d^2(G)$ are defined.

The domatic number was introduced by E. J. Cockayne and S. T. Hedetniemi in [1], the total domatic number by the same authors and R. M. Dawes in [2]. The doubly domatic number is a particular case of the k-ply domatic number introduced in [3].

We shall study Cartesian products of circuits. Let $G = H_1 \times H_2 \times ... \times H_n$, where $H_1, H_2, ..., H_n$ are circuits. The lengths of $H_1, H_2, ..., H_n$ will be $h_1, h_2, ..., h_n$

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respectively. The graph G may be considered as the Cayley graph of a direct product of finite cyclic groups of orders $h_1, h_2, ..., h_n$. (Among such direct products of groups there are all finite Abelian groups.)

We shall treat finite Abelian groups and thus we shall use the additive notation as it is usual in this case. The group operation is denoted by $+$ as an addition, the neutral element is denoted by 0 and called the zero element, the inverse element to x is denoted by $-x$.

Let A be a subset of a group $\mathcal{G}$ such that $0 \notin A$ and $x \in A$ implies $-x \in A$ for each $x \in \mathcal{G}$. The Cayley graph $G(\mathcal{G}, A)$ is the graph whose vertex set is $\mathcal{G}$ and in which two vertices x, y are adjacent if and only if $x - y \in A$.

By $\mathcal{H}$ we shall denote the Abelian group which is a direct product of finite cyclic subgroups $\mathcal{H}_1, ..., \mathcal{H}_n$. For $i = 1, ..., n$ let a_i be a generator of $\mathcal{H}_i$ and let h_i be its order. Each element of $\mathcal{H}$ can be expressed as $\sum_{i=1}^{n} \alpha_1 a_i$, where $\alpha_1, ..., \alpha_n$ are integers. The expressions $\sum_{i=1}^{n} \alpha_i a_i, \sum_{i=1}^{n} \beta_i a_i$ denote the same element of $\mathcal{H}$ if and only if $\alpha_i \equiv \beta_i \pmod{h_i}$ for $i = 1, ..., n$.

For each element of $\mathcal{H}$ there exists a unique expression $\sum_{i=1}^{n} \alpha_i a_i$ with $0 \leqq \alpha_i \leq h_i$ for $i = 1, ..., n$.

Let p be a positive integer. By $\mathcal{H}_0(p)$ we denote the subset of H consisting of the elements $\sum_{i=1}^{n} \alpha_i a_i$ such that $\sum_{i=1}^{n} i\alpha_i \equiv 0 \pmod{p}$. We shall prove a lemma.

Lemma 1. *The set $\mathcal{H}_0(p)$ is a subgroup of $\mathcal{H}$. If $h_i \equiv 0$ (mod p) for $i = 1, ..., n$, then the index of $\mathcal{H}_0(p)$ in $\mathcal{H}$ is p. In the case when p is a prime number and $n < p$, also the inverse implication holds.*

Proof. Evidently $\mathcal{H}_0(p)$ contains the zero element o of $\mathcal{H}$, for any two elements of $\mathcal{H}_0(p)$ their sum is in $\mathcal{H}_0(p)$ and for any element of $\mathcal{H}_0(p)$ its inverse is in $\mathcal{H}_0(p)$; therefore $\mathcal{H}_0(p)$ is a subgroup of $\mathcal{H}$. Suppose $h_i \equiv 0 \pmod{p}$ for $i = 1, ..., n$. If $\sum_{i=1}^{n} \alpha_i a_i = \sum_{i=1}^{n} \beta_i a_i$ then $\sum_{i=1}^{n} i\alpha_i \equiv \sum_{i=1}^{n} i\beta_i \pmod{p}$. For each integer j such that $0 \leqq j \leqq p - 1$ the set of all elements $\sum_{i=1}^{n} \alpha_i a_i$ with $\sum_{i=1}^{n} i x_i \equiv j \pmod{p}$ is evidently a class of $\mathcal{H}$ by $\mathcal{H}_0(p)$ and the index of $\mathcal{H}_0(p)$ in $\mathcal{H}$ is p. Now let there exist $k \in \{1, ..., n\}$ such that h_k is not divisible by p. Suppose that p is a prime number and $n < p$. Then there exists a solution x of the congruence $px \equiv 1 \pmod{h_k}$. We have $pa_k \in \mathcal{H}_0(p)$ by the definition and also $xpa_k = a_k \in \mathcal{H}_0(p)$. Let $b = \sum_{i=1}^{n} \beta_i a_i$ be an arbitrary element of $\mathcal{H}$ and let $\sigma = \sum_{i}^{n} = 1i\beta_i$. As p is prime, there exists τ such

that $k\tau \equiv \sigma \pmod p$. Let $c = b - \tau a_k$. Then $c = \sum_{i=1}^{n} \gamma_i a_i$, where $\gamma_k = \beta_k - \tau$ and $\gamma_i = \beta_i$ for $i \neq k$. We have $\sum_{i=1}^{n} i\gamma_i = \sum_{i=1}^{n} i\beta_i - k\tau \equiv 0 \pmod p$ and $c \in \mathcal{H}_0(p)$. As $a_k \in \mathcal{H}_0(p)$, also $\tau a_k \in \mathcal{H}_0(p)$ and $b = c + \tau a_k \in \mathcal{H}_0(p)$. As b was chosen arbitrarily, we have $\mathcal{H}_0(p) = \mathcal{H}$.

Now we prove a theorem.

Theorem 1.. *Let $G = H_1 \times ... \times H_n$, where $H_1, ..., H_2$ are circuits, let h_i be the length of H_i for $i = 1, ..., n$. If $h_i \equiv 0$ (mod $(2n+1)$) for $i = 1, ..., n$, then*

$$d(G) = 2n + 1,$$

$$\gamma(G) = (\Pi_{i=1}^{n} h_i)/(2n+1).$$

Proof. The graph G is a regular graph of degree $2n$, therefore by a result from [1] we have $d(G) \leqq 2n + 1$. Therefore it suffices to show a domatic partition of G having $2n+1$ classes. We may consider G as the Cayley graph $G(\mathcal{H}, A)$, where $\mathcal{H}$ is the above mentioned group and $A = \{a_1, ..., a_n, -a_1, ..., -a_n\}$ and its vertices may be considered as elements of $\mathcal{H}$. For $k = 0, ..., 2n$ put $D_k = \{\sum_{i=1}^{n} \alpha_i a_i \mid \sum_{i=1}^{n} i\alpha_i \equiv k \pmod{(2n+1)}\}$.

Denote $\mathcal{D} = \{D_, ..., D_{2n}\}$. The classes of $\mathcal{D}$ are classes of $\mathcal{H}$ by $\mathcal{H}_0(2n+1)$. We shall prove that D_0 is a dominating set in G. For each vertex $x = \sum_{i=1}^{n} \alpha_i a_i$ let $k(x)$ be the integer such that $0 \leqq k(x) \leqq 2n$ and $\sum_{i=1}^{n} i\alpha_i \equiv k(x) \pmod{(2n+1)}$: this number $k(x)$ is determined uniquely. If $k(x) = 0$, then $x \in D_0$. If $1 \leqq k(x) \leqq n$, then let $y = \sum_{i=1}^{n} \beta_i a_i$, where $\beta_{k(x)} = \alpha_{k(x)} - 1$ and $\beta_i = \alpha_i$ for $i \neq k(x)$. If $n+1 \leqq k(x) \leqq 2n$, then let $y = \sum_{i=1}^{n} \gamma_i a_i$, where $\gamma_{2n-k(x)+1} = \alpha_{2n-k(x)+1} + 1$, $\gamma_i = \alpha_j$ for $j \neq 2n - k(x) + 1$. In both the cases $y \in D_0$ and is adjacent to x. Analogously as for D_0 the proof can be done for any other class of $\mathcal{D}$. Therefore $\mathcal{D}$ is a domatic partition of G and $d(G) = 2n + 1$.

As G is regular of degree $2n$, each vertex of G is adjacent only to vertices of other classes of $\mathcal{D}$ than its own one and is not adjacent to two vertices of the same class. Therefore the system of sets $\{N[x] \mid x \in D_0\}$ is a partition of $V(G)$. As the number of vertices of G is $\Pi_{i=1}^{n} h_i$, we have $|D_0| = (\Pi_{i=1}^{n})/(2n+1)$ and evidently this is $\gamma(G)$. $\square$

In the following theorem we shall consider only $n = 2$.

Theorem 2. *Let $G = H_1 \times H_2$, where H_1, H_2 are circuits of lengths h_1, h_2 respectively. Then the following two assertions are equivalent:*

(i) $h_1 \equiv 0 \pmod 5$ and $h_2 \equiv 0 \pmod 5$;

(ii) $d(G) = 5$ and $\gamma(G) = h_1 h_2/5$.

Proof. The implication (i)$\Rightarrow$ (ii) follows from Theorem 1. Consider the groups $\mathcal{H}, \mathcal{H}_0(5)$ in this case. Let $\mathcal{D}$ be a domatic partition of G with 5 classes, let D_0 be the class of $\mathcal{D}$ which contains the vertex o (the zero of $\mathcal{H}$). As $d(G) = 5$, the closed neighbourhoods of any two distinct vertices of D_0 are disjoint and therefore any two distinct vertices of D_0 have distance at least 3 in G. Therefore the vertex $a_1 + a_2$ is not in D_0 and must be adjacent to a vertex of D_0. Such a vertex is neither a_1, or a_2 and therefore it is $a_1 + 2a_2$ or $2a_1 + a_2$. Suppos that $a_1 + 2a_2 \in D_0$. Also $-a_1 + a_2 \notin D_0$ and must be adjacent to a vertex of D_0. The vertices $-a_1, +a_2$ are adjacent to o and the vertex $2a, -a_2$ has the distance 2 from $a_1 + 2a_2$. Therefore $-2a_1 + a_2 \in D_0$. Silmilarly we prove that $-a_1 - 2a_2 \in D_0$ and $2a_1 - a_2 \in D_0$. Denote $b = a_1 + 2a_2, c = -2a_1 + a_2$. Therefore the assumption $o \in D_0$ and $b \in D_0$ implies $c \in D_0, -b \in D_0, -c \in D_0$. Analogously $b \in D$ and $o \in D$ implies $c \in D_0, -b \in D_0, -c \in D_0$. Now we may proceed further in such a way and prove that all elements of the subgroup $\mathcal{H}$ of generated by the elements b, c are in D_0. Here we use the symmetry of the graph G. We have $b \in \mathcal{H}_0(5), c \in \mathcal{H}_0(5)$. Any element of $\mathcal{H}_0(5)$ has the expression $\alpha_1 a_1 + \alpha_2 a_2$, where $\alpha_1 + 2\alpha_2 \equiv 0 \pmod 5$ and therefore it may be expressed as $\beta b + \gamma c$, where $\beta = (\alpha_1 + 2\alpha_2)/5, \gamma = \alpha_2 - 2(\alpha_1 + 2\alpha_2)/5$ and β, γ are integers, Therefore the subgroup of $\mathcal{H}$ generated by b, c is $\mathcal{H}_0(5)$. As $\mathcal{D}$ has to be a domatic partition, the index of $\mathcal{H}_0(5)$ in G must be 5 and (i) holds by Lemma 1. If $2a_1 + a_2 \in D_0$ instead of $a_1 + 2a_2 \in D_0$, then the proof is the same, only with interchanging a_1 and a_2. $\square$

Theorem 3. *Let G be the same graph as in Theorem 1. If $h_i \equiv 0$ (mod $(n+1)$) for $i = 1, ..., n$, then $d^2(G) = n + 1$.*

Proof. A result from [3] implies that $d^2(G) \leqq n + 1$. Therefore it suffices to show a doubly domatic partition $\mathcal{D}$ of G with $n + 1$ classes. For $k = 0, ..., n$ put $D_k = \{\sum_{i=1}^{n} \alpha_i a | \sum_{i=1}^{n} i\alpha_i \equiv k \pmod{(n+1)}\}$

Denote $\mathcal{D} = \{D_0, ..., D_n\}$. The classes of D are classes of $\mathcal{H}$ by $\mathcal{H}_0(n+1)$. We shall prove that D_0 is a doubly dominating set in G. For each vertex $x = \sum_{i=1}^{n} \alpha_i a_i$ let $k(x)$ be the integer such that $0 \leqq k(x) \leqq n$ and $\sum_{i=1}^{n} i\alpha_i \equiv k(x) \pmod{(n+1)}$: this number $k(x)$ is determined uniquely. If $k(x) = 0$, then $x \in D_0$. Otherwise let $y = \sum_{i=n}^{n} \beta_i a_i, z = \sum_{i=1}^{n} \gamma_i a_i$, where $\beta_{k(x)} - 1, \beta_j = \alpha_j$ for $j \neq k(x), \gamma_{n-k(x)+1} = \alpha_{n-k(x)+1} + 1, \gamma_j = \alpha_j$, for $j \neq n - k(x) + 1$. Evidently $y \in D_0, z \in D_0$ and both y, z are adjacent to x. Therefore D_0 is a doubly dominating set. Analogously as for D_0, the proof can be done for any other class of $\mathcal{D}$. Therefore $\mathcal{D}$ is a doubly domatic partition and $d^2(G) = n + 1$.$\square$

Lemma 2. *If $h_i \equiv 0$ (mod 2 $(n+1)$) for $i = 1, ..., n$, then there exists a doubly domatic partition $\mathcal{D} = \{D_0, ..., D_n\}$ in G such that for each $k = 0, ..., n$ the set D_k*

is the union of two disjoint non-empty sets D'_k, D''_k with the property that for each $x \in V(G) - D_k$ there exist vertices $y \in D'_k, z \in D''_k$ adjacent to x.

Proof. In this case we may take
$D'_k = \{\sum_{i=1}^{n} i\alpha_i \equiv k \pmod{2(n+1)}\}$ $D''_k = \{\sum_{i=1}^{n} \alpha_i a_i | \sum_{i=1}^{n} i\alpha_i \equiv n+1+k \pmod{2(n+1)}\}$ for $k = 0, ..., n$. If we put $D_k = D'_k \cup D''_k$, then $\{D_0, ..., D_n\}$ is the doubly domatic partition from Theorem 3. □

With help of this lemma we prove the following theorem.

Theorem 4. *Let G be the same graph as in Theorem 1. If $h_n \equiv 0$ (mod 4) and $h_i \equiv 0$ (mod $2n$) for $i < n$, then $d_t(G) = 2n$.*

Poof. As G is regular of degree $2n$, by a result from [2] we have $d_t(G) \leqq 2n$. Therefore it suffices to show a total domatic partition of G having $2n$ classes. If $n = 1$, then this is $\mathcal{D} = \{D_), D_1\}$, where $D_0 = \{\alpha a_1 | \alpha \equiv 0 \pmod 4\} \cup \{\alpha a_1 | \alpha \equiv 1 \pmod 4\}, D_1 = \{\alpha a_1 | \alpha \equiv 2 \pmod 4\} \cup \{\alpha a_1 | \alpha \equiv 3 \pmod 4\}$. If $n \geqq 2$, then let $G_o = H_1 \times ... \times H_{n-1}$. By Lemma 2 there exists a doubly domatic partition $\tilde{\mathcal{D}} = \{\tilde{D}_0, ..., \tilde{D}_{n-1}\}$ of G with n classes such that for each $k = 0, ..., n-1$ the set $\tilde{D}_k$ is the union of two subsets $\tilde{D}'_k, \tilde{D}''_k$ such that for each vertex $x \in V(G_0) - \tilde{D}_k$ there ewist vertices $y \in \tilde{D}_k, z \in \tilde{D}''_k$ adjacent to x. Now for $k = 0, ..., n-1$ put

$$\begin{aligned} D_k = \{b + \alpha a_n | b \in \tilde{D}'_k \ \& \ \alpha \equiv 0 (mod 4)\} \cup \{b + \alpha a_n | b \in \tilde{D}'_k \ \& \ \alpha \equiv 1 (mod 4)\} \cup \\ \cup \{b + \alpha a_n | b \in \tilde{D}''_k \ \& \ \alpha \equiv 2 (mod 4)\} \cup \{b + \alpha a_n | b \in \tilde{D}''_k \ \& \ \alpha \equiv 3 (mod 4)\}, \\ D_{n+k} = \{b + \alpha a_n | b \in \tilde{D}'_k \ \& \ \alpha \equiv 2 (mod 4)\} \cup \{b + \alpha a_n | b \in \tilde{D}'_k \ \& \ \alpha \equiv 3 (mod 4)\} \cup \\ \cup \{b + \alpha a_n | b \in \tilde{D}''_k \ \& \ \alpha \equiv 0 (mod 4)\} \cup \{b + \alpha a_n | b \in \tilde{D}''_k \ \& \ \alpha \equiv 1 (mod 4)\}. \end{aligned}$$

We prove that D_0 is a total dominating set. Let $x = b + \alpha a_n$ be a vertex of G. If $b \in \tilde{D}'_0$ and $\alpha \equiv 0 \pmod 4$ or $\alpha \equiv 3 \pmod 4$, then x is adjacent to $b + (\alpha + 1)a_n \in D_0$. If $b \in \tilde{D}'_0$ and $\alpha \equiv 1 \pmod 4$, or $\alpha \equiv 2 \pmod 4$, then x is adjacent to $b + (\alpha - 1)a_n \in D_0$. Analogously for $b \in D''_0$. If $b \notin \tilde{D}_0$, then there exist $y \in \tilde{D}'_0$ and $z \in \tilde{D}''_0$ adjacent to x. If $\alpha \equiv 0 \pmod 4$ or $\alpha \equiv 1 \pmod 4$ then x is adjacent to $y + \alpha a_n \in D_0$. For other classes of $\mathcal{D} = \{D_0, ..., D_{2n-1}\}$ other than D_0 the proof is analogous. Therefore $\mathcal{D}$ is a total domatic partition of G.

At the end we prove again a theorem concerning only $n = 2$.

Theorem 5. *Let $G = H_1 \times H_2$, where H_1, H_2 are circuits of lengths h_1, h_2 respectively. If at least one of the numbers h_1, h_2 is divisible by 4, then $d(G) \geqq 4$.*

Remark. In such a case, if G satisfies the conditions of Theorem 1, then $d(G) = 5$, otherwise $d(G) = 4$.

Proof. Without loss of generality let h_1 be divisible by 4. We shall consider again the vertices of G as elements of a group and express them in the form $\alpha_1 a_1 + \alpha_2 a_z$, where $0 \leqq \alpha_1 \leqq h_1 - 1, 0 \leqq \alpha_2 \leqq h_2 - 1$. For each $k \in \{0, 1, 2, 3\}$ let $D_k = \{\alpha_1 a_1 + \alpha_2 a_2 | \alpha_1 + 2\alpha_2 \equiv k \pmod 4\}$. The reader may verify himself that $\mathcal{D} = \{D_0, D_1, D_2, D_3\}$ is a domatic partition of G and this $d(G) \geqq 4$.□

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