Acta Univ. M. Belii
Math. no. 6(1998), pp. 63-75

EXISTENCE OF INVARIANT TORI OF CRITICAL DIFFERENTIAL-EQUATION SYSTEMS DEPENDING ON MORE-DIMENSIONAL PARAMETER. PART II

Rudolf Zimka

Abstract. In the paper a system of differential equations depending on more-dimensional parameter is studied. It is supposed that the matrix of the first linear approximation P has m pairs of pure imaginary eigenvalues while the others do not lie on the imaginary axis. Conditions under which such a system in the cases when $m = 3, 4$ has invariant tori are presented (in Part I the cases when $m = 1, 2$ were analysed).

Introduction

Consider the system of differential equations

$$\dot{x} = X(x, \mu) + X^*(x, \mu), \tag{1}$$

where $x \in R^n, \mu \in R^d, \dot{x} = \frac{dx}{dt}, X(x, \mu)$ - a vector polynomial with respect to $x, \mu, X(0, 0) = 0, X^*(x, \mu)$ - a continuous function in $\mathbb{M} = \{(x, \mu) : ||x|| < K, ||\mu|| < < L\}$ with the property:

$$X^*(\sqrt{\varepsilon}x, \varepsilon\mu_0) = (\sqrt{\varepsilon})^{3p+2}\tilde{X}(x, \varepsilon, \mu_0), \tag{2}$$

$\tilde{X}(x, \varepsilon, \mu_0)$ - a continuous function with respect to x, ε, μ_0 of the class $C_x^1(\mathbb{M}), \mu_0 = = \frac{\mu}{||\mu||}, 0 \leq \varepsilon < L, p$ - a natural number.

It is supposed that:

1. the matrix $P = \frac{\partial X(0,0)}{\partial x}$ has m pairs of pure imaginary eigenvalues $\pm i\lambda_1, ..., \pm i\lambda_m$ and the others $\lambda_{2m+1}, ..., \lambda_n$ have non-zero real parts
2. $\det P \neq 0$
3. $q_1\lambda_1 + ... + q_m\lambda_m \neq 0, 0 < |q| \leq 3p+2, |q| = |q_1| + ... + |q_m|, q_i$ - integer numbers, $i = 1, ..., m$.

The bifurcation equation of the system (1) is (see [6]):

1991 *Mathematics Subject Classification.* 53A55, 58A20.

Key words and phrases. Systems of differential equations, matrix of the first linear approximation, eigenvalues, critical and non-critical matrix, bifurcation equation, condition of positiveness and criticalness, domain of positiveness and criticalness, invariant torus, bifurcation.

$$(3) \qquad B\rho^2 + C\mu = 0,$$

$$\rho^2 = col(\rho_1^2, ..., \rho_m^2), \mu = col(\mu_1, ..., \mu_d),$$

$$B = \begin{pmatrix} B_{11} & ... & B_{1m} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ B_{m1} & ... & B_{mm} \end{pmatrix}, C = \begin{pmatrix} C_{11} & ... & C_{1d} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ C_{m1} & ... & C_{md} \end{pmatrix}.$$

Suppose that det $B \neq 0$. From (3) we have on the beams $\delta(\mu_0) = \{\varepsilon\mu_0 : \mu = = \frac{\mu}{||\mu||}, \mu \in \mathbb{M}, 0 \leq \varepsilon < L\}$:

$$\rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0),$$

$$\alpha^2(\mu_0) = \Lambda\mu_0, \Lambda = \begin{pmatrix} \alpha_{11} & ... & \alpha_{1d} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \alpha_{m1} & ... & \alpha_{md} \end{pmatrix}, \mu_0 = \frac{1}{||\mu||} col(\mu_1, ..., \mu_d)$$

(the notions and the notations in this article have the same meaning as in [6]).

It was shown in [6] that on the beams $\delta(\mu_0), \mu \in \mathcal{DP}$, the system (1) can be reduced to the system

$$\dot{x}_1 = \varepsilon X_1(x_1, \varepsilon, \mu_0) + X_1^0(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0) + (\sqrt{\varepsilon})^{3p+1}\tilde{X}_1(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0)$$

$$(4) \quad \dot{\varphi}_1 = \lambda_1(\varepsilon) + \varepsilon\Phi_1(x_1, \varepsilon, \mu_0) + \Phi_1^0(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0) + (\sqrt{\varepsilon})^{3p+1}\tilde{\Phi}_1(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0)$$

$$\dot{\nu}_1 = J\nu_1 + V_1^0(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0) + (\sqrt{\varepsilon})^{3p+1}\tilde{V}_1(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0),$$

where X_1, Φ_1 - vector polynomials with respect to $x_1, \varepsilon, X_1(0, 0, \mu_0) = 0$, $\Phi_1(0, \varepsilon, \mu_0) = 0, \lambda_1(0) = \lambda = col(\lambda_1, ..., \lambda_m), X_1^0, \Phi_1^0, V_1^0, \tilde{X}_1, \tilde{\Phi}_1, \tilde{V}_1$ - continuous 2π - periodic with respect to φ_1 functions in the domain $\mathbb{M}_1$ $\{(x_1, \varphi_1, \nu_1, \varepsilon, \mu_0) : x_1 \in \in R^m, ||x_1|| < K_1, \nu_1 \in R^{n-2m}, ||\nu_1|| < K_1, \varphi_1 \in R^m, 0 \leq \varepsilon < L, \mu \in \mathcal{DP}\}$ of the class $C^1_{x_1,\varphi_1,\nu_1}, X_1^0, \Phi_1^0, V_1^0$ - vanishing at $\nu_1 = 0, J$ - a Jordan canonical lower matrix.

It holds (see [1]):

$$(5) \qquad P_1(\mu_0) = \frac{\partial X_1(0, 0, \mu_0)}{\partial x_1} = 2[diag\ \alpha(\mu_0)]B[diag\ \alpha(\mu_0)].$$

Suppose that the domain of criticalness $\mathcal{DC}$ of the bifurcation equation (3) is non-empty set. Take $\mu \in \mathcal{DC}$. On the beam $\delta(\mu_0)$ the system (4) is the system with one dimensional positive parameter ε which was investigated in [1]. We can perform on the system (4) on the beam $\delta(\mu_0)$ the transformation procedure that was described in [1]. This procedure consists of p steps if the following conditions are satisfied:

1. $q_1\lambda_1^k + ... + q_{m_k}\lambda_{m_k}^k \neq 0, 0 < |q| \leq 3(p-k)+2$, where $\pm i\lambda_1^k, ..., \pm i\lambda_{m_k}^k$ are the pure imaginary eigenvalues of $P_k(\mu_0), k = 1, ..., p-1$.
2. det $B_k \neq 0, \beta_k^2(\mu_0) = -B_k^{-1}C_k(\mu_0) > 0$, where $B_k, C_k(\mu_0)$ are the matrices

of the bifurcation equation $B_k\rho_k^2+\varepsilon C_k(\mu_0)=0$ arising at the $(k+1)^{st}$ step, $k=1,...,p-2$.

3. $P_{k+1}(\mu_0)=2[diag\ \beta_k(\mu_0)]B_k[diag\ \beta_k(\mu_0)]$ is critical, $k=1,...,p-2$.

Performing this transformation procedure consisting of p steps on the beam $\delta(\mu_0)$ (the transformation of the system (1) to the system (4) is the 1^{st} step) the system (4) is reduced to the system

$$
\begin{aligned}
&\dot{x}_p=\varepsilon^p X_p(x_p,\varepsilon,\mu_0)+X_p^0(x_p,\varphi_1,...,\varphi_p,\nu_1,...,\nu_p,\varepsilon,\mu_0)+\\
&\quad+\varepsilon^{p+1}\tilde{X}_p(x_p,\varphi_1,...,\varphi_p,\nu_1,...,\nu_p,\varepsilon,\mu_0)\\
&\dot{\varphi}_k=\varepsilon^{k-1}\lambda_k(\varepsilon)+\varepsilon^p\Phi_k(x_k,\varepsilon,\mu_0)+\Phi_k^0(x_p,\varphi_1,...,\varphi_p,\nu_1,...,\nu_p,\varepsilon,\mu_0)+\\
&\quad+\varepsilon^{p+1}\tilde{\Phi}_k(x_p,\varphi_1,...,\varphi_p,\nu_1,...,\nu_p,\varepsilon,\mu_0)\\
&\dot{\nu}_k=\varepsilon^{k-1}J_{k-1}\nu_k+V_k^0(x_p,\varphi_1,...,\varphi_p,\nu_1,...,\nu_p,\varepsilon,\mu_0)+\\
&\quad+(\sqrt{\varepsilon})^{3p+2-k}\tilde{V}_k(x_p,\varphi_1,...,\varphi_p,\nu_1,...,\nu_p,\varepsilon,\mu_0),\ k=1,...,p,
\end{aligned}
\tag{6}
$$

where X_p,Φ_k - polynomials with respect to $x_p,\varepsilon, X_p(0,0,\mu_0)=0, \Phi_k(0,\varepsilon,\mu_0)=$ $=0, \lambda_k(0)=\lambda^{k-1}=col(\lambda_1^{k-1},...,\lambda_{m_{k-1}}^{k-1}), \pm i\lambda_1^{k-1},...,\pm i\lambda_{m_{k-1}}^{k-1}$ - the eigenvalues of the matrix $P_{k-1}, \lambda^0=\lambda, m_0=m, P_0=P, X_p^0,\Phi_k^0,V_k^0,\tilde{X}_p,\tilde{\Phi}_k,\tilde{V}_k$ - continuous functions 2π - periodic with respect to $\varphi_1,...,\varphi_p$ in the domain $\mathbb{M}_p=\{(x_p,\varphi_1,...,\varphi_p,$ $\nu_1,...,\nu_p,\varepsilon):||x_p||<K_p,||\nu_k||<K_p,\varphi_k\in R^{m_{k-1}},k=1,...,p,0\leq\varepsilon<L\}$ of the class C^1 with respect to all variables with the exception of ε, X_p^0, Φ_k^0, V_k^0 - vanishing at $\nu_1=...=\nu_p=0, P_p=\frac{\partial X_p(0,0,\mu_0)}{\partial x_p}$ - regular matrice, J_{k-1} - non-critical Jordan matrices, $J_0=J$.

In this article the existence of invariant tori of the system (1) is studied in the cases when the matrix P has three and four pairs of pure imaginary eigenvalues.

1. Three pairs of pure imaginary eigenvalues

Suppose that the matrix P of the system (1) has three pairs of pure imaginary eigenvalues $\pm i\lambda_1,\pm i\lambda_2,\pm i\lambda_3$ and the others $\lambda_7,...,\lambda_n$ have non-zero real parts.

The bifurcation equation (3) is:

$$B\rho^2+C\mu=0, \tag{1.1}$$

where $\rho^2=col(\rho_1^2,\rho_2^2,\rho_3^2), \mu=col(\mu_1,...,\mu_d)$,

$$B=\begin{pmatrix}B_{11}&B_{12}&B_{13}\\B_{21}&B_{22}&B_{23}\\B_{31}&B_{32}&B_{33}\end{pmatrix}, C=\begin{pmatrix}C_{11}&...&C_{1d}\\C_{21}&...&C_{2d}\\C_{31}&...&C_{3d}\end{pmatrix}.$$

Suppose that $\det B\neq 0$. Take $\mu\in\mathbb{M}$ and consider the beam $\delta(\mu_0)=\{\varepsilon\mu_0:0\leq$ $\leq\varepsilon<L\}$. The solution of (1.1) with respect to ρ^2 on the beam $\delta(\mu_0)$ is:

$$\rho^2=\varepsilon(-B^{-1}C\mu_0)=\varepsilon\alpha^2(\mu_0), \tag{1.2}$$

$$\alpha^2(\mu_0) = \begin{pmatrix} \alpha_1^2(\mu_0) \\ \alpha_2^2(\mu_0) \\ \alpha_3^2(\mu_0) \end{pmatrix} = \Lambda\mu_0, \Lambda = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & ... & \alpha_{1d} \\ \alpha_{21} & ... & \alpha_{2d} \\ \alpha_{31} & ... & \alpha_{3d} \end{pmatrix}.$$

The matrix $P_1(\mu_0)$ which is defined by (5) has the form:

$$P_1(\mu_0) = 2 \begin{pmatrix} \alpha_1^2(\mu_0)B_{11} & \alpha_1(\mu_0)\alpha_2(\mu_0)B_{12} & \alpha_1(\mu_0)\alpha_3(\mu_0)B_{13} \\ \alpha_1(\mu_0)\alpha_2(\mu_0)B_{21} & \alpha_2^2(\mu_0)B_{22} & \alpha_2(\mu_0)\alpha_3(\mu_0)B_{23} \\ \alpha_1(\mu_0)\alpha_3(\mu_0)B_{31} & \alpha_2(\mu_0)\alpha_3(\mu_0)B_{32} & \alpha_3^2(\mu_0)B_{33} \end{pmatrix},$$

where $\alpha_i(\mu_0) = \sqrt{\frac{1}{||\mu||}(\alpha_{i1}\mu_1 + ... + \alpha_{id}\mu_d)}$, $i = 1, 2, 3$.

Denote the rank of the matrix Λ in (1.2) by the symbol $h(\Lambda)$ and the domain of positiveness and the domain of criticalness of the bifurcation equation (1.1) by the symbols $\mathcal{DP}$ and $\mathcal{DC}$.

Lemma 1.1. *Let be* $h(\Lambda) = 1$. *Then* $\mathcal{DP} \neq \emptyset$ *if and only if* $\alpha_1 \neq 0, \alpha_i = k_i\alpha_1, k_i > > 0$, $i = 2, 3$.

Proof. $\mathcal{DP}$ of (1.1) is determined by the inequalities:

$$\alpha_1^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d) > 0$$

$$\alpha_2^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{21}\mu_1 + ... + \alpha_{2d}\mu_d) > 0 \tag{1.3}$$

$$\alpha_3^2(\mu_0) = \frac{1}{||\mu||}(\alpha_{31}\mu_1 + ... + \alpha_{3d}\mu_d) > 0.$$

The first inequality in (1.3) is satisfied at all parameters $\mu \in \mathbb{M}$ which belong to that half-sphere of the sphere $O = \{\mu = (\mu_1, ..., \mu_d) : 0 < ||\mu|| < L\}$ that is determined by the hyperplane $\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d = 0$ and by a point $\mu^* \in O$ at which $\alpha_1^2(\mu^*) > 0$. As $h(\Lambda) = 1$ and $\alpha_1 \neq 0$ so there exist $k_2 \in R, k_3 \in R$ such that $\alpha_2 = k_2\alpha_1, \alpha_3 = k_3\alpha_1$. Using this we can express the second and the third inequality in (1.3) in the form: $\frac{k_2}{||\mu||}(\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d) > 0, \frac{k_3}{||\mu||}(\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d) > 0$. From these inequalities it follows that $\mathcal{DP} \neq \emptyset$ only if $k_2 > 0, k_3 > 0$. If $\alpha_1 = 0$ then $\mathcal{DP} = \emptyset$. The proof is over.

Lemma 1.2. *Let be* $h(\Lambda) = 2$. *Let* $\alpha_{i_1}, \alpha_{i_2}$ *be the linear independent pair from the triad* $\{\alpha_1, \alpha_2, \alpha_3\}$. *If for the third member* α_{i_3} *from this triad it holds:* $\alpha_{i_3} = = k_1\alpha_{i_1} + k_2\alpha_{i_2}, k_1 \geq 0, k_2 \geq 0, k_1 + k_2 > 0$, *then* $\mathcal{DP} \neq \emptyset$.

The proof of Lemma 1.2. is similar to the proof of Lemma 1.1.

Lemma 1.3. *Let be* $h(\Lambda) = 3$. *Then* $\mathcal{DP} \neq \emptyset$.

Proof. The set $\mathcal{DP}$ consists of those parameters $\mu \in \mathbb{M}$ which satisfy the inequalities (1.3). Solving them we get:

$$
\begin{aligned}
&\alpha_{11}\mu_1 + ... + \alpha_{1d}\mu_d - t_1 &&= 0\\
(1.4)\qquad &\alpha_{21}\mu_1 + ... + \alpha_{2d}\mu_d \qquad - t_2 &&= 0\\
&\alpha_{31}\mu_1 + ... + \alpha_{3d}\mu_d \qquad\qquad - t_3 &&= 0,\ t_i > 0, i = 1,2,3.
\end{aligned}
$$

As $h(\Lambda) = 3$ the system (1.4) has solutions with d parameters, $d \geq 3$. Among these parameters the variables t_1, t_2, t_3 can always be. Those parameters $\mu \in \mathbb{M}$ corresponding to positive numbers t_1, t_2, t_3 create $\mathcal{D}P$. The proof is over.

Denote

$$
\begin{aligned}
&a_1(\mu_0) = \alpha_1^2(\mu_0)B_{11} + \alpha_2^2(\mu_0)B_{22} + \alpha_3^2(\mu_0)B_{33}\\
(1.5)\qquad &a_2(\mu_0) = \alpha_1^2(\mu_0)\alpha_2^2(\mu_0)|M_{33}| + \alpha_1^2(\mu_0)\alpha_3^2(\mu_0)|M_{22}| + \alpha_2^2(\mu_0)\alpha_3^2(\mu_0)|M_{11}|\\
&a_3(\mu_0) = \alpha_1^2(\mu_0)\alpha_2^2(\mu_0)\alpha_3^2(\mu_0) \quad \det B,
\end{aligned}
$$

where $|M_{ii}|$ is the minor of the element B_{ii} of det B, $i = 1,2,3, \mu \in \mathcal{D}P$.

Lemma 1.4. *The matrix $P_1(\mu_0)$ is critical at $\mu \in \mathcal{D}P$ if and only if the following two conditions are satisfied:*
1. $a_1(\mu_0)a_2(\mu_0) = a_3(\mu_0)$
2. $a_2(\mu_0) > 0$.
The eigenvalues $\pm i\lambda_1^1, \lambda_3^1$ of the matrix $P_1(\mu_0)$ are defined by the formulae:

$$\lambda_1^1 = 2\sqrt{a_2(\mu_0)}, \lambda_3^1 = 2a_1(\mu_0).$$

Proof. If λ is the eigenvalue of $P_1(\mu_0)$ then $\tilde{\lambda} = \frac{\lambda}{2}$ is the eigenvalue of $\frac{P_1(\mu_0)}{2}$. The characteristic equation of the matrix $\frac{P_1(\mu_0)}{2}$ is:

$$\lambda^3 - a_1(\mu_0)\lambda^2 + a_2(\mu_0)\lambda - a_3(\mu_0) = 0, \tag{1.6}$$

where $a_1(\mu_0), a_2(\mu_0), a_3(\mu_0)$ have the form (1.5). Comparing (1.6) with its expression by means of the roots $\pm i\tilde{\lambda}_1^1, \tilde{\lambda}_3$ of $\frac{P_1(\mu_0)}{2}$ what is

$$\lambda^3 - \tilde{\lambda}_3^1\lambda^2 + (\tilde{\lambda}_1^1)^2\lambda - (\tilde{\lambda}_1^1)^2\tilde{\lambda}_3^1 = 0,$$

we have:

$$a_1(\mu_0) = \tilde{\lambda}_3^1, a_2(\mu_0) = (\tilde{\lambda}_1^1)^2, a_3(\mu_0) = (\tilde{\lambda}_1^1)^2\tilde{\lambda}_3^1.$$

From this we get the assertion of lemma. The proof is over.

Lemma 1.5. *Let be $h(\Lambda) = 1$ and $\mathcal{D}P \neq \emptyset$. Then $\mathcal{D}C = \emptyset$ or $\mathcal{D}C \equiv \mathcal{D}P$.*

Proof. When $\mathcal{D}P \neq \emptyset$ then according to Lemma 1.1 $\alpha_2 = k_2\alpha_1, \alpha_3 = k_3\alpha_1$, $k_2 > 0, k_3 > 0$. The expressions $a_1(\mu_0), a_2(\mu_0), a_3(\mu_0)$ from (1.5) can be expressed in the following way:

$$
(1.7)\qquad \begin{aligned}
a_1(\mu_0) &= \alpha_1^2(\mu_0)(B_{11} + k_2B_{22} + k_3B_{33})\\
a_2(\mu_0) &= \alpha_1^4(\mu_0)(k_2|M_{33}| + k_3|M_{22}| + k_2k_3|M_{11}|)\\
a_3(\mu_0) &= \alpha_1^6(\mu_0)k_2k_3 \det B.
\end{aligned}
$$

According to Lemma 1.4 the conditions for criticalness are:

1. $a_1(\mu_0)a_2(\mu_0) = a_3(\mu_0)$.
2. $a_2(\mu_0) > 0$.

Putting the expressions (1.7) into these conditions we get the conditions for criticalness which do not depend on $\mu \in \mathcal{D}P$:

$$
(1.8)\qquad \begin{aligned}
&1.\ (B_{11} + k_2B_{22} + k_3B_{33})(k_2|M_{33}| + k_3|M_{22}| + k_2k_3|M_{11}|) = k_2k_3 \det B.\\
&2.\ k_2|M_{33}| + k_3|M_{22}| + k_2k_3|M_{11}| > 0.
\end{aligned}
$$

Suppose that $\mathcal{D}C \neq \emptyset$ and take $\mu^* \in \mathcal{D}C$. This means that the conditions (1.8) are satisfied at $\mu^* \in \mathcal{D}C$ and as they do not depend on $\mu \in \mathcal{D}P$ they are satisfied at every $\mu \in \mathcal{D}P$. The proof is over.

Consider now $\mathcal{D}P$ and $\mathcal{D}C$ of the bifurcation equation (1.1) and suppose that $\mathcal{D}P \neq \emptyset$. Then on $\mathcal{D}P$ the system (1) can be reduced to the system (4) with $x_1 \in R^3, \varphi_1 \in R^3, \nu_1 \in R^{n-6}$.

Theorem 1.1. *Let be $\mathcal{D}P \neq \emptyset$. Then to every small enough $\mu \in \mathcal{D}P\backslash\mathcal{D}C$ there exists the invariant manifold*

$$
(1.9)\qquad \begin{aligned}
x_1 &= ||\mu||\eta(\varphi_1, ||\mu||, \mu_0)\\
\nu_1 &= ||\mu||^2\Theta_1(\varphi_1, ||\mu||, \mu_0),
\end{aligned}
$$

where η, Θ_1 are continuous functions 2π - periodic in all components of $\varphi_1, \varphi_1 \in$ $\in R^3, x_1 \in R^3, \nu_1 \in R^{n-6}$. The natural number p can be taken $p = 1$.

Proof. Consider an arbitrary $\mu \in \mathcal{D}P\backslash\mathcal{D}C$. This parameter μ lies on the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 < \varepsilon < L\}$. On this beam the system (1) can be reduced to the system (4) what is the system with one positive parameter ε. According to Theorem from Section 3 of Chapter 1 in [1] the invariant manifold (1.9) exists. The proof is over.

Suppose that $\mu \in \mathcal{D}C$ of the bifurcation equation (1.1). On the beam $\delta(\mu_0)$ we can perform the second step of the transformation procedure. The bifurcation equation of the system (4) on the beam $\delta(\mu_0)$is

$$B_1(\mu_0)\rho_1^2 + \varepsilon C_1(\mu_0) = 0,$$

where $B_1(\mu_0) \in R$.

Assume that $B_1(\mu_0) \neq 0$ and $\beta_1^2(\mu_0) = -\frac{1}{B_1(\mu_0)}C_1(\mu_0) > 0$. Then the system (4) can be reduced to the system (6) with $p = 2, x_1 \in R, \varphi_1 \in R^3, \varphi_2 \in R, \nu_1 \in$ $\in R^{n-6}, \nu_2 \in R$ and $P_2(\mu_0) = 2\beta_1^2(\mu_0)B_1(\mu_0) \neq 0$.

Utilizing Theorem from Section 3 of Chapter 1 in [1] the following theorem can be formulated.

Theorem 1.2. *Let be $\mu \in \mathcal{D}C$. If $B_1(\mu_0) \neq 0$ and $\beta_1^2(\mu_0) > 0$ then to every small enough $\mu \in \delta(\mu_0)$ there exists the invariant manifold*

$$\begin{aligned} x_2 &= ||\mu||\eta(\varphi_1, \varphi_2, ||\mu||, \mu_0) \\ \nu_1 &= ||\mu||^3\Theta_1(\varphi_1, \varphi_2, ||\mu||, \mu_0) \\ \nu_2 &= ||\mu||^2\Theta_2(\varphi_1, \varphi_2, ||\mu||, \mu_0), \end{aligned}$$

where η, Θ_1, Θ_2 are continuous functions 2π - periodic in all components of φ_1, φ_2, $\varphi_1 \in R^3, \varphi_2 \in R, x_2 \in R, \nu_1 \in R^{n-6}, \nu_2 \in R$. The natural number p has the value $p = 2$.

2. Four pairs of pure imaginary eigenvalues

Suppose that the matrix P of the system (1) has four pairs of pure imaginary eigenvalues $\pm i\lambda_1, \pm i\lambda_2, \pm i\lambda_3, \pm i\lambda_4$ and the others $\lambda_9, ..., \lambda_n$ have non-zero real parts.

The bifurcation equation (3) of the system (1) is:

$$B\rho^2 + C\mu = 0, \tag{2.1}$$

where $\rho^2 = col(\rho_1^2, \rho_2^2, \rho_3^2, \rho_4^2), \mu = col(\mu_1, ..., \mu_d)$,

$$B = \begin{pmatrix} B_{11} & ... & B_{14} \\ ..... & ..... & ..... \\ B_{41} & ... & B_{44} \end{pmatrix}, C = \begin{pmatrix} C_{11} & ... & C_{1d} \\ ..... & ..... & ..... \\ C_{41} & ... & C_{1d} \end{pmatrix}.$$

Suppose that det $B \neq 0$. Take $\mu \in \mathbb{M}$ and consider the beam $\delta(\mu_0) = \{\varepsilon\mu_0 : 0 \leq \leq \varepsilon < L\}$. The solution of (2.1) with respect to ρ^2 on the beam $\delta(\mu_0)$ is:

$$\rho^2 = \varepsilon(-B^{-1}C\mu_0) = \varepsilon\alpha^2(\mu_0), \tag{2.2}$$

where $\alpha^2(\mu_0) = col(\alpha_1^2(\mu_0), ..., \alpha_4^2(\mu_0)) = \Lambda\mu_0, \Lambda = col(\alpha_1, ..., \alpha_4) =$

$$= \begin{pmatrix} \alpha_{11} & ... & \alpha_{1d} \\ ..... & ..... & ..... \\ \alpha_{41} & ... & \alpha_{4d} \end{pmatrix}.$$

The matrix $P_1(\mu_0)$ which is defined by (5) has the form:

$$P_1(\mu_0) = 2\begin{pmatrix} \alpha_1^2(\mu_0)B_{11} & \alpha_1(\mu_0)\alpha_2(\mu_0)B_{12} & \alpha_1(\mu_0)\alpha_3(\mu_0)B_{13} & \alpha_1(\mu_0)\alpha_4(\mu_0)B_{14} \\ \alpha_1(\mu_0)\alpha_2(\mu_0)B_{21} & \alpha_2^2(\mu_0)B_{22} & \alpha_2(\mu_0)\alpha_3(\mu_0)B_{23} & \alpha_2(\mu_0)\alpha_4(\mu_0)B_{24} \\ \alpha_1(\mu_0)\alpha_3(\mu_0)B_{31} & \alpha_2(\mu_0)\alpha_3(\mu_0)B_{32} & \alpha_3^2(\mu_0)B_{33} & \alpha_3(\mu_0)\alpha_4(\mu_0)B_{34} \\ \alpha_1(\mu_0)\alpha_4(\mu_0)B_{41} & \alpha_2(\mu_0)\alpha_4(\mu_0)B_{42} & \alpha_3(\mu_0)\alpha_4(\mu_0)B_{43} & \alpha_4^2(\mu_0)B_{44} \end{pmatrix},$$

where $\alpha_i(\mu_0) = \sqrt{\frac{1}{||\mu||}(\alpha_{i1}\mu_1 + ... + \alpha_{id}\mu_d)}, \quad i = 1, 2, 3, 4.$

The following 4 lemmas say how the existence of $\mathcal{D}P$ of the bifurcation equation (2.1) depends on the rank of the matrix Λ from (2.2). The proofs of these lemmas can be performed in the same way as they were done in the lemmas 1.1 - 1.3 of the section 1.

Lemma 2.1. *Let be $h(\Lambda) = 1$. Then $\mathcal{D}P \neq \emptyset$ if and only if $\alpha_1 \neq 0, \alpha_i = k_i\alpha_1$, $k_i > 0, i = 2, 3, 4$.*

Lemma 2.2. *Let be $h(\Lambda) = 2$. Let $\alpha_{i_1}, \alpha_{i_2}$ be a linearly independent pair from the tetrad $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. If for the third and the fourth members of this tetrad is holds:*

$$\alpha_{i_3} = k_1\alpha_{i_1} + k_2\alpha_{i_2}, \alpha_{i_4} = k_3\alpha_{i_1} + k_4\alpha_{i_2},$$

$$k_i \geq 0, i = 1, 2, 3, 4, \ \ k_1 + k_2 > 0, k_3 + k_4 > 0,$$

then $\mathcal{D}P \neq \emptyset$.

Lemma 2.3. *Let be $h(\Lambda) = 3$. Let $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}$ be a linearly independent triad from the tetrad $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. If for the fourth member α_{i_4} of this tetrad it holds:*

$$\alpha_{i_4} = k_1\alpha_{i_1} + k_2\alpha_{i_2} + k_3\alpha_{i_3}, k_i \geq 0, i = 1, 2, 3, k_1 + k_2, k_3 > 0,$$

then $\mathcal{D}P \neq \emptyset$.

Lemma 2.4. *Let be $h(\Lambda) = 4$. Then $\mathcal{D}P \neq \emptyset$.*

Now we shall deal with the question when the matrix $P_1(\mu_0)$ is critical. As at every $\mu \in \mathcal{D}P$ det $P_1(\mu_0) =$ det $\{2[diag \ \ \alpha(\mu_0)]B[diag \ \ \alpha(\mu_0)]\} \neq 0$ the eigenvalues of $P_1(\mu_0)$ are different from zero. Therefore $P_1(\mu_0)$ is critical at $\mu \in \mathcal{D}P$ only when its eigenvalues are one of the following kinds:

A. $\pm i\lambda_1^1, \pm i\lambda_2^1$
B. $\pm i\lambda_1^1, \lambda_3^1, \lambda_4^1 = -\lambda_3^1$
C. $\pm i\lambda_1^1, \lambda_3^1, Re\lambda_3^1 \neq 0, \lambda_4^1 \neq -\lambda_3^1$.

Consider the characteristic equation of the matrix $\frac{P_1(\mu_0)}{2}, \mu \in \mathcal{D}P$:

$$\lambda^4 - a_1(\mu_0)\lambda^3 + a_2(\mu_0)\lambda^2 - a_3(\mu_0)\lambda + a_4(\mu_0) = 0, \tag{2.3}$$

where $a_1(\mu_0) = Tr\frac{P_1(\mu_0)}{2}, a_2(\mu_0)$ - the sum of all principal minors of order 2 of $\frac{P_1(\mu_0)}{2}, a_3(\mu_0)$ - the sum of all principal minors of order 3 of $\frac{P_1(\mu_0)}{2}, a_4(\mu_0) =$
$=$ det $\frac{P_1(\mu_0)}{2}$.

Lemma 2.5. *The matrix $P_1(\mu_0)$ has at $\mu \in \mathcal{D}P$ the eigenvalues $\pm i\lambda_1^1, \pm i\lambda_2^1$ if and only if*

$$a_1(\mu_0) = 0, a_2(\mu_0) > 0, a_3(\mu_0) = 0, a_4(\mu_0) > 0, a_2^2(\mu_0) \geq 4a_4(\mu_0). \tag{2.4}$$

The values λ_1^1, λ_2^1 are determined by the formulae:

$$\begin{aligned} \lambda_1^1 &= \sqrt{2}\sqrt{|-a_2(\mu_0) + \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}|}, \\ \lambda_2^1 &= \sqrt{2}\sqrt{|-a_2(\mu_0) - \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}|}. \end{aligned} \tag{2.5}$$

Proof. Comparing (2.3) with its expression by means of its roots $\pm i\tilde{\lambda}_1^1, \pm i\tilde{\lambda}_2^1$ what is

$$\lambda^4 + [(\tilde{\lambda}_1^1)^2 + (\tilde{\lambda}_2^1)^2]\lambda^2 + (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_2^1)^2 = 0, \tag{2.6}$$

we get the assertion (2.4). The roots of the equation (2.6) using the notation $a_2(\mu_0) = (\tilde{\lambda}_1^1)^2 + (\tilde{\lambda}_2^1)^2, a_4(\mu_0) = (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_2^1)^2$ are determined by the equation $\lambda^4 + a_2(\mu_0)\lambda^2 + a_4(\mu_0) = 0$. Putting $u = \lambda^2$ we have: $u^2 + a_2(\mu_0)u + a_4(\mu_0) = 0$. The roots of this equation are given by the formula: $u_{12} = \frac{-a_2(\mu_0) \pm \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}}{2}$.

From this we have:

$$\lambda_1 = \pm i\frac{\sqrt{2}}{2}\sqrt{|-a_2(\mu_0) + \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}|},$$

$$\lambda_2 = \pm i\frac{\sqrt{2}}{2}\sqrt{|-a_2(\mu_0) - \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}|}.$$

Taking into account that $\lambda_i^1 = 2\tilde{\lambda}_i^1, i = 1, 2$, we get the assertion (2.5). The proof is over.

Note 2.1. If follows from (2.5) that when $a_2^2(\mu_0) = 4a_4(\mu_0)$ then $\lambda_1^1 = \sqrt{2a_2(\mu_0)}, \lambda_2^1 = \sqrt{2a_2(\mu_0)}$. This means that the eigenvalues $\pm i\lambda_1^1$ have the multiplicity two.

Lemma 2.6. *The matrix $P_1(\mu_0)$ has at $\mu \in \mathcal{D}P$ the eigenvalues $\pm i\lambda_1^1, \lambda_3^1, \lambda_4^1 = -\lambda_3^1$ if and only if*

$$a_1(\mu_0) = 0, a_3(\mu_0) = 0, a_4(\mu_0) < 0. \tag{2.7}$$

The values λ_1^1, λ_3^1 are determined by the formulae:

$$\lambda_1^1 = \sqrt{2}\sqrt{a_2(\mu_0) + \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}},$$

(2.8)

$$\lambda_2^1 = \sqrt{2}\sqrt{-a_2(\mu_0) + \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}}.$$

Proof. Comparing (2.3) with its expression by means of its roots $\pm i\tilde{\lambda}_1^1, \tilde{\lambda}_3^1, \tilde{\lambda}_4^1 = -\tilde{\lambda}_3^1$ what is

$$\lambda^4 + [(\tilde{\lambda}_1^1)^2 - (\tilde{\lambda}_3^1)^2]\lambda^2 - (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_3^1)^2 = 0, \tag{2.9}$$

we get the assertion (2.7). Putting $u = \lambda^2$ we have from (2.9): $u^2 + a_2(\mu_0)u + a_4(\mu_0) = 0, a_2(\mu_0) = (\tilde{\lambda}_1^1)^2 - (\tilde{\lambda}_3^1)^2, a_4(\mu_0) = -(\tilde{\lambda}_1^1)^2(\tilde{\lambda}_3^1)^2$. The roots of this equation are given by the formula: $u_{12} = \frac{-a_2(\mu_0) \pm \sqrt{a_2^2(\mu_0) - 4a_4(\mu_0)}}{2}$.From this we have:

$$\lambda_1 = \pm i\frac{\sqrt{2}}{2}\sqrt{|-a_2(\mu_0)-\sqrt{a_2^2(\mu_0)-4a_4(\mu_0)}|},$$

$$\lambda_2 = \tfrac{\sqrt{2}}{2}\sqrt{-a_2(\mu_0)+\sqrt{a_2^2(\mu_0)-4a_4(\mu_0)}}.$$

This gives (2.8). The proof is over.

Lemma 2.7. *The matrix $P_1(\mu_0)$ has at $\mu \in \mathcal{D}P$ the eigenvalues $\pm i\lambda_1^1, \lambda_3^1, \lambda_4^1$, $Re\lambda_3^1 \neq 0, \lambda_4^1 \neq -\lambda_3^1$, if and only if the following conditions are satisfied:*

(2.10)
$$a_1(\mu_0) \neq 0, a_3(\mu_0) \neq 0, a_1(\mu_0)a_3(\mu_0) > 0,$$

$$a_4(\mu_0) = \tfrac{a_2(\mu_0)a_3(\mu_0)}{a_1(\mu_0)} - \tfrac{a_3(\mu_0)}{a_1(\mu_0)}^2.$$

The values of $\lambda_1^1, \lambda_3^1, \lambda_4^1$ are given by the formulae:

$$\lambda_1^1 = 2\sqrt{\frac{a_3(\mu_0)}{a_1(\mu_0)}},$$

(2.11)
$$\lambda_3^1 = a_1(\mu_0) + \sqrt{a_1^2(\mu_0) - 4[a_2(\mu_0) - \frac{a_3(\mu_0)}{a_1(\mu_0)}}],$$

$$\lambda_4^1 = a_1(\mu_0) - \sqrt{a_1^2(\mu_0) - 4[a_2(\mu_0) - \tfrac{a_3(\mu_0)}{a_1(\mu_0)}}].$$

Proof. Comparing (2.3) with its expression by means of its roots $\pm i\tilde{\lambda}_1^1, \tilde{\lambda}_3^1, \tilde{\lambda}_4^1$ what is $\lambda^4 - (\tilde{\lambda}_3^1 + \tilde{\lambda}_4^1)\lambda^3 + [(\tilde{\lambda}_1^1)^2 + \tilde{\lambda}_3^1\tilde{\lambda}_4^1]\lambda^2 - (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_3^1 + \tilde{\lambda}_4^1)\lambda + (\tilde{\lambda}^1)^2\tilde{\lambda}_3^1\tilde{\lambda}_4^1 = 0$, we have:

(2.12)
$$a_1(\mu_0) = \tilde{\lambda}_3^1 + \tilde{\lambda}_4^1, a_2(\mu_0) = (\tilde{\lambda}_1^1)^2 + \tilde{\lambda}_3^1\tilde{\lambda}_4^1,$$

$$a_3(\mu_0) = (\tilde{\lambda}_1^1)^2(\tilde{\lambda}_3^1 + \tilde{\lambda}_4^1), a_4(\mu_0) = (\tilde{\lambda}_1^1)^2\tilde{\lambda}_3^1\tilde{\lambda}_4^1.$$

From (2.12) we get the assertion (2.10). Solving (2.12) with respect to $\tilde{\lambda}_1^1, \tilde{\lambda}_3^1, \tilde{\lambda}_4^1$ and taking into account the relation between the eigenvalues of the matrices $P_1(\mu_0)$ and $\frac{P_1(\mu_0)}{2}$ we get the assertion (2.11). The proof is over.

Lemma 2.8. *Let be $h(\Lambda) = 1$ and $\mathcal{D}P \neq \emptyset$. Then $\mathcal{D}C = \emptyset$ or $\mathcal{D}C \equiv \mathcal{D}P$.*

The proof of this lemma is similar to the proof of Lemma 1.5.

Consider $\mathcal{D}P$ and $\mathcal{D}C$ of the bifurcation equation (2.1). Suppose that $\mathcal{D}P$ is non-empty set. Then on $\mathcal{D}P$ the system (1) can be reduced to the system (4) with $x_1 \in R^4, \varphi_1 \in R^4, \nu_1 \in R^{n-8}, p = 1$.

Theorem 2.1. *Let be $\mathcal{DP} \neq \emptyset$. Then to every small enough $\mu \in \mathcal{DP} \backslash \mathcal{DC}$ there exists the invariant manifold*

$$x_1 = ||\mu|| \eta(\varphi_1, ||\mu||, \mu_0)$$
$$\nu_1 = ||\mu||^2 \Theta_1(\varphi_1, ||\mu||, \mu_0),$$

where η, Θ_1 are continuous functions 2π - periodic in all components of φ_1, $\varphi_1 \in R^4, x_1 \in R^4, \nu_1 \in R^{n-8}$. The natural number p can be taken $p = 1$.

The proof of this theorem is similar to the proof of Theorem 1.1.

Suppose that $\mathcal{DC} \neq \emptyset$. Take $\mu \in \mathcal{DC}$. We can perform on the beam $\delta(\mu_0)$ the second step of the transformation procedure. The bifurcation equation of the system (4) on the beam $\delta(\mu_0)$ is:

(2.13) $$B_1(\mu_0)\rho_1^2 + \varepsilon C_1(\mu_0) = 0,$$

where:

1. $B_1(\mu_0)$ is the matrix of the order 2 when the eigenvalues of $P_1(\mu_0)$ are of the kind A
2. $B_1(\mu_0) \in R$ when the eigenvalues of $P_1(\mu_0)$ are of the kind B, C.

Consider firstly the cases when the eigenvalues of $P_1(\mu_0)$ are of the type B, C. Suppose that $B_1(\mu_0) \neq 0$ and $\beta_1^2(\mu_0) = -\frac{1}{B_1(\mu_0)} C_1(\mu_0) > 0$. Then the system (4) can be reduced to the system (6) with $x_2 \in R, \varphi_1 \in R^4, \varphi_2 \in R, \nu_1 \in R^{n-8}, \nu_2 \in \in R^2, p = 2$ and $P_2(\mu_0) = 2\beta_1^2(\mu_0)B_1(\mu_0) \neq 0$. Utilizing Theorem from Section 3 of Chapter 1 in [1] we can formulate the following theorem.

Theorem 2.2. *Let be $\mu \in \mathcal{DC}$ and the eigenvalues of $P_1(\mu_0)$ of the kind B or C. If $B_1(\mu_0) \neq 0$ and $\beta_1^2(\mu_0) > 0$ then to every small enough $\mu \in \delta(\mu_0)$ there exists the invariant manifold*

$$x_2 = ||\mu|| \eta(\varphi_1, \varphi_2, ||\mu||, \mu_0)$$
$$\nu_1 = ||\mu||^3 \Theta_1(\varphi_1, \varphi_2, ||\mu||, \mu_0)$$
$$\nu_2 = ||\mu||^2 \Theta_2(\varphi_1, \varphi_2, ||\mu||, \mu_0),$$

where η, Θ_1, Θ_2 are continuous functions 2π - periodic in all components of φ_1, φ_2, $\varphi_1 \in R^4, \varphi_2 \in R, x_2 \in R, \nu_1 \in R^{n-8}, \nu_2 \in R^2$. The natural number p has the value $p = 2$.

Suppose now that the eigenvalues of $P_1(\mu_0)$ at $\mu \in \mathcal{DC}$ are $\pm i\lambda_1^1, \pm i\lambda_2^1$. Let the following conditions be satisfied:

(2.14)
1. $q_1\lambda_1^1 + q_2\lambda_2^1 \neq 0, 0 < |q| \leq 5$
2. $\det B_1(\mu_0) \neq 0$
3. $\beta_1^2(\mu_0) = -B_1^{-1}(\mu_0)C_1(\mu_0) > 0.$

Then on the beam $\delta(\mu_0)$ the system (4) can be reduced to the system (6) with $p = 2, x_2 \in R^2, \varphi_1 \in R^4, \varphi_2 \in R^2, \nu_1 \in R^{n-8}$ and

$$P_2(\mu_0) = \frac{\partial X_2(0, 0, \mu_0)}{\partial x_2} = 2[diag\ \ \beta_1(\mu_0)]B_1(\mu_0)[diag\ \ \beta_1(\mu_0)].$$

On the base of Theorem from Section 3 of Chapter 1 in [1] the following theorem is valid.

Theorem 2.3. *Let be $\mu \in \mathcal{D}C$, the eigenvalues of $P_1(\mu_0)$ of the kind A and the conditions (2.14) statisfied. If the matrix $P_2(\mu_0)$ in non-critical then to every small enough $\mu \in \delta(\mu_0)$ there exists the invariant manifold*

$$x_2 = ||\mu||\eta(\varphi_1, \varphi_2, ||\mu||, \mu_0)$$

$$\nu_1 = ||\mu||^2\Theta_1(\varphi_1, \varphi_2, ||\mu||, \mu_0),$$

where η, Θ_1 are continuous functions 2π - periodic in all components of $\varphi_1, \varphi_2, \varphi_1 \in \in R^4, \varphi_2 \in R^2, x_2 \in R^2, \nu_1 \in R^{n-8}$. The natural number p has the value $p = 2$.

Suppose that the matrix $P_2(\mu_0)$ is critical. Then the third step of the transformation procedure can be performed on the beam $\delta(\mu_0)$. The bifurcation equation of the system (6) is

$$B_2(\mu_0)\rho_2^2 + \varepsilon C_2(\mu_0) = 0, \tag{2.15}$$

where $B_2(\mu_0) \in R$.

Assume that $B_2(\mu_0) \neq 0$ and $\beta_2^2(\mu_0) = -\frac{1}{B_2(\mu_0)}C_2(\mu_0) > 0$. Then the system (6) with $p = 2$ can be reduced to the system (6) with $p = 3, x_3 \in R, \varphi_1 \in R^4, \varphi_2 \in \in R^2, \varphi_3 \in R, \nu_1 \in R^{n-8}$ and

$$P_3(\mu_0) = \frac{\partial X_3(0, 0, \mu_0)}{\partial x_3} = 2\beta_2^2(\mu_0)B_2(\mu_0) \neq 0.$$

On the base of Theorem from Section 3 of Chapter 1 in [1] the following theorem is valid.

Theorem 2.4. *If $B_2(\mu_0) \neq 0$ and $\beta_2^2(\mu_0) > 0$ then to every small enough $\mu \in \delta(\mu_0)$ there exists the invariant manifold*

$$x_3 = ||\mu||\eta(\varphi_1, \varphi_2, \varphi_3, ||\mu||, \mu_0)$$
$$\nu_1 = ||\mu||^4\Theta_1(\varphi_1, \varphi_2, \varphi_3, ||\mu||, \mu_0),$$

where η, Θ_1 are continuous functions 2π - periodic in all components of $\varphi_1, \varphi_2, \varphi_3$, $\varphi_1 \in R^4, \varphi_2 \in R^2, \varphi_3 \in R, x_3 \in R, \nu_1 \in R^{n-8}$. The natural number p has the value $p = 3$.

Note Many significant results in the bifurcation theory of dynamical systems were achieved during last three decades. A nice survey of them can be found in the books [4], [5] in which also the relations among reached results are discussed. The question of the existence of bifurcations in the case of two pairs of pure imaginary eigenvalues is for example studied in the articles [2], [3], [7], [8].

References

[1] Bibikov, Yu. N., *Multi-frequency non-linear oscillations and their bifurcations*, The Publishing House of the Saint Petersburg University, Saint Petersburg, 1991 (Russian).

[2] Carr, J., Chow, S.N., Hale, J. K., *Abelian integrals and bifurcation theory*, J. Differential Equations 59 (1985), 413-436.

[3] Chow, S. N., Li, C., Wang, D., *Uniqueness of periodic orbits of some vector fields with codimension two singularities*, J. Differential Equations 77 (1989), 231-253.

[4] Medveď, M., *Dynamical Systems*, Veda, Bratislava, 1988 (Slovak).
[5] Medveď, M., *Fundamentals of Dynamical Systems and Bifurcation Theory*, Adam Hilger, Bristol, New York, 1992.
[6] Zimka, R., *Existence of invariant tori of critical differential - equation systems depending on more - dimensional parameter. Part I.*, Acta Univ. M. Belii, Math. No. 5 (1997), 11–25.
[7] Zoladek, H., *On the versality of a family of symmetric vector-fields in the plane*, Math. USSR-Sb. 48 (1984), 463-492.
[8] Zoladek, H., *Bifurcations of certain family of planar vector fields tangent to axes*, J. Differential Equations 67 (1987), 1-55.

(Received July 17, 1998)

Dept. of Applied Informatics
Matej Bel University
Tajovského 10
974 01 Banská Bystrica
SLOVAKIA

E-mail address: zimka@econ.umb.sk