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FINITELY GENERATED FREE ORTHOMODULAR LATTICES IV

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Abstract. A full description of the finitely generated free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ with n generators ($n \geq 3$) in the varieties $\mathbf{V}(\mathbf{O}_k)$ ($k \geq 2$) of non-modular ortholattices generated by the orthomodular lattices $\mathbf{O}_k$ which are horizontal sums of k Boolean blocks $\mathbf{2}^3$ is presented.

1. Introduction

In [6] and [7] we described completely the finitely generated free modular ortholattices and gave formulas for their cardinalities. We recall that the lattice of subvarieties of the variety $\mathcal{MO}$ of all modular ortholattices is an infinite chain

$$\mathcal{T} \subsetneq \mathcal{B} \subsetneq \mathcal{MO}_2 \subsetneq \mathcal{MO}_3 \subsetneq \cdots \subsetneq \mathcal{MO}_k \subsetneq \mathcal{MO}_{k+1} \subsetneq \cdots \subsetneq \mathcal{MO} \tag{1}$$

of type $\omega + 1$ where $\mathcal{B}$ is the variety of Boolean algebras and the variety $\mathcal{MO}_k$ is generated by the modular ortholattice $\mathbf{MO}_k$ of height 2 which is a horizontal sum of k Boolean blocks $\mathbf{2}^2$ (see Figure 1 on the next page). Then in [8] we made our first attempt to achieve a similar goal in locally finite varieties of non-modular ortholattices. We described the finitely generated free algebras in the varieties $\mathbf{V}(\mathbf{L}_k)$ of orthomodular lattices generated by the horizontal sums $\mathbf{L}_k$ ($k \geq 2$) of one Boolean block $\mathbf{2}^3$ and $k-1$ Boolean blocks $\mathbf{2}^2$. These varieties form an another infinite chain of type $\omega + 1$ "parallel" to the chain in (1) in the sense that each $\mathbf{V}(\mathbf{L}_k)$ contains the variety $\mathcal{MO}_k$ and the variety $\mathbf{V}(\mathbf{L}_2)$ covers the variety $\mathcal{MO}_2$ (see [9]).

Stepping outside the variety $\mathcal{MO}$ not surprisingly increases the complexity of the description. This was already clearly seen in [8] though considering the varieties $\mathbf{V}(\mathbf{L}_k)$ meant the smallest possible step outside the varieties $\mathcal{MO}_k$ of modular ortholattices — in the generator $\mathcal{MO}_k$ we only replaced one of the blocks $\mathbf{2}^2$ by a larger block $\mathbf{2}^3$. In the present paper we pursue our investigation further and we

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consider the varieties of non-modular orthomodular lattices generated by the orthomodular lattices $\mathbf{O}_k$ which are horizontal sums of k Boolean blocks $\mathbf{2}^3$ (see Figure 1). We describe the finitely generated free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ with n generators in the varieties $\mathbf{V}(\mathbf{O}_k)$ for all $n \geq 3$ and $k \geq 2$. This more ambitious step outside the varieties $\mathcal{MO}_k$ of modular ortholattices results in a quite complex description (Theorem 3.11 and Corollary 3.12).

For a more detailed introduction to the topic as well as further discussion of our method and its tools we refer the reader to papers [6] and [7].

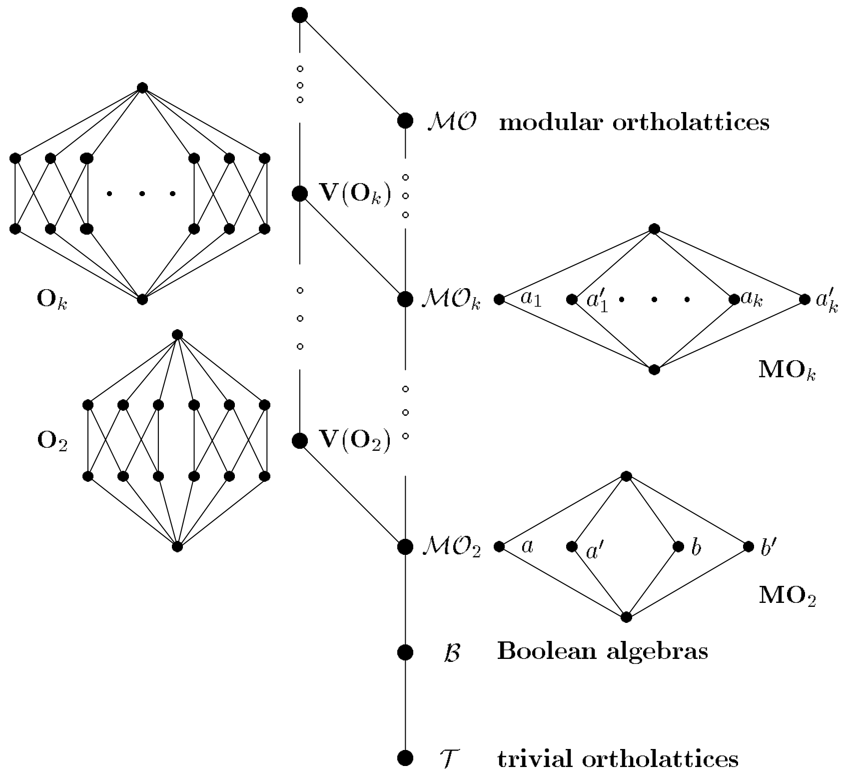

Figure 1

2. Preliminaries

In 1936 G. Birkhoff and J. von Neumann [3] suggested taking the lattice of closed subspaces of a Hilbert space as a suitable model for 'the logic of quantum mechanics'. This lattice equipped with the relation of orthogonal complement can be described as an ortholattice which in the case of a finite-dimensional Hilbert space is modular. It is not modular if the Hilbert space is infinite dimensional, but a weaker so-called orthomodular law is satisfied. Standard references for the algebraic aspects of the theory of orthomodular lattices are the monographs [9] and [2] and we refer the reader to [12] for 'quantum logics' aspects.

An *orthomodular lattice* is an (abstract) algebra $\mathbf{L} = (L; \vee, \wedge, ', 0, 1)$ with a bounded lattice reduct $(L; \vee, \wedge, 0, 1)$ and a unary operation of orthocomplementation $'$ such that for every $a, b \in L$

$$(a')' = a,\ a \wedge a' = 0,\ a \vee a' = 1,\ 0' = 1,\ 1' = 0,$$

$$(a \wedge b)' = a' \vee b',\ (a \vee b)' = a' \wedge b'$$

and the *orthomodular law*

$$b = (b \wedge a) \vee [b \wedge (b \wedge a)']$$

hold.

In any orthomodular lattice the n-ary *commutator* function is defined by

$$c(x_1, \ldots, x_n) = \bigvee_{(i_1, \ldots, i_n) \in \{0,1\}^n} x_1^{i_1} \wedge \cdots \wedge x_n^{i_n},$$

where $x_i^0 = x_i$ and $x_i^1 = x_i'$. The function $(c(x_1, \ldots, x_n))'$ will be denoted by $c'(x_1, \ldots, x_n)$. In particular, the binary commutator function, which plays an important role in our considerations, is given by

$$c(x, y) = (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y').$$

In this paper we focus on the orthomodular lattices $\mathbf{O}_k$ which are horizontal sums of k Boolean blocks $\mathbf{2}^3$ $(k \geq 2)$. We recall that a horizontal sum means here that any two distinct blocks intersect in $\{0, 1\}$. We start with the following easy lemma describing the commutator functions on $\mathbf{O}_k$.

2.1 Lemma. *Let $c(x_1, \ldots, x_n) : O_k^n \to O_k$ be the commutator function on the orthomodular lattice $\mathbf{O}_k$ $(n, k \geq 2)$. Then for any elements $a_1, \ldots, a_n \in O_k$,*

(1) *$c(a_1, \ldots, a_n) \in \{0, 1\}$ and*
(2) *$c(a_1, \ldots, a_n) = 0$ if and only if at least two of $a_1, \ldots, a_n$ are from different blocks of $\mathbf{O}_k$.*

Any interval of the form $[0, v]$ in an orthomodular lattice $\mathbf{L}$ $(v \in L)$ can be considered as an orthomodular lattice if one defines the orthocomplement of an element $a \in [0, v]$ in $[0, v]$ to be $a' \wedge v$, where a' is the orthocomplement of a in $\mathbf{L}$.

Elements $a \in L$ such that

$$a = (a \wedge x) \vee (a \wedge x') \tag{2}$$

for every $x \in L$ are called *central*. The set $Z(\mathbf{L})$ of all central elements of $\mathbf{L}$ is a Boolean subalgebra of $\mathbf{L}$, called the *centre* of $\mathbf{L}$. Moreover,

$$a \in Z(\mathbf{L}),\ v \in L \Rightarrow a \wedge v \in Z([0, v]).$$

The following fact concerning arbitrary orthomodular lattices $\mathbf{L}$ comes from [10] (see also [9; p. 20]).

(3) $$c \in Z(\mathbf{L}) \Leftrightarrow \mathbf{L} \cong [0, c] \times [0, c'].$$

We shall essentially employ this fact in the first step of our method where $\mathbf{L}$ will be the free orthomodular lattice $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ and c will be the commutator function $c(x_1, \ldots, x_n)$.

A variety $\mathbf{V}(\mathbf{A})$ generated by an algebra $\mathbf{A}$ is arithmetical (that is, congruence-distributive and congruence-permutable) if and only if $\mathbf{A}$ has a Pixley arithmeticity term function $p(x, y, z) : A^3 \to A$ satisfying

$$p(a, a, b) = p(b, a, b) = p(b, a, a) = b \quad \text{for all } a, b \in A$$

(see, for example, [4; p. 85]). The following result shows that the varieties $\mathbf{V}(\mathbf{O}_k)$ under consideration are arithmetical.

2.2 Proposition. *The varieties $\mathbf{V}(\mathbf{O}_k)$ are arithmetical with a Pixley arithmeticity term*

$$p(x, y, z) = (x \vee z) \wedge (x \vee y') \wedge (z \vee y') \\ \wedge [(c(x, y) \wedge z) \vee (c(y, z) \wedge x) \vee (c(x, z) \wedge x \wedge z)]$$

Proof. It is straightforward to verify that the identity

$$p(x, y, x) = x$$

holds in $\mathbf{O}_k$. Further, for any $a, b \in O_k$ we obtain

$$p(a, a, b) = (a \vee b) \wedge (a' \vee b) \wedge [b \vee (c(a, b) \wedge a)] = p(b, a, a).$$

If a and b belong to the same Boolean block of $\mathbf{O}_k$ then clearly $(a \vee b) \wedge (a' \vee b) = b$ and $c(a, b) = 1$; if a, b are from different blocks of $\mathbf{O}_k$ then evidently $(a \vee b) \wedge (a' \vee b) = 1$ and $c(a, b) = 0$. Thus we have

$$p(a, a, b) = b = p(b, a, a)$$

as required. □

3. Description of the free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$

A free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(1)$ with one generator in the varieties $\mathbf{V}(\mathbf{O}_k)$ $(k \geq 2)$ is isomorphic to the 1-generated free Boolean algebra

$$\mathbf{F}_{\mathcal{B}}(1) \cong \mathbf{2}^2,$$

which also is the free orthomodular lattice with one generator. A free algebra with two generators in all the varieties $\mathbf{V}(\mathbf{O}_k)$ $(k \geq 2)$ is a direct product of the free Boolean algebra with two generators $\mathbf{F}_{\mathcal{B}}(2)$ and the lattice $\mathbf{MO}_2$. Hence

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(2) \cong \mathbf{F}_{\mathcal{B}}(2) \times \mathbf{MO}_2 \cong \mathbf{2}^4 \times \mathbf{MO}_2,$$

and this also is isomorphic to the free orthomodular lattice with two generators (see [2; III.2]).

The free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ with $n \geq 3$ generators in the varieties $\mathbf{V}(\mathbf{O}_k)$ ($k \geq 2$) are certainly finite because the varieties $\mathbf{V}(\mathbf{O}_k)$ are locally finite (see [4; chapter 1.3]). A universal description of the free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ says that it is isomorphic to the algebra of all n-ary term functions on the generator $\mathbf{O}_k$. A more useful description, which has been a springboard for our work, can be according to Proposition 2.2 derived from Theorems 2.1, 2.2 of [7]. These two theorems in turn follow from the Arithmetic Strong Duality Theorem of Theory of natural dualitites [4; Theorem 3.11].

3.1 Theorem. ([7; Theorems 2.1,2.2]) *The free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ with n generators in the variety $\mathbf{V}(\mathbf{O}_k)$ ($k \geq 2, n \geq 3$) is isomorphic to the algebra of all functions from ${O_k}^n$ to O_k preserving the unary partial endomorphisms of $\mathbf{O}_k$.*

As for any $(a_1, \ldots, a_n) \in O_k^n$ we have $c(a_1, \ldots, a_n) \in \{0, 1\}$ by Lemma 2.1(1), it is easy to see that for every term function $t(x_1, \ldots, x_n) \in \mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$,

$$c(a_1, \ldots, a_n) = (c(a_1, \ldots, a_n) \wedge t(a_1, \ldots, a_n)) \vee (c(a_1, \ldots, a_n) \wedge t(a_1, \ldots, a_n)'),$$

hence by (2) it follows that the commutator $c(x_1, \ldots, x_n)$ is a central element of $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$. Thus according to (3), the free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ can be expressed as the product

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n) = [0, c(x_1, \ldots, x_n)] \times [0, c'(x_1, \ldots, x_n)].$$

Analogously as in [7; Theorem 3.1], the interval $[0, c(x_1, \ldots, x_n)]$ is isomorphic to the n-generated free Boolean algebra $\mathbf{F}_{\mathcal{B}}(n) \cong \mathbf{2}^{2^n}$. We can decompose the interval $[0, c'(x_1, \ldots, x_n)]$ by the commutators $c(x_i, x_j)$ $(i, j = 1, \ldots, n,\ i < j)$ as

$$[0, c'(x_1, \ldots, x_n)] \cong \prod_{\tilde{w} \in \{0,1\}^N} [0, \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)],$$

where the product is taken over all N-tuples $\tilde{w} = (w_{1,2}, \ldots, w_{n-1,n}) \in \{0, 1\}^N$, $N = \binom{n}{2}$ and

$$c^{w_{i,j}}(x_i, x_j) = \begin{cases} c(x_i, x_j), & \text{if } w_{i,j} = 0, \\ c'(x_i, x_j), & \text{if } w_{i,j} = 1. \end{cases}$$

The term function $t_{\tilde{w}}(x_1, \ldots, x_n) = \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)$ corresponds to a labelled unoriented graph $G_{\tilde{w}}$ (without multiple edges and loops) on the vertex set $\{x_1, \ldots, x_n\}$ with edges $x_i x_j$ whenever $w_{i,j} = 1$ for $i < j$. Any one of G, $\tilde{w}$ and $t_{\tilde{w}}(x_1, \ldots, x_n)$, the last denoted also by $C_G(x_1, \ldots, x_n)$, determines the other two. A necessary and sufficient condition on the structure of the graph G for the interval $[0, t_{\tilde{w}}(x_1, \ldots, x_n)] = [0, C_G(x_1, \ldots, x_n)]$ in $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ to be non-trivial is analogous to that in [7; Proposition 3.2].

3.2 Proposition. *The following conditions are equivalent:*

(a) $C_G(x_1, \ldots, x_n) : O_k^n \to O_k$ *is not identically equal to zero;*

(b) *there exist elements* $a_1, \ldots, a_n \in \mathbf{O}_k$ *with the following properties:*

   (i) $C_G(a_1, \ldots, a_n) = 1$,

   (ii) *the elements* $a_1, \ldots, a_n$ *are not all from the same block of* $\mathbf{O}_k$,

   (iii) $x_i x_j$ *is an edge of* G *if and only if* a_i, a_j *are elements of different blocks in* $\mathbf{O}_k$*;*

(c) *the graph* $G_p := G$ *consists of* l *isolated vertices* $(0 \le l \le n - p)$ *and one connected component which is a complete* p*-partite graph* $(2 \le p \le k)$.

Proof. If (a) holds then there are $a_1, \ldots, a_n \in O_k$ such that $C_G(a_1, \ldots, a_n) \neq 0$. This implies by Lemma 2.1(1) that $c^{w_{i,j}}(a_i, a_j)$ and $c'(a_1, \ldots, a_n)$ are equal to 1 for all $i < j$ and shows (b)(i). By Lemma 2.1(2), $c'(a_1, \ldots, a_n) = 1$ implies that there exist i and j such that a_i, a_j are from different blocks of $\mathbf{O}_k$ which shows (b)(ii). Further, as for all i, j, $i < j$ we have $c^{w_{i,j}}(a_i, a_j) = 1$, we get that a_i, a_j are from different blocks of $\mathbf{O}_k$ if and only if $w_{i,j} = 1$ if and only if $x_i x_j$ is an edge in G. This shows (b)(iii).

Let now $a_1, \ldots, a_n \in \mathbf{O}_k$ be as in condition (b). From the condition (b)(iii) it follows that x_i is an isolated vertex in G whenever $a_i \in \{0, 1\}$. If $a_i \notin \{0, 1\}$ then by (b)(ii) there is j such that a_j belongs to a different block than a_i and by (b)(iii) again, for all such i, j there is an edge $x_i x_j$ in G. Assume that the elements $a_i \notin \{0, 1\}$ belong to p different blocks of $\mathbf{O}_k$. By (b)(ii), $p \ge 2$. We conclude that G has isolated vertices x_i associated with $a_i \in \{0, 1\}$ and the remaining vertices x_i are partitioned according to which block of $\mathbf{O}_k$ the corresponding elements a_i come from, giving a complete p-partite graph. This proves (c).

We finally show that for any labelled graph $G = G_p$ satisfying (c) there are $a_1, \ldots, a_n \in O_k$ such that $C_G(a_1, \ldots, a_n)$ is equal to 1. The value of C_G at $(a_1, \ldots, a_n)$ equals 1 if and only if all the expressions $c^{w_{i,j}}(a_i, a_j)$ and $c'(a_1, \ldots, a_n)$ equal 1. If x_i, x_j $(i < j)$ are vertices from different blocks of the p-partite connected component of the graph $G_p = G$ then C_G contains the term $c'(x_i, x_j)$. By Lemma 2.1(2), such term will take value 1 at (a_i, a_j) if we choose a_i, a_j from different blocks of $\mathbf{O}_k$. If x_i, x_j $(i < j)$ lie in the same block of the p-partite graph G then C_G contains the term $c(x_i, x_j)$ which will take value 1 at (a_i, a_j) if we choose a_i, a_j from the same block of $\mathbf{O}_k$. If x_i is an isolated vertex in G then C_G contains the term $c(x_i, x_j)$ $(c(x_j, x_i))$ in case $i < j$ $(i > j)$. In order to have $c(x_i, x_j) = 1$ in this case, we must choose a_i from the same block as a_j for all j, which means we must choose a_i from the set $\{0, 1\}$. So we can get $C_G(a_1 \ldots, a_n) = 1$ if we allocate a unique block B_{i_j} $(i_j \in \{1, \ldots, k\})$ of $\mathbf{O}_k$ to the jth block of the p-partite component of G for $j = 1, \ldots, p$ and choose the coordinates of $(a_1, \ldots, a_n)$ corresponding to the vertices of the jth block of G to be (any) elements from $B_{i_j} \setminus \{0, 1\}$. For isolated vertices x_i we choose the corresponding a_i arbitrarily from $\{0, 1\}$. This proves (a). $\square$

According to Theorem 3.1, the interval $[0, C_G(x_1, \ldots, x_n)]$ in $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ consists of all functions from ${O_k}^n$ to O_k which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$ and preserve the unary partial endomorphisms of $\mathbf{O}_k$. Any such function takes value zero at $(a_1, \ldots, a_n) \in O_k^n$ whenever the term function C_G does.

Let T_G be the set of all $\mathbf{a} = (a_1, \ldots, a_n)$ from $(O_k)^n$ at which C_G is non-zero, that is $C_G(a_1, \ldots, a_n) = 1$. Let the coordinates $a_i \in \{0,1\}$ corresponding to isolated vertices of G be called *trivial.* Proposition 3.2 says that the non-trivial coordinates of $\mathbf{a} \in T_G$ lie in exactly p of the k Boolean blocks $B_1, \ldots, B_k$ of $\mathbf{O}_k$ corresponding to the blocks of the p-partite component of the graph $G = G_p$, $2 \leq p \leq k$. Let us assume from now that the blocks of the p-partite component of the graph $G = G_p$ have cardinalities $k_1, \ldots, k_p$, where $k_1 \geq k_2 \geq \cdots \geq k_p \geq 1$ and $\sum_{i=1}^{p} k_i \leq n$. We shall sometimes use the notation $G_p(k_1, \ldots, k_p)$ for the given graph G. We shall further call the elements $\mathbf{a} \in T_G$ whose $k_1, \ldots, k_p$ non-trivial coordinates are taken respectively from the first p blocks $B_1, \ldots, B_p$ of $\mathbf{O}_k$ *standard.*

In the first step we shall count the orbits of the automorphism group $\mathrm{Aut}(\mathbf{O}_k)$ on T_G. We shall denote the atoms and the coatoms of each block B_i by b_i, c_i, d_i and b'_i, c'_i, d'_i, respectively. Note that any automorphism f_i of the Boolean algebra $\mathbf{B}_i$ fixes 0 and 1 and is given by a permutation of the atoms of $\mathbf{B}_i$, $i \in \{1, 2, \ldots, k\}$. The following lemma describes the automorphisms of $\mathbf{O}_k$ and guarantees that when counting the orbits of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G, it suffices to focus only on standard elements $\mathbf{a} \in T_G$.

3.3 Lemma.
(i) For every automorphism $\alpha \in Aut(\mathbf{O}_k)$ there is a permutation ν on the index set $I_k := \{1, 2, \ldots, k\}$ and automorphisms $\{f_i : B_i \to B_i \mid i \in I_k\}$ of the Boolean algebras (blocks) $\mathbf{B}_i$ such that

$$(4) \qquad \alpha(x_i) = f_{\nu(i)}(x_{\nu(i)}) \text{ for every } i \in I_k \text{ and } x_i \in B_i.$$

(ii) Conversely, for every permutation ν on the index set $I_k := \{1, 2, \ldots, k\}$ and a set of automorphisms $\{f_i : B_i \to B_i \mid i \in I_k\}$ of the Boolean algebras (blocks) $\mathbf{B}_i$, a unary map $\alpha : O_k \to O_k$ defined by (4) is an automorphism of $\mathbf{O}_k$.
(iii) For every $\mathbf{b} \in T_G$ such that the non-trivial coordinates of $\mathbf{b}$ lie in p Boolean blocks $B_{i_1}, \ldots, B_{i_p}$ of $\mathbf{O}_k$, where $2 \leq p \leq k$ and $\{i_1, \ldots, i_p\} \subseteq I_k$, there is a standard element $\mathbf{a} \in T_G$ and an automorphism $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$ such that

$$(5) \qquad \alpha(a_1) = b_1, \ldots, \alpha(a_n) = b_n,$$

thus $\mathbf{b}$ belongs to the orbit $Orb(\mathbf{a})$ of $Aut(\mathbf{O}_k)$ on T_G.

Proof. The observations (i) and (ii) are easy and we leave them for the reader. To show (iii), we define a permutation ν on the set I_k such that $\nu(j) = i_j$ for $j = 1, \ldots, p$ and ν maps the set $I_k \setminus \{1, \ldots, p\}$ arbitrarily onto the set $I_k \setminus \{i_1, \ldots, i_p\}$. We take for the automorphisms $\{f_i : B_i \to B_i \mid i \in I_k\}$ of the Boolean algebras $\mathbf{B}_i$ just the identity maps. By this we define an automorphism $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$ such that $\alpha \restriction \mathbf{B}_j : \mathbf{B}_j \to \mathbf{B}_{i_j}$ is an isomorphism for every $j \in \{1, \ldots, p\}$. Let α^{-1} denote the inverse of α and let $a_i := \alpha^{-1}(b_i)$ for $i = 1, \ldots, n$. Now clearly, $\mathbf{a} = (a_1, \ldots, a_n)$ belongs to T_G by Proposition 3.2, $\mathbf{a}$ is standard and (5) holds. Hence $\mathbf{b}$ belongs to the orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G. $\square$

Let us for standard $\mathbf{a} \in T_G$ call the corresponding orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G *standard*, too. The part (iii) of previous lemma shows that when counting the orbits $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G for a p-partite graph $G = G_p(k_1, \ldots, k_p)$ where

$k_1 \geq k_2 \geq \cdots \geq k_p$ and $\sum_{i=1}^{p} k_i \leq n$, we can w.l.o.g. count only standard orbits Orb($\mathbf{a}$), that is, to assume that for $i = 1, \ldots, p$, the k_i non-trivial coordinates of $\mathbf{a}$ are taken from the block B_i. We shall therefore sometimes use the notation $\mathbf{a}(k_1, \ldots, k_p)$ for $\mathbf{a}$.

The part (i) of following lemma comes from [8].

3.4 Lemma.
(i) There are (up to the automorphism action)

$$P(k_i) = 2^{k_i - 1} + 6^{k_i - 1}$$

choices for the non-trivial k_i coordinates of $\mathbf{a}(k_1, \ldots, k_p) = (a_1, \ldots, a_n)$ to be selected from the block B_i, $i = 1, \ldots, p$.
(ii) There are

$$N(n, p; k_1, \ldots, k_p) = 2^{n-p} \cdot \prod_{i=1}^{p} (3^{k_i - 1} + 1)$$

orbits Orb($\mathbf{a}(k_1, \ldots, k_p)$) of Aut($\mathbf{O}_k$) on T_G.

Proof. (i) If the pair of the first two coordinates of $\mathbf{a}$ taken from B_i is one of the four pairs $(b, c), (b, c'), (b', c), (b', c')$, where the distinct elements $b, c \notin \{0, 1\}$ are not an atom of B_i and its complement, then any of the remaining $k_i - 2$ coordinates from B_i can be chosen arbitrarily from the six elements $\{b, b', c, c', d, d'\}$ of B_i giving $4 \cdot 6^{k_i - 2}$ choices for the k_i coordinates from the block B_i starting with such prescribed first two coordinates. In the other case the pair of the first two coordinates is one of $(b, b), (b, b'), (b', b), (b', b')$ for an atom b of B_i giving (up to the automorphism action) 2 choices b and b' for the first coordinate and, recursively, $P(k_i - 1)$ choices for the remaining $k_i - 1$ coordinates. Hence we arrive at the recursive formula

$$P(k_i) = 4 \cdot 6^{k_i - 2} + 2 \cdot P(k_i - 1).$$

By standard methods of solving such formulas we obtain

$$P(k_i) = \alpha \cdot 2^{k_i} + \beta \cdot 6^{k_i}, \ \alpha, \beta \in R.$$

One can check that $P(2) = 8$ and $P(3) = 40$, which leads to $\alpha = \frac{1}{2}$, $\beta = \frac{1}{6}$. Hence $P(k_i) = 2^{k_i - 1} + 6^{k_i - 1}$.

(ii) From (i) and the fact that there are $2^{n - \sum_{i=1}^{p} k_i}$ choices for the trivial coordinates of $\mathbf{a}$ to be selected from the set $\{0, 1\}$ it follows that the number of orbits Orb($\mathbf{a}(k_1, \ldots, k_p)$) of Aut($\mathbf{O}_k$) on T_G is

$$\begin{aligned} N(n, p; k_1, \ldots, k_p) &= (\prod_{i=1}^{p} (2^{k_i - 1} + 6^{k_i - 1})) \cdot 2^{n - \sum_{i=1}^{p} k_i} \\ &= 2^{(\sum_{i=1}^{p} k_i) - p} \cdot (\prod_{i=1}^{p} (3^{k_i - 1} + 1)) \cdot 2^{n - \sum_{i=1}^{p} k_i} \\ &= 2^{n-p} \cdot \prod_{i=1}^{p} (3^{k_i - 1} + 1). \quad \square \end{aligned}$$

Let us call the k_i non-trivial coordinates of $\mathbf{a}$ belonging to the block B_i to be *of type I* if they all are from the set $\{b, b'\}$ for an atom b and its orthocomplement b' in $\mathbf{B}_i$ and *of type II* otherwise. It is clear that among $P(k_i) = 2^{k_i-1} + 6^{k_i-1}$ choices for the non-trivial k_i coordinates of $\mathbf{a}$ coming from the block B_i there are exactly 2^{k_i} choices for the k_i coordinates of type I and the rest, $2^{k_i-1} + 6^{k_i-1} - 2^{k_i} = 6^{k_i-1} - 2^{k_i-1}$, are the choices for the k_i coordinates of type II. So that one can express the product $N(n, p; k_1, \ldots, k_p)$ in Lemma 3.4 as

$$N(n,p;k_1,\ldots,k_p) = (\prod_{i=1}^{p}[(6^{k_i-1} - 2^{k_i-1}) + 2^{k_i}]) \cdot 2^{n-\sum_{i=1}^{p} k_i} \tag{6}$$

from which it is clear how the coordinates of types I and II contribute to the resulting number.

We shall say that a standard orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G is *of type* $\{i_1, \ldots, i_s\}$ if the non-trivial coordinates of $\mathbf{a}$ of type II are exactly from the blocks $B_{i_1}, \ldots, B_{i_s}$ for a subset $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, p\}$.

In the second step of our method we determine the structure of the $\mathrm{Aut}(\mathbf{O}_k)$-preserving functions from ${O_k}^n$ to O_k which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$. We proceed as in [7] and [8]. We may extend the action of $\mathrm{Aut}(\mathbf{O}_k)$ on $\mathbf{O}_k$ pointwise to $(\mathbf{O}_k)^n$, so that for $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbf{O}_k)^n$ and $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$, $\alpha(\mathbf{a}) := (\alpha(a_1), \ldots, \alpha(a_n)) \in (\mathbf{O}_k)^n$ and a function $f\colon (\mathbf{O}_k)^n \to \mathbf{O}_k$ is α-preserving if for all $\mathbf{a} \in (\mathbf{O}_k)^n$, $f(\alpha(\mathbf{a})) = \alpha(f(\mathbf{a}))$. To define an $\mathrm{Aut}(\mathbf{O}_k)$-preserving function $f \le C_G$, we cannot freely choose images from $\mathbf{O}_k$ for representatives of the orbits $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G. The reason is that for $p < k$ there are automorphisms $\alpha \ne \beta$ in $\mathrm{Aut}(\mathbf{O}_k)$ such that for any representative $\mathbf{a}$ of orbit $\mathrm{Orb}\ \mathbf{a}$, $\alpha(\mathbf{a})$ is equal to $\beta(\mathbf{a})$, which restricts the choices for $f(\mathbf{a})$ to those which satisfy $f(\alpha(\mathbf{a})) = f(\beta(\mathbf{a}))$. Hence we only may freely choose the image $f(\mathbf{a})$ for each orbit-representative $\mathbf{a}$ within $\bigcap_{\gamma \in \mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{O}_k}(\gamma)$ (see [7]) while the values of the other elements $\alpha(\mathbf{a})$ in $\mathrm{Orb}(\mathbf{a})$ will be determined by

$$f(\alpha(\mathbf{a})) = \alpha(f(\mathbf{a})). \tag{7}$$

Consequently, the algebra $\mathbf{A}_G$ of the $\mathrm{Aut}(\mathbf{O}_k)$-preserving functions from ${O_k}^n$ to O_k which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$ is isomorphic to the product of the subalgebras $\bigcap_{\gamma \in \mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{O}_k}(\gamma)$ of $\mathbf{O}_k$ taken over all standard orbits $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G.

For any $s \in \{0, 1, \ldots, p\}$, let $\mathbf{L}_{(s,p-s)}$ denote an orthomodular lattice (a subalgebra of $\mathbf{O}_k$) consisting of s Boolean blocks $\mathbf{2}^3$ and $p - s$ Boolean blocks $\mathbf{2}^2$.

3.5 Proposition. *Let $G = G_p(k_1, \ldots, k_p)$ be a p-partite graph with blocks of cardinalitites $k_1, \ldots, k_p$ such that $k_1 \ge \cdots \ge k_p \ge 1$ and $\sum_{i=1}^{p} k_i \le n$. The algebra $\mathbf{A}_G$ of the $\mathrm{Aut}(\mathbf{O}_k)$-preserving functions from ${O_k}^n$ to O_k which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$ is*

$$\mathbf{A}_G \cong (\mathbf{MO}_p)^{2^n} \times \prod_{s=1}^{p} (\mathbf{L}_{(s,p-s)})^{N_A(n,p,s;k_1,\ldots,k_p)},$$

where

$$N_A(n,p,s;k_1,\dots,k_p) = 2^n \cdot \sum_{\substack{\{i_1,\dots,i_s\}\\ \subseteq\{1,\dots,p\}}} \prod_{r=1}^{s} \frac{6^{k_{i_r}-1}-2^{k_{i_r}-1}}{2^{k_{i_r}}}$$

Proof. We know that each standard orbit Orb($\mathbf{a}$) of Aut($\mathbf{O}_k$) on T_G contributes a factor $\bigcap_{\gamma\in\mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{O}_k}(\gamma)$ to the algebra $\mathbf{A}_G$. For each of possible types of the orbits Orb($\mathbf{a}$) we shall recognize the structure of $\bigcap_{\gamma\in\mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{O}_k}(\gamma)$ and determine the number of orbits Orb($\mathbf{a}$) of a given type.

Let us first consider a standard orbit Orb($\mathbf{a}$) of Aut($\mathbf{O}_k$) on T_G of type $I_p = \{1,\dots,p\}$, that is, all the non-trivial coordinates of $\mathbf{a}$ are of type II. Then the stabiliser of $\mathbf{a}$ consists of exactly those automorphisms in Aut($\mathbf{O}_k$) which fix all elements of the blocks $B_1,\dots,B_p$ in $\mathbf{O}_k$ and permute atoms (and consequently their complementary coatoms) in the remaining $k-p$ blocks of $\mathbf{O}_k$. Hence

$$\bigcap_{\gamma\in\mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{O}_k}(\gamma) \cong \mathbf{L}_{(p,0)} = \mathbf{O}_p$$

and the orbit Orb($\mathbf{a}$) contributes a factor $\mathbf{O}_p$ to the algebra $\mathbf{A}_G$. The number of standard orbits Orb($\mathbf{a}(k_1,\dots,k_p)$) of type I_p is

$$N(n,p;k_1,\dots,k_p;I_p) = \Big(\prod_{i=1}^{p}(6^{k_i-1}-2^{k_i-1})\Big)\cdot 2^{n-\sum_{i=1}^{p}k_i} = 2^n\cdot\prod_{i=1}^{p}\frac{6^{k_i-1}-2^{k_i-1}}{2^{k_i}},$$

so that the same is the number of factors $\mathbf{O}_p$ contributed by all standard orbits Orb($\mathbf{a}(k_1,\dots,k_p)$) of type $\{1,\dots,p\}$. Note that this number can be obtained from $N(n,p;k_1,\dots,k_p)$ in (6) by removing in each of the first p factors the term 2^{k_i} expressing the number of selections of the non-trivial coordinates of type I.

Let us now consider a standard orbit Orb($\mathbf{a}$) of type $S := \{i_1,\dots,i_s\}$ where $\varnothing \neq S \subsetneq I_p$, that is, $1 \le s \le p-1$. For each $j \in I_p \setminus S$, the k_j non-trivial coordinates of $\mathbf{a}$ taken from the block B_j are from the subset $\{b_j, b_j'\} \subset B_j$ for an atom $b_j \in B_j$. The stabiliser of $\mathbf{a}$ consists of exactly those automorphisms in Aut($\mathbf{O}_k$) which fix the elements of the $p-s$ subalgebras $\{0, b_j, b_j', 1\}$ of B_j for $j \in I_p \setminus S$, fix all elements of the s blocks B_j for $j \in S$ and permute atoms (and their complements) in the remaining $k-p$ blocks of $\mathbf{O}_k$. Hence

$$\bigcap_{\gamma\in\mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{O}_k}(\gamma) \cong \mathbf{L}_{(s,p-s)}$$

and each such orbit Orb($\mathbf{a}$) contributes a factor $\mathbf{L}_{(s,p-s)}$ to the algebra $\mathbf{A}_G$. The number of orbits Orb($\mathbf{a}(k_1,\dots,k_p)$) of type $S = \{i_1,\dots,i_s\}$ is

$$(8) \qquad N(n,p,s;k_1,\dots,k_p;\{i_1,\dots,i_s\}) = 2^n\cdot\prod_{r=1}^{s}\frac{6^{k_{i_r}-1}-2^{k_{i_r}-1}}{2^{k_{i_r}}}$$

so that the same is the number of factors $\mathbf{L}_{(s,p-s)}$ contributed by all standard orbits Orb($\mathbf{a}(k_1,\dots,k_p)$) of type $S = \{i_1,\dots,i_s\}$ where $\varnothing \neq S \subsetneq I_p$. Note that this number can be obtained from $N(n,p;k_1,\dots,k_p)$ in (6) by removing in the factors corresponding to $i \in S$ the term 2^{k_i} expressing the number of selections of the non-trivial coordinates of type I and removing in the factors corresponding to $i \in I_p \setminus S$ the term $6^{k_i-1} - 2^{k_i-1}$ expressing the number of selections of the non-trivial coordinates of type II.

It remains to consider the orbits $\mathrm{Orb}(\mathbf{a})$ of type $\varnothing$, that is, such that all the non-trivial coordinates of $\mathbf{a}$ are of type I. Each such orbit contributes a factor $\mathbf{L}_{(0,p)} \cong \mathbf{MO}_p$ and the number of such copies of $\mathbf{MO}_p$ is

$$N(n,p;k_1,\dots,k_p;\varnothing) = 2^{\sum_{i=1}^{p} k_i} \cdot 2^{n-\sum_{i=1}^{p} k_i} = 2^n. \quad \square$$

In the third step of our method we determine which of the different standard orbits $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G can be "glued together" by the action of the unary partial endomorphisms of $\mathbf{O}_k$. By this we mean the situation when

$$e(a_1) = b_1, \dots, e(a_n) = b_n \tag{9}$$

for representatives $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ of different standard orbits $\mathrm{Orb}(\mathbf{a})$, $\mathrm{Orb}(\mathbf{b})$ and a unary partial endomorphism e of $\mathbf{O}_k$. As the functions $f : O_k^n \to O_k$ that we consider must preserve all unary (partial) endomorphisms e of $\mathbf{O}_k$, we need to guarantee that the condition

$$(e(a_1) = b_1, \dots, e(a_n) = b_n) \Longrightarrow f(b_1, \dots, b_n) = e(f(a_1, \dots, a_n)) \tag{10}$$

holds for every unary partial endomorphism e of $\mathbf{O}_k$. The following few concepts will prove useful in our further analysis.

3.6 Definition. *A unary partial endomorphism e of $\mathbf{O}_k$ is said to be*

(i) **straight**, *if e maps all elements of $\mathrm{dom}(e) \cap B_i$ into B_i for all $i \in \{1, \dots, k\}$;*

(ii) **proper**, *if the domain $\mathrm{dom}(e)$ consists of elements from at least two different blocks of $\mathbf{O}_k$;*

(iii) $0,1$-**separating**, *if for any $x \in O_k$*

$$e(x) = 0 \text{ implies } x = 0 \text{ and } e(x) = 1 \text{ implies } x = 1.$$

3.7 Lemma. *Every unary partial endomorphism e of $\mathbf{O}_k$ can be expressed as*

$$e = \alpha \circ e'$$

for some automorphism $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$ and a straight partial endomorphism e' on $\mathbf{O}_k$ with domain $\mathrm{dom}(e') = \mathrm{dom}(e)$.

Proof. If a partial endomorphism e of $\mathbf{O}_k$ is straight, then the statement clearly holds for $e' = e$ and α being just the identity map on $\mathbf{O}_k$.

Let us now assume that e is not straight, that is, there exist elements $x_i \in B_i \setminus \{0,1\}$, $y_j \in B_j \setminus \{0,1\}$, $i,j \in \{1,\dots,k\}$, $i \neq j$, such that $e(x_i) = y_j$, and consequently, $e(x_i') = y_j'$. Suppose that e maps an element z_l of a block $B_l \neq B_i$ into the block B_j. Then the element $e(z_l)$ must be comparable to one of the elements

y_j, y'_j and w.l.o.g. we can assume that $e(z_l) \neq 0$. As $z_l \wedge x_i = z_l \wedge x'_i = 0$, we obtain that $e(z_l) \wedge y_j = e(z_l) \wedge y'_j = e(0) = 0$, a contradiction.

Hence any partial endomorphism e maps different blocks of $\mathbf{O}_k$ to mutually different blocks of $\mathbf{O}_k$. Therefore there is a permutation ν of the index set $I_k = \{1, 2, \ldots, k\}$ such that e maps every $x_i \in \mathrm{dom}(e) \cap B_i$ into $B_{\nu(i)}$. Let α be an automorphism of $\mathbf{O}_k$ determined by this permutation and by the identity maps $\{f_i : B_i \to B_i \mid i \in I_k\}$ (see Lemma 3.3). Let us define a partial endomorphism e' on $\mathbf{O}_k$ with domain $\mathrm{dom}(e') = \mathrm{dom}(e)$ by $e' := \alpha^{-1} \circ e$. Then clearly, e' is straight and $\alpha \circ e' = e$. □

Lemma 3.7 yields that a function $f : O_k^n \to O_k$ preserves all unary partial endomorphisms e of $\mathbf{O}_k$ provided it preserves the automorphisms of $\mathbf{O}_k$ and the straight partial endomorphisms e' of $\mathbf{O}_k$. Hence it is sufficient for us to consider the condition (10) only for straight partial endomorphisms e of $\mathbf{O}_k$.

We note that (9) is possible only if e is proper because the non-trivial coordinates of elements $\mathbf{a}, \mathbf{b} \in T_G$ always lie in at least two different blocks of $\mathbf{O}_k$. The following lemma shows that proper partial endomorphisms e must be $0, 1$-separating.

3.8 Lemma. *Every proper partial endomorphisms of $\mathbf{O}_k$ is $0, 1$-separating.*

Proof. Let $x_i \in \mathrm{dom}(e) \cap B_i$ and $y_j \in \mathrm{dom}(e) \cap B_j$ for different blocks B_i, B_j of $\mathbf{O}_k$. Note that then $\{x_i, y_j\} \cap \{0, 1\} = \varnothing$. W.l.o.g. suppose that $e(x_i) = 0$. As $y_j \vee x_i = 1 = y'_j \vee x_i$, we obtain

$$e(y_j) = e(y_j) \vee e(x_i) = e(y_j \vee x_i) = e(1) = 1$$

and analogously,

$$e(y'_j) = e(y'_j) \vee e(x_i) = e(y'_j \vee x_i) = e(1) = 1.$$

This leads to $e(0) = e(y_j \wedge y'_j) = e(y_j) \wedge e(y'_j) = 1$, a contradiction. The proof is complete. □

Hence it is sufficient for us to consider the condition (10) only for straight and proper (thus $0, 1$-separating) partial endomorphisms e of $\mathbf{O}_k$.

Let us call *primitive* any unary partial endomorphism u_j of $\mathbf{O}_k$ whose graph is

$$(u_j)^{\square} = \{(0, 0), (b_j, b'_j), (b'_j, b_j), (1, 1)\}$$

for an atom b_j of the block B_j of $\mathbf{O}_k$, $j \in I_k$. Let us further call $\{j_1, \ldots, j_s\}$-*primitive* any partial endomorphism u on $\mathbf{O}_k$ such that $u \restriction B_{j_r}$ is primitive for $r = 1, \ldots, s$ and $u(x) = x$ for all elements $x \in \mathrm{dom}(u) \setminus (B_{j_1} \cup \cdots \cup B_{j_s})$.

The following observation is now easy and we leave it for the reader.

3.9 Lemma. *Every straight and proper (thus $0, 1$-separating) partial endomorphism e of $\mathbf{O}_k$ which is not an automorphism can be expressed as*

$$e = \alpha \circ u$$

for some $\{j_1, \ldots, j_s\}$-primitive partial endomorphism u of $\mathbf{O}_k$ and a straight automorphism α of $\mathbf{O}_k$.

Hence our analysis finally shows us that it is sufficient to consider the condition (10) only for $\{j_1, \ldots, j_s\}$-primitive partial endomorphisms e of $\mathbf{O}_k$.

Let us consider a standard orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G of a type $S \subsetneq I_p$, where $s := |S| \geq 0$; hence for any $j \in I_p \setminus S$, the k_j non-trivial coordinates of $\mathbf{a}$ taken from the block B_j are from the subset $\{b_j, b'_j\} \subset B_j$ for an atom $b_j \in B_j$. Let $I_p \setminus S = \{j_1, \ldots, j_{p-s}\}$. Let e be an $\{j_1, \ldots, j_{p-s}\}$-primitive partial endomorphism of $\mathbf{O}_k$ and let (9) hold for e and $\mathbf{b} \in T_G \setminus \mathrm{Orb}(\mathbf{a})$. Then the orbit $\mathrm{Orb}(\mathbf{b})$ is also of type S. If the image $f(a_1, \ldots, a_n)$ of $\mathbf{a}$ in f is chosen from

$$\bigcap_{\gamma \in \mathrm{Stab}\, \mathbf{a}} \mathrm{fix}_{\mathbf{O}_k}(\gamma) \cong \mathbf{L}_{(s,p-s)},$$

then by the condition (10) the image $f(b_1, \ldots, b_n)$ of $\mathbf{b}$ in f is determined by $f(b_1, \ldots, b_n) = e(f(a_1, \ldots, a_n))$ and consequently, only one of the orbits $\mathrm{Orb}(\mathbf{a})$, $\mathrm{Orb}(\mathbf{b})$ contributes a factor $\mathbf{L}_{(s,p-s)}$ to the algebra of functions $f : (\mathbf{O}_k)^n \to \mathbf{O}_k$ which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$ and preserve the unary partial endomorphisms of $\mathbf{O}_k$. This means that for $0 \leq s \leq p-1$, the number of factors $\mathbf{L}_{(s,p-s)}$ in the structure of the interval $[0, C_G(x_1, \ldots, x_n)]$ in $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ will be obtained by dividing every $N(n, p, s; k_1, \ldots, k_p; S)$ in (8) by two for each $j \in I_p \setminus S = \{j_1, \ldots, j_{p-s}\}$, thus by dividing the exponents $N_A(n, p, s; k_1, \ldots, k_p)$ in Proposition 3.5 by 2^{p-s}. The number of factors $\mathbf{MO}_p$ in the structure of the interval $[0, C_G(x_1, \ldots, x_n)]$ in $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ will obviously be obtained by dividing $N(n, p; k_1, \ldots, k_p; \varnothing) = 2^n$ by 2^p, thus is equal to 2^{n-p}. Let us denote for $1 \leq s \leq p$,

$$N(n, p; s; k_1, \ldots, k_p) := \frac{N_A(n, p, s; k_1, \ldots, k_p)}{2^{p-s}}.$$

We have arrived to the following proposition.

3.10 Proposition. *The structure of the interval $[0, C_G(x_1, \ldots, x_n)]$ associated to a p-partite graph $G = G_p(k_1, \ldots, k_p)$ with blocks of cardinalitites $k_1, \ldots, k_p$ such that $k_1 \geq \cdots \geq k_p \geq 1$ and $\sum_{i=1}^{p} k_i \leq n$ is*

$$[0, C_G(x_1, \ldots, x_n)] \cong (\mathbf{MO}_p)^{2^{n-p}} \times \prod_{s=1}^{p} (\mathbf{L}_{(s,p-s)})^{N(n,p,s;k_1,\ldots,k_p)}$$

where

$$N(n, p, s; k_1, \ldots, k_p) = 2^{n-p+s} \cdot \sum_{\substack{\{i_1,\ldots,i_s\} \\ \subseteq \{1,\ldots,p\}}} \prod_{r=1}^{s} \frac{6^{k_{i_r}-1} - 2^{k_{i_r}-1}}{2^{k_{i_r}}} \qquad \square$$

Analogously as in [8], we use the fact that the number of the p-partite graphs $G = G_p(k_1, \ldots, k_p)$ on an n-element vertex set with blocks of cardinalities $k_1, \ldots, k_p$ ($k_1 \geq \cdots \geq k_p \geq 1, \sum_{i=1}^{p} k_i \leq n$) and with $l = n - \sum_{i=1}^{p} k_i$ isolated vertices is

(11) $$\phi(n; k_1, \ldots, k_p) = \binom{n}{\sum_{i=1}^{p} k_i} S\Big(\sum_{i=1}^{p} k_i; k_1, \ldots, k_p\Big)$$

where $S(n-l;k_1,\dots,k_p)$ is the number of partitions of a labelled $(n-l)$-element set $S=\{1,\dots,n-l\}$ into exactly p blocks $S^1,\dots,S^p$ of cardinalities $k_1,\dots,k_p$, respectively and is given by

$$(12)\quad S(n-l;k_1,\dots,k_p)=P(b_1,\dots,b_{n-l})=\frac{(n-l)!}{b_1!b_2!\dots b_{n-l}!(2!)^{b_2}\dots((n-l)!)^{b_{n-l}}}$$

where for $i=1,\dots,n-l$, b_i denotes the number of blocks of cardinality i among the blocks $S^1,\dots,S^p$ (see [1; 3.15]).

We further note that similarly to [7], $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)\cong\mathbf{F}_{\mathbf{V}(\mathbf{O}_n)}(n)$ if $n<k$. Hence in the following description of the finitely generated free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ it suffices to consider $k\le n$. Note that in the case $n=k=2$ we have $\phi(2;1,1)=1$ and we obtain the known description $\mathbf{F}_{\mathbf{V}(\mathbf{O}_2)}(2)\cong\mathbf{F}_{\mathcal{B}}(2)\times\mathbf{MO}_2$.

3.11 Theorem. *For any $2\le k\le n$, the finitely generated free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ is isomorphic to the product of the n-generated free Boolean algebra $\mathbf{F}_{\mathcal{B}}(n)$ with*

$$\prod_{p=2}^{k}\prod_{\substack{(k_1,\dots,k_p)\\ k_1\ge\dots\ge k_p\ge1\\ \sum_{i=1}^p k_i\le n}}[(\mathbf{MO}_p)^{2^{n-p}}\times\prod_{s=1}^{p}(\mathbf{L}_{(s,p-s)})^{N(n,p,s;k_1,\dots,k_p)}]^{\phi(n;k_1,\dots,k_p)}$$

where $\phi(n;k_1,\dots,k_p)$ are given by (11) and (12) and

$$N(n,p,s;k_1,\dots,k_p)=2^{n-p+s}\cdot\sum_{\substack{\{i_1,\dots,i_s\}\\ \subseteq\{1,\dots,p\}}}\prod_{r=1}^{s}\frac{6^{k_{i_r}-1}-2^{k_{i_r}-1}}{2^{k_{i_r}}}\qquad\square$$

It is easy to see (cf. [8]) that

$$\sum_{\substack{(k_1,\dots,k_p)\\ k_1\ge\dots\ge k_p\ge1\\ \sum_{i=1}^p k_i=n-l}} S(n-l;k_1,\dots,k_p)=S(n-l,p),$$

where the Stirling number $S(n-l,p)$ of the second kind is the number of partitions of a labelled $(n-l)$-element set into exactly p parts (see [1; 3.39]). This yields that

$$(13)\qquad \prod_{p=2}^{k}\prod_{\substack{(k_1,\dots,k_p)\\ k_1\ge\dots\ge k_p\ge1\\ \sum_{i=1}^p k_i\le n}}[(\mathbf{MO}_p)^{2^{n-p}}]^{\phi(n;k_1,\dots,k_p)}=\prod_{p=2}^{k}(\mathbf{MO}_p)^{(2^{n-p}\phi'(n,p))}$$

where

$$\phi'(n,p)=\sum_{l=0}^{n-p}\binom{n}{l}S(n-l,p).$$

Now on the right hand side of (13) we have an isomorphic copy of the n-generated free modular ortholattice $\mathbf{F}_{\mathcal{MO}_k}(n)$ in the variety $\mathcal{MO}_k$ (see [7]). Hence we can deduce the final result.

3.12 Corollary. *For any $2 \leq k \leq n$, the finitely generated free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ is isomorphic to*

$$\mathbf{F}_{\mathcal{MO}_k}(n) \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_p) \\ k_1 \geq \cdots \geq k_p \geq 1 \\ \sum_{i=1}^{p} k_i \leq n}} [\prod_{s=1}^{p} (\mathbf{L}_{(s,p-s)})^{N(n,p,s;k_1,\ldots,k_p)}]^{\phi(n;k_1,\ldots,k_p)}$$

where $\mathbf{F}_{\mathcal{MO}_k}(n)$ is the n-generated free modular ortholattice in the variety $\mathcal{MO}_k$, $\phi(n;k_1,\ldots,k_p)$ are given by (11) and (12) and

$$N(n,p,s;k_1,\ldots,k_p) = 2^{n-p+s} \cdot \sum_{\substack{\{i_1,\ldots,i_s\} \\ \subseteq \{1,\ldots,p\}}} \prod_{r=1}^{s} \frac{6^{k_{i_r}-1} - 2^{k_{i_r}-1}}{2^{k_{i_r}}}. \quad \square$$

3.13 Remark. We note that for $s = 1$,

$$N(n,p,1;k_1,\ldots,k_p) = 2^{n-p+1} \cdot \sum_{\{i\} \subseteq \{1,\ldots,p\}} \frac{6^{k_i-1} - 2^{k_i-1}}{2^{k_i}} = 2^{n-p}[(\sum_{i=1}^{p} 3^{k_i-1}) - p]$$

and the factor

$$\mathbf{F}_{\mathcal{MO}_k}(n) \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_p) \\ k_1 \geq \cdots \geq k_p \geq 1 \\ \sum_{i=1}^{p} k_i \leq n}} [(\mathbf{L}_{(1,p-1)})^{N(n,p,1;k_1,\ldots,k_p)}]^{\phi(n;k_1,\ldots,k_p)}$$

of the algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ in Corollary 3.12 is isomorphic to the n-generated free algebra $F_{\mathbf{V}(\mathbf{L}_{(1,p-1)})}(n)$ in the variety $V(\mathbf{L}_{(1,p-1)})$ described in [8] (where we used the notation $\mathbf{L}_p$ for the algebra $\mathbf{L}_{(1,p-1)}$).

We now illustrate the obtained results by presenting the structures of the free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ for $k = 2, 3, 4$ and $n = 3, 4, 5$.

The values of the coefficients $\phi(n;k_1,\ldots,k_p)$ are displayed in the first of the following tables:

n=3	n=4	n=5
$\phi(3;1,1) = 3$	$\phi(4;1,1) = 6$	$\phi(5;1,1) = 10$
$\phi(3;2,1) = 3$	$\phi(4;2,1) = 12$	$\phi(5;2,1) = 30$
$\phi(3;1,1,1) = 1$	$\phi(4;3,1) = 4$	$\phi(5;3,1) = 20$
	$\phi(4;2,2) = 3$	$\phi(5;2,2) = 15$
	$\phi(4;1,1,1) = 4$	$\phi(5;4,1) = 5$
	$\phi(4;2,1,1,) = 6$	$\phi(5;3,2) = 10$
	$\phi(4;1,1,1,1,) = 1$	$\phi(5;1,1,1) = 10$
		$\phi(5;2,1,1) = 30$
		$\phi(5;3,1,1) = 10$
		$\phi(5;2,2,1) = 15$
		$\phi(5;1,1,1,1) = 5$
		$\phi(5;2,1,1,1) = 10$

n=3	n=4	n=5
$N(3,2,1;2,1)=4$	$N(4,2,1;2,1)=8$	$N(5,2,1;2,1)=\ 16$
	$N(4,2,1;3,1)=32$	$N(5,2,1;3,1)=\ 64$
	$N(4,2,1;2,2)=16$	$N(5,2,1;2,2)=\ 32$
	$N(4,2,2;2,2)=16$	$N(5,2,2;2,2)=\ 32$
	$N(4,3,1;2,1,1)=4$	$N(5,3,1;2,1,1)=\ 8$
		$N(5,2,1;4,1)=\ 208$
		$N(5,2,1;3,2)=\ 80$
		$N(5,2,2;3,2)=\ 128$
		$N(5,3,1;3,1,1)=32$
		$N(5,3,1;2,2,1)=16$
		$N(5,3,2;2,2,1)=16$
		$N(5,4,1;2,1,1,1)=4$

The coefficients $N(n,p,s;k_1,\ldots,k_p)$ which are non-zero for $n=3,4,5$ are displayed in the second table above (all other coefficients $N(n,p,s;k_1,\ldots,k_p)$ for $n=3,4,5$ take value zero).

As a result, we obtain the following structures:

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_2)}(3) \cong \mathbf{F}_{\mathcal{B}}(3) \times (\mathbf{MO}_2)^{12} \times (\mathbf{L}_{1,1})^{12}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_2)}(4) \cong \mathbf{F}_{\mathcal{B}}(4) \times (\mathbf{MO}_2)^{100} \times (\mathbf{L}_{1,1})^{272} \times (\mathbf{L}_{2,0})^{48}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_2)}(5) \cong \mathbf{F}_{\mathcal{B}}(5) \times (\mathbf{MO}_2)^{720} \times (\mathbf{L}_{1,1})^{4080} \times (\mathbf{L}_{2,0})^{1760}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_3)}(3) \cong \mathbf{F}_{\mathcal{B}}(3) \times (\mathbf{MO}_2)^{12} \times (\mathbf{L}_{1,1})^{12} \times (\mathbf{MO}_3)^{1}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_3)}(4) \cong \mathbf{F}_{\mathcal{B}}(4) \times (\mathbf{MO}_2)^{100} \times (\mathbf{MO}_3)^{20} \times (\mathbf{L}_{1,1})^{272} \times (\mathbf{L}_{2,0})^{48} \times (\mathbf{L}_{1,2})^{24}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_3)}(5) \cong \mathbf{F}_{\mathcal{B}}(5) \times (\mathbf{MO}_2)^{720} \times (\mathbf{MO}_3)^{260} \times (\mathbf{L}_{1,1})^{4080} \times (\mathbf{L}_{2,0})^{1760} \times (\mathbf{L}_{1,2})^{800} \times (\mathbf{L}_{2,1})^{240}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_4)}(4) \cong \mathbf{F}_{\mathcal{B}}(4) \times (\mathbf{MO}_2)^{100} \times (\mathbf{MO}_3)^{20} \times (\mathbf{MO}_4)^{1} \times (\mathbf{L}_{1,1})^{272} \times (\mathbf{L}_{2,0})^{48} \times (\mathbf{L}_{1,2})^{24}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_4)}(5) \cong \mathbf{F}_{\mathcal{B}}(5) \times (\mathbf{MO}_2)^{720} \times (\mathbf{MO}_3)^{260} \times (\mathbf{MO}_4)^{30} \times (\mathbf{L}_{1,1})^{4080} \times (\mathbf{L}_{2,0})^{1760} \times (\mathbf{L}_{1,2})^{800} \times (\mathbf{L}_{2,1})^{240} \times (\mathbf{L}_{1,3})^{40}$$

References

[1] Aigner, M., *Combinatorial Theory*, Springer-Verlag, Berlin-Heidelberg-New York Grundlehren der mathematischen Wissenschaften 234, 1979.

[2] Beran, L., *Orthomodular lattices. Algebraic Approach*, Academia, Prague, 1984.
[3] Birkhoff, G. and von Neumann, J., *The logic of quantum mechanics*, Ann. of Math. **37** (1936), 823-843, In: "J. von Neumann Collected Works", Pergamon Press, Oxford, 1961, Vol IV, 105-125.
[4] Clark, D. M. and Davey, B. A., *Natural dualities for the working algebraist*, Cambridge University Press, 1998.
[5] Davey, B. A., Haviar, M. and Priestley, H.A. [1995], *The syntax and semantics of entailment in duality theory*, J. Symbolic Logic **60**, 1087-1114.
[6] Haviar, M., Konôpka, P., Priestley, H.A. and Wegener, C.B., *Finitely generated free modular ortholattices I*, International Journal of Theoretical Physics **36** (1997), 2639-2660.
[7] Haviar, M., Konôpka, P. and Wegener, C.B., *Finitely generated free modular ortholattices II*, International Journal of Theoretical Physics **36** (1997), 2661-2679.
[8] Haviar M. and Konôpka, P., *Finitely generated free orthomodular lattices III*, submitted.
[9] Kalmbach, G., *Orthomodular lattices*, London Mathematical Society Monographs, 18. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1983.
[10] MacLaren, M. D., *Atomic orthocomplemented lattices*, Pacific J. Math. **14** (1964), 597–612.
[11] Neumann, P. M., Stoy, G. A. and Thompson E. C., *Groups and Geometry*, Oxford University Press, Oxford-New York-Tokyo, 1994.
[12] Pták, P. and Pulmanová, S., *Orthomodular structures as quantum logics*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1991.

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