Acta Univ. M. Belii
Math. no. 7 (1999), pp. 31-34

ON COLOR-CLOSED MULTIPOLES

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Abstract. It is shown that for each color-complete k-pole there exists a (k+2)-pole which is color-closed but not color-complete.

In the present paper we use terminology and notation from [1]. We recall them. A multipole is a pair $M = (V(M), E(M))$ od disjoint finite sets, the vertex-set $V(M)$ and the edge-set $E(M)$ of M. Every edge $e \in E(M)$ has two ends and every end of e may or may not be incident with a vertex. If one end of e is incident with some vertex but the other is not, then e is a dangling edge. An end of an edge e that is incident with no vertex is called a semiedge.
Throughout the paper it is assumed that every vertex of every multipole is incident with precisely three edge ends. If a multipole M has k semiedges $e_1, e_2, \dots, e_k$, then it is called a k-pole and we write $M = M(e_1, e_2, \dots, e_k)$.
Let $M = M(e_1, e_2, \dots, e_k)$ and $N = N(f_1, f_2, \dots, f_k)$ be two k-poles. Then the junction $M \star N$ of M and N is the cubic graph that arises from the disjoint union $M \cup N$ by performing the junctions e_i with f_i, $i = 1, 2, \dots, k$.
Let M be a multipole and let $\varphi : E(M) \to \{1, 2, 3\}$ be a mapping assigning to each edge of M one of the elements $1, 2$, and 3 called colors. Then φ induces an assignment of colors to the ends of edges in M. The mapping φ is called a coloring of M, if for every vertex v of M the three ends incident with v are assigned pairwise distinct colors. A multipole will be called colorable if it admits a coloring. Otherwise we say that M is uncolorable. An uncolorable cubic graph will be called a snark.
Let $M = M(e_1, e_2, \dots, e_k)$ be a k-pole. The coloring set of M will be the set

$$\mathrm{Col}(M) = \{(\varphi(e_1), \varphi(e_2), \dots, \varphi(e_k)); \varphi \text{ is a coloring of } M\}$$

A coloring of semiedges of a k-pole is said to be admissible if it satisfies the conditions of Parity Lemma (see [1, lemma 2.2]), i.e. if $k_1 \equiv k_2 \equiv k_3 \equiv k (\mathrm{mod}\, 2)$ where k_i is the number of semiedges colored i, $i = 1, 2, 3$. A multipole M is said to be color-complete if every admissible coloring of its semiedges can be extended to a coloring of M.
We shall say that a k-pole M $(k \geq 2)$ can be extended to a snark if there exists a colorable k-pole N such that $M \star N$ is a snark. A multipole that cannot be extended to a snark is called color-closed.

One of the problems stated in [1] is the following

1991 *Mathematics Subject Classification.* 05C75.
Key words and phrases. Snark, multipole, color-complete k-pole, color-closed k-pole
Supported by grant no. 1/6132/99 of Slovak Grant Agency VEGA.

Problem 1. *Find a color-closed k-pole, $k \geq 5$, which is not color-complete.*

We show that such a multipole really exists. In fact, we prove more, namely

Theorem. *Let $N = N(e_1, e_2, e_3, e_4, e_5', e_6')$ be the 6-pole in Figure 1 and $M' = M'(e_5'', e_6'', e_5, e_6, \ldots, e_{k+2})$ be a color-complete k-pole, $k \geq 3$. Then the $(k+2)$-pole*

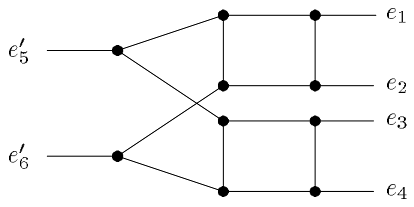

FIGURE 1.

$M = M(e_1, e_2, \ldots, e_{k+2})$ that arises from N and M' by performing the junctions e_5' with e_5'' and e_6' with e_6'' (see Figure 2) is color-closed but it is not color-complete.

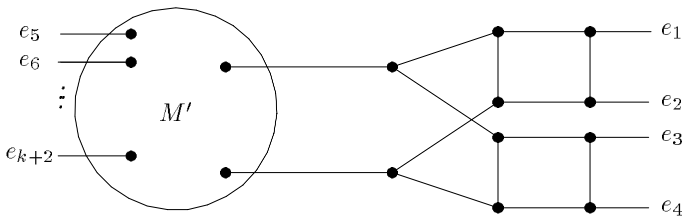

FIGURE 2.

From this theorem we easily get that the answer to Problem 1 is affirmative. In fact, since there are color-complete 3-poles and 4-pole (even infinite families of such multipoles) then by our Theorem there exist 5-poles and 6-poles which are color-closed but not color complete. Examples of such multipoles can be seen in Figure 3 (for 5-pole) and in Figure 4 (for 6-pole, cf.[1,Figure 17]).

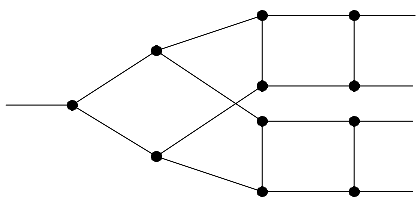

FIGURE 3.

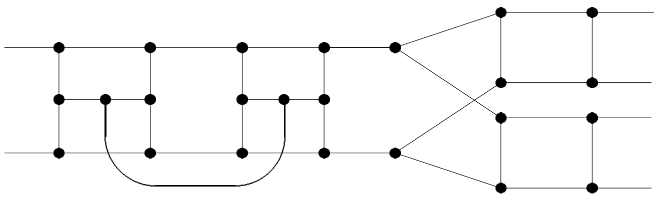

FIGURE 4.

These multipoles contain 4-cycles. Nevertheless, from the point of view of the theory of snarks (see [1]) it would be more interesting to find multipoles from Problem 1 which, additionally, have girth at least 5.

It remains to prove Theorem.

Proof of Theorem. It is easy to verify that $Col(N)$ does not contain the following types of element (up to permutation of colors): $1212\ldots$, $1213\ldots$, $1232\ldots$. Further, elements of these types , evidently, do not belong to $Col(M)$. Now we will show that every admissible coloring of semiedges of M different from the above types can be extended to a coloring of M.

Let φ be an admissible coloring of semiedges of M different from the above types. It suffices to consider two possibilities:

a) The number of occurrences of each of the colors $1, 2, 3$ in $(\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4))$ is even.
Obviously, there is a coloring φ' of N such that $\varphi'(e_i) = \varphi(e_i)$, $i = 1, 2, 3, 4$, and $\varphi'(e_5') = \varphi'(e_6')$. We will distinguish two subcases:

a_1) $k \equiv 0 (\mathrm{mod}\, 2)$
In this case the number of occurrences of each of the colors in the sequence $\varphi(e_5), \varphi(e_6), \ldots, \varphi(e_{k+2})$ is even. Then the coloring φ'' of semiedges of M' such that $\varphi''(e_i) = \varphi(e_i)$, $i = 5, 6, \ldots, k+2$ and $\varphi''(e_5'') = \varphi''(e_6'') = \varphi'(e_5')$ is admissible. Since M' is a color-complete multipole the coloring φ'' can be extended to a coloring of M'. Thus we have a coloring of M.

a_2) $k \equiv 1 (\mathrm{mod}\, 2)$
In this case the number of occurrences of each of the colors in the sequence $\varphi(e_5), \varphi(e_6), \ldots, \varphi(e_{k+2})$ is odd. Then, again, the coloring φ'' such that $\varphi''(e_i) = \varphi(e_i)$, $i = 5, 6, \ldots, k+2$ and $\varphi''(e_5'') = \varphi''(e_6'') = \varphi'(e_5')$ is admissible and we have the same situation as in the case a_1).

b) The number of occurrences of exactly two colors in $(\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4))$ is odd.
Without loss of generality we can assume that the number of occurrences of each of the colors $1, 2$ is odd. In this case there is a coloring φ' of N such that $\varphi'(e_i) = \varphi(e_i)$, $i = 1, 2, 3, 4$ and $\{\varphi'(e_5'), \varphi'(e_6')\} = \{1, 2\}$. Now it is easy to verify that the coloring φ'' of semiedges of M' such that $\varphi''(e_i) = \varphi(e_i)$, $i = 5, 6, \ldots, k+2$ and $\varphi''(e_5'') = \varphi'(e_5')$, $\varphi''(e_6'') = \varphi'(e_6')$ is admissible and hence it can be extended to a coloring of M'. Thus we have a coloring of M.
It remains to prove that the $(k+2)$-pole M is color-closed. Consider a snark

G expressible in the from $M \star K$. If K is colorable, then $\mathrm{Col}(K)$ can only contain elements of types (up to permutation of colors) $1212\ldots$, $1213\ldots$, $1232\ldots$ since M and K are color-disjoint. However, using a Kempe chain in K beginning and ending with a semiedge, we can alter such coloring to a coloring of one of the four types $11\ldots$, $2112\ldots$, $2113\ldots$, $2132\ldots$ present in $Col(M)$ and hence to color the whole snark. It follows that K cannot be colorable, i.e., M is color-closed and the proof is finished.

References

1. R. Nedela and M. Škoviera, *Decomposition and reductions of snarks*, J. Graph Theory **22** (1996), 253-279.

(Received July 7, 1999)

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