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CONGRUENCE CONDITIONS CONNECTED WITH DISTRIBUTIVITY AT 0

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ABSTRACT. An algebra $\mathcal{A}$ with 0 is distributive at 0 if for all $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$ it holds $[0]_{\Theta\cap(\Phi\vee\Psi)} = [0]_{(\Theta\cap\Phi)\vee(\Theta\cap\Psi)}$. We investigate how this property influenced direct decomposability of $[0]_\Theta$ for $\Theta \in Con\,\mathcal{A}_1 \times \mathcal{A}_2$ and what are the connections of this property to some local versions of the Chinese Remainder Theorem

The concept of distributivity at 0 was introduced in [2] as a "localization" of congruence distributivity in the vicinity of a constant 0 which is a nullary term of an algebra in question. Usefulness of it was recognized by G. Czédli [3] and J. Duda [5] for solving of some congruence problems. We proceed similar reasoning to show how distributivity at 0 influences decomposability of the congruence 0-class and how it is connected with some weakened versions of Chinese Remainder Theorem.

1 Basic concepts

Every algebra treated in this paper is considered to have a constant 0, i.e. 0 is a choosen element which is either a nullary operation (of the type of algebra) or a nullary term of this type. Some authors also consider 0 to be a constant unary term whose value is 0.

If R is a binary relation on a set A where $0 \in A$, we set

$$[0]_R = \{x \in A;\ \langle x, 0\rangle \in \Theta\}.$$

Definition 1. Let $\mathcal{A} = (A, F)$ be an algebra with 0 and $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$. We say that the (ordered) triplet (Θ, Φ, Ψ) is *distributive at* 0 if

(d) $$[0]_{\Theta\cap(\Phi\vee\Psi)} = [0]_{(\Theta\cap\Phi)\vee(\Theta\cap\Psi)}$$

holds in $\mathcal{A}$. We say that (Θ, Φ, Ψ) is *dually distributive at* 0 if

(d*) $$[0]_{\Theta\vee(\Phi\cap\Psi)} = [0]_{(\Theta\vee\Phi)\cap(\Theta\vee\Psi)}$$

holds in $\mathcal{A}$.

An algebra $\mathcal{A}$ is called (*dually*) *distributive at* 0 if (Θ, Φ, Ψ) has this property for every $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$.

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Lemma 1. *Let $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$ and the triplets $(\Theta \cap \Psi, \Theta, \Phi)$ and (Φ, Θ, Ψ) be dually distributive at 0. Then the triplet (Θ, Φ, Ψ) is distributive at 0. Hence, if $\mathcal{A}$ is dually distributive at 0 then it is distributive at 0.*

Proof. We can count directly

$$[0]_{(\Theta\cap\Phi)\vee(\Theta\cap\Psi)} = [0]_{(\Theta\vee(\Theta\cap\Psi))\cap(\Phi\vee(\Theta\cap\Psi))} = [0]_{\Theta\cap(\Phi\vee(\Theta\cap\Psi))} = [0]_\Theta \cap [0]_{\Phi\vee(\Theta\cap\Psi)} =$$
$$= [0]_\Theta \cap [0]_{\Phi\vee\Theta} \cap [0]_{\Phi\vee\Psi} = [0]_\Theta \cap [0]_{\Phi\vee\Psi} = [0]_{\Theta\cap(\Phi\vee\Psi)}\,.$$

The second assertion follows immediately. □

Remark. It was shown by G. Czédli [3] that the converse of Lemma 1 does not hold. He found a $\wedge$-semilattice with 0 which is not dually distributive at 0 although the whole variety of $\wedge$-semilattices with 0 is distributive at 0, see [2].

Recall that $\mathcal{A}$ is *permutable at* 0 if $[0]_{\Theta\circ\Phi} = [0]_{\Phi\circ\Theta}$ holds for every $\Theta, \Phi \in Con\,\mathcal{A}$. For these algebras, we can prove

Lemma 2. *Let $\mathcal{A} = (A, F)$ be an algebra with 0 which is permutable at 0. Then*
(a) $[0]_{\Theta\vee\Phi} = [0]_{\Theta\circ\Phi}$ *for each* $\Theta, \Phi \in Con\,\mathcal{A}$*;*
(b) *If $\mathcal{A}$ is distributive at 0 then $\mathcal{A}$ is dually distributive at 0.*

Proof. For (a), it is enough to prove $[0]_{\Theta\vee\Phi} \subseteq [0]_{\Theta\circ\Phi}$. Suppose $x \in [0]_{\Theta\vee\Phi}$. Then

$$\langle x, 0\rangle \in \Theta \vee \Phi$$

thus there exist an odd integer n and elements $y_1, \dots, y_n$ of A such that

$$\langle x, y_1\rangle \in \Theta, \quad \langle y_1, y_2\rangle \in \Phi, \dots, \ \langle y_{n-1}, y_n\rangle \in \Theta, \quad \langle y_n, 0\rangle \in \Phi\,.$$

Hence, $y_{n-1} \in [0]_{\Theta\circ\Phi}$ thus also $y_{n-1} \in [0]_{\Phi\circ\Theta}$, i.e. there is $z_1 \in A$ with $\langle y_{n-1}, z_1\rangle \in \Phi$ and $\langle z_1, 0\rangle \in \Theta$. Applying transitivity, $\langle y_{n-2}, z_1\rangle \in \Phi$ whence

$$y_{n-2} \in [0]_{\Phi\circ\Theta} = [0]_{\Theta\circ\Phi}\,.$$

We can repeat this processing and, after $n-1$ steps, we conclude $x \in [0]_{\Theta\circ\Phi}$.
The assertion (b) was proven by J. Duda [5]. □

2 Decomposability of 0-classes

Definition 2. Let $\mathcal{A}_1, \mathcal{A}_2$ be algebras of the same type and $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. Let $\Theta \in Con\,\mathcal{A}$. We say that the 0-class $[0]_\Theta$ is *directly decomposable* if there exist $\Theta_1 \in Con\,\mathcal{A}_1$ and $\Theta_2 \in Con\,\mathcal{A}_2$ such that

$$[0]_\Theta = [0]_{\Theta_1} \times [0]_{\Theta_2}\,.$$

Recall that for $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, by *projection congruences* Π_1, Π_2 we mean congruences on $\mathcal{A}$ induced by the projections pr_1, pr_2 of $\mathcal{A}$ onto $\mathcal{A}_1$, $\mathcal{A}_2$, respectively.

Denote by ω_i or ω_A the least congruence on $\mathcal{A}_i$ or $\mathcal{A}$ and by ι_i or ι_A the greatest congruence on $\mathcal{A}_i$ or $\mathcal{A}$, respectively. Of course, $\Pi_1 \cap \Pi_2 = \omega_A$.

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and $\Theta \in Con\ \mathcal{A}$. Introduce the relations Θ^1 and Θ^2 on A by setting

$$\Theta^1 = \{\langle (x_1, x_2), (y_1, y_2)\rangle \in \Theta;\ x_2 = 0 = y_2\}$$
$$\Theta^2 = \{\langle (x_1, x_2), (y_1, y_2)\rangle \in \Theta;\ x_1 = 0 = y_1\}.$$

Theorem 1. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be algebras of the same type τ satisfying $f(0,\dots,0)=0$ for each n-ary ($n \geq 1$) operation f of τ. Let $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$ and $\Theta \in Con\,\mathcal{A}$. Then $[0]_\Theta$ is directly decomposable if and only if it holds*

$$\text{(P)} \qquad [0]_\Theta = [0]_{\Pi_1\circ\Theta^1} \cap [0]_{\Pi_2\circ\Theta^2}.$$

Proof. (a) Suppose that $[0]_\Theta = [0]_{\Theta_1}\times[0]_{\Theta_2}$ for some $\Theta_1 \in Con\ \mathcal{A}_1$ and $\Theta_2 \in Con\ \mathcal{A}_2$. Let $x=(x_1,x_2)$ and $x \in [0]_{\Pi_1\circ\Theta^1} \cap [0]_{\Pi_2\circ\Theta^2}$. Then

$$(x_1,x_2)\Pi_1(x_1,0)\Theta^1(0,0) \quad \text{and} \quad (x_1,x_2)\Pi_2(0,x_2)\Theta^2(0,0),$$

thus $(x_1,0)\in[0]_\Theta$, $(0,x_2)\in[0]_\Theta$ and, with respect to direct decomposability of $[0]_\Theta$, also $x\in[0]_\Theta$. The converse inclusion is evident, i.e. the condition (P) holds.

(b) Suppose (P) and let $y=(y_1,y_2)\in[0]_\Theta$. Then $\langle y,0\rangle \in \Pi_1\circ\Theta^1$ and $\langle y,0\rangle \in \Pi_2\circ\Theta^2$. We define

$$\Theta_1 = \{\langle x_1,y_1\rangle \in \mathcal{A}_1\times\mathcal{A}_1;\ \langle (x_1,0),(y_1,0)\rangle \in\Theta\}$$
$$\Theta_2 = \{\langle x_2,y_2\rangle \in \mathcal{A}_2\times\mathcal{A}_2;\ \langle (0,x_2),(0,y_2)\rangle \in\Theta\}.$$

Obviously, Θ_i is an equivalence on $\mathcal{A}_i$ and, due to $f(0,\dots,0)=0$, we can easily show that $\Theta_i \in Con\ \mathcal{A}_i$, $i=1,2$. Moreover, $\langle y,0\rangle\in\Pi_1\circ\Theta^1$ gives $\langle y_1,0\rangle\in\Theta_1$, analogously, $\langle y,0\rangle\in\Pi_2\circ\Theta^2$ implies $\langle y_2,0\rangle\in\Theta_2$ thus $y=(y_1,y_2)\in[0]_{\Theta_1}\times[0]_{\Theta_2}$ proving $[0]_\Theta\subseteq[0]_{\Theta_1}\times[0]_{\Theta_2}$.

Suppose now $x\in[0]_{\Theta_1}\times[0]_{\Theta_2}$. Then $x\in[0]_{\Pi_1\circ\Theta^1}$ and $x\in[0]_{\Pi_2\circ\Theta^2}$, and, by (P), $x\in[0]_\Theta$ giving the converse inclusion. Thus $[0]_\Theta$ is directly decomposable. □

Corollary. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be algebras of the same type τ satisfying $f(0,\dots,0)=0$ for every $n-$ary ($n\geq 1$) operation f of τ. Let $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$ and $\Theta\in Con\,\mathcal{A}$. If it holds*

$$\text{(Q)} \qquad [0]_\Theta \subseteq [0]_{\Pi_1\circ\Theta^1} \cap [0]_{\Pi_2\circ\Theta^2}$$

and the triplet (Θ,Π_1,Π_2) is dually distributive at 0 then $[0]_\Theta$ is directly decomposable.

Proof. Suppose that the triplet (Θ,Π_1,Π_2) is dually distributive at 0. Then

$$\begin{aligned}[0]_{\Pi_1\circ\Theta^1}\cap[0]_{\Pi_2\circ\Theta^2} &= [0]_{\Pi_1\circ\Theta^1\cap\Pi_2\circ\Theta^2} \subseteq [0]_{\Pi_1\circ\Theta\cap\Pi_2\circ\Theta} \subseteq [0]_{(\Pi_1\vee\Theta)\cap(\Pi_2\vee\Theta)}\\ &= [0]_{\Theta\vee(\Pi_1\cap\Pi_2)} = [0]_\Theta.\end{aligned}$$

Together with (Q), this gives (P). □

Remark. If a variety $\mathcal{V}$ is permutable at 0, $\mathcal{A}_1$, $\mathcal{A}_2\in\mathcal{V}$ and $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$ is distributive at 0, then $[0]_\Theta$ is directly decomposable for each $\Theta\in Con\,\mathcal{A}$. This result was proved by P. Agliano and A. Ursini [1], see Proposition 4.1. Namely, if $\mathcal{V}$ is permutable at 0 (i.e. $\mathcal{V}$ is subtractive in the terminology of [1]) then $\mathcal{A}\in\mathcal{V}$ has distributive lattice of all its ideals. Applying our Lemma 2 and (1) $\Leftrightarrow$ (3) in Proposition 4.1 of [1], we have the proof of the mentioned assertion.

3 Modifications of the Chinese Remainder Theorem

It is well-known that the Chinese Remainder Theorem is closely connected with congruence distributivity. In the sequel we illustrate similar connections for other (local) versions of CRT.

Recall that an algebra $\mathcal{A}$ satisfies *Chinese Remainder Theorem* (briefly *CRT*) if for any $n \in \boldsymbol{N}$ and every $\Theta_1, \dots, \Theta_n \in Con\,\mathcal{A}$, if $\langle w_i, w_j \rangle \in \Theta_i \vee \Theta_j$ for each $i, j \in \{1, \dots, n\}$ then there exists $z \in A$ such that $\langle z, w_i \rangle \in \Theta_i$ for $i = 1, \dots, n$.

Definition 3. An algebra $\mathcal{A}$ with 0 satisfies the 0-*Chinese Remainder Theorem* (0-*CRT*) if for each $c \in A$ and every $\Theta, \Phi_1, \dots, \Phi_n \in Con\,\mathcal{A}$ it holds: if $[0]_{\Phi_i} \cap [c]_\Theta \neq \emptyset$ for $i = 1, \dots, n$ then $[0]_{\Phi_1} \cap \dots \cap [0]_{\Phi_n} \cap [c]_\Theta \neq \emptyset$.

It is almost evident that every algebra $\mathcal{A}$ with 0 satisfying CRT satisfies also 0-CTR.

Lemma 3. *An algebra $\mathcal{A}$ satisfies 0-CRT if and only if for each $c \in A$ and every Θ, Φ_1, Φ_2 of $Con\,\mathcal{A}$ it holds:*

$$[0]_{\Phi_1} \cap [c]_\Theta \neq \emptyset \neq [0]_{\Phi_2} \cap [c]_\Theta \Rightarrow [0]_{\Phi_1} \cap [0]_{\Phi_2} \cap [c]_\Theta \neq \emptyset\,.$$

The proof is straightforward. □

Theorem 2. *For an algebra $\mathcal{A}$ permutable at 0, the following conditions are equivalent:*

(1) *$\mathcal{A}$ is distributive at 0;*
(2) *$\mathcal{A}$ satisfies 0-CRT.*

Proof. (1) $\Rightarrow$ (2): Suppose $[0]_{\Phi_1} \cap [c]_\Theta \neq \emptyset \neq [0]_{\Phi_2} \cap [c]_\Theta$. This and Lemma 2 imply $c \in [0]_{\Theta \circ \Phi_1} = [0]_{\Theta \vee \Phi_1}$. Similarly, $c \in [0]_{\Theta \vee \Phi_2}$. Hence

$$c \in [0]_{(\Theta \vee \Phi_1) \cap (\Theta \vee \Phi_2)} = [0]_{\Theta \vee (\Phi_1 \cap \Phi_2)} = [0]_{\Theta \circ (\Phi_1 \cap \Phi_2)}$$

i.e. there exists $d \in A$ such that $\langle c, d \rangle \in \Theta$, $\langle d, 0 \rangle \in \Phi_1$, $\langle d, 0 \rangle \in \Phi_2$ whence $d \in [c]_\Theta \cap [0]_{\Phi_1} \cap [0]_{\Phi_2}$. By Lemma 3, $\mathcal{A}$ satisfies 0-CRT.

(2) $\Rightarrow$ (1): Let $\mathcal{A}$ be permutable at 0 and satisfies 0-CRT. Let $\Theta, \Phi_1, \Phi_2 \in Con\,\mathcal{A}$ and $c \in [0]_{(\Theta \vee \Phi_1) \cap (\Theta \vee \Phi_2)}$. Then $c \in [0]_{\Theta \vee \Phi_1}$, $c \in [0]_{\Theta \vee \Phi_2}$. Since $\mathcal{A}$ is permutable at 0, we have also $c \in [0]_{\Theta \circ \Phi_1} \cap [0]_{\Theta \circ \Phi_2}$. Hence, there exist $a, b \in A$ such that $\langle c, a \rangle \in \Theta$, $\langle a, 0 \rangle \in \Phi_1$ and $\langle c, b \rangle \in \Theta$, $\langle b, 0 \rangle \in \Phi_2$, i.e. $a \in [0]_{\Phi_1} \cap [c]_\Theta$, $b \in [0]_{\Phi_2} \cap [c]_\Theta$. By 0-CRT, there exists $d \in A$ such that

$$d \in [0]_{\Phi_1} \cap [0]_{\Phi_2} \cap [c]_\Theta\ ,$$

i.e. $\langle c, d \rangle \in \Theta$, $\langle d, 0 \rangle \in \Phi_1 \cap \Phi_2$. We conclude $c \in [0]_{\Theta \circ (\Phi_1 \cap \Phi_2)} \subseteq [0]_{\Theta \vee (\Phi_1 \cap \Phi_2)}$. Hence, $\mathcal{A}$ is dually distributive at 0. By Lemma 1, we have (1). □

Definition 4. An algebra $\mathcal{A}$ is $\circ$-*distributive* if

$$\Theta \cap (\Phi \circ \Psi) = (\Theta \cap \Phi) \circ (\Theta \cap \Psi) \quad \text{for all} \quad \Theta, \Phi, \Psi \in Con\,\mathcal{A}.$$

If $\mathcal{A}$ is permutable at 0 and $\circ$-distributive then $\mathcal{A}$ is also distributive at 0.

Definition 5. An algebra $\mathcal{A}$ satisfies the *block-CRT* if for every $\Theta_1, \Theta_2, \Theta_3 \in Con\,\mathcal{A}$ and any class X_j of Θ_j $(j = 1, 2, 3)$ we have:

$$\text{if } X_1 \cap X_2 \neq \emptyset\,,\ X_2 \cap X_3 \neq \emptyset\,,\ X_1 \cap X_3 \neq \emptyset \ \text{ then } X_1 \cap X_2 \cap X_3 \neq \emptyset\,.$$

Of course, if $\mathcal{A}$ satisfies CRT, then $\mathcal{A}$ satisfies also the block-CRT. Conversely, we can show

Lemma 4. *Let $\mathcal{A}$ be a congruence-permutable algebra. If $\mathcal{A}$ satisfies the block-CRT then $\mathcal{A}$ satisfies also CRT.*

Proof. It is enough to prove CRT only for three congruences. Let $\Theta_1, \Theta_2, \Theta_3 \in Con\,\mathcal{A}$ and $\langle w_i, w_j\rangle \in \Theta_i \vee \Theta_j$ for $i, j \in \{1, 2, 3\}$. By congruence-permutability,

$$\langle w_i, w_j\rangle \in \Theta_i \circ \Theta_j\,.$$

Hence $[w_i]_{\Theta_i} \cap [w_j]_{\Theta_j} \neq \emptyset$ and the block CTR gives an element z with $\langle w_i, z\rangle \in \Theta_i$ for all i proving CRT. □

Theorem 3. *For an algebra $\mathcal{A}$, the following conditions are equivalent:*

(1) *$\mathcal{A}$ satisfies block-CRT;*
(2) *$\mathcal{A}$ is $\circ$-distributive.*

Proof. (1) $\Rightarrow$ (2): Evidently, for each $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$ we have $(\Theta \cap \Phi) \circ (\Theta \cap \Psi) \subseteq \Theta \cap (\Phi \circ \Psi)$. Let us prove the converse inclusion. If $\langle a, b\rangle \in \Theta \cap (\Phi \circ \Psi)$ then there exists $c \in A$ with $\langle a, c\rangle \in \Phi$, $\langle c, b\rangle \in \Psi$, hence

$$a \in [a]_\Theta \cap [c]_\Phi, \quad b \in [a]_\Theta \cap [b]_\Psi, \quad c \in [c]_\Phi \cap [b]_\Psi\ .$$

By (1), there is $d \in [a]_\Theta \cap [b]_\Psi \cap [c]_\Phi$, i.e. $\langle a, d\rangle \in \Theta \cap \Phi$ and $\langle d, b\rangle \in \Theta \cap \Psi$ proving $\langle a, b\rangle \in (\Theta \cap \Phi) \circ (\Theta \cap \Psi)$.

(2) $\Rightarrow$ (1): Let X_1, X_2, X_3 be blocks of $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$, respectively. Suppose $a \in X_1 \cap X_2$, $b \in X_2 \cap X_3$ and $c \in X_1 \cap X_3$. Then $\langle a, c\rangle \in \Theta$, $\langle a, b\rangle \in \Phi$, $\langle b, c\rangle \in \Psi$, whence $\langle a, c\rangle \in (\Theta \cap \Phi) \circ (\Theta \cap \Psi)$, i.e. there exists $d \in A$ with $\langle a, d\rangle \in \Theta \cap \Phi$, $\langle d, c\rangle \in \Theta \cap \Psi$. Hence $d \in X_1 \cap X_2 \cap X_3$. □

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