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MOORE-OSGOOD THEOREM FOR FUZZY FUNCTIONS

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Abstract. Moore-Osgood theorem is the tool that enables us to show important corollaries of the uniform convergence for a sequence of real functions. We show that in the literal transcription of the uniform convergence notion for fuzzy functions is not appropriate, as we would lose this important statement. We show that the uniform convergence of pseudoinverses is the sufficient and necessary condition to save Moore-Osgood theorem for fuzzy functions.

The concept of uniform convergence is one of the basic principles in the study of real functions, or more generally of the mappings in metric spaces. One of the main reasons of its importance is the Moore-Osgood theorem. Here is its formulation for real functions:

Proposition 1 (Moore-Osgood). *If the sequence of real functions $\{f_n\}_{n=1}^{\infty}$ uniformly converges to a real function f on a set A, if for a given $x_0 \in A$ and for each natural number n there is $\lim_{x\to x_0} f_n(x) = a_n$, then $\lim_{n\to\infty} a_n = f(x_0)$.*

There are many important corollaries of this statement, like hereditary continuity of the limit function, possibility (perhaps under some additional conditions) of termwise differentiation and integration and other.

In concepts that generalize either the one of a real function or the uniform convergence, or both, we have to be aware that in fact it is exactly the Moore-Osgood theorem (and its corollaries) that makes the uniform convergence so important. The literal transcription may sometimes be contraproductive, as we will show on the case of fuzzy functions.

We will start introducing the notion of a fuzzy number. Though usually a fuzzy number is understood as a fuzzy set with the unique modal value, in a lot of works we can find also a different definition. Here and in the rest of this work the symbol R denotes the set of all real numbers:

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Definition 1. A *real fuzzy number* is a fuzzy set $\rho : R \to [0;1]$ for which

(1) there is an $x_0 \in R$ for which $\rho(x_0) = 0$,
(2) there is an $x_1 \in R$ for which $\rho(x_1) = 1$,
(3) $\rho(r) = \sup\{\rho(s), s < r\}$ for each $r \in R$.

This definition resembles in a way the density function, hence this representation of a fuzzy number is also sometimes called a statistical one. Note that due to this definition a fuzzy number is a left-continuous function.

The set of all fuzzy real numbers will be denoted by $F(R)$.

The set of all (crisp) real numbers is embedded into $F(R)$ in the following way: a crisp real number t is represented by a function $\delta_t \in F(R)$ for which $\delta_t(r) = 0$ if $r \leq t$ and $\delta_t(r) = 1$ if $r > t$.

The partial ordering in $F(R)$ is given by the following way:

$$\rho \leq \sigma \text{ if and only if } \rho(r) \geq \sigma(r) \text{ for each } r \in R.$$

If α is a number in the interval $]0,1]$, then the α-cut of a fuzzy number ρ is the set $(\rho)_\alpha = \{x \in R; \rho(x) \geq \alpha\}$. For $\alpha = 0$ we define $(\rho)_0$ as the closure of the set $\bigcup_{\alpha>0}(\rho)_\alpha$.

These cuts are evidently intervals with the right endpoint ∞. Clearly we cannot use Hausdorff metric for measuring their distances, as these intervals are not compact, but still the following is quite natural: The distance of two cuts will be the distance of their left endpoints (note that due to Definition 1 this is always a real number). As no confusion can emerge (we do not use Hausdorff metric in this work), we will use the symbol h for this distance. In this sense we will also speak about the convergence of the cuts.

Definition 2. A *real fuzzy function* is a mapping $f : R \to F(R)$.

Hence a fuzzy function assigns a fuzzy real number to a crisp real number.

If f is a fuzzy function and $0 < \alpha \leq 1$ then the α-level function of f is the function $(f)_\alpha$ defined in the following way:

$$(f)_\alpha(x) = \inf\{y \in R; f(x)(y) \geq \alpha\}$$

Another useful tool will be the pseudoinverse of a fuzzy number introduced by Sherwood and Taylor in [4] and studied in many other works, among which the closest relation with our topic is in the paper [3] by Klement. Though this term is used quite frequently, we remind its definition (we use the usual convention $\sup\emptyset = -\infty$):

Definition 3. Let $\rho \in F(R)$. Its *pseudoinverse* is the function $\rho^{(-1)} : [0;1] \to [-\infty;\infty]$ for which $\rho^{(-1)}(\alpha) = \sup\{r; \rho(r) < \alpha\}$.

The notion of pseudoinverse and its relation to quasiinverse is thoroughly explained in [5].

The set of all pseudoinverses of the fuzzy numbers in $F(R)$ will be denoted by $F^{(-1)}(R)$. It has been shown in [3] that $F^{(-1)}(R)$ is exactly the set of left-continuous, non-decreasing functions defined on $[0,1]$ with the value $-\infty$ at the point zero. Moreover, the mapping $p : \rho \mapsto \rho^{(-1)}$ is an involutive order-preserving isomorphism from $(F(R), +_{min})$, where $+_{min}$ is the addition of fuzzy numbers with respect to the minimum triangular norm, onto $(F^{(-1)}(R), +)$, where the addition is the usual addition of real functions.

Definition 4. The pseudoinverse of a fuzzy function f will be understood as the mapping $f^{(-1)} : R \to F^{(-1)}(R)$ such that $f^{(-1)}(x) = (f(x))^{(-1)}$.

The relationship between pseudoinverses and α-cuts can be seen from the following statement:

Proposition 2. *If ρ is a fuzzy number, $\alpha \in]0,1]$, then*

$$(\rho)_\alpha = \{y \in R; y \geq \rho^{(-1)}(\alpha)\}.$$

Proof. Let $y \in (\rho)_\alpha$. Then according to the definition of a cut $\rho(y) \geq \alpha$.

If r is an arbitrary real number for which $\rho(r) < \alpha$, then, as ρ is a non-decreasing function, $y \geq r$. Hence

$$y \geq \sup\{r \in R; \rho(r) < \alpha\} = \rho^{(-1)}(\alpha).$$

In a similar way we can show the remaining inclusion. □

We will deal with a sequence of fuzzy functions $\{f_n\}_{n=1}^{\infty}$. A literal transcription of uniform convergence from the crisp case to the fuzzy one would provide the following definition: A sequence $\{f_n\}_{n=1}^{\infty}$ of fuzzy functions defined on a set $A \subset R$ converges uniformly to a fuzzy function f defined on A, if for each positive ϵ there exists a natural number n_0 such that for all $n \geq n_0$ and $x \in A$ the inequality

$$\sup\{|f_n(x)(y) - f(x)(y)|, y \in R\} < \epsilon$$

holds.

The following example, introduced also in [1], shows that this definition is not an appropriate one for a sequence of fuzzy functions.

Example 1. Let for a natural number n the fuzzy function f_n be the following: at each $x \in R$ the value $f_n(x)$ is the fuzzy number

$$f_n(x)(y) = \begin{cases} 0 & \text{if } y \leq 0, \\ \frac{n-1}{n} y & \text{if } 0 < y \leq 1, \\ \frac{y}{n} + 1 - \frac{2}{n} & \text{if } 1 < y < 2, \\ 1 & \text{if } y \geq 2. \end{cases}$$

This sequence fulfills the requirement of the above mentioned definition, if for the limit function f we take the fuzzy function whose value at any $x \in R$ is the following fuzzy number:

$$f(x)(y) = \begin{cases} 0 & \text{if } y \leq 0, \\ y & \text{if } 0 < y < 1, \\ 1 & \text{if } y \geq 1, \end{cases}$$

as for each $x \in R$ there is

$$\sup\{|f_n(x)(y) - f(x)(y)|, y \in R\} = \frac{1}{n}.$$

On the other hand the 1-level function (the level function for $\alpha = 1$) of f_n for each natural n is the constant function $(f_n)_1(x) = 2$, while the 1-level function for the limit is the constant function $(f)_1(x) = 1$. Thus the sequence of 1-level functions does not converge uniformly to the 1-level function of the limit function (not even pointwise).

As we can see, defining uniform continuity this way does not allow us to use the Moore-Osgood theorem. Namely, the 1-cuts of $f_n(x)$ do not converge to the 1-cut of $f(x)$ for any real x in the previous example.

The following proposition expresses the proper condition to enable the use of Moore-Osgood theorem. We have to note that speaking about the convergence of a fuzzy functions pseudoinverse sequence we have to keep in mind the supremum metric in the space of values.

Proposition 3. *If for the sequence of fuzzy functions $\{f_n\}_{n=1}^{\infty}$ there exists a fuzzy function f such that the sequence of pseudoinverses $\{f_n^{(-1)}\}_{n=1}^{\infty}$ converges to $f^{(-1)}$ on a set A, then the sequence $\{f_n\}_{n=1}^{\infty}$ with the function f fulfill the statement of the Moore-Osgood theorem.*

Proof. Suppose the sequence of pseudoinverses $f_n^{(-1)}$ uniformly converges to the pseudoinverse $f^{(-1)}$ on the set A. Then for each $\epsilon > 0$ there is a natural number n_0 such that for all $n \geq n_0$ we have

$$\sup |f_n^{(-1)}(x) - f^{(-1)}(x)| < \epsilon \quad \text{for each } x \in A,$$

or, using the definition of a pseudoinverse to a fuzzy function,

$$\sup |(f_n(x))^{(-1)} - (f(x))^{(-1)}| < \epsilon \quad \text{for each } x \in A.$$

As we use the supremum metric in the space of values, this inequality implies that for any $\alpha \in]0, 1]$ we have

$$|(f_n(x))^{(-1)}(\alpha) - (f(x))^{(-1)}(\alpha)| < \epsilon \quad \text{for each } x \in A.$$

By Proposition 2 we get

$$h((f_n(x))_\alpha, (f(x))_\alpha) < \epsilon \quad \text{for each } x \in A,$$

but this means that the distance of the left endpoints of these intervals is less than ϵ for any $x \in A$, which is exactly the uniform convergence of level functions $(f_n)_\alpha$ to the corresponding level functions f_α. □

Hence using this type of uniform convergence for fuzzy functions we have e.g. the corollary that if in a uniformly convergent sequence of fuzzy functions all the level functions are continuous, then also the level functions of the limit are continuous as the level functions in the sequence uniformly converge to the corresponding level function of the limit. In a similar way, perhaps with additional conditions, we can obtain other similar results.

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