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POSITIVE LINEAR MAPS ON TRIGONOMETRIC POLYNOMIALS

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Abstract. Positive linear maps on trigonometric polynomials with values in a partially ordered vector space are investigated.

Introduction

The classical Herglotz theorem states that a sequence $(z_k)_{k=-\infty}^{\infty}$ of complex numbers is the sequence of the Fourier-Stieltjes coefficients of a nondecreasing function g defined on the interval $[0, 2\pi]$ if and only if the sequence $(z_k)_{k=-\infty}^{\infty}$ is positive defined. Recall that $z_k = \int_0^{2\pi} e^{-ikt}\, dg(t)$ is said to be the k-th Fourier-Stieltjes coefficient of a function g of bounded variation defined on the interval $[0, 2\pi]$. A sequence $(z_k)_{k=-\infty}^{\infty}$ of complex numbers is said to be positive defined if and only if $0 \le \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \overline{c_k} z_{k-j}$ for any finite sequence of complex numbers $(c_j)_{j=-n}^{n}$. A generalization of Herglotz theorem for vector lattices was given in the paper [1]. Paper [2] generalizes this theorem for a sequence $(z_k)_{k=-\infty}^{\infty}$, elements of which belong to the complexification Z of a monotone σ-complete partially ordered vector space Y. For such a sequence it was constructed a positive linear map Φ defined on trigonometric polynomials with values in Y. For the proof of positivity of the map Φ we have used a limit process. In this paper we show that this limit process may be omitted (Theorem 2.3). The present paper deals with a partially ordered vector space Y and we present those statements of paper [2] which may be proved without the assumption of monotone σ-completeness of Y. However, this assumption is necessary for an extension of the map Φ onto the set of all continuous 2π-periodic functions.

1. Preliminaries

In the whole paper we denote by the symbol Y a real partially ordered vector space and Z the complexification of Y. A real vector space Y is called a partially ordered vector space if it has a partial ordering $\le$ such that:

$$\forall x, y, z \in Y : x \le y \implies x + z \le y + z$$

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$$\forall x, y \in Y \ \forall \lambda \geq 0 : x \leq y \implies \lambda x \leq \lambda y \ .$$

If a sequence $(y_n)_{n=1}^{\infty}$ of elements of Y has a least upper (greatest lower) bound, then this element is denoted by $\bigvee_{n=1}^{\infty} y_n$ ($\bigwedge_{n=1}^{\infty} y_n$). A partially ordered vector space Y is called archimedean if $\bigwedge_{n=1}^{\infty} \frac{1}{n} y = 0$ for any positive $y \in Y$. A partially ordered vector space is said to be monotone σ-complete if any increasing sequence $(a_n)_{n=1}^{\infty}$, which is bounded above, has a least upper bound $\bigvee_{n=1}^{\infty} a_n$, see [5]. Any monotone σ-complete partially ordered vector space is archimedean.

Example 1.1. Any real vector space with a trivial ordering (i.e. $x \leq y$ if and only if $x = y$) is monotone σ-complete partially ordered vector space and archimedean. Let Y be a set of real polynomials with natural operations and ordering defined by $P \leq Q$ if and only if $P(t) \leq Q(t)$ for all $t \in [a, b]$. Then Y is not monotone σ-complete, but Y is still archimedean. Let Y_n be subspace of Y consisting of all polynomials of degree $\leq n$. Then Y_n is monotone σ-complete.

Take a function $h : [a, b] \to Y$ of the form $h(t) = \sum_{j=1}^{n} \varphi_j(t) y_j$, where $\varphi_j : [a, b] \to \mathbf{R}$ are continuous and $y_j \in Y$. Then the integral

$$\int_a^b h(t)dt = \sum_{j=1}^{n} \Big(\int_a^b \varphi_j(t)dt \Big) y_j$$

is correctly defined and

$$\int_a^b h(t)dt \geq 0 \text{ whenever } h(t) \geq 0 \text{ for all } t \in [a, b],$$

see [3,p. 253].

A function p of the form $p(t) = \sum_{j=-n}^{n} c_j e^{ijt}$ is called a (complex) trigonometric polynomial. Obviously, the function p is real if and only if $c_{-j} = \overline{c_j}$ for any integer $j \in \{-n, ..., 0, ..., n\}$. The set of all real (complex) trigonometric polynomials is denoted by $T_{2\pi}(\mathbf{R})$ (by $T_{2\pi}(\mathbf{R}, \mathbf{C})$).

Clearly, any linear map $\Phi : T_{2\pi}(\mathbf{R}) \to Y$ may be extended onto a linear map $\Phi : T_{2\pi}(\mathbf{R}, \mathbf{C}) \to Z$ by the formula $\Phi(p + i\, q) = \Phi(p) + i\Phi(q)$. For any integer j put $\chi_j(t) = e^{-ijt}$. For a linear map $\Phi : T_{2\pi}(\mathbf{R}, \mathbf{C}) \to Z$ the element $z_j = \Phi(\chi_j)$ is said to be the j-th Fourier coefficient of Φ.

2. Fourier coefficients of a positive linear map

We now give a characterization of a positive linear map $\Phi : T_{2\pi}(\mathbf{R}) \to Y$ in terms of its Fourier coefficients.

Definition 2.1. Let $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ be a sequence of elements of Z.

(i) The sequence $\mathbf{z}$ is said to be positive defined if $0 \leq \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \overline{c_k} z_{k-j}$ for any finite sequence $(c_j)_{j=-n}^{n}$ of complex numbers.

(ii) The sum

$$\sum_{j=-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) z_j e^{ijt}$$

is denoted by $\sigma_N(\mathbf{z}, t)$ and is called the Cesaro sum of the sequence $\mathbf{z}$.

Lemma 2.1. *Let* $\Phi \ : T_{2\pi}(\mathbf{R}) \to Y$ *be a positive linear map. Then the sequence* $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ *of the Fourier coefficients of* Φ *is positive defined.*

Proof. Let $(c_j)_{j=-n}^{n}$ be a sequence of complex numbers. Put

$$f(t) = \sum_{j=-n}^{n} c_j e^{ijt} \text{ and } g(t) = \left|f(t)\right|^2 = \sum_{j=-n}^{n}\sum_{k=-n}^{n} c_j \overline{c_k} e^{i(j-k)t}.$$

Then

$$0 \le \Phi(g) = \sum_{j=-n}^{n}\sum_{k=-n}^{n} c_j \overline{c_k} z_{k-j}.$$

So, the sequence $\mathbf{z}$ is positive defined.

Lemma 2.2. *Let* $p(t) = \sum_{j=-n}^{n} c_j e^{ijt}$ *be a nonnegative trigonometric polynomial. Then there is a complex trigonometric polynomial* $q(t) = \sum_{j=0}^{n} d_j e^{ijt}$ *such that* $p(t) = \left|q(t)\right|^2$.

Proof. Put $P(\zeta) = \sum_{j=-n}^{n} c_j \zeta^j$. Then $p(t) = P(e^{it})$. Write

$$w = \frac{i(1+\zeta)}{1-\zeta}, \text{ or equivalently } \zeta = \frac{w-i}{w+i} \ .$$

For real w we have $|\zeta| = 1$. Put

$$R(w) = (w^2+1)^n P(\frac{w-i}{w+i}) \ .$$

Then

$$R(w) = \sum_{j=-n}^{n} c_j (w-i)^{n+j}(w+i)^{n-j} \ ,$$

which means that R is a polynomial of degree $\le 2n$. Moreover,

$$R(w) \ge 0 \text{ for any real } w \ .$$

It means that the polynomial R has only real zeros of even multiplicity and pairs of nonreal conjugated zeros of the same multiplicity. In other words,

$$R(w) = A \prod_{k=1}^{m} (w - w_k)(w - \overline{w_k}), \text{ where } A \ge 0 \ .$$

Therefore,

$$P(\frac{w-i}{w+i}) = \frac{A}{(w^2+1)^n} \prod_{k=1}^{m} (w-w_k)(w-\overline{w_k}) = \frac{A}{(w^2+1)^{n-m}} \prod_{k=1}^{m} \frac{w-w_k}{w+i}\frac{w-\overline{w_k}}{w-i} \ .$$

Hence,

$$P(\zeta)=\frac{A}{\left(1-\left(\frac{1+\zeta}{1-\zeta}\right)^2\right)^{n-m}}\prod_{k=1}^{m}\frac{\frac{i(1+\zeta)}{1-\zeta}-w_k}{\frac{i(1+\zeta)}{1-\zeta}+i}\,\frac{\frac{i(1+\zeta)}{1-\zeta}-\overline{w_k}}{\frac{i(1+\zeta)}{1-\zeta}-i}=$$

$$=\frac{A}{4^{n-m}}(1-\zeta)^{n-m}\left(1-\frac{1}{\zeta}\right)^{n-m}\prod_{k=1}^{m}\frac{(1+\zeta)+iw_k(1-\zeta)}{2}\,\frac{(1+\zeta)+i\overline{w_k}(1-\zeta)}{2\zeta}=$$

$$=\frac{A}{4^{n}}(1-\zeta)^{n-m}\left(1-\frac{1}{\zeta}\right)^{n-m}\prod_{k=1}^{m}(1+\zeta+iw_k(1-\zeta))\,\left(1+\frac{1}{\zeta}-i\overline{w_k}\left(1-\frac{1}{\zeta}\right)\right)\,.$$

Put

$$Q(\zeta)=\sqrt{\frac{A}{4^n}}(1-\zeta)^{n-m}\prod_{k=1}^{m}(1+\zeta+iw_k(1-\zeta))\,.$$

Then $P(\zeta)=Q(\zeta)\overline{Q(\zeta)}$ whenever $\overline{\zeta}=\frac{1}{\zeta}$, i.e. $|\zeta|=1$. The trigonometric polynomial $q(t)=Q(e^{it})$ has desired properties.

Remark. For another proof see [4, p.92, 294-295].

The converse of Lemma 2.1 is also true:

Theorem 2.3. *Let $\mathbf{z}=(z_j)_{j=-\infty}^{\infty}$ be a positive defined sequence of elements of Z. Then the map $\Phi\ : T_{2\pi}(\mathbf{R},\mathbf{C})\to Z$ defined by the formula*

$$\Phi(p)=\sum_{j=-n}^{n}c_jz_{-j}\ \text{ for } p(t)=\sum_{j=-n}^{n}c_je^{ijt}$$

is a unique positive linear map for which the sequence of the Fourier coefficients is the sequence $\mathbf{z}$.

Proof. Linearity and uniqueness of the map Φ is obvious. We shall show positivity. Let $p(t)=\sum_{j=-n}^{n}c_je^{ijt}\geq 0$ for all real t. By Lemma 2.1,

$$p(t)=\left|q(t)\right|^2\ ,\ \text{where } q(t)=\sum_{j=0}^{n}d_je^{ijt}\ .$$

Hence,

$$p(t)=\Big(\sum_{j=0}^{n}d_je^{ijt}\Big)\Big(\sum_{k=0}^{n}\overline{d_k}e^{-ikt}\Big)=\sum_{j=-n}^{n}\Big(\sum_{k=\max(0,j)}^{\min(n,n+j)}d_k\overline{d_{k-j}}\Big)e^{ijt}\ .$$

It means

$$c_j=\sum_{k=\max(0,j)}^{\min(n,n+j)}d_k\overline{d_{k-j}}\ \text{ and}$$

$$\sum_{j=-n}^{n}c_jz_{-j}=\sum_{j=-n}^{n}\Big(\sum_{k=\max(0,j)}^{\min(n,n+j)}d_k\overline{d_{k-j}}\Big)z_{-j}=\sum_{k=0}^{n}\sum_{j=0}^{n}d_k\overline{d_j}z_{j-k}\geq 0\ ,$$

because $\mathbf{z}=(z_j)_{j=-\infty}^{\infty}$ is a positive defined sequence.

So, we have the following characterization of positive linear maps by their Fourier coefficients.

Theorem 2.4. *For any sequence* $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ *of elements of* Z *the following properties are equivalent.*

(i) *There is a unique positive linear map* $\Phi : T_{2\pi}(\mathbf{R}) \to Y$ *for which* $\mathbf{z}$ *is the sequence of the Fourier coefficients.*

(ii) *The sequence* $\mathbf{z}$ *is positive defined.*

Now, we consider positive defined sequences in connection with their Cesaro sums.

Lemma 2.5. *Let a sequence* $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ *of elements of* Z *be positive defined. Then*

(i) $\sigma_N(\mathbf{z}, t) \geq 0$ *for all real* t *and natural* N.

(ii) $0 \leq z_0 \in Y$, $z_{-n} = \overline{z_n}$, $\pm \mathrm{Re}(z_n) \leq z_0$ *and* $\pm \mathrm{Im}(z_n) \leq z_0$ *for any integer* n.

Proof. (i). Put $c_j = e^{-ijt}$ for $0 \leq j \leq N$. Then

$$0 \leq \sum_{j=0}^{N} \sum_{k=0}^{N} c_j \overline{c_k} z_{k-j} = \sum_{j=-N}^{N} (N + 1 - |j|) z_j e^{ijt} = (N + 1)\sigma_N(\mathbf{z}, t).$$

(ii). Put $c_0 = 1$. Then

$$0 \leq \sum_{j=0}^{0} \sum_{k=0}^{0} c_j \overline{c_k} z_{k-j} = c_0 z_0 = z_0,$$

i.e., $0 \leq z_0 \in Y$. Put $c_0 = c_n = 1$ and $c_j = 0$ for $0 < j < n$. Then

$$0 \leq \sum_{j=0}^{n} \sum_{k=0}^{n} c_j \overline{c_k} z_{k-j} = 2z_0 + z_n + z_{-n}.$$

It means that $\mathrm{Im}(z_n + z_{-n}) = 0$ and $-\mathrm{Re}(z_n + z_{-n}) \leq 2z_0$. Replacing c_n by -1, we obtain $\mathrm{Re}(z_n + z_{-n}) \leq 2z_0$. Put $c_n = \pm i$, then we have $\pm \mathrm{Im}(z_n) \leq z_0$ and $\mathrm{Re}(z_n - z_{-n}) = 0$. Therefore, $z_{-n} = \overline{z_n}$, $\pm \mathrm{Re}(z_n) \leq z_0$ and $\pm \mathrm{Im}(z_n) \leq z_0$.

Lemma 2.6. *Let* $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ *be a sequence of elements of* Z *such that* $z_j = 0$ *for* $|j| > m$. *If* $\sum_{j=-m}^{m} z_j e^{ijt} \geq 0$ *for all* t, *then the sequence* $\mathbf{z}$ *is positive defined.*

Proof. Put

$$f(t) = \sum_{j=-n}^{n} c_j e^{ijt},\ g(t) = |f(t)|^2 = \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \overline{c_k} e^{i(j-k)t} \text{ and } h(t) = \sum_{j=-n}^{n} z_j e^{ijt} .$$

Then

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} g(t)h(t)dt = \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \overline{c_j} z_{k-j} .$$

Lemma 2.7. *Let a sequence* $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ *of elements of* Z *be such that* $\sigma_N(\mathbf{z}, t) \geq 0$ *for all real* t *and natural* N. *Then* $0 \leq z_0 \in Y$, $z_{-j} = \overline{z_j}$, $\pm\operatorname{Re}(z_j) \leq 2z_0$ *and* $\pm\operatorname{Im}(z_j) \leq 2z_0$ *for any integer* j.

Proof. For any integer $N \geq 0$ put

$$\zeta_j = \Big(1 - \frac{|j|}{N+1}\Big) z_j \text{ for } -N-1 < j < N+1 \text{ and } \zeta_j = 0 \text{ otherwise.}$$

Since $\sigma_N(\mathbf{z}, t) \geq 0$ for all t, the sequence $(\zeta_j)_{j=-\infty}^{\infty}$ is positive defined by Lemma 2.6. Now, Lemma 2.5 implies $0 \leq \zeta_0 \in Y$, $\zeta_{-j} = \overline{\zeta_j}$, $\pm\operatorname{Re}(\zeta_j) \leq \zeta_0$ and $\pm\operatorname{Im}(\zeta_j) \leq \zeta_0$ for $-N-1 < j < N+1$. It means

$$0 \leq z_0 \in Y,\ z_{-j} = \overline{z_j}\ ,$$

$$\pm\Big(1 - \frac{|j|}{N+1}\Big)\operatorname{Re}(z_j) \leq z_0 \text{ and } \pm\Big(1 - \frac{|j|}{N+1}\Big)\operatorname{Im}(z_j) \leq z_0$$

for $-N-1 < j < N+1$. Now, for any integer j take $N = 2|j| - 1$. Then

$$\pm\operatorname{Re}(z_j) \leq 2z_0 \text{ and } \pm\operatorname{Im}(z_j) \leq 2z_0 \text{ for any integer } j\ .$$

Theorem 2.8. *Let a sequence* $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ *of elements of* Z *be such that* $\sigma_N(\mathbf{z}, t) \geq 0$ *for all real* t *and natural* N. *If*

$$\bigwedge_{N=1}^{\infty} \frac{z_0}{N} = 0\ ,$$

then the linear map $\Phi\ : T_{2\pi}(\mathbf{R}, \mathbf{C}) \to Z$ *defined by the formula*

$$\Phi(f) = \sum_{j=-n}^{n} c_j z_{-j} \text{ for } p(t) = \sum_{j=-n}^{n} c_j e^{ijt}$$

is a unique positive linear map for which the sequence of the Fourier coefficients is the sequence $\mathbf{z}$.

Proof. Clearly, it suffices to prove positivity of Φ. So, assume

$$p(t) = \sum_{j=-n}^{n} c_j e^{ijt} \geq 0\ .$$

We have to prove that

$$\sum_{j=-n}^{n} c_j z_{-j} \geq 0\ .$$

Since $p(t) \geq 0$ and $\sigma_N(\mathbf{z}, t) \geq 0$ for all t, we have

$$0 \leq \frac{1}{2\pi}\int_0^{2\pi} f(t)\sigma_N(\mathbf{z}, t)dt = \sum_{j=-n}^{n} c_j \Big(1 - \frac{|j|}{N+1}\Big) z_{-j}$$

whenever $n \le 2N+1$. Put

$$S = \sum_{j=-n}^{n} c_j z_{-j} \text{ and } S_N = \sum_{j=-n}^{n} c_j \Big(1 - \frac{|j|}{N+1}\Big) z_{-j} \ .$$

Then

$$S \ge S - S_N = \sum_{j=-n}^{n} c_j \frac{|j|}{N+1} z_{-j} = \sum_{j=1}^{n} 2 \ \mathrm{Re}(c_j \frac{|j|}{N+1} z_{-j}) \ ,$$

because the elements c_j, c_{-j} and z_j, z_{-j} are conjugated. Let $c_j = a_j + ib_j$, where a_j and b_j are real, and $z_j = u_j + iv_j$, where $u_j, v_j \in Y$. Then

$$z_{-j} = u_j - iv_j \text{ and } S \ge 2 \sum_{j=1}^{n} \frac{j}{N+1} (a_j \ u_j + b_j \ v_j) \ .$$

Lemma 2.7 implies $\pm u_j \le 2z_0$ and $\pm v_j \le 2z_0$. The inequalities $\pm a_j \le c_0$ and $\pm b_j \le c_0$ follow from Lemma 2.6 and (ii) of Lemma 2.5. Therefore,

$$S \ge -8 \sum_{j=1}^{n} \frac{j}{N+1} c_0 z_0 = -\frac{4n(n+1)}{N+1} c_0 z_0 \ .$$

It means

$$S \ge \bigvee_{N=1}^{\infty} -\frac{4n(n+1)}{N+1} c_0 z_0 = -4n(n+1) c_0 \bigwedge_{N=1}^{\infty} \frac{z_0}{N+1} = 0 \ .$$

So, Φ is positive.

Corollary 2.9. *Let a sequence $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ of elements of Z be such that $\sigma_N(\mathbf{z}, t) \ge 0$ for all real t and natural N. If Y is archimedean, then there is a positive linear map $\Phi : T_{2\pi}(\mathbf{R}) \to Y$ for which the sequence of the Fourier coefficients is the sequence $\mathbf{z}$.*

The last theorem summarizes our previous results.

Theorem 2.10. *For any sequence $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ of elements of Z with*

$$0 \le z_0 \in Y \text{ and } \bigwedge_{N=1}^{\infty} \frac{z_0}{N} = 0$$

the following properties are equivalent.

(i) *There is a unique positive linear map $\Phi : T_{2\pi}(\mathbf{R}) \to Y$ for which $\mathbf{z}$ is the sequence of the Fourier coefficients.*

(ii) *The sequence $\mathbf{z}$ is positive defined.*

(iii) *The Cesaro sums $\sigma_N(\mathbf{z}, t)$ are nonnegative for all natural N.*

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