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# A NOTE TO THE TRANSFORMATION OF T-NORMS BASED ON THE LIMIT T-NORMS

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Abstract. This is a continuation of the work in Smutná, A note to the construction of t-norms based on the limit t-norms, EUROFUSE-SIC'99, Budapest, Hungaria, 1999, 408-411. Transformation of the basic t-norms $T_D$ and $T_M$ by means of a transformation based on a non-decreasing transformation $\varphi$ of the unit interval $[0,1]$ yielding triangular norms are studied.

## 1. Introduction

Let us recall the well-known definition of the triangular norm.

**Definition 1.** A *triangular norm* (*t-norm* for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T:[0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]$ the following four axioms are satisfied:

(T1) *Commutativity*
$$T(x,y) = T(y,x),$$

(T2) *Associativity*
$$T(x,T(y,z)) = T(T(x,y),z),$$

(T3) *Monotonicity*
$$T(x,y) \leq T(x,z) \quad \text{whenever } y \leq z,$$

(T4) *Boundary Condition*
$$T(x,1) = x.$$

**Example 1.** The operators $T_D$, $T_M$ defined by
$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$T_M(x,y) = \min(x,y)$$
are t-norms. They belong to the basic t-norms.

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**Definition 2.** Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function. The function $\varphi^{(-1)}$ which is defined by

$$\varphi^{(-1)}(x) = sup\{z \in \ [0,1]; \varphi(z) < \ x\},$$

is called the pseudo-inverse of the function $\varphi$, with the convention $\sup \emptyset = 0$

*Remark 1.* Note that for the function $\varphi$ and its pseudo-inverse the inequality $\varphi^{(-1)}(\varphi(x)) \leq x$ for all $x \in [0,1]$ is satisfied.

Now, we deal with the formula $\varphi^{(-1)}[T(\varphi(x), \varphi(y))]$ where $\varphi$ is a function from the unit interval to itself, and $T$ is a t-norm. We are interested in the conditions under which this formula yields a t-norm. There are several methods in the literature concerning the construction of a new t-norm $T$, see [1],[2],[3]. We cite here one of them, see [7].

**Proposition 1.** *Let $\varphi : [0,1] \to [0,1]$ be non-decreasing on the interval $[0,1]$ and continuous on the open unit interval $]0,1[$ function and $T$ be a t-norm. Then the operator $T_\varphi$, which is defined by*

$$T_\varphi(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ \varphi^{(-1)}[T(\varphi(x), \varphi(y))] & \text{otherwise,} \end{cases}$$

*is a t-norm.*

## 2. The t-norms generated by $T_D$

In this part we investigate the t-norms, which are produced by construction from Proposition 1, when we consider Drastic product $T_D$ and any function $\varphi$ non-decreasing on the unit interval $[0,1]$.

**Example 2.** For illustration we show next examples of generated t-norms.

(i) Let $\varphi_1 : [0,1] \to [0,1]$ be defined by

$$\varphi_1(x) = x^2,$$

then

$$(T_D)_{\varphi_1}(x,y) = T_D(x,y).$$

(ii) Let $\varphi_2 : [0,1] \to [0,1]$ be defined by

$$\varphi_2(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}] \cup [\frac{3}{4}, 1] \ , \\ \frac{3}{4} & \text{otherwise,} \end{cases}$$

then

$$(T_D)_{\varphi_2}(x,y) = T_D(x,y).$$

(iii) Let $\varphi_3 : [0,1] \to [0,1]$ be defined by

$$\varphi_3(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 1 & \text{otherwise,} \end{cases}$$

then

$$(T_D)_{\varphi_3}(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) \in [\frac{1}{2}, 1[^2, \\ T_D & \text{if } (x,y) \in [0, \frac{1}{2}[^2, \\ T_M & \text{otherwise.} \end{cases}$$

(iv) Let $\varphi_4 : [0,1] \to [0,1]$ be defined by

$$\varphi_4(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{4}], \\ x & \text{if } x \in ]\frac{1}{4}, \frac{1}{2}], \\ x + \frac{1}{4} & \text{if } x \in ]\frac{1}{2}, \frac{3}{4}], \\ 1 & \text{otherwise,} \end{cases}$$

then

$$(T_D)_{\varphi_4}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0, \frac{3}{4}[^2 \cup [\frac{3}{4}, 1[ \times [0, \frac{1}{4}] \cup [0, \frac{1}{4}] \times [\frac{3}{4}, 1[, \\ \frac{3}{4} & \text{if } (x,y) \in [\frac{3}{4}, 1[^2, \\ T_M & \text{otherwise.} \end{cases}$$

(v) Let $\varphi_5 : [0,1] \to [0,1]$ be defined by

$$\varphi_5(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{4}], \\ \frac{1}{2} & \text{if } x \in ]\frac{1}{4}, \frac{1}{2}], \\ 1 & \text{otherwise,} \end{cases}$$

then

$$(T_D)_{\varphi_5}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0, \frac{3}{4}[^2, \\ \frac{1}{4} & \text{if } (x,y) \in [\frac{1}{2}, 1[ \times [\frac{1}{4}, \frac{1}{2}] \cup [\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{2}, 1[, \\ \frac{1}{2} & \text{if } (x,y) \in ]\frac{1}{2}, 1[^2, \\ T_M & \text{otherwise.} \end{cases}$$

As far as $T_D$ is a special t-norm, Proposition 1 holds true for the all functions $\varphi$.

**Proposition 2.** *Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function and an operator $(T_D)_\varphi : [0,1]^2 \to [0,1]$ be defined by*

$$(T_D)_\varphi(x,y) = \begin{cases} \min(x,y) & \textit{if } \max(x,y) = 1, \\ \varphi^{(-1)}[T_D(\varphi(x), \varphi(y))] & \textit{otherwise.} \end{cases}$$

*Then the operator* $(T_D)_\varphi$ *is a t-norm and* $(T_D)_\varphi = T_D$ *if and only if* $\varphi^{(-1)}(1) = 1$ *or* $\varphi^{(-1)}(1) = 0$.

*Proof.* Let $\varphi^{(-1)}(1) = 1$, then from Definition 2 we have $\varphi(c) < 1$ for $c \in [0, 1[$. Suppose $x, y \in [0, 1[$, then $\varphi(x) < 1, \varphi(y) < 1$ and $T_D(\varphi(x), \varphi(y)) = 0$. Therefore $(T_D)_\varphi(x, y) = \varphi^{(-1)}(0) = 0$. If $x = 1$ or $y = 1$, then $(T_D)_\varphi(x, y) = \min(x, y)$ by Proposition 1.

Let $\varphi^{(-1)}(1) = 0$, then for all $x, y \in [0, 1[, T_D(\varphi(x), \varphi(y)) \leq 1$ and consequently $0 \leq (T_D)_\varphi(x, y) \leq \varphi^{-1}(1) = 0$, i.e., $(T_D)_\varphi(x, y) = (T_D)(x, y) = 0$.

Let $\varphi^{(-1)}(1) \neq 1$ and $\varphi^{(-1)}(1) \neq 0$. Then there exists $c \in ]0, 1[$ such that $\varphi(c) = 1$ and there exists $d \in ]0, c[$ such that $\varphi(d) < 1$. Then $(T_D)_\varphi(c, c) = \varphi^{(-1)}(1) \neq 0 = T_D(c, c)$.

**Proposition 3.** *Let* $\varphi : [0, 1] \to [0, 1]$ *be a non-decreasing function, such that there exists* $c \in [0, 1[$*, such that* $\varphi(c) = 1$ *and for all* $c^* < c$, $\varphi(c^*) < 1$. *Let* $\{[a_i, b_i]\}_{i \in I}$ *be a family of subintervals of* $[0, c[$*, such that* $\varphi(x) = c_i$ *whenever* $x \in [a_i, b_i]$ *(on the set* $[0, c[\setminus \bigcup_{i \in I} [a_i, b_i]$ *the function* $\varphi$ *is strictly increasing) . Then for the t-norm* $(T_D)_\varphi$ *the equality*

$$(T_D)_\varphi(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, c[^2, \\ c & \text{if } (x, y) \in [c, 1[^2, \\ \varphi^{(-1)}(c_i) = a_i & \text{if } (x, y) \in [c, 1[ \times [a_i, b_i] \text{ or } (x, y) \in [a_i, b_i] \times [c, 1[, \\ T_M(x, y) & \text{otherwise} \end{cases}$$

*is satisfied.*

*Proof.*

(i) Let $(x, y) \in [0, c[^2$, then $\varphi(x) < 1, \varphi(y) < 1$ and $T_D(\varphi(x), \varphi(y)) = 0$. Therefore $(T_D)_\varphi(x, y) = 0$.

(ii) Let $(x, y) \in [c, 1[^2$, then $\varphi(x) = 1, \varphi(y) = 1$ and $T_D(\varphi(x), \varphi(y)) = 1$ and $(T_D)_\varphi(x, y) = \varphi^{(-1)}(1) = c$.

(iii) Let $(x, y) \in [c, 1[ \times [a_i, b_i]$, then $\varphi(x) = 1, \varphi(y) = c_i$ and $T_D(\varphi(x), \varphi(y)) = c_i$. Therefore $(T_D)_\varphi(x, y) = \varphi^{(-1)}(c_i) = a_i$.
If $(x, y) \in [a_i, b_i] \times [c, 1[$ then $(T_D)_\varphi(x, y) = \varphi^{(-1)}(c_i) = a_i$ from commutativity.

(iv) Let $x \in [c, 1]$ and $y \in [0, 1] \setminus [a_i, b_i]$ for all $i \in I$, then $\varphi(y) < \varphi(x) = 1$ and $T_D(\varphi(x), \varphi(y)) = \varphi(y)$. Therefore $(T_D)_\varphi(x, y) = \varphi^{(-1)}(\varphi(y)) = y = T_M(x, y)$.
If $x \in [0, 1] \setminus \bigcup_{i \in I} ]a_i, b_i]$ and $y \in [c, 1]$ then $(T_D)_\varphi(x, y) = T_M(x, y)$ from commutativity.

## 3. The t-norms generated by $T_M$

In this part we investigate the t-norms, which are produced by construction from Proposition 1, when we consider the t-norm $T_M$. Because of specific form of this t-norm, again we can strenghthen Proposition 1.

**Proposition 4.** *Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function and the operator $(T_M)_\varphi : [0,1]^2 \to [0,1]$ be defined by*

$$(T_M)_\varphi(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ \varphi^{(-1)}[T_M(\varphi(x), \varphi(y))] & \text{otherwise.} \end{cases}$$

*Then $(T_M)_\varphi = T_M$ if and only if $\varphi$ is a strictly increasing function.*

**Proposition 5.** *Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function continuous on the open unit interval $]0,1[$. Let $\{[a_i, b_i]\}_{i \in I}$ be a family of subintervals of $[0,1]$, such that $\varphi(x) = c_i$ whenever $x \in [a_i, b_i]$ (on interval $[0,1] \setminus \bigcup_{i \in I} [a_i, b_i]$ function $\varphi$ is strictly increasing). Then for the operator $(T_M)_\varphi$ is a t-norm given by*

$$(T_M)_\varphi(x,y) = \begin{cases} \varphi^{(-1)}(c_i) = a_i & \text{if } (x,y) \in [a_i, 1[ \times [a_i, b_i] \text{ or } (x,y) \in [a_i, b_i] \times [b_i, 1[, \\ T_M(x,y) & \text{otherwise.} \end{cases}$$

*Remark 2.* Note that the Proposition 4 and Proposition 5 can be proved likewise as Proposition 2 and Proposition 3.

**Example 3.**

(i) Let $\varphi_1 : [0,1] \to [0,1]$ be defined by

$$\varphi_1(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}] , \\ \frac{x}{2} + \frac{1}{2} & \text{otherwise,} \end{cases}$$

then

$$(T_M)_{\varphi_1}(x,y) = T_M(x,y).$$

(ii) Let $\varphi_2 : [0,1] \to [0,1]$ be defined by

$$\varphi_2(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{4}], \\ x & \text{if } x \in ]\frac{1}{4}, \frac{1}{2}], \\ x + \frac{1}{4} & \text{if } x \in ]\frac{1}{2}, \frac{3}{4}], \\ 1 & \text{otherwise,} \end{cases}$$

then

$$(T_M)_{\varphi_2}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0, \frac{1}{4}[ \times [\frac{1}{4}, 1[ \cup [0,1[ \times [0, \frac{1}{4}[, \\ \frac{3}{4} & \text{if } (x,y) \in [\frac{3}{4}, 1[^2, \\ T_M & \text{otherwise.} \end{cases}$$

(iii) Let $\varphi_3 : [0,1] \to [0,1]$ be defined by

$$\varphi_3(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{4}], \\ \frac{3}{4} & \text{if } x \in ]\frac{1}{4}, \frac{1}{2}], \\ 1 & \text{otherwise,} \end{cases}$$

then

$$(T_M)_{\varphi_3}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[\times[0,\frac{1}{4}] \cup [0,\frac{1}{4}] \times [\frac{1}{4},1[ \\ \frac{1}{4} & \text{if } (x,y) \in ]\frac{1}{4},1[\times]\frac{1}{4},\frac{1}{2}]\cup]\frac{1}{4},\frac{1}{2}]\times]\frac{1}{4},1[, \\ \frac{1}{2} & \text{if } (x,y) \in ]\frac{1}{2},1[^2, \\ T_M & \text{otherwise.} \end{cases}$$

The function $\varphi_3$ satisfy neither Proposition 4 nor Proposition 5 and resulting operator $(T_M)_{\varphi_3}$ is not a t-norm. It is easy to see that for the operator $(T_M)_{\varphi_3}$ is satisfied next inequality

$$(T_M)_{\varphi_3}\left(\frac{3}{4},(T_M)_{\varphi_3}\left(\frac{3}{4},\frac{1}{2}\right)\right) = 0 \neq \frac{1}{4} = (T_M)_{\varphi_3}\left((T_M)_{\varphi_3}\left(\frac{3}{4},\frac{3}{4}\right),\frac{1}{2}\right),$$

which is violation of associativity.

**Corollary 1.** *Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function which is either right-continuous or strictly monotone. Then the mapping $(T_M)_\varphi : [0,1]^2 \to [0,1]$ given by*

$$(T_M)_\varphi = \begin{cases} \varphi^{(-1)}(T_M(\varphi(x),\varphi(y)) & \text{if } \max(x,y) < 1, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

*is a t-norm.*

Note that a general characterization of all functions $\varphi$ such that $(T_M)_\varphi$ is a t-norm is still an open problem. The right-continuity or strict-monotonicity of the function $\varphi$ are not necessary. We can see it in Example 3, case (ii), where the operator $(T_M)_{\varphi_2}$ is a t-norm.

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