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THE DIMENSION OF ORTHOMODULAR POSETS CONSTRUCTED BY PASTING BOOLEAN ALGEBRAS II

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Abstract. In [1], the dimension of certain atomic amalgams of Boolean algebras, so-called loops, was found. In this paper (which is a continuation of [1]) we calculate the dimension of atomic amalgams of Boolean algebras with no loop as a subposet.

In [1] some results on the dimension of atomic amalgams of Boolean algebras are presented. For instance, an atomic amalgam of Boolean algebras 2^3 can, in general, have an arbitrarily dimension. In [1] we mainly investigated the dimension of two extreme cases: the dimension of loops and the dimension of loopless atomic amalgams. The basic results for loops have been presented in [1]. Here we present the basic results concerning the loopless case.

For preliminary definitions and results we refer the reader to [1], while necessary rudiments of dimension theory of ordered sets can be found in [4].

We consider only the finite atomic amalgams such that every pair of blocks either intersects trivially in the bounds 0 and 1 or the intersection consists of the bounds, an atom and its complement. Every such atomic amalgam of Boolean algebras can be represented by the Greechie diagram, which we can consider as a graph. One can represent a given amalgam by several Greechie diagrams mutually different as graphs. For instance, in Fig. 1a and 1b the Greechie diagrams of the same atomic amalgam of three Boolean algebras 2^3 are depicted. For an atomic amalgam $\mathcal{L} = (L, \leq, 0, 1, ')$ of Boolean algebras we will always choose some fixed diagram as its graph and denote it by $G(\mathcal{L})$.

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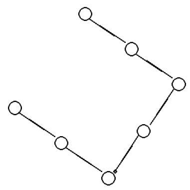

Fig. 1a

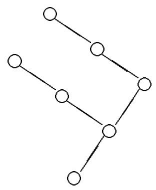

Fig. 1b

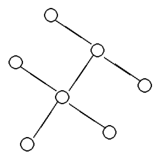

Fig. 1c

Throughout this paper we consider only atomic amalgams of Boolean algebras which do not contain a loop of order $n \geq 3$ as a subposet (i.e. their Greechie diagram is a tree). We also suppose that some vertex of $G(\mathcal{L})$ is fixed - this is called the *root* of the graph $G(\mathcal{L})$. We will always suppose that the vertices of every block (in chosen Greechie's diagram considered as a graph) have mutually different distances from the root. For instance, in this paper we do not consider the diagram in Fig. 1c to be a Greechie diagram. The atoms of the atomic amalgam $\mathcal{L}$ and the corresponding vertices of the graph $G(\mathcal{L})$ will often be identified. We will denote the set of all vertices of the graph G by V(G).

Theorem 1. *Let* $\mathcal{L} = (L, \leq, 0, 1, ')$ *be an atomic amalgam of Boolean algebras* 2^3 *having the following two properties:*
(i) *every block has at most two pasting atoms;*
(ii) *no subposet of* $\mathcal{L}$ *is a loop of order* $n \geq 3$.
Then $dim\mathcal{L} = 3$.

Proof. Let $G(\mathcal{L})$ be a fixed graph of $\mathcal{L}$ with a root c satisfying the requirements above. Throughout the proof we also suppose that every pasting atom has an even distance from the root c (i.e. in no block the 'midle' vertex can be a pasting atom). We denote by $d(u, v)$ the usual distance between vertices u and v in the graph $G(\mathcal{L})$. Without loss of generality, we can suppose that $G(\mathcal{L})$ is a connected graph (the opposite case is an easy consequence).

Let v be a pasting atom (vertex) of blocks. For every vertex w adjacent to v such that $d(c, w) = d(c, v) + 1$ there is the subgraph of $G(\mathcal{L})$ induced by the set
$\{u \in V(G(\mathcal{L}));\ u = v \quad \text{or} \quad w \text{ lies on the path } c - u\}$.
We call this induced subgraph *the branch (determined by w) with the starting point v*. We will denote the branches with the fixed starting point v by (arbitrarily but fixed) $B_1^{(v)}, ..., B_{k_v}^{(v)}$. Suppose that such notation is established for every pasting atom (vertex) v of $G(\mathcal{L})$.

Obviously, $dim\mathcal{L} \geq 3$. So it suffices to prove that $dim\mathcal{L} \leq 3$. To show this, it is sufficient to find three subsets A_0, A_1, A_2 of the set $Crit\mathcal{L}$ of all critical pairs $[x', y]$ (i.e. of the set of all ordered pairs $[x', y]$ where $x = y$ is an arbitrary atom, or x, y are atoms not belonging to the same block and x' is the complement of x) satisfying
(a) $A_0 \cup A_1 \cup A_2 = Crit\mathcal{L}$ and
(b) each of the subsets A_0, A_1, A_2 is cycle-free
(i.e. it does not contain a sequence $[x'_1, y_1], [x'_2, y_2], \ldots, [x'_n, y_n]$ with $y_1 \leq x'_2,\ y_2 \leq x'_3, \ldots, y_{n-1} \leq x'_n,\ y_n \leq x'_1$).

Define the sets A_0, A_1, A_2 as follows:

1. $[x', x] \in A_0$ iff $d(c, x) = 4k$,
   $[x', x] \in A_1$ iff $d(c, x) = 2k + 1$,
   $[x', x] \in A_2$ iff $d(c, x) = 4k + 2$,

where $d(c, x)$ is the usual distance between the vertices c and x in $G(\mathcal{L})$.

2. If the atoms x, y do not belong to the same block and if y lies on the path $c - x$ and $[y', y] \in A_i$, then we put

$[x', y] \in A_i$ and $[y', x] \in A_i$.

3. If the atoms x, y belong to different branches with the same starting point z and if

$x \in V(B_i^{(z)})$, $y \in V(B_j^{(z)})$, $i < j$,

then

$[x', y] \in A_2$ and $[y', x] \in A_1$ provided $[z', z] \in A_0$,
$[x', y] \in A_0$ and $[y', x] \in A_1$ provided $[z', z] \in A_2$

(see Fig. 2). Note that $d(c, z)$ is an even number.

Obviously, every critical pair $[x', x]$ belongs to $A_0 \cup A_1 \cup A_2$. If $[x', y]$ is a critical pair and $x \neq y$, then there are the following possibilities: the atom x lies on the path $c - y$ or the atom y lies on the path $c - x$ or the atoms x, y belong to different branches with the same starting point z. Hence again $[x', y] \in A_0 \cup A_1 \cup A_2$. Therefore,

$A_0 \cup A_1 \cup A_2 = Crit\mathcal{L}$.

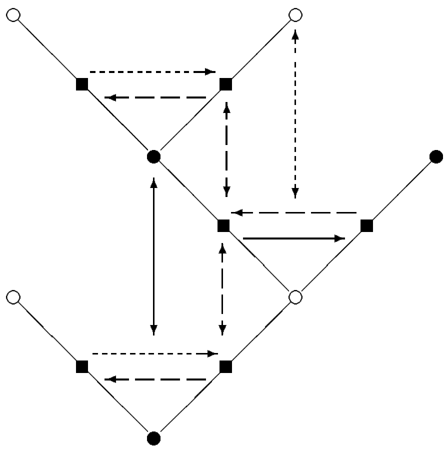

Fig. 2

Now we are going to prove that the sets A_0, A_1, A_2 are cycle-free.

Let $[x'_1, y_1], [x'_2, y_2], \ldots, [x'_n, y_n]$ be a sequence of elements of A_r, $r \in \{0, 1, 2\}$, such that

$y_1 < x'_2$, $y_2 < x'_3$, $\ldots$ $, y_{n-1} < x'_n$. We need to show that $y_n < x'_1$ is impossible. For this purpose we will prove that $[x'_1, y_n] \in A_r$ (i.e. $[x'_1, y_n]$ is a critical pair). Thus, to complete the proof it is sufficient to prove the statement:

If $[x_1', y_1], [x_2', y_2] \in A_r$, $r \in \{0,1,2\}$ and $y_1 < x_2'$ then $[x_1', y_2] \in A_r$.

We will distinguish several cases.

1. The atoms x_1, y_1 belong to different branches with the same starting point z_1.

a)The atom x_2 lies on the path $c - y_2$, $x_2 \neq y_2$.

By assumption $y_1 < x_2'$ and therefore y_1, x_2 belong to the same block. The equality $x_2 = z_1$ implies that $[x_2', x_2]$ and $[x_2', y_2]$ belong to the same set A_j but $[x_1', y_1]$ and $[z_1', z_1] = [x_2', x_2]$ belong to different sets A_i, A_j (by the definition of A_0, A_1 and A_2) and this contradicts our assumptions. If $x_2 \neq z_1$ then the ordered pairs $[x_1', y_2]$ and $[x_1', y_1]$ belong (according to the part 3 of the definition) to the same set A_r.

b)The atom y_2 lies on the path $c - x_2$, $x_2 \neq y_2$.

We distinguish three subcases.

b1) The atom y_2 lies on the path $c - z_1$ and $y_2 \neq z_1$.

The ordered pairs $[y_2', y_2]$, $[x_1', y_2]$ and $[x_2', y_2]$ belong to the same set A_r by the definition (the part 2) of A_0, A_1 and A_2.

b2) $y_2 = z_1$.

The ordered pairs $[y_2', y_2]$ and $[x_2', y_2]$ belong to the same set A_i (the part 2 of the definition) but $[x_1', y_1]$ does not belong to A_i (the part 3 of the definition) and this contradicts our assumptions.

b3) The atom z_1 lies on the path $c - y_2$ and $z_1 \neq y_2$.

The ordered pair $[x_1', y_2]$ is a critical pair, $[x_1', y_2]$ and $[x_1', y_1]$ belong to the same set A_r (by the definition).

c) The atoms x_2, y_2 belong to different branches with the same starting point z_2.

If $d(c, z_1) < d(c, z_2)$ then y_1 and y_2 belong to the same branch with the starting point z_1. Hence, we have that $[x_1', y_2]$ is a critical pair and $[x_1', y_2]$, $[x_1', y_1]$ belong to the same set A_r. If the converse inequality holds then the atoms x_1, x_2 belong to the same branch with the starting point z_2 and the assertion is true again.

Let $d(c, z_1) = d(c, z_2)$ (i.e. $z_1 = z_2$). If x_1 and y_2 belong to the same branch with the starting point z_1 then the ordered pairs $[x_1', y_1]$ and $[x_2', y_2]$ do not belong to the same set A_i (the part 3 of the definition), which contradicts our assumption. In the opposite case the statement follows by transitivity of the ordering of the set of all indices of the branches with the same starting point.

2. The atom x_1 lies on the path $c - y_1$ and $x_1 \neq y_1$.

a) The atoms x_2, y_2 belong to different branches with the same starting point z_2.

a1) Let z_2 lies on the path $c - x_1$, $z_2 \neq x_1$.

The ordered pairs $[x_1', y_2]$ and $[x_2', y_2]$ belong to the same set A_r by the definition.

a2) $z_2 = x_1$.

The ordered pairs $[x_1', x_1]$ and $[x_1', y_1]$ belong to the same set A_i (by the part 2 of the definition) but $[x_2', y_2]$ does not belong to A_i (the part 3 of the definition), a contradiction.

a3) Let x_1 lies on the path $c - z_2$, $x_1 \neq z_2$.

The ordered pair $[x_1', y_2]$ is obviously a critical pair and the ordered pairs $[x_1', x_1]$, $[x_1', y_1]$ and $[x_1', y_2]$ belong to the same set.

a4) The case that x_1 and z_2 belong to different branches (with the same starting point) contradicts our assumption.

b) The atom x_2 lies on the path $c - y_2$ and $x_2 \neq y_2$.
The ordered pairs $[x'_1, x_1]$, $[x'_1, y_1]$ and $[x'_1, y_2]$ belong to the same set A_r by the definition.

c) Let the atom y_2 lies on the path $c - x_2$ and $x_2 \neq y_2$.
Firstly, we claim that $x_1 = y_2$ or the atoms x_1, y_2 do not belong to the same block. Indeed, if $x_1 \neq y_2$ belong to the same block then $[x'_1, x_1] \in A_i$ and $[y'_2, y_2] \in A_j$, for some $i \neq j$. By the definition it follows that $[x'_1, y_1] \in A_i$ and $[x'_2, y_2] \in A_j$, which contradicts our assumption. If x_1 lies on the path $c - y_2$, then the ordered pairs $[x'_1, x_1]$, $[x'_1, y_2]$ and $[x'_1, y_1]$ belong to the same set A_r by the definition. In opposite case $[y'_2, y_2]$, $[x'_1, y_2]$ and $[x'_2, y_2]$ belong to the same set A_r again by the definition.

3. The atom y_1 lies on the path $c - x_1$ and $x_1 \neq y_1$. This case can be handled analogously as the previous case 2.

4. Let $x_1 = y_1$ or $x_2 = y_2$.
Both equalities $x_1 = y_1$ and $x_2 = y_2$ can not hold simultaneosly, since $[x'_1, y_1]$, $[x'_2, y_2]$ belong to the same set A_r and $x_1 = y_1 < x'_2$.

Let $x_1 = y_1$ and $x_2 \neq y_2$. In this case the atoms x_2, y_2 do not belong to the same block.

a) The atom y_2 lies on the path $c - y_1$.
If y_2 and x_1 belong to the same block then $[y'_2, y_2]$, $[x'_2, y_2]$ belong to the same set A_i, but $[x'_1, y_1]$ is not an element A_i (the part 1 of the definition), a contradiction. If y_2 and x_1 do not belong to the same block, then $[x'_1, y_2]$ is a critical pair and $[y'_2, y_2]$, $[x'_2, y_2]$, $[x'_1, y_2]$ belong to the same set A_r.

b) The atom y_1 lies on the path $c - y_2$.
If x_2 lies on the path $c - y_2$ then the ordered pairs $[x'_2, x_2]$ and $[x'_2, y_2]$ belong to the same set A_i but $[x'_1, y_1]$ is not an element A_i (the part 1 of the definition), a contradiction. If x_2 does not lie on the path $c - y_2$ then the ordered pairs $[x'_2, y_2]$ and $[x'_1, y_1]$ do not belong to the same set (the part 3 of the definition) and this again contradicts our assumptions.

c) The atoms y_1, y_2 belong to different branches with the same starting point z.
Let $x_2 = z$. The ordered pairs $[x'_2, x_2]$ and $[x'_2, y_2]$ belong to the same set A_i (the part 2 of the definition) but the ordered pairs $[x'_2, x_2]$ and $[x'_1, y_1]$ do not belong to the same set (the part 1 of the definition), a contradiction.

Let $x_2 \neq z$. In this case x_2 and x_1 belong to the same branch with the starting point z, therefore $[x'_1, y_2]$ and $[x'_2, y_2]$ belong to the set A_r (the part 3 of the definition).

The case $x_2 = y_2$ and $x_1 \neq y_1$ is left to the reader. □

Theorem 2. *Let $\mathcal{L} = (L, \leq, 0, 1, ')$ be an atomic amalgam of Boolean algebras 2^3 containing no loop of order $n \geq 3$ as a subposet. Then $dim\mathcal{L} \leq 4$ and $dim\mathcal{L} = 4$ is possible.*

Proof. a) First we show that $dim\mathcal{L} \leq 4$. Similarly as in the proof of Theorem 1 it is sufficient to divide the critical pairs of $\mathcal{L}$ into cycle-free subsets A_0, A_1, A_2, A_3 such that $A_0 \cup A_1 \cup A_2 \cup A_3 = Crit\mathcal{L}$.

Let $G(\mathcal{L})$ be a fixed graph of $\mathcal{L}$ with a root c such that the vertices of the same block have from the root c mutually different distances. We divide the critical pairs of $\mathcal{L}$ into the sets A_0, A_1, A_2, A_3 according to the following rules:

1. $[x', x] \in A_r$ iff $d(c, x) = 4k + r$, $0 \leq r \leq 3$,
for some natural numbers k, r.
2. Let $[x', y]$ be a critical pair, $x \neq y$. If y lies on the path $c - x$ and $[y', y] \in A_i$, then we put $[x', y] \in A_i$ and $[y', x] \in A_i$.
3. If the atoms x, y belong to different branches with the starting point z, $x \in V(B_i^{(z)})$, $y \in V(B_j^{(z)})$, $i < j$ and $[z', z] \in A_k$,
then we put
$[x', y] \in A_{k+1}$, $[y', x] \in A_{k+2}$
(we compute modulo 4).

It is easy to check that $A_0 \cup A_1 \cup A_2 \cup A_3 = Crit\mathcal{L}$. Similarly as in the proof of Theorem 1 one can show that $[x'_1, y_1] \in A_r$, $[x'_2, y_2] \in A_r$, $r \in \{0, 1, 2, 3\}$ and $y_1 < x'_2$ imply $[x'_1, y_2] \in A_r$. Consequently, each of the sets A_0, A_1, A_2, A_3 is cycle-free.

b) Now we find an atomic amalgam $\mathcal{L}$ of Boolean algebras 2^3 such that $dim\mathcal{L} = 4$.

Let $\mathcal{L}$ be the atomic amalgam represented by the Greechie diagram (the graph $G(\mathcal{L})$) in Fig. 3. It is pasted from fifteen Boolean algebras 2^3. By the previous part of the proof, $dim\mathcal{L} \leq 4$. We assert that $dim\mathcal{L} = 4$. Suppose on the contrary that $dim\mathcal{L} = 3$, i.e. the set of all critical pairs of $\mathcal{L}$ can be divided into sets A_1, A_2, A_3 such that $A_1 \cup A_2 \cup A_3 = Crit\mathcal{L}$ and each of the sets A_1, A_2, A_3 is cycle-free .

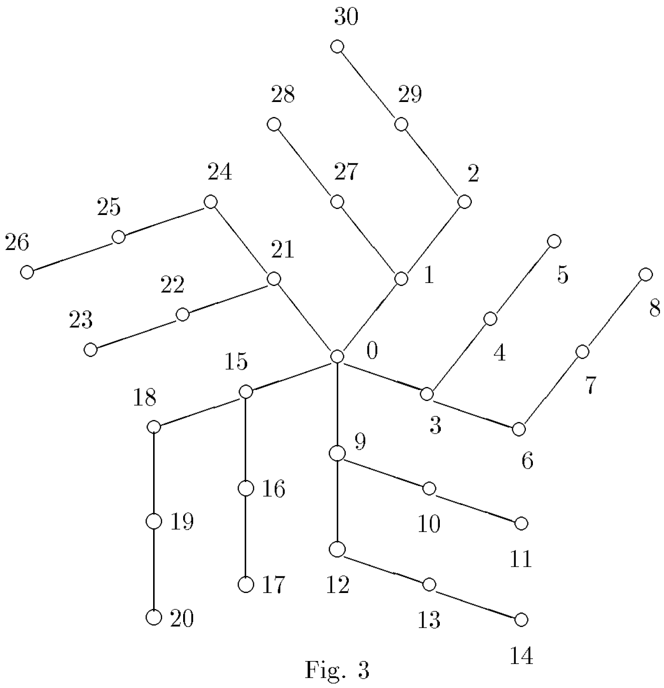

Fig. 3

Without loss of generality we may suppose that
$[0', 0] \in A_1$

$[1', 1], [3', 3], [9', 9], [15', 15], [21', 21] \in A_2$
$[2', 2], [6', 6], [12', 12], [18', 18], [24', 24] \in A_3$.
No pairs among $[4', 0], [5', 0], [10', 0]$ and $[11', 0]$ belong to A_2 (otherwise we would have a cycle with $[3', 3]$ or with $[9', 9]$ in A_2) and no pairs among $[7', 0], [8', 0], [13', 0], [14', 0]$ belong to A_3.

Now we show that if $[7', 0]$ or $[8', 0]$ belongs to A_2 then the pairs $[13', 0]$, $[14', 0]$ (and by symmetry also $[19', 0], [20', 0], [25', 0], [26', 0], [29', 0], [30', 0]$) do not belong to A_2. To verify it, take, for instance, $[8', 0] \in A_2$ and $[13', 0] \in A_2$. Then we have $[9', 6] \in A_3$ (otherwise we would have the cycle with $[0', 0]$ in A_1 or the cycle with $[8', 0]$ in A_2) and similarly $[3', 12] \in A_3$. So the cycle $[9', 6], [3', 12]$ belongs to A_3 , a contradiction. Analogously, it can be shown that if $[4', 0] \in A_3$ or $[5', 0] \in A_3$ then the pairs $[10', 0]$, $[11', 0]$ do not belong to A_3. If in the above assertions we change every pair of type $[x', 0]$ by $[0', x]$ we also obtain true assertions. This means that there exists a set S such that

$S \in \{\{4, 5, 7, 8\}, \{10, 11, 13, 14\}, \{16, 17, 19, 20\}, \{22, 23, 25, 26\}, \{27, 28, 29, 30\}\}$.
and $[x', 0] \in A_1$, $[0', x] \in A_1$ for each x belonging to the set S. Without loss of generality we may suppose that
(1) $[0', 4], [4', 0], [0', 5], [5', 0], [0', 7], [7', 0], [0', 8], [8', 0] \in A_1$
This implies that
$[3', 7], [7', 3], [3', 8], [8', 3] \in A_2$
(otherwise we would have a cycle with $[6', 6]$ in A_3 or with some pair from the list (1) in A_1). Similarly,
$[6', 5], [5', 6], [6', 4], [4', 6] \in A_3$.
This yields $[4', 7] \in A_1$ (in A_2 we would have a cycle with $[8', 3]$ and in A_3 with $[6', 5]$) and $[8', 5] \in A_1$ which contradicts to the assumption that A_1 is cycle-free ($[4', 7]$ and $[8', 5]$ form the cycle in A_1). The proof is complete. □

Theorem 3. . *Let* $\mathcal{L} = (L, \leq, 0, 1, ')$ *be an atomic amalgam of finite Boolean algebras of cardinality at least 8 and let it not contain any loop of order* $n \geq 3$ *as a subposet. Let* $\mathcal{L}$ *consist of blocks (Boolean algebras)* $B_1, \ldots, B_k$ *and let* $dim B_i = p_i$ *for* $i = 1, \ldots, k$, *and* $max\{p_1, \ldots, p_k\} = p \geq 4$.
Then $dim\mathcal{L} = p$.

Proof. It is obvious that $dim\mathcal{L} \geq p$. We are going to show that $dim\mathcal{L} \leq p$. It suffices to divide the set of all critical pairs into sets $A_0, A_1, \ldots, A_{p-1}$ such that $A_0 \cup A_1 \cup ... \cup A_{p-1} = Crit\mathcal{L}$ and each of the sets $A_0, A_1, \ldots, A_{p-1}$ is cycle-free.

Let $G(\mathcal{L})$ be a fixed graph of $\mathcal{L}$ with a root c such that the vertices of the same block have mutually different distances from the root c. Similarly as in the proof of Theorem 2 we divide the critical pairs of $\mathcal{L}$ into the sets $A_0, A_1, \ldots, A_{p-1}$ according to the following rules:
1. $[x', x] \in A_r$ iff $d(c, x) = p.k + r$, $0 \leq r \leq p-1$
for some natural numbers k, r.
2. Let $[x', y]$ be a critical pair, $x \neq y$. If y lies on the path $c - x$ and $[y', y] \in A_i$, then we put
$[x', y] \in A_i$ and $[y', x] \in A_i$.
3. If the atoms x, y belong to different branches with the starting point z, $x \in V(B_i^{(z)})$, $y \in V(B_j^{(z)})$, $i < j$ and $[z', z] \in A_k$, then we put

$[x', y] \in A_{k+1}$, and $[y', x] \in A_{k+2}$
(we compute modulo p).

Analogously as in the proof of Theorem 1 one can show that $A_0 \cup A_1 \cup ... \cup A_{p-1} = Crit\mathcal{L}$ and each of the sets $A_0, A_1, \dots, A_{p-1}$ is cycle-free. □

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