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TANGENTIAL PROLONGATION OF SURFACES IN E_3- CLASSIFICATION OF PARAMETRIC NETS

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Abstract. The paper is devoted to some applications of the classical differential geometry of surfaces in E_3 in the computer grafics. It contains a classification of parametric nets on surfaces by a tangential prolongation of nets. This classification gives possibilities of the choice of a suitable parametrisation from the point of view drawing of surfaces in computer grafics.

Introduction

In the computer grafics of surfaces in E_3 a very useful frame tool is a suitable parametric net. Let us introduce basic notions on parametric representations of surfaces in E_3, see for example [1], [2]. Let

$$\bar{r}(u,v) = (x(u,v), y(u,v), z(u,v)), (u,v) \in \Omega \subset R^2 \tag{1}$$

be an equation of a surface $\mathcal{P}$ in a cartesian coordinate frame. Let functions $x(u,v)$, $y(u,v)$, $z(u,v)$ be differentiable up to second order. Both the point $(u,v) \in \Omega$ and its image $\bar{r}(u,v)$ on the surface $\mathcal{P}$ will be called regular if the vector product $\bar{r}_u \times \bar{r}_v$ is not equal to zero, where we use the shortened notations

$$\bar{r}_u = \frac{\partial \bar{r}}{\partial u},\ \bar{r}_v = \frac{\partial \bar{r}}{\partial v},\ \bar{r}_{uv} = \frac{\partial^2 \bar{r}}{\partial u \partial v}.$$

In the opposite case we will say that points are singular. We suppose that the surface has only a finite number of the singular points.

The curve $\bar{r}(u,v_0)$ or $\bar{r}(u_0,v)$ on the surface $\mathcal{P}$ will be called the parametric $u-$curve going through a point $\bar{r}(u_0,v_0)$ on $\mathcal{P}$. We are interested in surfaces of lines wich are determined by the tangent vectors of the certain parametric curves (for example of the $v-$curves) at the points of a parametric curve of the other type (an $u-$curve). Their equations are as follows

(2) $\bar{R}(u,t,v_0) = \bar{r}(u,v_0) + t\bar{r}_v(u,v_0)$ *or*
(3) $\bar{R}(v,t,u_0) = \bar{r}(u_0,v) + t\bar{r}_u(u_0,v)$.

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The surface (2) will be called the tangential $v=$ surface along the $u-$curve for $v=v_0$. Analogously we will say that the surface (3) is the tangential $u-$surface along the $v-$curve for $u=u_0$. We introduce some applications of the tangential $v-$surfaces:

a) $\bar{R}(u,t,v_0)$, $t\in<-a,a>\subset R$ is the so-called tangential $v-$belt. It can be used as a grafic information about the surface $\mathcal{P}$ along the $u-$curve for $v=v_0$.

b) The map $(u,v)\mapsto\bar{r}(u,v)$ determines a two-parametric mouvment in E_3. Then the mouvments given by the parametric curves can be called "basic mouvments". Then $\bar{R}(u,t,v_0)$, $t\in<0,1>$, is the surface of $v-$velocities along the $u-$curve $v=v_0$.

c) $\bar{R}(u,t,v_0)$, $t\in<0,a>$ or $t\in<-a,0>$, will be called the tangential $v-$prolongation of the surface $\mathcal{P}$ along the $u-$curve for $v=v_0$. Its technical application is clear from the definition.

Analogously it can be said in the case of $u-$surfaces.
The main goal of the paper is the classification of the parametric nets on surfaces in E_3 based on the tangential $v-$ or $u-$ surfaces.

Classification of the parametric nets on surfaces in E_3

The tangential $v-$ or $u-$ surfaces which we have introduced are line surfaces. These ones can by classify as developable (surfaces of tangents of curves, cone - surface of one-parameter family of lines with a steady vertex, cylindre-surface of one-parameter family of parallel lines) and as undevelopable.

Proposition 1. *The tangential $v-$ surface along an $u-$curve is developable if and only if $\bar{r}_{uv}=\alpha\bar{r}_{\hat{u}}+\beta\bar{r}_v$ at every point of the $u-$curve.*

Proof. Arbitrary line surface $\bar{r}=\bar{a}(u)+t\bar{b}(u)$ is developable if and only if the tangential planes at points of any line are identified, i.e if and only if $\bar{b}_u\cdot(\bar{b}\times\bar{a}_u)=0$. Considering our $v-$ surface in the form (2), i.e. $\bar{a}(u)=\bar{r}(u,v_0)$, $\bar{b}=\bar{r}_v(u,v_0)$, we inmediately get that $\bar{r}_{uv}=\alpha\bar{r}_u+\beta\bar{r}_v$ is the necessary and sufficient condition for the surface $\bar{R}(u,t,v_0)$ to be developable.

Let (2) be the developable tangential $v-$surface, $\bar{r}_{uv}=\alpha\bar{r}_u+\beta\bar{r}_v$. Then its curve

$$\bar{R}^*(u)=\bar{r}(u,v_0)+g(u)\bar{r}_v(u,v_0) \tag{4}$$

is its edge of regression, i.e. the surface $\bar{R}(u,t,v_0)$ is the surface of the tangents of this curve, if and only if $\bar{R}^*_u=k\bar{r}_v(u,v_0), i.e.iff$

$$1+g\alpha=0, g_u+g\beta=k. \tag{5}$$

It immediately gives 1.) If $\alpha=0$ then the tangent $v-$surface is a cylinder surface.

2.) If $\alpha \neq 0$, (then $g = -1/\alpha$), and $k = 0, (i.e. \alpha_u = \beta\alpha)$, then the tangent $v-$surface is a cone surface.

3.) If $\alpha \neq 0, (g = -1/\alpha)$ and $\alpha_u \neq \beta\alpha$ then the tangent $v-$surface is the surface of all tangents of the curve (4). The analogous assertions are right for the tangent $u-$surfaces.

We obtain the following classification of parametric nets of surfaces in E_3:

I. Tangent $v-$ and $u-$ surfaces are developable, i.e. $\bar{r}_{uv} = \alpha\bar{r}_u + \beta\bar{r}_v$.

a) $\alpha = 0,\ \beta = 0, i.e., \bar{r}_{uv} = \bar{0}$. Then $\bar{r}(u,v) = \bar{r}_1(u) + \bar{r}_2(v), i.e.$ the surface $\mathcal{P}$ can be created by the translation of a parametric curve along the other one. Both tangent $v-$and $u-$ surface are cylinder surfaces.

b) $\alpha \neq 0,\ \beta = 0$. The tangential $u-$surfaces along $v-$curves are cylinder surfaces. If $\alpha_u = 0$ then the tangent $v-$surfaces along $u-$curves are cones. If $\alpha_u \neq 0$ then the tangent $v-$surfaces along $u-$curves are surfaces of tangents of curves.

c) $\alpha = 0,\ \beta \neq 0$. Analogously the tangent $v-$surfaces along the $u-$curves are of cylinder types. If $\beta_v = 0$ then the tangent $u-$ surfaces along the $v-$ curves are of cone types. In the case when $\beta_v \neq 0$ the tangent $u-$surfaces are determined by the tangents of a curve.

Remark. The relation $\bar{r}_{uv} = \alpha\bar{r}_u$ is a partial differential equation for the unknown coordinate functions $x(u,v)$, $y(u,v)$, $z(u,v)$. If we limite ourselves on the separable product form of these functions we get the class of surfaces $\mathcal{P}_{kv}$ with the coordinate expression:

$$(*) \qquad \bar{r}(u,v) = (f_1(u)g(v),\ f_2(u)g(v),\ g_3(v)),\ \alpha = g_v/g.$$

These surfaces can be created from a plane curve $(f_1(u), f_2(u))$ by homotheties with the coefficient $g(v)$ and with the center in the origin and by translations $g_3(v)\bar{k}$, where $\bar{k}$ is the unit vector of the third axis.
As an example we introduce the sphere S^2 $\bar{r}(u,v) = (r\cos u\cos v, r\sin u\cos v, r\sin v)$

d) $\bar{r}_{uv} = \alpha\bar{r}_u + \beta\bar{r}_v,\ \alpha \cdot \beta \neq 0$.

If $\alpha_u = \beta\alpha$ or $\alpha_u \neq \beta\alpha$ then the tangent $v-$surface along an $u-$curve is a cone surface or a surface of all tangents of a curve respectively.

If $\beta_v = \beta\alpha$ or $\beta_v \neq \beta\alpha$ then the tangent $u-$surface along an $v-$curve is a cone surface or a surface of all tangent of a curve respectively.

If we limite ourselves on the separable product form of coordinate functions of surfaces satisfying the linear partial diferential equation $\bar{r}_{uv} = \alpha\bar{r}_u + \beta\bar{r}_v$ of second order we get the following example

$$\bar{r}(uv) = \left(c_1 exp\left(-\int \frac{\lambda_1 \beta(u)}{\alpha_1(u)-\lambda_1} du \right) exp\lambda_1 \int \alpha_2\,(v)dv, \right.$$
$$c_2 exp\left(-\int \frac{\lambda_2 \beta(u)}{\alpha_1(u)-\lambda_2} du \right) exp\lambda_2 \int \alpha_2(v)dv,$$
$$\left. c_3 exp\left(-\int \frac{\lambda_3 \beta(u)}{\alpha_1(u)-\lambda_3} du \right) exp\lambda_3 \int \alpha_2 dv \right),$$

where $c_1, c_2, c_3\lambda_1, \lambda_2, \lambda_3$ are arbitrary constants and $\beta(u), \alpha_1(u), \alpha_2(v)$ are arbitrary functions.

Remark 2. Every line surface $\mathcal{P}$ can be parametrized by the equation

$$\bar{r}\,(u,v) = \bar{a}\,(u) + v\bar{b}\,(u), \tag{6}$$

where $\bar{a}\,(u)$ is the so-called determining curve and $\bar{b}\,(u)$ is the vector of the surface line going cross the point $\bar{a}(u)$. This parametric net we will called natural. Let us recall that this parametrization does not to be suitable from the point of the picture of this surface but can be useful from the applicability point of wiew. It is clear that the tangent $v-$surfaces are identified with $\mathcal{P}$. The tangent $u-$surfaces of the surface (6) are presented by the equation

$$\bar{R}\,(v,t,u_0) = \bar{a}\,(u_0) + v\bar{b}(u_0) + t\left(\bar{a}_u\,(u_0) + v\bar{b}_u\,(u_0)\right).$$

They are developable iff the tangent $v-$surfaces are developable, i.e. iff the surface (6) is developable, i.e. iff $\bar{b}_u(\bar{a}_u \times \bar{b}) = 0$. In this case $\bar{r}_{uv} = \bar{b}_u, \bar{r}_u = \bar{a}_u + v\bar{b}_u, \bar{r}_v = \bar{b}$.

There are two cases:

a) $\bar{b} = c(u)\bar{a}_u$, i.e. the surface (6) is the surface of tangents of the curve $\bar{a}(u)$. Then

$$\bar{b}_u = c_u\bar{a}_u + c\bar{a}_{uu} = \frac{1}{v}\left(\bar{a}_u + v(c_u\bar{a}_u + c\bar{a}_{uu})\right) - \frac{1}{cv}\,c\,\bar{a}_u,$$

i.e. $\alpha = \frac{1}{v}, \beta = -\frac{1}{cv}$. Then $\beta_v = \frac{1}{cv^2} \neq \beta\alpha$, i.e. the tangent $u-$surfaces are also surfaces of tangents of curves.

b) $\bar{b} \neq c\,(u)\,\bar{a}_u$. Then $\bar{b}_u = c_1\bar{a}_u + c_2\bar{b} = \frac{c_1}{1+vc_1}\left(\bar{a}_u + v\,(c_1\bar{a}_u + c_2\bar{b})\right) + c_2 \cdot \frac{1}{1+vc_1}\right)\bar{b}$,

i.j. $\alpha = \frac{c_1}{1+vc_1}, \quad \beta = c_2\frac{1}{1+vc_1}$.

We are looking for a such function $g\,(u)$, the curve $\mathcal{R}\,(u) = \bar{a}\,(u) + g\,(u)\bar{b}(u)$ to be the edge of regression of the surface (6). It satisfies

$$\bar{a}_u + g_u\,\bar{b} + g\,\bar{b}_u = c\bar{b},\ i.e.\quad \bar{a}_u + g_u\,\bar{b} + g\,(c_1\bar{a}_u + c_2\bar{b}) = c\bar{b}.$$

It is true iff

$$1 + g\,c_1 = 0 \quad , \quad g_u + g\,c_2 = c.$$

If $c_1 = 0$ then the surface (6) is a cylinder. Then $\beta = c_2$. If $c_2 = 0$, i.e. if $\bar{b}_u = \bar{0}$ then the tangent $u-$surfaces are also cylinders. If $c_2\,(u) \neq 0$ then $\beta_v = c_{2\,v} = 0$ and so the tangent $u-$surfaces are cones.

If $c_1 \neq 0$ and $c = 0$ then the surface (6) is a cone. In this case

$$g = -\frac{1}{c_1}\ ,\ c_2 = \frac{c'_{1u}}{c_1}\ ,\ \beta = \frac{c'_{1u}}{c_1}\,\frac{1}{1+vc_1}\ ,\ \beta_v = -\frac{c'_{1u}}{(1+vc_1)^2}\ ,\ \alpha\beta\frac{c'_{1u}}{(1+vc_1)^2}.$$

We get. If $c'_1 = 0$ then the tangent $u-$surface is a cylinder. If $c'_1 \neq 0$ then the tangent $u-$surface is a surface of tangents of a curve.

If $c_1 \cdot c \neq 0$ then the surface (6) is a surface of tangents of a curve. Then $\alpha\beta \neq \beta_v$ and so the tangent $u-$surfaces are also surfaces of tangents of curves.

II. If $\bar{r}_{uv}, \bar{r}_u, \bar{r}_v$ are linearly independent then both the tangent $v-$ and $u-$ surfaces of the surface $\mathcal{P}$ are undevelopable. If the surface $\mathcal{P}$ is a line surface then it has this property if and only if is undevelopable.

Remark 3. (About graphs of the functions $z = f(x, y)$ of two variable.) Let

$$\bar{r}\,(u, v) = (u, v, f(u, v))$$

be the natural parametrization of the surface given by a function $z = f(x, y)$. In this case $\bar{r}_{uv} = (0, 0, f_{uv}), \bar{r}_u = (1, 0, f_u), \bar{r}_v = (0, 1, f_v)$. Therefore both the tangent $u-$ and $v-$ surfaces along those parametric curves are developable iff $f_{uv} = 0$. In this case $\alpha = 0 = \beta$ and both the tangent $u-$ and $v-$surfaces are cylinders. If $f_{uv} \neq 0$ then the tangent surfaces are undevelopable.

PARAMETRIZATION OF THE SURFACES GIVEN BY AN EQUATION $F(x, y, z) = 0$

Let a surface $\mathcal{P}$ is given by an equation $F(x, y, z) = 0$ and let $\bar{r}\,(u, v) = (x(u, v),\ y\,(u, v),\ z\,(u, v))$ be its parametric representation, i.e. let the equation

$$F\ (x\,(u, v),\ y\,(u, v), z\,(u, v)) = 0 \tag{7}$$

is the identity for $(u, v) \in \Omega$. Derivating the identity (7) with respect to u and v we get

$$F_x\, x_u + F_y y_u + F_z z_u = 0\ ,\ F_x x_v + F_y y_v + F_z z_v = 0. \tag{8}$$

Then the derivative of the first part of (8) with respect to v gives

$$F_{xx}x_u x_v + F_{xy}(x_u y_v + x_v y_u) + F_{xz}(x_u z_v + x_v z_u) + F_{yy}y_u y_v + F_{yz}(y_u z_v + z_u y_v) +$$
$$+ f_{zz} z_u z_v + F_x x_{uv} + F_y y_{uv} + F_z z_{uv} = 0. \tag{9}$$

The equation (8) and (9) can by shortly rewrite in the forms

$$dF\,(\bar{r}_u) = 0\,, dF\,(\bar{r}_v) = 0\,, d^2F\,(\bar{r}_u, \bar{r}_v) + dF\,(\bar{r}_{uv}) = 0.$$

Proposition 2. *Let $\bar{r}(u, v)$ be a parametric representation of the surface $\mathcal{P}$ given by the equation $F(x, y, z) = 0$. Then the tangent $v-$ and $u-$ surfaces are developable iff $d^2F(\bar{r}_u, \bar{r}_v) = 0$, i.e. iff the tangent vector $\bar{r}_u, \bar{r}_v$ at any point of $\mathcal{P}$ vanish the differential d^2F of second order.*

Proof. If $\bar{r}(u, v)$ is a parametric representation of $\mathcal{P}$ then the equation (8) and (9) are satisfied. Let $d^2F(\bar{r}_u, \bar{r}_v) = 0$. Then the relation (9) gives $dF(\bar{r}_{uv}) = 0$, i.e. $\bar{r}_{uv}$ is a tangent vector of the surface $\mathcal{P}$, i.e. $\bar{r}_{uv} = \alpha\bar{r}_u + \beta\bar{r}_v$. Conversaly if $\bar{r}_{uv} = \alpha\bar{r}_u + \beta\bar{r}_v$ then $dF(\bar{r}_{uv}) = 0$ and then the relation (9) completes our proof.

Remark 4. It is easy to see that a parametric representation of the sphere $x^2 + y^2 + z^2 = r^2$ has the tangent $v-$ and $u-$ surfaces which are developable if and only if it is orthogonal.

Proposition 3. *Let a surface $\mathcal{P}$ is given by the equation $F(x, y, z) = 0$. Then there exists a such parametric representation of the surface $\mathcal{P}$ that its tangent $v-$ and $u-$ surfaces are developable if and only if there exist such two vector fields X_1, X_2 which at points of $\mathcal{P}$ satisfy the equations $dF(X_1) = 0$, $dF(X_2) = 0$, $d^2F(X_1, X_2) = 0$ and $[X_1, X_2]$=0, where $[X_1, X_2]$ is the Lie bracket of the vector fields X_1, X_2.*

Proof. The necessary condition is clear. It is well known that if $[X_1, X_2] = 0$ then there exists a such parametric representation $\bar{r} = \bar{r}(u, v)$ that $\bar{r}_u = X_1$, $\bar{r}_v = X_2$, see [3]. If $dF(X_1) = 0$, $dF(X_2) = 0$ then $\bar{r} = \bar{r}(u, v)$ is a parametric representation of $\mathcal{P}$. If $d^2F(X_1, X_2) = 0$ then by Proposition 2 the tangent of $v-$surfaces are developable.

Remark 5. From the graphic point of view the most suitable are the parametric representation the tangent $v-$ and $u-$ surfaces of which are cylinders or cones. So we prefer the parametric representations of the classes Ia, Ib, Ic.

Example. Consider the sphere $x^2 + y^2 + z^2 = r^2$. Look for a parametric representation of the type (*) from the class $Ib, \bar{r}(u, v) = (f_1(u)g(v), f_2(u)g(v), g_3(v))$. Using the relations (8) and (9) it is easy to infer the following conditions for the functions $f_1(u), f_2(u)$, $g(v)$, $g_3(v)$: $f_1^2 + f_2^2 = k^2$, k is constant

$$k^2g^2 + g_3^2 = r^2.$$

For $d = r > 0, f_1 = r\cos u$, $f_2 = r\sin u$, $g = \cos v$, $g_3 = r\sin v$ we get the spherical representation of the consider sphere which is a global parametric representation with two singular isolated points.

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