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THE LATTICE OF VARIETIES OF GRAPHS

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Abstract. In the paper we investigate classes of graphs closed under isomorphic images, subgraph identifications and contractions and we study the lattice of these classes.

0. Introduction

By a graph $\mathcal{G} = (V, E)$ we mean an undirected connected finite graph without loops and multiple edges. We denote the set of all vertices of a graph $\mathcal{G}$ by $V(\mathcal{G})$ and the set of all edges by $E(\mathcal{G})$. An edge $\{u, v\}$ is briefly denoted by uv. We denote the complete n-vertices graph by $\mathcal{K}_n$ and the n-vertices circle (in which every vertex is of degree two) by $\mathcal{C}_n$.

A class of all graphs closed under isomorphic images is called a property of graphs (for example in [1]) or a variety of graphs (in [5]). To put considerations in the right context within set theory, we will assume that the vertex sets of all considered graphs are subsets of a fixed countable infinite set W, and we talk about graphs *over* W.

The set of all varieties of graphs for which vertex sets are subsets of W with set inclusion as the partial ordering is a complete lattice isomorphic to the Boolean lattice $P(W)$ of all subsets of the set W. The atoms of this lattice are the varieties which are generated generated by only one graph. In theory of graphs we are interested in varieties of graphs closed under more closed operators, for example varieties closed under induced subgraphs [11], varieties closed under induced subgraphs and identifications [5], varieties closed under generalized hereditary operators [1], [2], [9], etc.

One of the most important operators in theory of graphs is the operator of contraction (of edges). It produces "smaller" graphs. A natural operator producing "bigger" graphs is the operator of identification in (connected) induced subgraphs. In this paper we pay attention to varieties of graphs closed under identifications and contractions.

A set of all varieties of graphs closed under given closure operators with set inclusion as the partial ordering is a complete lattice ([3], Theorem 5.2, p. 18]). The smallest variety containing a set $\mathbb{K}$ of graphs is denoted by $V(\mathbb{K})$ and we call it the variety generated by $\mathbb{K}$. If $\mathbb{K} = \{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$ we simply denote it by $V(\mathcal{G}_1, \ldots, \mathcal{G}_n)$.

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1. Preliminary results.

Our aim in this paper is to investigate varieties of graphs closed under sub-orgraph identifications and contractions. The following operation of *a subgraph identification* of graphs in a connected induced subgraph generalizes the operation of the union of graphs and was introduced in [5].

Definition 1.1. Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be disjoint graphs. Let $\mathcal{G}'_1 = (V'_1, E'_1)$ and $\mathcal{G}'_2 = (V'_2, E'_2)$ be connected induced subgraphs of $\mathcal{G}_1$, $\mathcal{G}_2$, respectively and let $f : G'_1 \to G'_2$ be an isomorphism. The subgraph identification of $\mathcal{G}_1$ with $\mathcal{G}_2$ under f is the graph $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2 = (V, E)$, where

$$V = V_1 \cup (V_2 - V'_2),$$
$$E = \{uv \mid u, v \in V \quad \text{and} \quad uv \in E_1 \cup E_2 \, \text{or} \, f(u)v \in E_2\}.$$

If graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are not disjoint we may take instead of the graph $\mathcal{G}_2$ a graph $\mathcal{G}_3$ isomorphic with $\mathcal{G}_2$ and disjoint with $\mathcal{G}_1$ (for details see [5]). When no confusion can arise we will simply talk about the subgraph identification under the induced subgraph $\mathcal{G}'_1$ or about gluing in the induced subgraph $\mathcal{G}'_1$.

The fact that $f : G'_1 \to G'_2$ is an isomorphism of the connected induced subgraph $\mathcal{G}'_1 \subseteq \mathcal{G}_1$ onto the connected induced subgraph $\mathcal{G}'_2 \subseteq \mathcal{G}_2$ will be denoted by $f : G_1 \rightarrowtail G_2$.

It is easy to see that $\mathcal{G}_1 \cup^f \mathcal{G}_2 \cong \mathcal{G}_2 \cup^{f^{-1}} \mathcal{G}_1$ and if f is an authomorphism of a graph $\mathcal{G}$ then $\mathcal{G} \cup^f \mathcal{G} = \mathcal{G}$. Clearly, a subgraph identification of connected graphs is again a connected graph.

Lemma 1.1 (see [5]). *Let $\mathcal{G} = (V, E)$ be a connected graph, which is neither a complete graph nor a circle. Then there are two nonadjacent vertices $u, v \in V(G)$ such that $\mathcal{G} - \{u, v\}$ is a connected graph.*

Corollary 1.2. *If $\mathcal{G}$ is a graph which is neither a circle nor a complete graph, then $\mathcal{G}$ contains proper connected induced subgraphs $\mathcal{G}_1$, $\mathcal{G}_2$ such that $\mathcal{G} \cong \mathcal{G}_1 \cup^f \mathcal{G}_2$, where $f : G_1 \rightarrowtail G_2$.*

Definition 1.2 ([4]). We say that a graph $\mathcal{G}_2$ is a contraction of a graph $\mathcal{G}_1$ if there exists a one-to-one correspondence between $V(G_2)$ and the elements of a partition of $V(G_1)$ such that each element of the partition induces a connected subgraph of $\mathcal{G}_1$, and two vertices of $\mathcal{G}_2$ are adjacent if and only if the subgraph induced by the union of the corresponding subsets is connected.

If adjacent vertices $u, v \in V(G_1)$ belong to the same block of the partition of the set $V(G_1)$ we will say that the vertices u, v have been identifying by the contraction.

If a graph $\mathcal{G}_2$ is a contraction of a graph $\mathcal{G}_1$, we write $\mathcal{G}_2 \lhd \mathcal{G}_1$.

Let $\mathbb{K}$ be a family of graphs. Denote

$$\gamma(\mathbb{K}) = \{\mathcal{G}_1 \cup^f \mathcal{G}_2; \mathcal{G}_1, \mathcal{G}_2 \in \mathbb{K}, \quad f : G_1 \rightarrowtail G_2\},$$

$$C(\mathbb{K}) = \{\mathcal{G} : \mathcal{G} \lhd \mathcal{G}' \text{ for some graph } \mathcal{G}' \in \mathbb{K}\},$$

$I(\mathbb{K})$ – the set of all isomorphic images of graphs in $\mathbb{K}$.

Since $G \cup^{id} G = G$ and $\mathcal{G} \lhd \mathcal{G}$ we have
$\mathbb{K} \subseteq \gamma(\mathbb{K}) \subseteq \gamma^2(\mathbb{K}) \subseteq \cdots \subseteq \gamma^n(\mathbb{K}) \subseteq \dots,$
$\mathbb{K} \subseteq C(\mathbb{K}) \subseteq C^2(\mathbb{K}) \subseteq \cdots \subseteq C^n(\mathbb{K}) \subseteq \dots,$
for any set $\mathbb{K}$ of graphs. Note that $O^n(\mathbb{K}) = O^{n-1}(O(\mathbb{K}))$, for each $n > 1$.

Definition 1.3. A set $\mathbb{K}$ of graphs over W is said to be a variety of graphs closed under subgraph identifications and contractions if

$$I(\mathbb{K}) \subseteq \mathbb{K} \quad \& \quad \gamma(\mathbb{K}) \subseteq \mathbb{K} \quad \& \quad C(\mathbb{K}) \subseteq \mathbb{K}.$$

It is obvious that the operators C and γ are closure operators on the system of all sets of graphs over W. Thus, the next statement holds.

Proposition 1.3. *The set of all varieties of graphs over W closed under subgraph identifications and contractions with the set inclusion as the partial ordering is a complete lattice.*

Let $\mathbb{K}$ be a set of graphs. Define the operator σ by
$\sigma(\mathbb{K}) = (C\gamma)(\mathbb{K}) \cup (C\gamma)^2(\mathbb{K}) \cup \cdots = \cup_{n=1}^{\infty}(C\gamma)^n(\mathbb{K}),$
where $(C\gamma)(\mathbb{K}) = C(\gamma(\mathbb{K}))$ and $(C\gamma)^n(\mathbb{K}) = C(\gamma((C\gamma)^{n-1}(\mathbb{K}))$ if $n > 1$.

Theorem 1.4. *For every set $\mathbb{K}$ of graphs*

$$V(\mathbb{K}) = \sigma(\mathbb{K}).$$

Proof. Let $\mathcal{G}_1, \mathcal{G}_2 \in \sigma(\mathbb{K})$ and let $f : G_1 \rightarrowtail G_2$. Then there exist m, n such that $\mathcal{G}_1 \in (C\gamma)^n(\mathbb{K})$ and $\mathcal{G}_2 \in (C\gamma)^m(\mathbb{K})$. We see at once that $n < m$ implies $\mathcal{G}_1, \mathcal{G}_2 \in (C\gamma)^m(\mathbb{K})$ and so $\mathcal{G}_1 \cup^f \mathcal{G}_2 \in (C\gamma)^{m+1}(\mathbb{K})$. Similarly, $\mathcal{G} \in (C\gamma)^n(\mathbb{K})$ and $\mathcal{G}_1 \lhd \mathcal{G}$ yields $\mathcal{G}_1 \in (C\gamma)^{n+1}(\mathbb{K})$. Thus, we have shown that $\sigma(\mathbb{K})$ is a variety of graphs closed under subgraph identification and contraction and it contains the set $\mathbb{K}$. Consequently $V(\mathbb{K}) \subseteq \sigma(\mathbb{K})$. The opposite inclusion is obvious. □

2. The lattice of varieties of graphs

In this section we investigate the lattice of all varieties of graphs closed under identifications and contractions. This lattice is denoted by $\mathcal{L}$.

Clearly, the least element of the lattice $\mathcal{L}$ is the vatiety $V(\mathcal{K}_1)$, where $\mathcal{K}_1$ is a one-vertex graph. We will denote it by $\mathbf{0}$.

Proposition 2.1. *The variety $V(\mathcal{K}_2)$ generated by the two-vertex graph is the variety of all trees. Moreover, it is the only atom of the lattice $\mathcal{L}$.*

Proof. Let $\mathbb{V} \neq \mathbf{0}$ be an element of the lattice $\mathcal{L}$ and let $\mathcal{G} \in \mathbb{V}$, $\mathcal{G} \neq \mathcal{K}_1$. It is easy to see that the graph $\mathcal{K}_2$ is a contraction of $\mathcal{G}$ and so $\mathcal{K}_2 \in \mathbb{V}$, which implies $V(K_2) \subseteq \mathbb{V}$. Using an induction on a number of vertices we see that every tree belongs to $V(K_2)$. On the other hand, no graph $\mathcal{G}$ in $V(K_2)$ contains a circle (a contraction of a tree is again a tree and a subgraph identification of trees is a tree, too).

Lemma 2.2. *The only variety covering the variety of all trees in $\mathcal{L}$ is the variety $V(\mathcal{C}_3)$.*

Proof. Let $\mathbb{V}$ be an element of $\mathcal{L}$ for which $V(\mathcal{K}_2) < \mathbb{V}$. The variety $\mathbb{V}$ contains a graph $\mathcal{G}$ containing a circle $\mathcal{C}_n$. This clearly forces $\mathcal{C}_3 \in \mathbb{V}$. (It is obvious that $\mathcal{C}_3$ is a contraction of the circle $\mathcal{C}_n$ and so $\mathcal{C}_3 \triangleleft \mathcal{G}$.) Therefore $V(\mathcal{C}_3) \subseteq \mathbb{V}$. □

Lemma 2.3. *Let $\mathcal{G}$ be a graph belonging to a variety $\mathbb{V} \geq V(\mathcal{C}_3)$. If the set $E(\mathcal{G})$ contains edges uv and uw but it does not contain the edge vw then the variety $\mathbb{V}$ also contains the graph $\tilde{\mathcal{G}}$ given by*

$$V(\tilde{\mathcal{G}}) = V(\mathcal{G}) \quad \text{and} \quad E(\tilde{\mathcal{G}}) = E(\mathcal{G}) \cup \{vw\}.$$

Proof. By Lemmas 2.1 and 2.2 the variety $\mathbb{V}$ contains the graph $\mathcal{C}_3$, hence it contains the identification $\mathcal{H}$ of two copies of $\mathcal{C}_3$, where $V(\mathcal{H}) = \{u'.v', w', x\}$ and $E(\mathcal{H}) = \{u'v', u'w', u'x, v'w', w'x\}$ (see Fig. 1). Let $f : G \rightarrowtail H$ be given by

$f(u) = u'$, $f(v) = v'$ and $f(w) = x$.

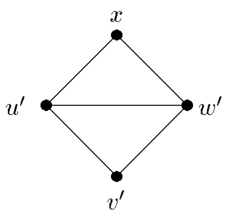

Fig. 1

Now, the graph $\tilde{\mathcal{G}}$ is obtained by contraction of the edge ww' of the graph $\mathcal{G} \cup^f \mathcal{H}$. This yields $\tilde{\mathcal{G}} \in \mathbb{V}$.

Corollary 2.4. *The variety $V(\mathcal{C}_3)$ contains all complete graphs.*

Definition 2.1. We will say that a graph $\tilde{\mathcal{G}}$ is a triangular cover of a graph $\mathcal{G}$ if $\tilde{\mathcal{G}}$ can be obtained from $\mathcal{G}$ by adding edges as in Lemma 2.3.

Theorem 2.5. *If a circle $\mathcal{C}_m$ is a contraction of an identification $\mathcal{G}_1 \cup^f \mathcal{G}_2$ then $\mathcal{C}_m$ is a contraction of a graph $\tilde{\mathcal{G}}_1$ or $\tilde{\mathcal{G}}_2$, where $\tilde{\mathcal{G}}_i$ is a suitable triangular cover of the graph $\mathcal{G}_i$, $i \in \{1, 2\}$.*

Proof. Let $\mathcal{C}_m \triangleleft \mathcal{G}_1 \cup^f \mathcal{G}_2$, and let $\{A_1, A_2, \ldots, A_m\}$ be a partition of the set $V(\mathcal{G}_1 \cup^f \mathcal{G}_2)$ corresponding to the above contraction of $\mathcal{G}_1 \cup^f \mathcal{G}_2$ to $\mathcal{C}_m$.

a) Let there exist a block A_i disjoint with the set $V(\mathcal{G}'_1)$ and let $A_i \subseteq V(\mathcal{G}_1)$. Let $A_j \cap V(\mathcal{G}'_1) \neq \emptyset$, $A_k \cap V(\mathcal{G}'_1) \neq \emptyset$, $1 \leq j \leq i \leq k \leq m$ and $A_q \cap V(\mathcal{G}'_1) = \emptyset$ for each $q \in \{j+1, \ldots, k-1\}$ (recall that the graphs are glued in the subgraph $\mathcal{G}'_1$). For vertices $w_1 \in A_j \cap V(\mathcal{G}'_1)$, $w_2 \in A_k \cap V(\mathcal{G}'_1)$, there is a path

(p_1) $\quad w_1 = v_0, v_1, \ldots, v_r = w_2$

in the subgraph $\mathcal{G}'_1$, therefore $A_p \cap V(\mathcal{G}'_1) \neq \emptyset$ for each $p \in \{1, \ldots, j, k, \ldots, m\}$. There exist also a path

(p_2) $\quad w_2 = u_0, u_1, \ldots, u_s = w_1$

in the graph $\mathcal{G}_1$ disjoint with (p_1), i.e. there exists a circle $\mathcal{C}$ in the graph $\mathcal{G}_1$ for which $V(\mathcal{C}) \cap A_q \neq \emptyset$ for each $q \in \{1, \ldots, m\}$. Denote

$B_1 = A_1 \cap V(\mathcal{G}_1), \quad \ldots, \quad B_m = A_m \cap V(\mathcal{G}_1)$.
If the subgraphs of the graph $\mathcal{G}_1$ induced by the sets $B_1, \ldots, B_m$ are connected then $\mathcal{C}_m \lhd \mathcal{G}_1$.

Let there exist (for example) vertices $x \in B_1$, $y \in B_1 \cap V(\mathcal{C})$ for which there is no path from x to y in the subgraph of $\mathcal{G}_1$ induced by B_1. Denote by
$x = z_0, z_1, \ldots, z_t = y$
a path from x to y in the graph $\mathcal{G}_1$. If the distance of vertices z_p, z_q or z_p, v, $p, q \in \{0, 1, \ldots, t\}$, $v \in V(\mathcal{C})$ is two and these vertices belong to the same block B_l, $l \in \{1, \ldots, m\}$, or belong to adjacent blocks B_l, B_{l+1}, $l \in \{1, \ldots m-1\}$, then we can add the edge $z_p z_q$ and $z_p v$ to a vertex set obtained from $E(\mathcal{G}_1)$ (by Lemma 2.3). After finitely many steps we obtain a graph $\mathcal{G}_1^\star$ such that in the subgraph of $\mathcal{G}_1^\star$ induced by the set B_1 there is a path from x to y. Repeating this proces we can obtain a triangular cover $\tilde{\mathcal{G}}_1$ of the graph $\mathcal{G}_1$ such that the subgraphs of $\tilde{\mathcal{G}}_1$ induced by the sets $B_1, \ldots, B_m$ are connected and so $\mathcal{C}_m \lhd \tilde{\mathcal{G}}_1$.

b) Let $A_q \cap V(\mathcal{G}_1') \neq \emptyset$ for each $q \in \{1, \ldots, m\}$ and let
$v_1 \in A_1 \cap V(\mathcal{G}_1'), \ldots, v_m \in A_m \cap V(\mathcal{G}_1')$.
There exists a path from v_l to v_{l+1} in the graph $\mathcal{G}_1'$ for each $l \in \{1, \ldots, m\}$ (we compute modulo m). Hence there exists a circle $\mathcal{C}$ of the graph $\mathcal{G}_1$ or of the graph induced by the set $V(\mathcal{G}_1') \cup (V(\mathcal{G}_2) - V(\mathcal{G}_2'))$ (the natural copy of the graph $\mathcal{G}_2$) in the graph $\mathcal{G}_1 \cup^f \mathcal{G}_2$ for which $V(\mathcal{C}) \cap A_q \neq \emptyset$ for each $q \in \{1, \ldots, m\}$. Thus, the next part of the proof runs in the same way as the above corresponding part of the proof.

Theorem 2.6. *If a graph $\mathcal{G}$ belongs to $V(\mathcal{C}_m)$ then with each circle $\mathcal{C}$ of $\mathcal{G}$ contains a plane subgraph with the exterior face $\mathcal{C}$ and regions $\mathcal{C}_n$, $3 \leq n \leq m$.*

Proof. Let $\mathcal{G} \in V(\mathcal{C}_m)$.

a) The statement holds if $\mathcal{G} = \mathcal{C}_m$.

b) Let graphs $\mathcal{G}_1, \mathcal{G}_2 \in V(\mathcal{C}_m)$ contain with each circle $\mathcal{C}$ also a plane subgraph with the exterior $\mathcal{C}$ and regions $\mathcal{C}_n$, $3 \leq n \leq m$ and let $\mathcal{C} = v_1 v_2 \ldots v_n v_1$ be a circle of $\mathcal{G}_1 \cup^f \mathcal{G}_2$. If $\mathcal{C}$ is a subgraph of $\mathcal{G}_1$ or $\mathcal{G}_2$ then $\mathcal{G}_1$ or $\mathcal{G}_2$ and so also $\mathcal{G}$ contains a plane subgraph with the exterior $\mathcal{C}$ and regions $\mathcal{C}_n$, $3 \leq n \leq m$. Let $\mathcal{C} = v_1 v_2 \ldots v_n v_1$ be a subgraph neither $\mathcal{G}_1$ nor $\mathcal{G}_2$. Let $v_i, v_{i+1}, \ldots, v_{i+j} \in V(\mathcal{C})$; we will say that $v_i \curvearrowright v_{i+j}$ is a *jump* in $\mathcal{G}$ if $v_i \in V(\mathcal{G}_1) - V(\mathcal{G}_1')$, $v_{i+1}, \ldots, v_{i+j-1} \in V(\mathcal{G}_1')$, $v_{i+j} \in V(\mathcal{G}_2) - V(\mathcal{G}_2')$ or $v_i \in V(\mathcal{G}_2) - V(\mathcal{G}_2')$, $v_{i+1}, \ldots, v_{i+j-1} \in V(\mathcal{G}_1')$, $v_{i+j} \in V(\mathcal{G}_1) - V(\mathcal{G}_1')$ (see Fig. 2).

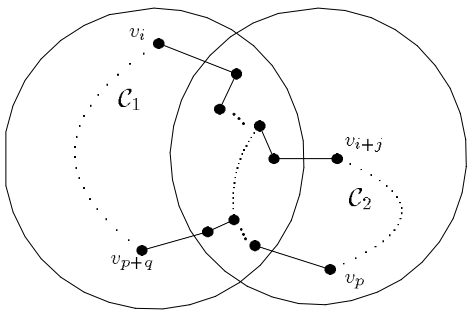

Fig. 2

We proceed by induction on the number of jumps of the circle $\mathcal{C}$. Firstly, we suppose that there are only two jumps in $\mathcal{G}_1 \cup^f \mathcal{G}_2$, $v_i \curvearrowright v_{i+j}$ and $v_p \curvearrowright v_{p+q}$, $i < p$. Since $\mathcal{G}_1 \cup^f \mathcal{G}_2$ is the subgraph identification under a connected subgraph $\mathcal{G}_1'$, there exists a path $v_{i+1}, w_1, w_2, \ldots, w_k, v_{p+q-1}$ in $\mathcal{G}_1'$. If this path is disjoint with the circle $\mathcal{C}$, we get a circle $\mathcal{C}^{(1)}$ of the graph $\mathcal{G}_1$ and a circle $\mathcal{C}^{(2)}$ of the graph $\mathcal{G}_2$ which both contain the path $(v_{i+1}, w_1, w_2, \ldots, w_k, v_{p+q-1})$ or its part (see Fig. 2). By assumptions there exist plane subgraphs with exteriors faces $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ and regions $\mathcal{C}_n$, $3 \le n \le m$. If the mentioned path is not disjoint with the circle $\mathcal{C}$ we get circles $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(k)}$ such that there exist plane subgraphs with exteriors faces $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(k)}$ and regions $\mathcal{C}_n$, $3 \le n \le m$, each of them belongs to either $\mathcal{G}_1$ or $\mathcal{G}_2$ and each of them contains a part of the path $(v_{i+1}, w_1, w_2, \ldots, w_k, v_{p+q-1})$ (see Fig. 3).

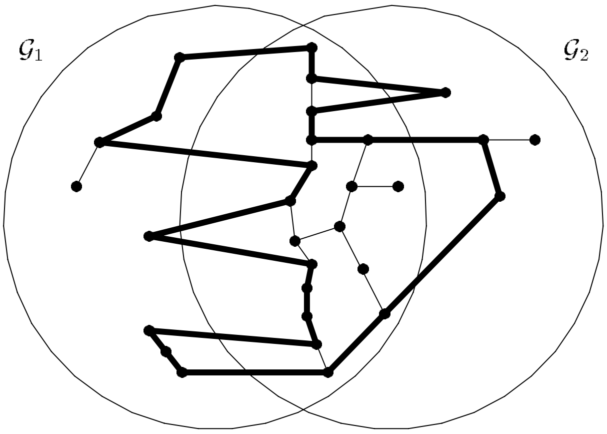

Fig. 3

Since one can get the circle $\mathcal{C}$ by successive gluing the circles $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(k)}$, there exists plane subgraph with exterior face $\mathcal{C}$ and regions $\mathcal{C}_n$, $3 \le n \le m$, too. Assuming the statement for circles with less than $2r$ jumps, we will prove it for $2r$ jumps. Without loss of generality we can assume that $v_i \curvearrowright v_{i+j}$, $v_p \curvearrowright v_{p+q}$ are jumps and that for each jump $v_k \curvearrowright v_{k+s}$ of the circle $\mathcal{C}$, $i \le k \le p$ holds. Analogously as in the case of two jumps we can get circles $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(k)}$ having less than $2r$ jumps. By assumption there exist plane subgraphs with exteriors faces $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(k)}$ and regions $\mathcal{C}_n$, $3 \le n \le m$, therefore there exists plane subgraph with exterior face $\mathcal{C}$ and regions $\mathcal{C}_n$, $3 \le n \le m$, too.

c) Let $\mathcal{G} \triangleleft \mathcal{G}'$, $\mathcal{G}' \in V(\mathcal{C}_m)$ and let $\mathcal{C} = v_1 v_2 \ldots v_n v_1$ be a circle of $\mathcal{G}$. Let $A_1, A_2, \ldots, A_s$ be a partition of $V(G')$ corresponding to the contraction of $\mathcal{G}'$ to $\mathcal{G}$. Without loss of generality we can assume that

$v_1 \in A_1$, …, $v_n \in A_n$.

For any blocks A_i, A_{i+1}, $1 \le i \le n$ (we compute modulo n) there are vertices $w_i \in A_i$ and $w_{i+1} \in A_{i+1}$ for which $w_i w_{i+1} \in E(G')$. The subgraphs induced by sets A_i and A_{i+1} are connected, hence there exists a path from v_i to v_{i+1} in $\mathcal{G}'$. It implies that there exists (in $\mathcal{G}'$) a circle $\mathcal{C}'$ with vertices from A_1, … , A_n having a contraction the circle $\mathcal{C}$. By assumption there is a plane subgraph of $\mathcal{G}'$ with exterior face $\mathcal{C}'$ and regions C_n, $3 \le n \le m$. and by contracting edges we can get from it the plane subgraph with exterior face $\mathcal{C}$ and regions C_n, $3 \le n \le m$.

Corollary 2.7. *The lattice $\mathcal{L}$ contains the infinite chain*
$\mathbf{0} < V(\mathcal{K}_2) < V(\mathcal{C}_3) < V(\mathcal{C}_4) < \cdots < V(\mathcal{C}_n) < \cdots < \mathbf{1}$
where the variety $\mathbf{1}$ is generated by the set of all circles.

Proof. By Theorem 2.6 we have $\mathcal{C}_{n+1} \notin V(\mathcal{C}_n)$ for each $n \ge 3$. It follows from Corollary 1.2 that the variety $\mathbf{1}$ is the greatest element of $\mathcal{L}$.

Theorem 2.8. *The variety $V(\mathcal{C}_4)$ does not cover the variety $V(\mathcal{C}_3)$ and the variety $V(\mathcal{C}_5)$ does not cover the variety $V(\mathcal{C}_4)$*

Proof. Let us denote by $\mathcal{G}_{3-4}$ the graph in Fig. 4

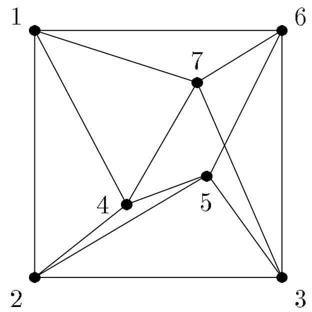

Fig. 4

It is obvious that $V(\mathcal{C}_3) \le V(\mathcal{G}_{3-4}) \le V(\mathcal{C}_4)$. We can check that a plane subgraph of $\mathcal{G}_{3-4}$ with the exterior face $C = (1, 2, 3, 6)$ and regions $\mathcal{C}_3$ does not exist, therefore $\mathcal{G}_{3-4} \notin V(\mathcal{C}_3)$. On the other hand it can be checked that
a) if we add any edge to $E(G_{3-4})$ or
b) make any contraction of the graph $\mathcal{G}_{3-4}$,
we obtain a graph belonging to the variety $V(\mathcal{C}_3)$. Hence $\mathcal{C}_4 \notin V(G_{3-4})$. It implies $V(\mathcal{C}_3) < V(\mathcal{G}_{3-4}) < V(\mathcal{C}_4)$.

We can analogously prove that $V(\mathcal{C}_4) < V(\mathcal{G}_{4-5}) < V(\mathcal{C}_5)$, where $\mathcal{G}_{4-5}$ is the graph in Fig. 5a.

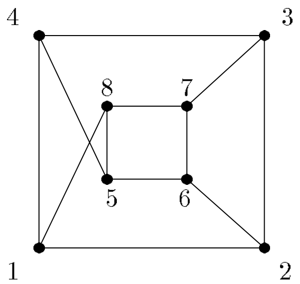

Fig. 5a

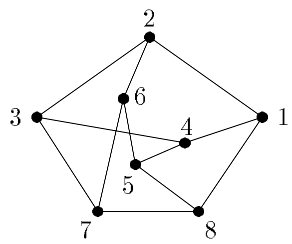

Fig. 5b

Note that the graph depicted in Fig. 5a is depicted in Fig. 5b, too.

The graphs in Fig. 4 and 5 indicate that the structure of the lattice of varieties is not trivial. We will give some problems referring to the lattice $\mathcal{L}$ of varieties.

1. What is the width of the lattice $\mathcal{L}$? (By results of Robertson and Seymour [10], $\mathcal{L}$ does not contain an infinite antichain.)

2. How many varieties cover the variety $V(\mathcal{C}_3)$?

3. What is the length of the interval $[V(\mathcal{C}_3), V(\mathcal{C}_4)]$?

4. Assume $\mathcal{H}$ is the graph in Fig. 6. Is the variety $V(\mathcal{H})$ noncomparable with the variety $V(\mathcal{C}_5)$?

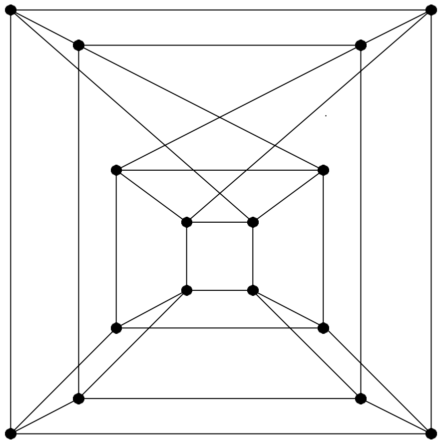

Fig. 6

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