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FUZZY DIVERGENCE MEASURES

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Abstract. A divergence measure is a tool than can be used to measure how two fuzzy sets differ from each other. Particularly it can be used to estimate the fuzziness measure of a fuzzy set. The existing concepts of a divergence measure do not reflect on what membership values the given fuzzy sets are different. We use a divergence measure whose output is not a real number but a fuzzy quantity and show that this quantity is able to distinguish those pairs of fuzzy sets, for which the classical divergence measure gives identical results.

In recent years several attempts to compare pairs of fuzzy sets have been done either measuring their similarity ([2], [9], [11]) or difference between them ([1]). In relation with the latter paper, Montes et. al ([8]) introduced the definition of a divergence measure. This concept generalizes, except for the symmetry property (that could be excluded from the set of axiom in some particular cases) the concept of dissimilarity measures proposed by Bouchon-Meunier et al. in [1].

Assigning a real number as the value of the difference between two fuzzy subsets allows us to define fuzziness measures by comparing a fuzzy subset with its complement, with the closest (in some sense) crisp set or with the equilibrium ([6]). However, this restriction to the set of real numbers can lead to the loss of some important information about this difference, namely it does not distinguish whether differences occur in low or high membership degrees.

For the reader's convenience we introduce some basic definitions.

Definition 1. By a fuzzy subset A of the universe Ω we understand a mapping $A : \Omega \to [0; 1]$.

If no confusion can arise, we speak simply of fuzzy sets rather than fuzzy subsets of a given universe. The set of all fuzzy subsets of Ω will be denoted by $F(\Omega)$. We say that a fuzzy set A is a subset of a fuzzy set B if for these functions the inequality $A \leq B$ holds. In the usual way we understand the α-cuts of sets in $F(\Omega)$, i.e. if $\alpha \in (0; 1]$, then the α-cut of a fuzzy set A is the (crisp) set

$$A_\alpha = \{x \in \Omega; A(x) \geq \alpha\}.$$

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If we assume some kind of topology on Ω, then the zero cut of A is the set

$$A_0 = cl\{x \in \Omega; A(x) \geq 0\},$$

where cl is the closure operator.

The intersection and union of fuzzy sets can be defined using an arbitrary triangular norm. Nevertheless, we will work only with the minimum triangular norm here, that means, by the intersection of fuzzy sets A and B we will understand the fuzzy set $A \cap B = \min\{A, B\}$, and by their union the fuzzy set $A \cup B = \max\{A, B\}$.

In several previous papers ([5], [7], [8]) we have introduced and studied a way to quantify the degree of difference between two fuzzy sets by a real function called a divergence measure, which has as its particular cases the usual distances between fuzzy sets already known and used ([4]).

The measure of difference between two fuzzy sets was defined on the basis of the following natural properties:
1) It should be a nonnegative and symmetric function of two fuzzy sets,
2) it should become zero if the two sets coincide,
3) it should decrease if the two sets become more similar in some sense.

While it is easy to formulate analytically the first and the second condition, the third one depends on the formalization of the similarity concept. A possible approach is based on the fact that if a fuzzy set C is added (in the sense of a union) to both A and B, two sets which are closer to each other are obtained; the same should hold for the intersection.

Definition 2. Let Ω be a universe. A mapping $D : F(\Omega)^2 \to R$ is called a divergence measure if for each $A, B, C \in F(\Omega)$ there is:

$$D(A, B) = D(B, A), D(A, A) = 0$$

and

$$\max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B).$$

It is also possible that a divergence measure is not defined on the whole set $F(\Omega)^2$, but just on some subset of this product.

The nonnegativity of D follows from the second and the third property, where for the fuzzy set C we put the empty set, i.e. $C(x) = 0$ for each $x \in \Omega$.

A natural candidate for the divergence measure in case of a finite universe (or in case we work only with finite fuzzy sets) is the Hamming distance defined by the formula

$$D(A, B) = \sum_{x \in \Omega} |A(x) - B(x)|.$$

If (Ω, μ) is a measurable space, than another example of a divergence measure which works with integrable fuzzy sets is

$$D(A, B) = \int_{\Omega} |A - B| d\mu.$$

A common disadvantage of this approach is that the divergence between fuzzy sets is expressed by a single real number not accounting on which level the difference between given fuzzy sets is realized.

Example 1. Let $\Omega = \{x; y\}$, let A, B and C be the following fuzzy subsets of Ω:

$$A(x) = 0, A(y) = 1, B(x) = \frac{1}{2}, B(y) = 1, C(x) = 0, C(y) = \frac{1}{2}.$$

Using the Hamming distance we obtain

$$D(A, B) = D(A, C) = \frac{1}{2}.$$

The divergence between A and B in the previous example is the same as the one between A and C. On the other hand, the α-cuts of A and B for $\alpha > \frac{1}{2}$ (i.e. the "important" α-cuts) are the same, which is not true for the fuzzy sets A and C. From this point of view we can require that the divergence between A and B should be smaller than between A and C.

We try to introduce a fuzzy quantity that would reflect the above mentioned difference as well as fulfill properties analogous to those from Definition 2.

Let A be a fuzzy set, let $\alpha \in [0; 1]$. By the symbol A^α we will denote the fuzzy set

$$A^\alpha(x) = \begin{cases} A(x) & \text{if } x \in A_\alpha \\ 0 & \text{otherwise,} \end{cases}$$

where A_α is the α-cut of A. Note that $A = A^0$.

We will also need the definition of a pseudoinverse to a non-increasing function defined on the unit interval.

Definition 3. Let f be a nonnegative non-increasing real function defined on the interval $[0; 1]$. Its pseudoinverse is the function $f^{(-1)} : [0; \infty) \to [0; 1]$ for which $f^{(-1)}(x) = \sup\{r; f(r) > x\}$, with the convention $\sup \emptyset = 0$.

The notion of pseudoinverse can be defined in much more general context (see [10]). For our purpose this definition will be sufficient.

Let now D be an arbitrary divergence measure, let A, B be fuzzy sets such that $D(A^\alpha, B^\alpha)$ exists for each $\alpha \in [0; 1]$. For these sets we can construct the function $\varphi_{A,B}$ in the following way:

$$\varphi_{A,B}(\alpha) = \sup\{D(A^\omega, B^\omega); \omega \geq \alpha\}$$

Obviously the function $\varphi_{A,B}$ depends also on the chosen divergence measure D and this should be reflected in the notation. As we work with a fixed divergence measure, to keep the notation simple we allow this small inaccuracy.

Proposition 1. *For any fuzzy sets A and B the function $\varphi_{A,B}$ is non-increasing.*

Proof. If $\alpha < \beta$, then evidently

$$\{D(A^\omega, B^\omega); \omega \geq \alpha\} \supseteq \{D(A^\omega, B^\omega); \omega \geq \beta\},$$

and the least upper bounds of these sets are therefore in the same order. This yields the required inequality $\varphi_{A,B}(\alpha) \geq \varphi_{A,B}(\beta)$. $\square$

So we can apply the operation of a pseudoinverse to this function.

Definition 4. The fuzzy set $\Delta(A,B)=\varphi_{A,B}^{(-1)}$ will be called the fuzzy divergence measure between A and B,

This function can be considered as a fuzzy quantity $\overline{a}$ corresponding to the linguistic construction "a number not much greater than a". Applying the transformation $\tau(\Delta(A,B))=1-\Delta(A,B)$ we obtain exactly the well-known statistical representation of a positive fuzzy number (see e.g. [3]).

In the following we will show that Δ has similar properties to those of a (crisp) divergence measure from Definition 2, Therefore it can be considered as its generalization. In the following we suppose that all the fuzzy sets we work with admit their mutual fuzzy divergence measure.

Proposition 2. *For all* $A,B\in F(\Omega)$ *there is* $\Delta(A,B)=\Delta(B,A)$.

This statement is a direct consequence of the symmetry from the definition of a divergence measure. Therefore also $\varphi_{A,B}=\varphi_{B,A}$ holds.

The following property expresses the fact that the fuzzy divergence of two sets that coincide is a fuzzy quantity corresponding to the representation of zero.

Proposition 3. *For any* $A\in F(\Omega)$ *there is* $\Delta(A,A)(0)=1, \Delta(A,A)(x)=0$ *for all* $x>0$.

Proof. As $D(A^\omega,A^\omega)=0$ for all $\omega\in[0;1]$ we have $\varphi_{A,A}(\alpha)=0$ for all $\alpha\in[0;1]$. Then

$$\Delta(A,A)(0)=\varphi_{A,A}^{(-1)}(0)=\sup\{\alpha\in[0;1];\varphi_{A,A}(\alpha)\geq 0\}=1.$$

If $x>0$, then

$$\{\alpha;\varphi_{A,A}(\alpha)\geq x\}=\emptyset,$$

and hence $\Delta(A,A)(x)=0$. □

Proposition 4. *For all* $A,B,C\in F(\Omega)$ *there is*

$$\max\{\Delta(A\cap C,B\cap C),\Delta(A\cup C,B\cup C)\}\subseteqq\Delta(A,B).$$

Proof. As the pseudoinverse for a non-increasing function is an order-preserving operation, it is sufficient to show that there is

$$\varphi_{A\cap C,B\cap C}\leq\varphi_{A,B}\quad\text{and}\quad\varphi_{A\cup C,B\cup C}\leq\varphi_{A,B}.$$

If $\omega\in[0;1]$ and $x\in\Omega$, then clearly $(A\cap B)(x)=\min\{A(x),B(x)\}\geq\omega$ if and only if both $A(x)\geq\omega$ and $B(x)\geq\omega$. Hence $(A\cap B)^\omega=A^\omega\cap B^\omega$.

This means that for any $\alpha\in[0;1]$ we have

$$\varphi_{A\cap C,B\cap C}(\alpha)=\sup\{D((A\cap C)^\omega,(B\cap C)^\omega);\omega\geq\alpha\}=$$
$$=\sup\{D(A^\omega\cap C^\omega,B^\omega\cap C^\omega);\omega\geq\alpha\}.$$

As D is a divergence measure, due to its properties the last term is less or equal to

$$\sup\{D(A^\omega,B^\omega);\omega\geq\alpha\}=\varphi_{A,B}(\alpha).$$

Thus the first required inequality is shown. The other can be proved the same way, using the property $(A\cup B)^\omega=A^\omega\cup B^\omega$. □

Now we will return to Example 1 and show that a fuzzy divergence measure provides us with more information comparing to the crisp one.

Example 2. Let $\Omega = \{x; y\}$, let A, B and C have the same meaning as in Example 1. We will find $\Delta(A, B)$ and $\Delta(A, C)$ with Δ based on the Hamming distance.

It is easy to verify that the functions $\varphi_{A,C}, \varphi_{B,C}$ are the following:

$$\varphi_{A,C}(\alpha) = 1 \quad \text{for all} \quad \alpha \in [0; 1],$$

$$\varphi_{B,C}(\alpha) = \begin{cases} \frac{1}{2} & \text{if } \alpha \in [0; \frac{1}{2}] \\ 0 & \text{if } \alpha \in (\frac{1}{2}; 1]. \end{cases}$$

Then using the pseudoinverses of these functions we have

$$\Delta(A, C)(x) = \varphi_{A,C}^{(-1)}(x) = \begin{cases} 1 & \text{if } x \in [0; 1) \\ 0 & \text{if } x \in [1; \infty), \end{cases}$$

$$\Delta(B, C)(x) = \varphi_{B,C}^{(-1)}(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0; \frac{1}{2}) \\ 0 & \text{if } x \in [\frac{1}{2}; \infty). \end{cases}$$

We see that while the divergence measure based on the Hamming distance is the same (Example 1), their fuzzy divergence measures are different. Moreover, $\Delta(B, C) \leq \Delta(A, C)$, what reflects the fact, that the differences between A and C are on higher membership degree, i.e. in most applications should be considered as more remarkable.

References

[1] Bouchon-Meunier, B., Rifqui, M., Bothorel, S., *Towards general measures of comparisons of objects*, Fuzzy Sets and Systems **84** (1996), 143-153.

[2] Chen, S.M., Yeh, M.S., Hsiao, P.Y., *A comparison of similarity measures of fuzzy values*, Fuzzy Sets and Systems **72** (1995), 79-89.

[3] Klement, E. P., *Integration of fuzzy-valued functions*, Rev. Roumaine Math. Pures Appl. **30** (1985), 375-384.

[4] Klir, G.J., Fogler, T.A., *Fuzzy Sets, Uncertainity and Information*, Prentice Hall International, 1998.

[5] Montes, S., *Partitions and divergence measures in fuzzy models*, Ph.D. Thesis, University of Oviedo (1998).

[6] Montes, S., Couso, I., Gil, P., *Relationships between Uncertainity Measures, Fuzziness Measures and Divergence Measures*, Proceedings of EUSFLAT-ESTYLF Joint Conference, Palma de Mallorca (1999), 227-230.

[7] Montes, S., Gil, P., *Some classes of divergence measures between fuzzy sets and fuzzy partitions*, Mathware and Soft Computing **5** (1999), 253-265.

[8] Montes, S., Gil, P. Bertoluzza, C., *Divergence between fuzzy sets and fuzziness*, Proceedings of IPMU'98 Conference, Paris (1998), 1030-1035.

[9] Pappis, C.P., Karacapilidis, N.I., *A comparative assessment of measures of similarity of fuzzy values*, Fuzzy Sets and Systems **56** (1993), 171-174.

[10] Viceník, P., *A note to a construction of t-norms based on pseudo-inverses of monotone functions*, Fuzzy Sets and Systems **104** (1999), 15-18.

[11] Wang, X., De Baets, B., Kerre, E., *A comparative study of similarity measures*, Fuzzy sets and systems **73** (1995), 259-268.

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