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STRICT ORDER–BETWEENNESSES

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ABSTRACT. In this note necessary and sufficient conditions for a strick ternary relation to be a strict order–betweenness are given. As an application, a characterization of lattices of convex subsets of posets is obtained.

INTRODUCTION

In every poset, we can introduce a ternary relation r called order–betweenness in the following way:

$(a, b, c) \in r$ if and only if $a \leq b \leq c$ or $c \leq b \leq a$.

G. Birkloff proposed (cf. the 2nd edition of his Lattice theory, 1948, Problem 1) to search for axioms for a ternary relation to be an order - betweenness. Such system of axioms was found by M. Altwegg [1], M. Sholander [2] gave an alternative system of axioms.

In certain cases, it is more convenient to handle with strict ternary relations, as sets of triples of different elements. E.g., if we define a ternary relation r' in a poset $\mathbb{A} = (A, \leq)$ to be a strict order - betweenness provided that

$(a, b, c) \in r'$ if and only if $a < b < c$ or $c < b < a$,

the convexity of a subset X of A means that

$(a, b, c) \in r', a, c \in X$ imply $b \in X$.

In this note we give necessary and sufficient conditions for a strict ternary relation to be a strict order–betweenness. As a consequence, a characterization of lattices of convex subsets of posets is obtained. An alternative characterization can be found in [4].

1. STRICT ORDER–BETWEENNESSES

M. Altwegg proved the following theorem (cf. [1]):

1.1. Theorem. *Let M be a nonempty set, ζ a ternary relation in M. Then there exists a partial order $\leq$ in M with*

$(a, b, c) \in \zeta$ *iff* $a \leq b \leq c$ *or* $a \geq b \geq c$

if and only if ζ satisfies:

(Z_1) $(x, x, x) \in \zeta$ *for each* $x \in M$,

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(Z_2) $(x,y,z) \in \zeta$ *implies* $(z,y,x) \in \zeta$,
(Z_3) $(x,y,z) \in \zeta$ *implies* $(x,x,y) \in \zeta$,
(Z_4) $(x,y,x) \in \zeta$ *implies* $x = y$,
(Z_5) $(x,y,z),(y,z,u) \in \zeta, y \neq z$ *imply* $(x,y,u) \in \zeta$,
(Z_6) *if* $x_0, x_1, ..., x_n, x_{n+1} = x_0, x_{n+2} = x_1$ *is a sequence of elements of* M *such that* $(x_{i-1}, x_{i-1}, x_i) \in \zeta$ *and* $(x_{i-1}, x_i, x_{i+1}) \notin \zeta$ *for each* $i \in \{1, ..., n+1\}$ *(with a positive integer* n*), then* n *is odd.*

Let M be a nonempty set, ξ a ternary relation in M. We will refer to ξ as a strict ternary relation, if it contains only triples of different elements.

Consider the following conditions concerning a strict ternary relation ξ:

(S) $(x,y,z) \in \xi$ implies $(z,y,x) \in \xi$,
(T) $(x,y,z) \in \xi, (y,z,u) \in \xi$ imply $(x,y,u) \in \xi$,
(R) $(x,y,z) \in \xi, (x,z,t) \in \xi$ imply $(y,z,t) \in \xi$,
(X) $(x,y,z) \in \xi, (x',y,z') \in \xi$ imply $(x,y,z') \in \xi$ or $(x,y,x') \in \xi$,
(F) $(x,y,z) \in \xi, (y,u,v) \in \xi$ imply $(x,y,u) \in \xi$ or $(z,y,u) \in \xi$,
(C) $(x,y,z) \in \xi, (y,u,v) \in \xi$ imply $(v,y,x) \in \xi$ or $(z,y,u) \in \xi$,
(I) $(x,y,z) \in \xi$ implies $(x,z,y) \notin \xi$,
(O) if $(a_0,y_1,a_1),(a_1,y_2,a_2),...,(a_{n-1},y_n,a_n)$ is a sequence of elements of ξ such that $(a_{i-1},a_i,a_{i+1}) \notin \xi$ for each $i \in \{1,...,n-1\}$ and $(a_0,y_1,a_1) = (a_{n-1},y_n,a_n)$, then n is odd.

1.2. Definition. *By a strict order–betweenness in a set* M *a strict ternary relation* ξ *in* M *satisfying*

$$(a,b,c) \in \xi \text{ iff } a < b < c \text{ or } a > b > c$$

for a partial order $\leq$ *in* M*, will be meant.*

We will prove the following theorem.

1.3. Theorem. *Let* ξ *be a strict ternary relation in a nonempty set* M*. The following conditions are equivalent:*

(1) ξ *is a strict order–betweenness,*
(2) ξ *satisfies* $(S),(T),(X),(F)$ *and* (O),
(3) ξ *satisfies* $(S),(X),(C)$ *and* (O).

First we want to clear the relations between the conditions $(S)-(O)$*. The following statements are easy to prove for any strict ternary relation* ξ*.*

1.4. Lemma. *If* ξ *satisfies (R), then it satisfies (I).*

1.5. Lemma. *If* ξ *satisfies* $(S),(F),(I)$*, then it satisfies also* (R).

1.6. Lemma. *If* ξ *satisfies* $(S),(C)$*, then it satisfies* $(R),(F),(T)$*, too.*

1.7. Lemma. *If* ξ *satisfies* $(S),(T),(F)$*, then it satisfies also* (C).

1.8. Corollary. *The following assertions are equivalent:*

(a) ξ *satisfies* (S) *and* (C),
(b) ξ *satisfies* $(S),(T)$ *and* (F),
(c) ξ *satisfies* $(S),(T),(R),(F),(C),(I)$.

In the following two lemmas we suppose that ξ is a strict ternary relation in a set M satisfying all conditions $(S)-(O)$. Let us define a ternary relation ζ by

$$\zeta = \xi \cup \{(u,u,u) : u \in M\} \cup \{(u,v,w) : u = v \neq w \text{ or } u \neq v = w$$
$$\text{and } u, w \in \{x,y,z\} \text{ for some } (x,y,z) \in \xi\}.$$

The aim is to show that ζ fulfils (Z_6).

1.9. Lemma. *Let $x_0, x_1, ..., x_k \in M$ (k is a prositive integer) and $(x_{i-1}, x_{i-1}, x_i) \in \zeta$ for each $i \in \{1,...,k\}$, $(x_{i-1}, x_i, x_{i+1}) \notin \zeta$ for each $i \in \{1,...,k-1\}$, $x_0 \neq x_1$. Then there exist $(a_0, y_1, a_1), (a_1, y_2, a_2), ..., (a_{k-1} y_k, a_k) \in \xi$ such that $(a_{i-1}, a_i, a_{i+1}) \notin \xi$ for each $i \in \{1,...,k-1\}$, further either $x_0 = a_0, x_1 \in \{y_1, a_1\}$ or $x_0 = y_1, x_1 = a_1$ and simultaneously either $x_{k-1} = a_{k-1}, x_k \in \{y_k, a_k\}$ or $x_{k-1} = y_k, x_k = a_k$ holds.*

Proof. We will proceed by induction on k. If $k = 1$, the assertion is evident. Let us remark that we need here the assumption $x_0 \neq x_1$. If $k > 1$, then the relation $x_0 \neq x_1$ is implied by $(x_0, x_1, x_2) \notin \zeta$. Suppose that the assertion is true for a positive integer k. We will prove it for $k+1$. So let $x_0, x_1, ..., x_k, x_{k+1} \in M$ satisfy the above assumptions and let $(a_0, y_1, a_1), ..., (a_{k-1}, y_k, a_k)$ be a sequence corresponding to $x_0, x_1, ..., x_k$ by induction hypothesis. We have the following possibilities:

1) $x_{k-1} = a_{k-1}, x_k = y_k$,
2) $x_{k-1} = a_{k-1}, x_k = a_k$,
3) $x_{k-1} = y_k, x_k = a_k$.

Further, since $(x_k, x_k, x_{k+1}) \in \zeta$ and $x_k \neq x_{k+1}$, because $(x_{k-1}, x_k, x_{k+1}) \notin \zeta$, there exists $t \in M$ with

I) $(x_k, t, x_{k+1}) \in \xi$ or
II) $(t, x_k, x_{k+1}) \in \xi$ or
III) $(t, x_{k+1}, x_k) \in \xi$.

We proceed combining the cases 1) - 3) with I) - III). Let us suppose that 1) and I) occur. The relations $(x_{k-1}, x_k, a_k) = (a_{k-1}, y_k, a_k) \in \xi, (x_k, t, x_{k+1}) \in \xi$ imply $(a_k, x_k, t) \in \xi$ by (C). Then using (T) we obtain $(a_k, x_k, x_{k+1}) \in \xi$. If we show that $(a_{k-1}, a_k, x_{k+1}) \notin \xi$, then adding (a_k, x_k, x_{k+1}) to the sequence $(a_0, y_1, a_1), ..., (a_{k-1}, y_k, a_k)$ we obtain such a sequence as we need. If it were $(a_{k-1}, a_k, x_{k+1}) \in \xi$, we would have $(x_k, a_k, x_{k+1}) = (y_k, a_k, x_{k+1}) \in \xi$ by (R), contrary to the above proved $(a_k, x_k, x_{k+1}) \in \xi$. Combining 1) with II) or III), we can proceed analogously. It is easy to verify that if I) and any of the cases 2), 3) occur, we can add (x_k, t, x_{k+1}), while if III) and simultaneously 2) or 3) occur, we can add (x_k, x_{k+1}, t) to the sequence $(a_0, y_1, a_1), ..., (a_{k-1}, y_k, a_k)$. Now let us suppose that II) and 2) occur. Since $(x_{k-1}, x_k, x_{k+1}) \notin \xi$, we have also $(y_k, x_k, x_{k+1}) \notin \xi$, by (T). Using (C) we obtain $(a_{k-1}, x_k, t) \in \xi$. Now consider the sequence $(a_0, y_1, a_1), ..., (a_{k-2}, y_{k-1}, a_{k-1})$,
$(a_{k-1}, x_k, t), (t, x_k, x_{k+1})$ of elements of ξ. We will show that $(a_{k-2}, a_{k-1}, t) \notin \xi$, $(a_{k-1}, t, x_{k+1}) \notin \xi$. The first relation follows from $(a_{k-2}, a_{k-1}, a_k) \notin \xi$ by (R). If it were $(a_{k-1}, t, x_{k+1}) \in \xi$, we would have $(a_{k-1}, t, x_k) \in \xi$ by (R), contrary to $(a_{k-1}, x_k, t) \in \xi$. Let us notice that in this case we change the last member of the sequence $(a_0, y_1, a_1), ...,$
(a_{k-1}, y_k, a_k) and add a new one. If $k = 2$, we have $(x_0, x_1, t), (t, x_1, x_2)$, as we need. The remaining case, if II) and 3) occur, can be analysed analogously.

1.10 Lemma. The relation ζ fulfils (Z_6).

Proof. Let $x_0, x_1, ..., x_n,\ x_{n+1} = x_0,\ x_{n+2} = x_1$ be a sequence of elements of M such that $(x_{i-1},\ x_{i-1},\ x_i) \in \zeta$ and $(x_{i-1},\ x_i,\ x_{i+1}) \notin \zeta$ for each $i \in \{1, ..., n\}$. The previous lemma ensures the existence of a sequence $(a_0, y_1, a_1), ..., (a_n, y_{n+1}, a_{n+1})$ of elements of ξ such that $(a_{i-1}, a_i, a_{i+1}) \notin \xi$ for each $i \in \{1, ..., n\}$ and one of the following conditions $a), b), c)$ is satisfied and simultancously one of the possibilities $\alpha), \beta), \gamma)$, occurs:

$a)$ $x_0 = a_0,\ x_1 = y_1,$
$b)$ $x_0 = a_0,\ x_1 = a_1,$
$c)$ $x_0 = y_1,\ x_1 = a_1,$

$\alpha)$ $x_n = a_n,\ x_0 = y_{n+1},$
$\beta)$ $x_n = a_n,\ x_0 = a_{n+1},$
$\gamma)$ $x_n = y_{n+1},\ x_0 = a_{n+1}.$

In each of the cases $a)$ and $\beta)$, $a)$ and $\gamma)$, $b)$ and $\beta)$, $b)$ and $\gamma)$, we take the sequence $(a_0, y_1, a_1), ..., (a_n, y_{n+1}, a_{n+1}), (a_{n+1}, y_1, a_1) = (a_0, y_1, a_1)$. In the cases $a)$ and $\alpha)$, $b)$ and $\alpha)$ we take the sequence $(a_0, y_1, a_1), ..., (a_n, y_{n+1}, a_{n+1}), (a_{n+1}, a_0, a_1)$, $(a_1, y_1, a_0), (a_0, y_1, a_1)$. If $c)$ and $\beta)$ or $c)$ and $\gamma)$ occur, we take $(a_0, y_1, a_1), ...,$ (a_{n-1}, y_n, a_n), $(a_n, a_{n+1}, a_0), (a_0, y_1, a_1)$. Finally, if $c)$ and $\alpha)$ occur, we take the sequence $(a_0, y_1, a_1), ..., (a_{n-1}, y_n, a_n), (a_n, y_{n+1}, a_0), (a_0, y_1, a_1)$. Each of these sequences satisfies the assumptions of the condition (O). We will show it, e.g., in the last case. We have $(a_n, y_{n+1}, a_{n+1}) \in \xi, (a_0, y_{n+1}, a_1) = (a_0, y_1, a_1) \in \xi$, so that $(a_n, y_{n+1}, a_0) \in \xi$ by (X), because $(a_n, y_{n+1}, a_1) = (x_n, x_0, x_1) = (x_n, x_{n+1}, x_{n+2}) \notin \xi$. Further we will show $(a_{n-1}, a_n, a_0) \notin \xi, (a_n, a_0, a_1) \notin \xi$. If it were $(a_n, a_0, a_1) \in \xi$, we would have $(y_{n+1}, a_0, a_1) \in \xi$ by (R), which contradicts $(a_0, y_{n+1}, a_1) = (a_0, y_1, a_1) \in \xi$. Let us suppose that $(a_{n-1}, a_n, a_0) \in \xi$. Using $(a_n, y_{n+1}, a_{n+1}) \in \xi$ we obtain (a_0, a_n, y_{n+1}) by (C), because $(a_{n+1}, a_n, a_{n-1}) \notin \xi$. But this is a contradiction, as we have proved $(a_n, y_{n+1}, a_0) \in \xi$. Now using (O) we conclude that n is odd.

Proof of theorem 1.3. It is easy to see that (1) implies (2). Further, (2) implies (3) by 1.8. Now let ξ satisfy $(S), (X), (C)$ and (O). Then ξ satisfies all conditions $(S) - (O)$, again by 1.8. Let ζ be defined as before 1.9. It is easy to verify that ζ satisfies $(Z_1) - (Z_5)$. By lemma 1.10 it satisfies (Z_6), too. Theorem 1.1 ensures the existence of a partial order $\leq$ in M such that $(a, b, c) \in \zeta$ if and only if either $a \leq b \leq c$ or $a \geq b \geq c$ holds. Obviously $(a, b, c) \in \xi$ is equivalent to $a < b < c$ or $a > b > c$, so that ξ is a strict order–betweenness.

The following examples show that the system of conditions given in (2) of 1.3 and (3) of 1.3, respectively, is independent. In each of these examples we point out, which of the conditions $(S) - (O)$ are not satisfied.

1.11 Example. Let $M = \{a, b, c\}$, $\xi = \{(a, b, c)\}$. Then ξ doesn't satisfy (S).

1.12 Example. Let $M = \{a, b, c\}$, $\xi = \{(a, b, c), (a, c, b), (c, b, a), (b, c, a)\}$. Then ξ doesn't satisfy $(T), (R), (C), (I)$.

1.13 Example. Let $M = \{a, b, c, d, e\}$, $\xi = \{(b, c, d), (a, c, e), (d, c, b), (e, c, a)\}$. Then ξ doesn't satisfy (X).

1.14 Example. Let $M = \{a, b, c, d, e\}$, $\xi = \{(a, b, c), (b, d, e), (c, b, a), (e, d, b)\}$. Then ξ doesn't satisfy (F) and (C).

1.15 Example. Let $M = \{a, b, c, d, e, f\}$, $\xi = \{(a, b, c), (c, d, e), (e, f, a), (c, b, a), (e, d, c), (a, f, e)\}$. Then ξ doesn't satisfy (O).

2. Characterization of Conv $\mathbb{A}$

Theorem 1.3 enables us to give a characterization of lattices of convex subsets of partially ordered sets. For a partially ordered set $\mathbb{A} = (A, \leq)$ let Conv $\mathbb{A}$ denote the system of all convex subsets of A. It is easy to see that (Conv $\mathbb{A}, \subseteq$) is a complete atomistic lattice (atomisticity means that every element is a join of atoms).

2.1 Theorem. *Let $\mathbb{L} = (L, \wedge, \vee, \leq)$ be a complete atomistic lattice, card $L > 1$. Further, let M be the set of all atoms of $\mathbb{L}$, ξ the ternary relation in M defined by*

$$(a, b, c) \in \xi \iff b < a \vee c,\ b \neq a,\ b \neq c.$$

The following conditions are equivalent:

(I) $\mathbb{L}$ is isomorphic to Conv $\mathbb{A}$ for a partially ordered set $\mathbb{A}$;
(II) ξ satisfies $(T), (X), (F), (O)$ and

(K) $a \leq \sup X,\ X \subseteq M,\ a \in M - X$ imply $(x_1, a, x_2) \in \xi$ for some $x_1, x_2 \in X$;

(III) ξ satisfies $(X), (C), (O)$ and (K).

Proof. Since the relation ξ is evidently symmetric, the conditions $(II), (III)$ are equivalent by 1.8. To prove $(I) \Rightarrow (II)$, let φ be an isomorphism of $\mathbb{L}$ onto Conv $\mathbb{A}$ for a partially ordered set $\mathbb{A} = (A, \leq^*)$. As atoms of the lattice Conv $\mathbb{A}$ are just the one-element subsets of A, the mapping $\varphi' : M \to A$ defined by

$$\varphi'(x) = a \iff \varphi(x) = \{a\}$$

is a bijection of M onto A. Evidently $(x, y, z) \in \xi$ means that either $\varphi'(x) <^* \varphi'(y) <^* \varphi'(z)$ or $\varphi'(z) <^* \varphi'(y) <^* \varphi'(x)$ holds. Consider the partial order $\leq'$ in M defined in such a way that φ' is an isomorphism of $(M, \leq')$ onto $\mathbb{A}$. Then we have $(x, y, z) \in \xi$ if and only if $x <' y <' z$ or $z <' y <' x$ holds, so that ξ is a strict order–betweeenness. Using theorem 1.3 we obtain that ξ satisfics $(T), (X), (F)$ and (O). It remains to show that (K) is satisfied. So let $a \leq \sup X,\ X \subseteq M,\ a \in M - X$. Then $\varphi'(a)$ belongs to the convex hull of $\{\varphi'(x) : x \in X\}$ in $\mathbb{A}$. Consequently there exist $x_1, x_2 \in X$ with $\varphi'(x_1) \leq^* \varphi'(a) \leq^* \varphi'(x_2)$. Since $a \notin X$, we have $x_1 <' a <' x_2$ and hence $(x_1, a, x_2) \in \xi$. We are going to prove $(II) - (I)$. So let ξ satisfy $(T), (X), (F), (O)$ and (K). Since ξ is also symmetric, we have $(a, b, c) \in \xi$ if and only if $a <^* b <^* c$ or $c <^* b <^* a$ for a partial order $\leq^*$ in M, by 1.3. We will show that $\mathbb{L}$ is isomorphic to Conv $(M, \leq^*)$. Notice that a subset X of M is convex if and only if $(x_1, a, x_2) \in \xi,\ x_1, x_2 \in X$ imply $a \in X$. If $a \in L$, let M_a denote the set $\{p \in M : p \leq a\}$. To verify that the set M_a is convex in $(M, \leq^*)$, let $(u, x, v) \in \xi,\ u, v \in M_a$. But then $x < u \vee v \leq a$, hence $x \in M_a$. Obviously $a \leq b$

implies $M_a \subseteq M_b$. Since $a = \sup M_a$, the converse implication holds, too. Finally, let X be any subset of M, convex in $(M, \leq^*)$ and let $a = \sup X$. We will prove $M_a = X$. The inclusion $X \subseteq M_a$ is evident. Let us suppose that there exists an element $p \in M_a - X$. The condition (K) ensures the existence of $x_1, x_2 \in X$ with $(x_1, p, x_2) \in \xi$. In view of the fact that X is convex we have $p \in X$, a contradiction. The proof is complete.

To show that no of the conditions given in (II) and (III), respectively, can be omitted, consider the following examples.

2.2 Example. Let $\mathbb{L}$ be as in Fig. 1. Then evidently $\xi = \{(a, b, c), (c, b, a)\}$ and it satisfies all conditions $(S)-(O)$, but it doesn't satisfy (K). Namely $a \leq \sup\{b, c, d\}$, while $(b, a, c), (b, a, d), (c, a, d) \notin \xi$.

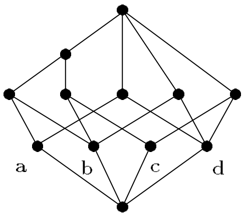

Fig. 1

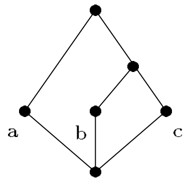

Fig. 2

2.3 Example. Let $\mathbb{L}$ be as in Fig. 2. Then evidently ξ is that of example 1.12. Hence it satisfies $(X), (F), (O)$ and also (K), while $(T), (R), (C)$ and (I) are not satisfied.

2.4 Example. Let $\mathcal{L}$ be the system of all subsets X of the set $\{a, b, c, d, e\}$ satisfying

$$b, d \in X \text{ or } a, e \in X \Rightarrow c \in X.$$

Then $(\mathcal{L}, \subseteq)$ is an atomistic lattice and the relation ξ corresponds to that of example 1.13. So it satisfies (K) and all $(S) - (O)$, besides (X).

2.5 Example. Let $\mathcal{L}$ be the system of all subsets X of $\{a, b, c, u, v\}$ with

$$a, c \in X \Rightarrow b \in X,$$

$$v \in X \text{ and } (b \in X \text{ or } c \in X) \Rightarrow u \in X.$$

Then $(\mathcal{L}, \subseteq)$ is an atomistic lattice, $\xi = \{(\{a\}, \{b\}, \{c\}), (\{c\}, \{b\}, \{a\}), (\{b\}, \{u\}, \{v\}), (\{v\}, \{u\}, \{b\}), (\{c\}, \{u\}, \{v\}), (\{v\}, \{u\}, \{c\})\}$. It can be seen easily that from among the conditions $(S) - (O)$ and (K), just (F) and (C) are not satisfied.

2.6 Example. Let $\mathcal{L}$ be the system of all subsets X of the set $\{a, b, c, d, e, f\}$ satisfying

$$a, c \in X \Rightarrow b \in X,$$

$$c, e \in X \Rightarrow d \in X,$$

$$e, a \in X \Rightarrow f \in X.$$

Then $(\mathcal{L}, \subseteq)$ is an atomistic lattice. The relation ξ corresponds to that of example 1.15, hence it satisfies all conditions $(S)-(I)$ and it doesn't satisfy (O). Evidently (K) holds, too.

Another characterization of lattices of convex subsets of partially ordered sets is given in [4]. We refer to such lattices as c-lattices there. It is also proved that each c-lattice is a direct product of directly irreducible c-lattices and directly irreducible c-lattices are described. The construction of all partially ordered sets $\mathbb{B}$ with Conv $\mathbb{B}$ isomorphic to Conv $\mathbb{A}$ for any given partially ordered set $\mathbb{A}$ can be found in [3].

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