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EXTREMUM CONDITIONS FOR A DEGENERATED CRITICAL POINT

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Abstract. For a degenerated critical point of a function of two variables are given necessary and sufficient conditions for a local extremum.

It is well known that a type of a critical point (point of minimum, maximum or saddle point) for a function f of two variables may be determined using partial derivatives of the second order whenever the hessian is nonzero. The case of the zero hessian is considered as complicated and using of derivatives of higher order is recommended. The present paper shows how to use the derivatives of higher order. It was motivated by paper [1] which contains some necessary condition.

For the simplicity we shall assume that the origin $(0,0)$ is a critical point of the function f and we write

$$a_{ij} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0).$$

We have

$$a_{10} = a_{01} = 0. \tag{1}$$

We assume the zero hessian

$$a_{20}a_{02} - a_{11}^2 = 0. \tag{2}$$

We first consider the case, when one of the derivatives of the second order is nonzero. So, we have

$$a_{20} \neq 0 \text{ or } a_{02} \neq 0.$$

For the simplicity we assume

$$a_{20} > 0 \tag{3}$$

and we are interested in conditions under which the origin $(0,0)$ is a point of local minimum of f (maximum is impossible). Let the function f has continuous partial derivatives to the fourth order in a neighbourhood of $(0,0)$. Put

$$g(x,y) = f(x + p_1 y + \frac{p_2}{2} y^2, y)$$

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Functions f and g have a local minimum at the point $(0,0)$ simultaneously, because the maps

$$\Phi : (x, y) \mapsto (x + p_1 y + \frac{p_2}{2} y^2, y)$$

and

$$\Psi : (x, y) \mapsto (x - p_1 y - \frac{p_2}{2} y^2, y)$$

are mutually inverse homeomorphisms which preserve the origin. We denote

$$\frac{\partial^{i+j} g}{\partial x^i \partial y^j}(0,0) = b_{ij} \text{ for } i + j \leq 4.$$

Then we have

(4) $\quad b_{10} = a_{10} = 0$

(5) $\quad b_{20} = a_{20} > 0$

(6) $\quad b_{01} = a_{01} + a_{10} p_1 = 0$

(7) $\quad b_{02} = a_{02} + 2a_{11} p_1 + a_{20} p_1^2 + a_{10} p_2 = a_{02} + 2a_{11} p_1 + a_{20} p_1^2$

(8) $\quad b_{03} = a_{03} + 3a_{12} p_1 + 3a_{21} p_1^2 + a_{30} p_1^3 + 3a_{11} p_2 + 3a_{20} p_1 p_2$

(9) $\quad b_{04} = a_{04} + 4a_{13} p_1 + 6a_{22} p_1^2 + 4a_{31} p_1^3 + a_{40} p_1^4 + 6a_{12} p_2 +$
$\quad\quad + 12a_{21} p_1 p_2 + 6a_{30} p_1^2 p_2 + 3a_{20} p_2^2$

(10) $\quad b_{11} = a_{11} + a_{20} p_1$

(11) $\quad b_{12} = a_{12} + 2a_{21} p_1 + a_{30} p_1^2 + a_{20} p_2$

Choose p_1 and p_2 such that

(12) $\quad b_{11} = 0$

(13) $\quad b_{12} = 0$

It means

(14) $\quad p_1 = -\frac{a_{11}}{a_{20}}$

(15) $\quad p_2 = \frac{-1}{a_{20}}(a_{12} + 2a_{21} p_1 + a_{30} p_1^2)$

Put $H_3 = b_{03}$ and $H_4 = b_{04}$, where p_1 and p_2 are defined by (14) and (15). Then (8), (9), (10), (11), (14) and (15) imply

(16) $\quad H_3 = a_{03} + 3a_{12} p_1 + 3a_{21} + a_{30} p_1^3$

(17) $\quad H_4 = a_{04} + 4a_{13} p_1 + 6a_{22} p_1^2 + 4a_{31} p_1^3 + a_{40} p_1^4 - 3a_{20} p_2^2$

By (7), (14) and (2) we have

$$b_{02} = a_{02} - 2a_{11} \frac{a_{11}}{a_{20}} + a_{20} \frac{a_{11}^2}{a_{20}^2} = a_{02} - 2\frac{a_{02} a_{20}}{a_{20}} + a_{20} \frac{a_{02} a_{20}}{a_{20}^2} = a_{02} - 2a_{02} + a_{02} = 0 \ .$$

Theorem 1. *Let f be a function of two variables which has continuous partial derivatives to the fourth order in some neighbourhood of the origin which is a degenerated critical point of f. Let*

$$a_{20} > 0 \ .$$

Conditions $H_3 = 0$ and $H_4 \geq 0$ (resp. $H_3 = 0$ and $H_4 > 0$) are necessary (resp. sufficient) for a local minimum of f at the origin.

Proof. Let f has a local minimum at the point $(0,0)$. Define

$$\varphi(y) = g(0,y)$$

or equivalently

$$\varphi(y) = f(p_1 y + \frac{p_2}{2} y^2, y) \ .$$

Then φ has a local minimum at 0. We have

$$\varphi^{(k)}(0) = \frac{\partial^k g}{\partial y^k}(0,0) = b_{0k} \text{ for } k = 1,2,3,4 \ .$$

By (6) and (16)

$$\varphi'(0) = \varphi''(0) = 0 \ .$$

Therefore conditions

$$H_3 = b_{03} = \varphi'''(0) = 0$$

and

$$H_4 = b_{04} = \varphi^{(4)}(0) \geq 0$$

are necessary. Now, we prove sufficiency. By Taylor's formula we have

$$g(x,y) = g(0,0) + \sum_{1 \leq i+j \leq 4} \frac{b_{ij}}{i!j!} x^i y^j + r_4(x,y), \text{ where}$$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{r_4(x,y)}{(x^2+y^2)^2} = 0$$

which implies

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{r_4(x,y)}{x^2+y^4} = 0,$$

The sum $\sum_{1 \leq i+j \leq 4} \frac{b_{ij}}{i!j!} x^i y^j$ does not contain the terms x, y, y^2, xy, y^3 and xy^2 , because the corresponding b_{ij} are zero. The inequality

$$|xy^3| = |y|\sqrt{x^2 y^4} \leq |y| \frac{x^2+y^4}{2} \leq |y|(x^2+y^4)$$

shows that the term xy^3 is negligible with respect to (x^2+y^4). Since $x^2 \le x^2+y^4$, terms x^3, x^2y, x^2y^2, x^3y and x^4 are also negligible with respect to (x^2+y^4). Therefore,

$$g(x,y) = g(0,0) + \frac{b_{20}}{2}x^2 + \frac{b_{04}}{24}y^4 + s(x,y),$$

where

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{s(x,y)}{x^2+y^4} = 0.$$

It proves sufficiency.

Example 1. Put

$$f_1(x,y) = x^2 + 4xy + 4y^2 + 6xy^2 + 9y^4$$
$$f_2(x,y) = x^2 + 4xy + 4y^2 + 6xy^2 + 12y^3 - 6xy^3 - 2y^4$$
$$f_3(x,y) = x^2 + 4xy + 4y^2 + 6xy^2 + 12y^3 + 8y^4$$

In all cases $p_1 = -2,\ p_2 = -6$. We have $H_3 = -72$ for f_1 , $H_3 = 0$ and $H_4 = 24$ for f_2 and $H_3 = 0$ and $H_4 = -24$ for f_3 . So, only the function f_2 has a local minimum at $(0,0)$.

Of course, it may happen that $H_3 = H_4 = 0$.

Example 2. Put

$$f_4(x,y) = x^2 + 4xy + 4y^2 + 12xy^2 + 24y^3 + 2xy^3 + 40y^4 + 13y^5 + y^6$$
$$f_5(x,y) = x^2 + 4xy + 4y^2 + 12xy^2 + 24y^3 + 2xy^3 + 40y^4 + 12y^5 + 2y^6$$
$$f_6(x,y) = x^2 + 4xy + 4y^2 + 12xy^2 + 24y^3 + 2xy^3 + 40y^4 + 12y^5$$

Then $p_1 = -2,\ p_2 = -12$ and $H_3 = H_4 = 0$ in all cases. Put

$$g_i(x,y) = f_i(x - 2y - 6y^2, y) \text{ for } i = 4,5,6\ .$$

Then

$$g_4(x,y) = x^2 + 2xy^3 + y^5 + y^6 = (x+y^3)^2 + y^5$$
$$g_5(x,y) = x^2 + 2xy^3 + 2y^6 = (x+y^3)^2 + y^6$$
$$g_6(x,y) = x^2 + 2xy^3 = (x+y^3)^2 - y^6$$

Now, define

$$h_i(x,y) = g_i(x - y^3, y) \text{ for } i = 4,5,6\ .$$

Then

$$h_4(x,y) = x^2 + y^5$$
$$h_5(x,y) = x^2 + y^6$$
$$h_6(x,y) = x^2 - y^6$$

So, only f_5 has a local minimum at the origin.

If $H_3 = H_4 = 0$, then the previous example indicates that a type of a critical point may be determined by the function h defined by

$$h(x,y) = g(x + \frac{p_3}{6}y^3, y) = f(x + p_1 y + \frac{p_2}{2}y^2 + \frac{p_3}{6}y^3, y).$$

In fact, in this case it is possible to define characteristics H_5 and H_6 (if the function f is six times continuously differentiable in some neighbourhood of the origin) and an analog of Theorem 1 in terms of H_5 and H_6 may be proved. However, we omit the details, because H_5 and H_6 contain 16 and 23 terms respectively.

Example 3. Put

$$f_7(x,y) = \begin{cases} x^2 + e^{-\frac{1}{y^2}} \text{ for } y \neq 0 \\ x^2 \text{ for } y = 0 \end{cases}$$

$$f_8(x,y) = \begin{cases} x^2 + e^{-\frac{1}{y^2}} \text{ for } y > 0 \\ x^2 - e^{-\frac{1}{y^2}} \text{ for } y < 0 \\ x^2 \text{ for } y = 0 \end{cases}$$

Functions f_7 and f_8 have the same partial derivatives (of all orders) at the origin, but only f_7 has a local minimum at the origin. It shows that values of partial derivatives of all orders at a critical point need not determine its type.

Now, assume that

$$a_{02} = a_{20} = 0 = a_{02}a_{20} - a_{11}^2$$

Then also

$$a_{11} = 0.$$

Put

$$P_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3 \ .$$

and

$$P_4(x,y) = a_{40}x^4 + 4a_{31}x^3y + 6a_{22}x^2y^2 + 4a_{13}xy^3 + a_{04}y^4 \ .$$

Theorem 2. *Let f be a function of two variables which has continuous partial derivatives to the fourth order in some neighbourhood of the origin which is a critical point of f. Let*

$$a_{02} = a_{20} = a_{11} = 0 \ .$$

Conditions

$$a_{30} = a_{21} = a_{12} = a_{03} = 0$$

and

$$P_4(x,y) \geq 0 \ \ (\textit{resp. } P_4(x,y) > 0 \ \ \textit{whenever } x^2 + y^2 \neq 0\ )$$

are necessary (resp. sufficient) for a local minimum of f at the origin.

Proof. Let the origin be a point of local minimum of f. For arbitrary reals α and β define

$$\varphi(t) = f(\alpha t, \beta t) \ .$$

Then

$$\varphi'(0) = \varphi''(0) = 0 \ .$$

Since the function φ has a local minimum at 0, we have

$$\varphi'''(0) = a_{30}\alpha^3 + 3a_{21}\alpha^2\beta + 3a_{12}\alpha\beta^2 + a_{03}\beta^3 = P_3(\alpha, \beta) = 0$$

and

$$\varphi^{(4)}(0) = P_4(\alpha, \beta) \geq 0 \ .$$

So, the polynomial P_3 is identically zero. Therefore, all its partial derivatives (of all orders) are identically zero. Particularly, all coefficients of P_3 are zero. It proves necessity. Now, we prove sufficiency. By Taylor's formula we have

$$f(x, y) = f(0, 0) + \frac{1}{4!}P_4(x, y) + r_4(x, y) \ ,$$

where

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{r_4(x, y)}{(x^2 + y^2)^2} = 0.$$

Put

$$c = \min_{x^2+y^2=1} P_4(x, y).$$

Clearly,

$$c > 0$$

and

$$P_4(x, y) \geq c(x^2 + y^2)^2 \text{ whenever } x^2 + y^2 = 1.$$

Since $P_4(x, y)$ and $(x^2+y^2)^2$ are homogeneous polynomials of the same degree, the last inequality holds for all x and y. It means

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{r_4(x, y)}{P_4(x, y)} = 0.$$

Therefore, $r_4(x, y)$ is negligible with respect to $P_4(x, y)$ and f has a local minimum at the origin.

References

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