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GENERIC CHAOS IN METRIC SPACES

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Abstract. A dynamical system given by a continuous map f from a metric space X into itself is called generically ε-chaotic if the set of Li-Yorke pairs, i.e., the set of points $[x,y] \in X^2$ for which $\liminf_{n\to\infty} \varrho(f^n x, f^n y) = 0$ and simultaneously $\limsup_{n\to\infty} \varrho(f^n x, f^n y) > \varepsilon$ is residual in X^2. If $\varepsilon = 0$, f is called generically chaotic. It is shown that the characterization of generically ε-chaotic maps given by Ľ. Snoha in the interval case can be extended to a large class of metric spaces. While on the interval generic chaos implies generic ε-chaos for some $\varepsilon > 0$, in the paper an example of a convex continuum in the plane is given on which generic chaos does not imply generic ε-chaos for any $\varepsilon > 0$.

1. Introduction.

We will study a dynamical system $(X; f)$ given by a metric space (X, ϱ) and a continuous map $f : X \to X$ (in written $f \in C(X)$). Usually when studying chaoticity of such systems the authors assume that X is compact. Instead, we will only assume that X is complete (even less, see below).

The notion of *chaos* in connection with a map was first used by Li and Yorke [LY] without giving any formal definition. Since then many definitions of chaos appeared, most of them being surveyed in [KS]. Each of them reflects some aspects of the dynamics of those systems which are generally considered to be really 'chaotic'.

The notion of *generic chaos* was introduced by A. Lasota (see [P]). A system $(X; f)$ is generically chaotic if the set of so called Li-Yorke pairs of points, i.e., the set of points $[x,y] \in X^2$ for which $\liminf_{n\to\infty} \varrho(f^n x, f^n y) = 0$ and $\limsup_{n\to\infty} \varrho(f^n x, f^n y) > 0$ is residual in X^2 (i.e., its complement is a first category set in X^2).

J. Piórek [P] in 1985 found examples of generically chaotic interval maps, so it became clear that maps satisfying such a strong definition of chaoticity exist.

Ľ. Snoha [S1] in 1990 gave a full characterization of generically chaotic self-maps of a real compact interval I in terms of behaviour of subintervals of I as well as in terms of topological transitivity. He also introduced the notion of *dense chaos* by requiring that the set of Li-Yorke pairs be dense instead of residual. In [S2] he found a full characterization of densely chaotic interval maps and proved that in the class of piecewise monotone maps with finite number of pieces of monotonicity

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the notion of generic chaos and that of dense chaos coincide. Finally, in [S3] he generalized some results from [S1] to what he called two-parameter chaos.

The inspiration for the present paper was a concluding remark of Ľ. Snoha from [S2] saying that "Some results concerning the generic chaos can be carried over to the case of continuous self-maps of the compact metric spaces. For example, if for every two balls B_1 and B_2, $\liminf_{n\to\infty} \operatorname{dist}(f^n(B_1), f^n(B_2)) = 0$ and if there is an $a > 0$ such that for every ball B, $\limsup_{n\to\infty} \operatorname{diam} f^n(B) > a$, then f is generically chaotic." The main aim of the present paper is to develop this idea of Ľ. Snoha and to check to what extent his characterization of generically chaotic maps from [S1][1] and that of generically (α, β)-chaotic maps from [S3] can be carried over from the interval to metric spaces.

Before going further we need to discuss the question which metric spaces will be appropriate for us to work with.

First of all, note that the definition of generic chaos has a good sense only if the space X^2 is of second category (i.e., not of first category) in itself because only then a residual set in the space X^2 can reasonably be considered to form a 'majority' of it (usually, in spaces of first category the residuality is not being defined at all). Still, a residual set in a space of second category need not be dense in the space (e.g., take the space $[0,1] \cup (\mathbb{Q} \cap [2,3])$ with the metric inherited from the real line and the set $[0,1]$). But in the definition of generic chaos the residual set of Li-Yorke pairs should be required to be automatically dense, we believe.

Therefore we will require that X^2 be a Baire space — then X^2 is of second category in itself and any residual set in X^2 is automatically dense. (Recall that a space Y is Baire if every open set in Y is of second category in Y or, equivalently, in itself. This is equivalent with the property that the intersection of any countable collection of open dense sets is dense in Y. Another equivalent definition is that any residual set in Y is dense in Y. See, e.g., [HMcC].)

Of course, it could seem more reasonable to assume something on the space X itself rather than on X^2. First, we should realize that a necessary condition for X^2 to be Baire is that X be Baire. Unfortunately, this is not a sufficient condition — the square of a metric Baire space need not be Baire (see [Kr] or [HMcC]).

The question therefore is what assumptions on X ensure that X^2 be Baire. Here we wish to mention at least that, among others, any one of the following three conditions is sufficient for X^2 to be Baire (see [HMcC, Theorem 2.4, Proposition 1.23, Theorem 5.1]):

(A1) X is a complete metric space.
(A2) X is a G_δ set in a complete metric space.
(A3) X is Baire and separable metric space.

We thus finish our discussion about the assumptions on X: we will assume that *X is a metric space whose square is Baire.* In particular, it is sufficient to assume that X satisfies any one of the above three conditions.

Now let us go to our results. But first recall some definitions. Consider a

[1]He brought the attention of the author of the present paper to a misprint in [S1, Theorem 1.2] — the condition "(h-1) f has a unique ..." should read as "(h-1) f is not constant in any subinterval of I and has a unique ...".

dynamical system $(X;f)$ and $\varepsilon > 0$ and denote

$$C(f) = \left\{ [x,y] \in X^2 : \liminf_{n\to\infty} \varrho\left(f^n x, f^n y\right) = 0 \text{ and } \limsup_{n\to\infty} \varrho\left(f^n x, f^n y\right) > 0 \right\},$$
$$C(f,\varepsilon) = \left\{ [x,y] \in X^2 : \liminf_{n\to\infty} \varrho\left(f^n x, f^n y\right) = 0 \text{ and } \limsup_{n\to\infty} \varrho\left(f^n x, f^n y\right) > \varepsilon \right\} .$$

We say that f is *generically* or *densely chaotic* if the set $C(f)$ is residual or dense in X^2, respectively. Similarly, f is *generically* or *densely ε-chaotic* if the set $C(f,\varepsilon)$ is residual or dense in X^2, respectively.

In [S1] it is among others proved that if $f \in C(I)$ where I is a real compact interval then the following are equivalent:

(a) f is generically chaotic,
(b) for some $\varepsilon > 0$, f is generically ε-chaotic,
(c) for some $\varepsilon > 0$, f is densely ε-chaotic,
(d) the following two conditions are fulfilled simultaneously:
  (d1) for every two intervals J_1, J_2, $\liminf_{n\to\infty} \varrho(f^n(J_1), f^n(J_2)) = 0$,
  (d2) there is $\varepsilon > 0$ such that for every interval J, $\limsup_{n\to\infty} \operatorname{diam} f^n(J) > \varepsilon$.

(Moreover, the equivalences $(b) \Leftrightarrow (c) \Leftrightarrow (d)$ hold with the same ε. Further, any generically chaotic function is densely chaotic but not conversely.)

We show that this result can be extended to metric spaces, though not completely (the implication $(c) \Rightarrow (a)$ in the next theorem does not hold with the same ε, contrary to the interval case).

Theorem A. *Let (X,ϱ) be a metric space whose square X^2 is a Baire space and let $f \in C(X)$. Then the following three conditions are equivalent:*

(a) *for some $\varepsilon > 0$, f is generically ε-chaotic,*
(b) *for some $\varepsilon > 0$, f is densely ε-chaotic,*
(c) *the following two conditions are fulfilled simultaneously:*
  (c1) *for every two balls B_1, B_2, $\liminf_{n\to\infty} \varrho(f^n(B_1), f^n(B_2)) = 0$,*
  (c2) *there exists some $\varepsilon > 0$ such that for every ball B, $\limsup_{n\to\infty} \operatorname{diam} f^n(B) > \varepsilon$.*

Moreover, the implications $(a) \Rightarrow (b) \Rightarrow (c)$ hold with the same ε. The implication $(c) \Rightarrow (a)$ does not hold with the same ε, in general. Nevertheless, one can claim that the condition (c) implies that f is generically ε^*-chaotic for any $\varepsilon^* < \varepsilon/2$.

We also show that, contrary to the interval case, in metric spaces generic chaos does not imply generic ε-chaos. Recall that a metric space is called a continuum if it is compact and connected.

Theorem B. *There is a continuum X in the euclidean plane and a map $f \in C(X)$ such that f is generically chaotic but is not generically ε-chaotic for any $\varepsilon > 0$. The continuum X can even be taken to be convex.*

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2. Proof of Theorem A and a generalization.

Being inspired by [S3] we are going to prove a result which is more general than Theorem A.

For a dynamical system $(X; f)$ and real numbers α, β define the following subsets of the square X^2:

$$C_1(f,\alpha) = \left\{[x,y] \in X^2 : \liminf_{n\to\infty} \varrho(f^n x, f^n y) \leqslant \alpha\right\},$$
$$C_2(f,\beta) = \left\{[x,y] \in X^2 : \limsup_{n\to\infty} \varrho(f^n x, f^n y) > \beta\right\},$$
$$C(f,\alpha,\beta) = C_1(f,\alpha) \cap C_2(f,\beta) \ .$$

Since we speak on chaos, it would be reasonable to consider only $0 \leqslant \alpha \leqslant \beta < \operatorname{diam} X$ (in particular, $C_1(f,\alpha) = \emptyset$ for $\alpha < 0$ and $C_2(f,\beta) = \emptyset$ for $\beta \geqslant \operatorname{diam} X$). Nevertheless, the results will work for any α, β and therefore we will not assume any restrictions on them.

According to [S3] a map $f \in C(X)$ is called *generically* or *densely* (α,β)*-chaotic* if the set $C(f,\alpha,\beta)$ is residual or dense in X^2, respectively.

If $\alpha = 0$ or $\alpha = \beta = 0$ we sometimes omit them. More precisely, instead of generic or dense $(0,\varepsilon)$-chaos we also shortly speak on generic or dense ε-chaos, respectively and instead of generic or dense $(0,0)$-chaos we simply speak on generic or dense chaos, respectively. Thus, this terminology is in accordance with the fact that for above defined sets $C(f)$ and $C(f,\varepsilon)$ we have $C(f) = C(f,0,0)$ and $C(f,\varepsilon) = C(f,0,\varepsilon)$.

The following lemma is a direct analogue of [S3, Lemma 2] and so we give the proof only for completeness.

Lemma 2.1. *Let (X,ϱ) be a metric space whose square X^2 is a Baire space. Let $f \in C(X)$ and $\alpha \in \mathbb{R}$. Then the following three conditions are equivalent:*

(i) *$C_1(f,\alpha)$ is residual in X^2,*
(ii) *$C_1(f,\alpha)$ is dense in X^2,*
(iii) *for every two balls B_1, B_2, $\liminf_{n\to\infty} \varrho(f^n(B_1), f^n(B_2)) \leqslant \alpha$.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. We are going to prove $(iii) \Rightarrow (i)$. So let (iii) be fulfilled. We have $C_1(f,\alpha) = \bigcap_{n=1}^{\infty} L(n, \alpha + \frac{1}{n})$ where

$$L\left(n, \alpha + \frac{1}{n}\right) = \left\{[x,y] \in X^2 : \inf_{k\geqslant n} \varrho(f^k x, f^k y) < \alpha + \frac{1}{n}\right\} .$$

For every n, $L(n, \alpha + \frac{1}{n})$ is obviously an open set in X^2. To show that $C_1(f,\alpha)$ is residual it is thus sufficient to prove that for every n, $L(n, \alpha + \frac{1}{n})$ is dense in X^2. So fix n and balls B_1, B_2. We prove that $L(n, \alpha + \frac{1}{n}) \cap (B_1 \times B_2) \neq \emptyset$. From (iii) it follows that there exists $k \geq n$ with $\varrho(f^k(B_1), f^k(B_2)) < \alpha + \frac{1}{n}$. This implies the existence of points $x \in B_1$, $y \in B_2$ such that $\varrho(f^k x, f^k y) < \alpha + \frac{1}{n}$. Hence $[x,y] \in L(n, \alpha + \frac{1}{n})$ and the proof is complete. □

Next lemma shows that in case of the set $C_2(f,\beta)$ the situation in metric spaces is more complicated than the one on the interval and one can get only a weaker result than that from [S1, Lemma 4.16].

Lemma 2.2. *Let (X, ϱ) be a metric space whose square X^2 is a Baire space. Let $f \in C(X)$ and $\beta \in \mathbb{R}$. Consider the following conditions:*

(i) *$C_2(f, \beta)$ is residual,*
(ii) *$C_2(f, \beta)$ is dense,*
(iii) *for every ball B, $\limsup_{n\to\infty} \operatorname{diam} f^n(B) > \beta$,*
(iv) *$C_2(f, \beta^*)$ is residual for every $\beta^* < \frac{\beta}{2}$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. We are going to prove $(iii) \Rightarrow (iv)$. So let (iii) be fulfilled. Fix $\beta^* < \frac{\beta}{2}$. Put

$$C_2\left(f, n, \frac{\beta}{2}\right) = \left\{[x, y] \in X^2 : \sup_{k \geqslant n} \varrho(f^k x, f^k y) > \frac{\beta}{2}\right\} .$$

Then $\bigcap_{n=1}^{\infty} C_2\left(f, n, \frac{\beta}{2}\right) \subset C_2(f, \beta^*)$. For any n, $C_2\left(f, n, \frac{\beta}{2}\right)$ is open. Therefore to get (iv), it is sufficient to prove that for any n, the set $C_2\left(f, n, \frac{\beta}{2}\right)$ is dense in X^2. To this end, fix n and balls B_1, B_2. We need to show that $C_2\left(f, n, \frac{\beta}{2}\right) \cap (B_1 \times B_2) \neq \emptyset$.

Distinguish two cases.

Case 1. For some r, $f^r(B_1) \subset f^r(B_2)$. Since $\limsup_{i\to\infty} \operatorname{diam} f^i(B_1) > \beta$ we can take $k \geqslant \max\{r, n\}$ with $\operatorname{diam} f^k(B_1) > \beta$. Since $f^k(B_1) \subset f^k(B_2)$ there are $x \in B_1$, $y \in B_2$ with $\varrho(f^k x, f^k y) > \beta$ whence $[x, y] \in C_2(f, n, \beta) \subset C_2(f, n, \frac{\beta}{2})$.

Case 2. For every r, $f^r(B_1) \setminus f^r(B_2) \neq \emptyset$. Now take $k \geqslant n$ with $\operatorname{diam} f^k(B_2) > \beta$ and a point $u \in f^k(B_1) \setminus f^k(B_2)$. Then there is a point $v \in f^k(B_2)$ such that $\varrho(u, v) > \frac{\beta}{2}$, since otherwise for any two points $v_1, v_2 \in f^k(B_2)$ we would have $\varrho(v_1, v_2) \leqslant \varrho(v_1, u) + \varrho(u, v_2) \leqslant \beta$ and hence $\operatorname{diam}(B_2) \leqslant \beta$, a contradiction. Now take f^k-preimages $x \in B_1$ and $y \in B_2$ of u and v, respectively. Then $\varrho(f^k x, f^k y) > \frac{\beta}{2}$ and again $[x, y] \in C_2\left(f, n, \frac{\beta}{2}\right)$. $\square$

From Lemma 2.1 and Lemma 2.2 we get

Theorem 2.3. *Let (X, ϱ) be a metric space whose square X^2 is a Baire space. Let $f \in C(X)$ and α, $\beta \in \mathbb{R}$. Consider the following four conditions:*

(a) *f is generically (α, β)-chaotic,*
(b) *f is densely (α, β)-chaotic,*
(c) *the following two conditions are fulfilled simultaneously:*
   (c1) *for every two balls B_1, B_2, $\liminf_{n\to\infty} \varrho(f^n(B_1), f^n(B_2)) \leqslant \alpha$,*
   (c2) *for every ball B, $\limsup_{n\to\infty} \operatorname{diam} f^n(B) > \beta$,*
(d) *f is generically (α, β^*)-chaotic for every $\beta^* < \frac{\beta}{2}$.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Now we are ready to prove Theorem A.

Proof of Theorem A. By putting $\alpha = 0$ and $\beta = \varepsilon$ in Theorem 2.3, we get Theorem A except for the claim that, in general, the condition (c) from Theorem A does not imply the generic ε-chaoticity of the map f.

To prove this claim fix $\varepsilon > 0$ and $a > 0$ and consider the following seven points in the euclidean plane: $V = [0,0]$, $A = [\frac{\varepsilon}{2} + a, 0]$, $B = [-\frac{\varepsilon}{2} - a, 0]$, $C = [0, -a]$, $C_1 = [0, -\frac{a}{4}]$, $C_2 = [0, -\frac{a}{2}]$, $C_3 = [0, -\frac{3}{4}a]$. Let X be the subspace of the euclidean plane defined as the union of the straight line segments AB and VC (i.e., X has the form of the letter T). Define a map $f : X \to X$ as follows. Let $f(V) = f(C_2) = f(C) = V$, $f(C_1) = A$, $f(C_3) = B$, $f(A) = f(B) = C$ and let f be affine on each of the straight line segments VC_1, C_1C_2, C_2C_3, C_3C, VA, VB. Then f is continuous, $f(AB) = VC$, $f(VC) = AB$ and for every ball G in X there is some n with $f^n(G) \supset AB$. Then $f^{n+1}(G) \supset VC$, $f^{n+2}(G) \supset AB$, etc. Hence the condition (c) from Theorem A is fulfilled.

On the other hand, repeat that $f(VC) = AB$ and $f(AB) = VC$ and notice that

$$L := \max\{\varrho(x,y) : x \in VC,\ y \in AB\} = \sqrt{a^2 + \left(\frac{\varepsilon}{2} + a\right)^2}\ .$$

Thus $\limsup_{n\to\infty} \varrho(f^n x, f^n y) \leqslant L$ whenever $x \in VC$ and $y \in AB$. For sufficiently small a we get $L \leqslant \varepsilon$ and in such a case $(VC \times AB) \cap C_2(f, \varepsilon) = \emptyset$. Consequently, f is not generically ε-chaotic. $\square$

Remark 2.4. Since in the proof of Theorem A we have $\lim_{a\to 0} L = \frac{\varepsilon}{2}$, the constant $\frac{\varepsilon}{2}$ at the very end of Theorem A cannot be replaced by any larger number — such a 'universal' (i.e., depending only on ε and not on the space under consideration) 'constant' larger than $\frac{\varepsilon}{2}$ does not exist. Nevertheless, for a *particular* space X it can happen that the constant $\frac{\varepsilon}{2}$ *can* be replaced by a larger number (and, even, a question is whether there is a space where this does not happen). For instance, in case of our 'letter T' space with fixed a we can replace $\frac{\varepsilon}{2}$ by any number smaller than $\frac{\varepsilon}{2} + a$. Moreover, using the idea from the proof of [S1, Lemma 4.15] one can even prove that this is the case when X is any finite graph. Still, our 'letter T' spaces show that there is no number larger than $\frac{\varepsilon}{2}$ which could serve as the mentioned 'universal' (depending only on ε) 'constant' for the class of all finite graphs.

3. Proof of Theorem B.

By ΔABC we will denote the triangle with vertices A, B, C (here we think of a triangle as a convex subset of the plane).

Recall that a map $f \in C(X)$ is called *exact* if for any ball B in X there exists $n \in \mathbb{N}$ with $f^n(B) = X$. An example of such a map is the standard tent map $\tau(x) = 1 - |2x - 1|$ defined on the unit interval $I = [0,1]$.

A continuous map $F \in C(I^2)$ is called triangular if it is of the form $F(x,y) = (f(x), g(x,y))$. Instead of $g(x,y)$ we also write $g_x(y)$. Here $\{g_x,\ x \in I\}$ is a family of continuous maps from $C(I)$ depending continuously on $x \in I$.

The following lemmas are intuitively obvious but for completeness we give proofs.

Lemma 3.1. *There is a triangular map $F(x,y) = (f(x), g_x(y))$ in $C(I^2)$ such that F is exact, g_0 and g_1 are the identity maps $I \to I$ and the set $I \times \{0\}$ is F-invariant.*

Proof. Put $f = \tau$, $g_0 = g_1 = \mathrm{id}$. For every $x \in [\frac{1}{4}, \frac{1}{2}]$ let g_x be the map such that $g_x(0) = g_x(\frac{2}{3}) = 0$, $g_x(\frac{1}{3}) = g_x(1) = 1$ and g_x is linear on each of the intervals $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$. Further, for $x \in [0, \frac{1}{4}]$ let g_x be the map uniquely determined by the following conditions:

- $g_x(0) = 0$, $g_x(\frac{1}{2}) = \frac{1}{2}$, $g_x(1) = 1$,

- g_x is piecewise linear with three pieces of linearity,
- the slope of g_x in a right neighbourhood of 0 as well as that in a left neighbourhood of 1 is $8x+1$ and the slope of g_x in a neighbourhood of $\frac{1}{2}$ is -3.

Finally, for every $x \in (\frac{1}{2}, 1]$ put $g_x = g_{1-x}$.

Obviously, F is well defined, continuous and the set $I \times \{0\}$ is F-invariant. Notice that for any x and any interval $J \subset I$, $\operatorname{diam} g_x(J) \geqslant \frac{1}{3} \operatorname{diam} J$.

We are going to prove that F is exact. So, take nondegenerate intervals $J_1, J_2 \subset I$. We need to show that there exists N with $F^N(J_1 \times J_2) = I^2$.

First take n with $\tau^n(J_1) = I$ and denote $S_x = \{y \in I : [x, y] \in F^n(J_1 \times J_2)\}$. If we denote $\delta = (\frac{1}{3})^n \operatorname{diam} J_2$ then one can see that for every x and every component s_x of S_x we have $\operatorname{diam} s_x \geqslant \delta > 0$.

Now take k such that for every interval $J \subset I$ whose length is at least δ, $(g_{\frac{2}{3}})^k(J) = I$. This together with the facts that the point $\frac{2}{3}$ is fixed for τ and for all x sufficiently close to $\frac{2}{3}$ we have $g_x = g_{\frac{2}{3}}$, imply that $F^k\left(F^n(J_1 \times J_2)\right) \supset [\frac{2}{3} - \varepsilon, \frac{2}{3} + \varepsilon] \times I$ for some $\varepsilon > 0$.

Finally, take r with $\tau^r([\frac{2}{3} - \varepsilon, \frac{2}{3} + \varepsilon]) = I$. Since all the maps g_x, $x \in I$ are onto, it is sufficient to put $N = n + k + r$. $\square$

Lemma 3.2. *Given a triangle $T = \Delta ABC$, there is an exact map $f \in C(T)$ such that all the points from $AB \cup AC$ are fixed points of f.*

Proof. By Lemma 3.1 there is an exact triangular map $F(\varphi, r) = (\tau(\varphi), g_\varphi(r))$, $\varphi \in I$, $r \in I$ such that $g_0 = g_1 = \mathrm{id}$. Since the set $I \times \{0\}$ is F-invariant, we can think of φ and r as of polar coordinates. In such a way F becomes a continuous map from a disc sector $\{[\varphi, r] : \varphi \in [0,1], r \in [0,1]\}$ into itself. Obviously, F is exact and all the points of the form $[0, r]$ and $[1, r]$, $r \in [0,1]$ are fixed points of F. Using the topological conjugacy via an appropriate homeomorphism from the disc sector onto T we get a map $f \in C(T)$ with all the required properties. $\square$

Now we are ready to prove Theorem B.

Proof of Theorem B. In the plane take the points given in polar coordinates φ, r by $V = [0,0]$ and $A_n = \left[\frac{\pi}{2^n}, \frac{1}{2^{n-1}}\right]$, $n = 1, 2, \ldots$. Consider the set $X = \bigcup_{n=1}^{\infty} VA_n$, i.e. a union of straight line segments, endowed with the metric inherited from the euclidean plane. Obviously, X is a continuum.

Define $f \in C(X)$ as follows. Let $f(V) = V$ and for any n, let $f|_{VA_n}$ be topologically conjugate to the tent map.

Since $f(VA_n) = VA_n$ and the set $VA_n \setminus \{V\}$ is open in X, the fact that $\operatorname{diam}(VA_n) \to 0$ when $n \to \infty$ shows that the condition (c2) from Theorem A is not fulfilled for any $\varepsilon > 0$. Hence f is not generically ε-chaotic for any $\varepsilon > 0$.

We are going to show that f is generically chaotic. To this end denote $X_k = \bigcup_{n=1}^{k} VA_n$, $k \in \mathbb{N}$ and realize that the exactness of the tent map gives the exactness of $f|_{VA_n}$ for every n. This implies that for any ball B in X, $f^r(B) \supset VA_s$ for some r and s. Hence, by Theorem A, for any fixed k the map $f|_{X_k}$ is generically ε_k-chaotic for some $\varepsilon_k > 0$. Therefore the set M_k of points from X_k^2 which are not Li-Yorke pairs, is of first category in X_k^2 and hence of first category in X^2. Since any point from X^2 belongs to X_k^2 for some k, we then get that the set of points from X^2 which are not Li-Yorke pairs is the first category set $\bigcup_{k=1}^{\infty} M_k$. Thus f is generically chaotic.

Now we are going to modify the described example in order that the space be convex.

Denote $T_n = \Delta V A_n A_{n+1}$, $n = 1, 2, \dots$. Then $Y = \bigcup_{n=1}^{\infty} T_n$ is a convex continuum in the plane. By Lemma 3.2, for every n there is an exact map $g_n \in C(T_n)$ such that every point from $VA_n \cup VA_{n+1}$ is a fixed point of g_n. Let g be a self-map of Y defined as follows. For $y \in Y$ put $g(y) = g_k(y)$ where k is such that $y \in T_k$. It is easy to see that g is well defined and continuous. To prove that g is generically chaotic but not generically ε-chaotic for any $\varepsilon > 0$, repeat the above proof that the map f has these properties (just replace VA_n by T_n and X_k by $Y_k = \bigcup_{n=1}^{k} T_n$). $\square$

Added in proof. After submitting the paper the author learned about the recent preprint [HY] which is written in the setting of compact metric spaces and surjective maps and which partially overlaps with the present paper (cf. our Theorem A and the equivalence of (2), (3) and (4) in the 'sensitive' case of Theorem 3.5 from [HY]).

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