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ON RELATIONS SATISFYING SOME HORN FORMULAS

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Abstract. A general approach to relations usually considered on a set is presented. Relations are supposed to satisfy particular Horn formulas. It is proved that this approach is equivalent to the particular framework developed for investigation of compatible relations on algebras. Collections of such relations are algebraic lattices under inclusion, with an ideal being isomorphic with the power set of A. Conditions are presented under which such a lattice consists of all relations which are reflexive on subsets of A. These conditions turn out to be closely connected with lattice properties of the diagonal relation on A.

1 Introduction

It is known that an algebra $\mathcal{A}$ determines particular lattices of $\mathcal{A}$-compatible binary relations, such as congruence and tolerance lattices, not only on $\mathcal{A}$, but also on each subalgebra of $\mathcal{A}$. In [2], a framework for the generation of such lattices was introduced.

In the present paper, another approach to these algebraic lattices is developed, and some new results are proved. Starting with a set A, we consider all binary relations on A which satisfy a set of particular Horn formulas. We prove that relations satisfying these Horn formulas on a set are precisely those which are introduced in [2] for algebras. Conditions which should be satisfied by the Horn formulas, in order that diagonal relations and also some other connected relations belong to the collection are given. Further, there is a Horn formula whose presence provides the existence of a congruence on the lattice of relations, such that its blocks consist of reflexive relations on subsets of A. We prove that properties of the diagonal relation yield some structural properties of the corresponding lattice.

2 Results

Let $\mathcal{L}$ be a first order language with only one relational symbol α which is binary, and with no functional symbols. Let $\mathcal{S}$ be a set of universal formulas of the type $\mathcal{L}$ over a set of variables X, such that each $\varphi \in \mathcal{S}$ is as follows:

$$\varphi \equiv (\forall x_1)\dots(\forall x_k)(F_1\&\dots\&F_n \Longrightarrow G_1\&\dots\&G_m), \tag{1}$$

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where F_i, G_j, $i = 1, \ldots, n, j = 1, \ldots, m$ are atomic formulas (i.e., of the form $\alpha(x_p, x_q)$, $p, q \in \{1, \ldots, k\}$) and the set of variables occurring in $G_1, \ldots, G_m$ is a subset of the set of variables occurring in $F_1, \ldots, F_n$.

If $\mathcal{S}$ is a foregoing set of formulas and A a nonempty set, then denote by $\mathcal{R}_{\mathcal{S}}^A$ the set of all relations ρ on A, such that $(A, \rho) \models \mathcal{S}$.

Let $\Sigma = \{(\sigma_i, \sigma_i') : i \in I\}$ where I is an index set and for each $i \in I$ both σ_i and σ_i' are relations on the same set V_i, satisfying:

(2) $\quad \{x : x\sigma_i' y \text{ or } y\sigma_i' x, \text{ for some } y \in V_i\} \subseteq \{x : x\sigma_i y \text{ or } y\sigma_i x, \text{ for some } y \in V_i\}.$

Denote by R_Σ^A the set of all relations ρ such that for all $i \in I$

(3) $$Hom(\sigma_i, \rho) \subseteq Hom(\sigma_i', \rho),$$

(where Hom (γ, δ) is the set of all relational homomorphisms from a relation γ on C into a relation δ on D; i.e., maps $f : C \to D$ such that from $a\gamma b$ it follows that $f(a)\delta f(b)$).

Theorem 1. *Let A be a set. If R_Σ^A is the collection of relations described as above, where $|\sigma_i| < \aleph_0$, then there is a set of Horn formulas $\mathcal{S}$, such that the set $\mathcal{R}_{\mathcal{S}}^A$ coincides with R_Σ^A. Conversely, if $\mathcal{S}$ is a set of Horn formulas described at the beginning, then there are sets V_i and Σ defined above, such that the collection $\mathcal{R}_{\mathcal{S}}^A$ coincides with R_Σ^A.*

Proof. From the condition $|\sigma_i| < \aleph_0$, it follows that $|\sigma_i'| < \aleph_0$ (by (2)). We can also assume that each V_i is finite. To every ordered pair (σ_i, σ_i') of relations on a set V_i, for $|V_i| < \aleph_0$ there corresponds a Horn formula, as described in the sequel, such that a relation ρ on the set A satisfies that formula if and only if it satisfies (3).

Let h be a bijection between V_i and a set of variables $X = \{x_1, x_2, \ldots, x_k\}$. Further, for each pair $(a_j, b_j) \in \sigma_i$, $j \in J = \{1, \ldots, n\}$ consider atomic formula $F_j = \alpha(h(a_j), h(b_j))$. Similarly, for each pair $(c_l, d_l) \in \sigma_i'$, $l \in K = \{1, \ldots, m\}$, consider atomic formula $G_l = \alpha(h(c_l), h(d_l))$.

Now, let A be a set and $\rho \subseteq A^2$ a relation which satisfies condition (3). If $f : V_i \longrightarrow A$, and $(a_j, b_j) \in \sigma_i$ implies that $(f(a_j), f(b_j)) \in \rho$, then for the same f, from $(a_j, b_j) \in \sigma_i'$ it follows that $(f(a_j), f(b_j)) \in \rho$. The mapping $\mathcal{V} = h^{-1} \circ f$ maps X to A i.e., it is a valuation. Thus, every mapping f corresponds to a valuation. On the other hand, if $\mathcal{V} : X \longrightarrow A$ is a valuation, then $f = h \circ \mathcal{V}$ is the corresponding mapping from V_i to A. From the previous consideration it follows that every f from Hom(σ_i, ρ) also belongs to Hom(σ_i', ρ) if and only if the Horn formula that corresponds to (3) for $\alpha = \rho$, is true in every valuation.

Further, observe a Horn formula φ as in (1), where V is a set of variables appearing in it. Let σ and σ' be relations on V defined by:

$(x, y) \in \sigma$ if and only if there is an atomic formula $F_i = \alpha(x, y)$ in the antecedent of φ and

$(x, y) \in \sigma'$ if and only if there is an atomic formula $G_i = \alpha(x, y)$ in the consequent of φ.

By the consideration as above we conclude that the relations on A satisfying the formula φ and the corresponding inclusion (3) coincide. $\square$

Corollary 1. *The set* $\mathcal{R}^A_\mathcal{S}$ *is an algebraic lattice under inclusion.*

Proof. Consider a trivial algebra $\mathcal{A} = (A, f)$ $(f(x) = x)$ on $A \neq \emptyset$. Then, $\mathcal{R}^A_\mathcal{S}$ contains compatible relations. Since the corresponding (according to Theorem 1) collection of relations R^A_Σ is an algebraic lattice under inclusion by Proposition 2 in [2] (because it coincides with R^A_Σ), $\mathcal{R}^A_\mathcal{S}$ is also an algebraic lattice. □

In the sequel, we consider a set of Horn formulas $\mathcal{S}$ such that the diagonal relation on a given set satisfies each of them.

Let $\mathcal{S}$ be a set of Horn formulas of the type (1), such that

$$(A, \Delta_A) \models \mathcal{S} \tag{4}$$

holds for a nonempty set A (Δ_A, or Δ is the diagonal relation on A).

Next we describe Horn formulas which satisfy (4).

Observe that every formula defined by (1) over a set of variables X is equivalent to the finite conjunction of formulas ϕ being of the form

$$\phi \equiv (\forall x_1, \dots, x_k)(\alpha(x_{i_1}, x_{i_2}) \& \dots \& \alpha(x_{i_{p-1}}, x_{i_p}) \Longrightarrow \alpha(x_1, x_2)), \tag{5}$$

where $\{x_{i_1}, \dots, x_{i_p}\} = X_\phi = \{x_1, \dots, x_k\} \subseteq X$.

Denote by $\mathcal{T}_\phi$ the set of atomic formulas figuring in the antecedent of ϕ:

$\mathcal{T}_\phi = \{\alpha(x_{i_1}, x_{i_2}), \dots, \alpha(x_{i_{p-1}}, x_{i_p})\}$.

Further on, for every $x \in X_\phi$, we define $U_\phi(x)$, as follows:

$y \in U_\phi(x)$ if and only if there are $n \in N$ and $u_0, \dots, u_n \in X_\phi$, such that $\alpha(u_j, u_{j+1}) \in \mathcal{T}_\phi$ or $\alpha(u_{j+1}, u_j) \in \mathcal{T}_\phi$, for $j = 0, ..., n-1$ and $x = u_0$, $y = u_n$.

A part of the following proposition is a consequence of Lemma 2 in [2], but we provide another proof.

Proposition 1. *Let* A *be a set, such that* $|A| > 1$ *and* ϕ *a Horn formula defined by (5) over a set of variables* X*. Then,* $\Delta_B \models \phi$ *for all* $\emptyset \neq B \subseteq A$ *if and only if* $U_\phi(x_1) \cap U_\phi(x_2) \neq \emptyset$.

Proof. Suppose that $U_\phi(x_1) \cap U_\phi(x_2) = \emptyset$. Then, let B be a nonempty subset of A and $a, b \in B$, $a \neq b$. Consider the valuation $\mathcal{V} : X_\phi \mapsto A$, such that $\mathcal{V}(u) = a$ for every $u \in U_\phi(x_1)$, and $\mathcal{V}(v) = b$ for every $v \notin U_\phi(x_1)$. Then obviously Δ_B is not a model for ϕ, since all atomic formulas from $\mathcal{T}_\phi$ in this interpretation are associated to ordered pairs with equal coordinates ((a, a) or (b, b)) belonging to Δ_B, and $\alpha(x_1, x_2)$ is interpreted by $(a, b) \notin \Delta_B$.

Conversely, let $U_\phi(x_1) \cap U_\phi(x_2) \neq \emptyset$. Then the diagonal relation of any nonempty subset B of A is a model of ϕ. Indeed, for any valuation $\mathcal{V} : X_\phi \to A$ which assigns different values $a, b \in B$ to x_1 and x_2, both antecedent and consequent of ϕ are false if α is interpreted by Δ_B, hence ϕ is satisfied. Obviously, ϕ is satisfied also in the case when the same value from B is assigned to x_1 and x_2. Thus, $\Delta_B \models \phi$.

Observe that the empty set trivially satisfies any set of Horn formulas of this type, hence $\emptyset \in \mathcal{R}^A_\mathcal{S}$, for every nonempty set A. □

Proposition 2. *Let A, $\mathcal{S}$ and $\mathcal{R}_{\mathcal{S}}^{A}$ be as above. Then, the principal ideal $\Delta\downarrow$ in $\mathcal{R}_{\mathcal{S}}^{A}$ is isomorphic with the power set of A.*

Proof. By the condition (4), diagonal relations of all subsets of A belong to $\mathcal{R}_{\mathcal{S}}^{A}$. Hence, the mapping $f : \Delta_B \mapsto B$, $B \subseteq A$ (with $\Delta_\emptyset = \emptyset$) is obviously the required isomorphism. □

Proposition 3. *If B is a nonempty subset of A, then (i) B^2 and (ii) $B^2 \cup \Delta$ are relations from $\mathcal{R}_{\mathcal{S}}^{A}$*

Proof. Let $\phi \in \mathcal{S}$, as described by (5):
$\phi \equiv (\forall x_1, \ldots, x_k)(\alpha(x_{i_1}, x_{i_2})\& \ldots \&\alpha(x_{i_{p-1}}, x_{i_p}) \Longrightarrow \alpha(x_1, x_2))$.

(i) If the antecedent of ϕ is satisfied by B^2, then obviously the consequent also holds. Indeed, by the assumption the variables x_1 and x_2 appear also in the antecedent of ϕ. Therefore, B^2 is a model of ϕ.

(ii) Suppose that for any valuation, $B^2 \cup \Delta$ satisfies the antecedent of ϕ, i.e., that the interpretation of $\alpha(x_{i_m}, x_{i_{m+1}})$ is either $(a, a) \in \Delta$, $a \in A$, or $(b, c) \in B^2$, $b, c \in B$. If the interpretation of $\alpha(x_1, x_2)$ is an ordered pair (d, e) from B^2, then $B^2 \cup \Delta \models \phi$. If one of these coordinates, e.g. d, is not an element from B, then, since $\Delta \in \mathcal{R}_{\mathcal{S}}^{A}$, by Proposition 1 it follows that $d = e$. Thus again $B^2 \cup \Delta \models \phi$.

Hence, $B^2 \cup \Delta \in \mathcal{R}_{\mathcal{S}}^{A}$. □

In the sequel, Δ is supposed to belong to $\mathcal{R}_{\mathcal{S}}^{A}$, for every A. We discuss particular cases of such lattices, examples of which are well known.

If a relation $\rho \subseteq A^2$ satisfies the formula

$$\Phi \equiv (\forall x)(\forall y)(\alpha(x, y) \Rightarrow \alpha(x, x)\&\alpha(y, y)), \tag{6}$$

then it is called a **weakly reflexive** relation on A.

Some particular known cases are as follows. Let A be a nonempty set and $Rw\,A$ the set of all weakly reflexive relations on A; $Qw\,A$ the set of all relations on A which are reflexive and transitive on subsets of A (weak quasi-orders on A); $Tw\,A$ the set of all relations on A which are reflexive and symmetric on subsets of A (weak tolerances on A); $Ew\,A$ the set of all relations on A which are symmetric and transitive on subsets of A (weak equivalences on A). Obviously, all these relations satisfy the formula (6).

It is easy to see that all the mentioned sets are algebraic lattices of the form $\mathcal{R}_{\mathcal{S}}^{A}$, for a suitable set of Horn formulas $\mathcal{S}$. Hence, in all these lattices the principal ideal $\Delta\downarrow$ generated by the diagonal relation Δ on A is isomorphic with the power set $\mathcal{P}(A)$ of A. However, these lattices have some additional properties, as follows. The filter $\Delta\uparrow$ (i.e., the interval-sublattice $[\Delta, A^2]$) is the lattice of the corresponding reflexive relations on the whole set A. Each of these is a disjoint union of interval lattices $[\Delta_B, B^2]$, $B \subseteq A$.

Next we give conditions under which $\mathcal{R}_{\mathcal{S}}^{A}$ has the foregoing properties.

Recall that $a \in L$ is said to be **codistributive** if for all $x, y \in L$, $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$. Such element induces a homomorphism n_a of L onto $a\downarrow$, defined by $n_a(x) = x \wedge a$.

Observe that in an algebraic lattice every codistributive element is infinitely codistributive ([5]). In this case, the congruence classes induced by n_a have maximal elements.

In all algebraic lattices listed above, the diagonal relation Δ is an (infinitely) codistributive element. Moreover, maximal elements of the congruence classes induced by n_Δ are squares of subsets of $\mathcal{A}$.

Theorem 2. *Let $\mathcal{S}, A$ and $\mathcal{R}_\mathcal{S}^A$ be as above. The following are equivalent:*

(i) Δ is a codistributive element in $(\mathcal{R}_\mathcal{S}^A, \subseteq)$, and maximal elements of the congruence classes induced by n_Δ are squares of subsets of A;

(ii) $\mathcal{S} \vdash \Phi$, where Φ is given by (6);

(iii) $\mathcal{R}_\mathcal{S}^A$ is a disjoint union of lattices consisting of reflexive relations on subsets of A, which satisfy $\mathcal{S}$.

Proof. $(ii) \Rightarrow (i)$ follows by Proposition 3 in [2].

$(i) \Rightarrow (ii)$ Suppose that there is a relation $\rho \in \mathcal{R}_\mathcal{S}^A$ which does not satisfy (ii), i.e., such that for some $a, b \in A$

$(a, b) \in \rho$ and at most one of two pairs (a, a), (b, b) is in ρ.

Take $(a, b) \in \rho$ and $(a, a) \notin \rho$, and let $B = \{x \in A \mid (x, x) \in \rho\}$.

Now, if Δ is a codistributive element in $\mathcal{R}_\mathcal{S}^A$, then, since $\rho \wedge \Delta = \Delta_B$, it follows that ρ belongs to the same class of the congruence induced by n_Δ as B^2. However, $\rho \not\leq B^2$ and B^2 is not the greatest element of the class.

$(ii) \Rightarrow (iii)$ Suppose that every $\rho \in \mathcal{R}_\mathcal{S}^A$ satisfies the formula Φ, i.e., that

$$(x, y) \in \rho \text{ implies } (x, x) \in \rho \text{ and } (y, y) \in \rho.$$

Then, for $B = \{x \mid (x, x) \in \rho\}$, $\Delta_B \in \mathcal{R}_\mathcal{S}^A$, and $B^2 \in \mathcal{R}_\mathcal{S}^A$. In addition,

$$\Delta_B \leq \rho \leq B^2 \tag{7}$$

i.e., $\mathcal{R}_\mathcal{S}^A = \bigcup([\Delta_B, B^2] \mid B \subseteq A)$.

$(iii) \Rightarrow (i)$ If $\rho, \theta \in \mathcal{R}_\mathcal{S}^A$, then there are $B, C \subseteq A$, such that $\rho \in [\Delta_B, B^2]$, $\theta \in [\Delta_C, C^2]$. Now,

$$\rho \vee \theta \in [\Delta_{B\cup C}, (B \cup C)^2]$$

(since $\Delta_B \vee \Delta_C \leq \rho \vee \theta \leq B^2 \vee C^2 \leq (B \cup C)^2$).

Hence, $(\rho \vee \theta) \wedge \Delta = \Delta_{B\vee C} = \Delta_B \vee \Delta_C = (\rho \wedge \Delta) \vee (\theta \wedge \Delta)$, which proves that Δ is a codistributive element of $(\mathcal{R}_\mathcal{S}^A, \subseteq)$.

By (7), B^2 is the greatest element of the class to which ρ belongs, since $\rho \wedge \Delta = B^2 \wedge \Delta = \Delta_B$. $\square$

From now on, we assume that formula Φ given by (6) (describing the weak reflexivity) is a consequence of formulas in $\mathcal{S}$.

Proposition 4. *If $\rho \in \mathcal{R}_\mathcal{S}^A$, then also $\rho \cup \Delta \in \mathcal{R}_\mathcal{S}^A$.*

Proof. Let $\rho \in \mathcal{R}_\mathcal{S}^A$ and $\rho \cap \Delta = \Delta_B$, $B \subseteq A$. By Theorem 2, $\rho \leq B^2$. We have to prove that $\rho \vee \Delta = \rho \cup \Delta$ in $\mathcal{R}_\mathcal{S}^A$. Observe that $\rho \vee \Delta \leq B^2 \vee \Delta = B^2 \cup \Delta$, by Proposition 3. Now, if $\rho \cup \Delta \notin \mathcal{R}_\mathcal{S}^A$, then there is a formula (5), which is not satisfied by $\rho \cup \Delta$, i.e., there is a valuation $\mathcal{V}$ such that the antecedent is true, while the consequent is false. Let $\mathcal{V}(x_1) = b$, and $\mathcal{V}(x_2) = c$, in this valuation. Now, $b, c \in B$, $b \neq c$ and $(b, c) \notin \rho$. Since $\Delta \in \mathcal{R}_\mathcal{S}^A$, by Proposition 1, we have that $U_\phi(x_1) \cap U_\phi(x_2) \neq \emptyset$. In other words, in this valuation all elements from $U_\phi(x_1)$ and $U_\phi(x_2)$ have values from B. Now, starting with $\mathcal{V}$, we consider another valuation $\mathcal{V}'$ on B, as follows. Values of all variables which in $\mathcal{V}$ are elements from B remain the same while all other variables take the same value (from B). In this valuation ρ does not satisfy the formula, which gives a contradiction.

Hence, $\rho \cup \Delta \in \mathcal{R}_{\mathcal{S}}^{A}$, whenever $\rho \in \mathcal{R}_{\mathcal{S}}^{A}$. □

Next we prove that some properties of Δ enable structural decomposition of the lattice $\mathcal{R}_{\mathcal{S}}^{A}$.

As it is known, an element a of a bounded lattice L is **neutral** if the mappings $x \mapsto x \wedge a$ and $x \mapsto x \vee a$ are homomorphisms on L, and $x \mapsto (x \wedge a, x \vee a)$ is an embedding from L into $a{\downarrow} \times a{\uparrow}$.

Theorem 3. *A lattice identity holds on the lattice $\mathcal{R}_{\mathcal{S}}^{A}$ if and only if it holds on its sublattice $\Delta{\uparrow}$ of all reflexive relations from $\mathcal{R}_{\mathcal{S}}^{A}$.*

Proof. In every lattice $\mathcal{R}_{\mathcal{S}}^{A}$, Δ is a neutral element. This is an easy consequence of Proposition 4. The proof of the Theorem is now straightforward, by the definition of a neutral element, and by the fact that $\Delta{\downarrow} = \mathcal{P}(A)$. □

If $\mathcal{A} = (A, F)$ is an algebra, then $\mathcal{R}_{\mathcal{S}}^{\mathcal{A}}$ is the set of all relations from $\mathcal{R}_{\mathcal{S}}^{A}$ which are compatible with all fundamental operations on $\mathcal{A}$.

The following are almost immediate consequences of the above results.

Corollary 2. *Let $\mathcal{A} = (A, F)$ be an algebra and $\mathcal{S}$ a set of formulas as previously defined. Let also $\mathcal{R}_{\mathcal{S}}^{\mathcal{A}}$ be the set of all compatible relations on $\mathcal{A}$ which satisfy $\mathcal{S}$. Then $\mathcal{R}_{\mathcal{S}}^{\mathcal{A}}$ is an algebraic lattice under inclusion whose ideal $\Delta{\downarrow}$ is isomorphic with the lattice $Sub\mathcal{A}$.* □

Corollary 3. *If $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{R}_{\mathcal{S}}^{\mathcal{A}}$ are as in Corollary 2, then the following are equivalent:*

(i) Δ is a codistributive element of the lattice $\mathcal{R}_{\mathcal{S}}^{\mathcal{A}}$ and the maximal elements of congruence classes induced by n_Δ are squares of subalgebras;

(ii) every $\rho \in \mathcal{R}_{\mathcal{S}}^{\mathcal{A}}$ is weakly reflexive;

(iii) $\mathcal{R}_{\mathcal{S}}^{\mathcal{A}}$ is a disjoint union of lattices consisting of reflexive, compatible relations on subalgebras of $\mathcal{A}$, which satisfy $\mathcal{S}$. □

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