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LINE DIGRAPHS OF COMPLETE BIPARTITE SYMMETRIC DIGRAPHS ARE RATIONAL

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ABSTRACT. A digraph D is divisible by t if its arc set can be partitioned into t subsets, such that the sub–digraphs (called factors) induced by the subsets are all isomorphic. If D has q arcs, then it is t-rational if it is divisible by t or t does not divide q. D is rational if it is t-rational for all $t \geqq 2$. In this note, we show that graphs $L(K^*_{n,n})$ are rational.

1. INTRODUCTION

An isomorphic factorization of a digraph D is a partition of its arc set into subsets such that the sub–digraphs (called factors) induced by the subsets are mutually isomorphic. If there exists an isomorphic factorization D into t factors, we say that D is divisible by t. For given t and a given digraph D having precisely q arcs, an obvious necessary condition for the divisibility of D by t is that t divides q. This is called the divisibility condition for D and t. D is t-rational if D is divisible by t or the divisibility condition for D and t is not satisfied, otherwise D is t-irrational; D is rational if it is t-rational for all $t \geqq 2$, otherwise D is irrational, in which case D is t-irrational for some $t \geqq 2$.

The problem which concerns us is to find values of r and t for which all r-regular digraphs are t-rational. Wormald [6] has shown that for fixed t and r such that $2 \leqq t \leqq r$, almost all r-regular digraphs are not divisible by t, and for fixed $t \geqq 2$ almost all regular tournaments also are not divisible by t. Further, in [6] it was proved that all 1–regular digraphs are rational. For r-regular graphs, some results in the direction were achieved in [1, 2, 3, 4, 5, 6].

The line digraph $L(D)$ of a digraph $D(V, A)$ has the arc set of D as its vertex set, and there is an arc from xy to zw in $L(D)$ if $y = z$. The aim of this paper is to prove the divisibility of digraphs $L(K^*_{n,n})$ by t for any t dividing the number of its arcs.

2. RESULT

Let $K^*_{n,n}$ be a complete bipartite symmetric digraph with partite sets V_1 and V_2, where $|V_1| = |V_2| = n$. Then the line graph digraph $L(K^*_{n,n})$ is a digraph with $2n^2$ vertices, $2n^3$ arcs and regular of degree n. We shall prove the following theorem:

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Theorem. *Let $n > 1$ be any positive integer. Then the line graph $L(K^*_{n,n})$ is rational.*

Proof. Assume $t|2n^3$ for any positive integer $t > 1$. We show that then $L(K^*_{n,n})$ is divisible by t.

Let $K^*_{n,n}$ denote the complete bipartite symmetric digraph with partite sets $V_1 = \{x_1, x_2, \dots, x_n\}$ and $V_2 = \{y_1, y_2, \dots, y_n\}$, and let

$$\begin{aligned}\alpha_1 =&(x_1y_1x_2y_2\dots x_ny_n)(x_1y_2x_2y_3\dots x_ny_1)\dots(x_1y_nx_2y_1\dots x_ny_{n-1})\\ &(y_1x_1y_2x_2\dots y_nx_n)(y_2x_1y_3x_2\dots y_1x_n)\dots(y_nx_1y_1x_2\dots y_{n-1}x_n) =\\ =&\varepsilon_1\varepsilon_2\dots\varepsilon_n\gamma_1\gamma_2\dots\gamma_n\end{aligned}$$

be the vertex permutation of $L(K^*_{n,n})$. Let α_2 denote a permutation of arcs of $L(K^*_{n,n})$ that is induced by the permutation α_1. The induced arc permutation α_2 is seen to have the property that the length of every cycle is n, and that the number of these cycles is equal to $2n^2$. Thus induced permutation α_2 has the expression of the form of a product of cycles

$$\alpha_2 = \prod_{i=1}^{n}\prod_{j=1}^{n}\varepsilon_i\gamma_j \cdot \prod_{j=1}^{n}\prod_{i=1}^{n}\gamma_j\varepsilon_i.$$

Define now a new digraph $K^*(A,B)$ with partite sets $A = \{u_1, u_2, \dots, u_n\}$ and $B = \{v_1, v_2, \dots, v_n\}$. Let every vertex $u_i(v_i)$ correspond to the cycle $\varepsilon_i(\gamma_i)$, $i = 1, 2, \dots, n$, and let the vertex $u_i(v_j)$ be connected by an arc with the vertex $v_j(u_i)$ if and only if a cycle $\varepsilon_i\gamma_j(\gamma_i\varepsilon_i)$ belongs to α_2. It is evident that the digraph $K^*(A,B)$ is isomorphic to $K^*_{n,n}$. Next, let $\overrightarrow{K}(X,Y)$ denote a complete bipartite digraph which contains all arcs of which start–vertex is from X and end–vertex is from Y.

The exact construction of t isomorphic factors of $L(K^*_{n,n})$ depends on the parity of t.

Case 1. Let t be even and let $t|2n^3$. Then $t = 2r$ for some positive integer r, and therefore r divides n^3. Let $\gcd(r, n^2) = r_1$. Consequently, there exist positive integers b and c, such that $b|n$, $c|n$, and $r_1 = bc$. Next, let $r|r_1 = d$. Then obviously $d|n$.

Divide $K^*(A,B)$ into two isomorphic digraphs $\overrightarrow{K}(A,B)$ and $\overrightarrow{K}(B,A)$. Owing to this it is sufficient to construct a decomposition of $\overrightarrow{K}(A,B)$ into r isomorphic sub–digraphs.

Firstly, construct the decomposition of $\overrightarrow{K}(A,B)$ into r_1 isomorphic sub–digraphs. Let $A = \bigcup_{k=1}^{b} A_k$, $B = \bigcup_{s=1}^{c} B_s$, $|A_k| = n/b$, and $|B_s| = n/c$, where the sets A_k and B_s are mutually disjoint. Define r_1 sub–digraphs of $\overrightarrow{K}(A,B)$ in the following way: $G_{ks} = \overrightarrow{K}(A_k, B_s)$ for every ordered couple $(k,s) \in \{1,2,\dots,b\} \times \{1,2,\dots,c\}$. Sub–digraphs G_{ks} are all isomorphic because there exists an isomorphism between $G_{k_1s_1}$ and $G_{k_2s_2}$ induced by mapping $(k_1,s_1) \to (k_2,s_2)$. By the backward application of the previous correspondence on the digraphs G_{ks}, we get the decomposition of $L(K^*_{n,n})$ into r_1 isomorphic sub–digraphs, denote them by F_{ks}. To complete the proof it now suffices, without loss of generality, to decompose F_{11} into d isomorphic sub–digraphs. Let one part of the complete bipartite digraph F_{11} contains vertices of cycles $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n/b}$ and second part contains vertices of cycles $\gamma_1, \gamma_2, \ldots, \gamma_{n/c}$. Denote by α_2/F_{11} the reduced induced arc permutation which contains only those cycles in α_2 which correspond to arcs belonging to F_{11}. Then

$$\alpha_2/F_{11} = \prod_{i=1}^{n/b} \prod_{j=1}^{n/c} \varepsilon_i \gamma_j$$

is the product cycles having lengths which are multiples of d as $d|n$. Choose now from each cycle of the permutation α_2/F_{11} an arc h_{ij} that occupies the first place in given cycle and put

$$E = E(F_{111}) = \{(\alpha_2/F_{11})^{ud}(h_{ij}); u \geqq 0\}.$$

Then $\{E, (\alpha_2/F_{11})(E), \ldots, (\alpha_2/F_{11})^{d-1}(E)\}$ is a partition of the arcs of F_{11}. This constitutes an isomorphic factorization of F_{11}, as the sub–digraph F_{111} induced by E is isomorphic to the sub–digraphs of F_{11} induced by each of $(\alpha_2/F_{11})(E), \ldots$ Isomorphisms between F_{111} and these sub–digraphs are provided by the corresponding powers of α_1. Hence F_{11} is divisible by d. In consequence of preceding follows that the digraph $L(K^*_{n,n})$ is divisible by t.

Case 2. Let t be odd and let $t|2n^3$. Then t divides n^3. Let $(t, n^2) = t_1$ and let $t/t_1 = d$. Then obviously d must divide n. Since $t_1|n^2$, then there exist positive integers b and c such that $b|n$, $c|n$, and $t_1 = bc$. Consider subsets A_k and B_s that have the same meaning as in the Case 1.

Suppose $b > 1$ and $c > 1$. Define for every ordered couple $(k, s) \in \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$ digraphs $G_{ks} = \overrightarrow{K}(A_k, B_s) \cup \overrightarrow{K}(B_{s+1}, A_k)$ where the addition $s+1$ is taken modulo c with residues $1, 2, \ldots, c$. It is seen that every digraph G_{ks} is isomorphic to the directed "path" $\overrightarrow{P}_3$ with "vertices" B_{s+1}, A_k and B_s, whereupon these paths are arc–disjoint. Then digraphs F_{ks} obtained from G_{ks} by analogous fashion as stated above provide an isomorphic factorization of $L(K^*_{n,n})$ into t_1 factors. Take now the digraph F_{11} and decompose it into d isomorphic sub–digraphs. Let one part of the bipartite digraph F_{11} contains vertices that are elements of cycles $\varepsilon_1, \varepsilon_2, \ldots, e_{n/b}$ and the second part contains vertices that are elements of cycles $\gamma_1, \gamma_2, \ldots, \gamma_{n/c}, \gamma_{n/c+1}, \ldots, \gamma_{2n/c}$, and let the induced reduced permutation α_2/F_{11} contains only those cycles from α_2 for which there exists a corresponding arc in F_{11}. Then

$$\alpha_2/F_{11} = \prod_{i=1}^{n/b} \prod_{j=1}^{n/c} \varepsilon_i \gamma_j \cdot \prod_{j=n/c+1}^{2n/c} \prod_{i=1}^{n/b} \gamma_j \varepsilon_i,$$

where all cycles of α_2/F_{11} have lengths which are multiples of d. From each cycle $\varepsilon_i \gamma_j$ and $\gamma_j \varepsilon_i$ in α_2/F_{11} choose the first arc h_{ij} and e_{ji}, respectively, and put

$$E = E(F_{111}) = \{(\alpha_2/F_{11})^{ud}(h_{ij}), (\alpha_2/F_{11})^{ud}(e_{ji}); u \geqq 0\}.$$

The system $\{E, (\alpha_2/F_{11}(E), \ldots, (\alpha_2/F_{11})^{d-1}(E)\}$ is a partition of the arc set of F_{11} and sub–digraphs induced by these subsets provides an isomorphic factorization of F_{11}. Thus F_{11} is divisible by d und in consequence of the preceding the digraph $L(K^*_{n,n})$ is divisible by t.

Suppose now $b > 1$ and $c = 1$. Then $d = 1$ and $t = b$. Define in this case b sub–digraphs of $K^*(A, B)$ in this way: $G_k = \overrightarrow{K}(A_k, B) \cup \overrightarrow{K}(B, A_{k+1})$ for $k = 1, 2, \ldots, b$. The addition $k + 1$ is taken modulo b with residues $1, 2, \ldots, b$. Note, that subsets A_k have the same meaning as in Case 1. As above, the gained sub–digraphs F_k are all isomorphic because each of them is isomorphic to the directed "path" $\overrightarrow{P}_3$ with "vertices" A_k, B and A_{k+1}, and these directed "paths" are arc–disjoint. Hence $L(K^*_{n,n})$ is divisible by t. Since for every t dividing $2n^3$ the digraph $L(K^*_{n,n})$ is divisible by t, then $L(K^*_{n,n})$ is rational, which completes the proof.

In conclusion, we note that analogous theorem for non-oriented graphs $L(K_{n,n})$ will be proved in the forthcoming paper.

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A NOTE ON THE IMPROPER KURZWEIL-HENSTOCK INTEGRAL

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ABSTRACT. A connection is studied between the improper Kurzweil-Henstock integral on the real line and the integral over a compact space.

INTRODUCTION

In [5] two possibilities are mentioned of defining the improper Kurzweil-Henstock integral on the real line (see also [2] for a more general range). In [1] and [6] the Kurzweil-Henstock construction has been examined for a general compact range. It is natural to consider one-point compactification of the real line. Therefore we work with the compactification and we prove a convergence theorem in compact spaces describing the situation from the real case.

KURZWEIL-HENSTOCK INTEGRAL IN COMPACT TOPOLOGICAL SPACES

Let $\mathbb{N}$ be the set of all strictly positive integers, $\mathbb{R}$ the set of the real numbers, $\mathbb{R}^+$ be the set of all strictly positive real numbers. Let X be a Hausdorff compact topological space. If $A \subset X$, then the interior of the set A is denoted by $int\, A$.

We shall work with a family $\mathcal{F}$ of compact subsets of X closed under the intersection and a monotone and additive mapping $\lambda : \mathcal{F} \to [0, +\infty]$. The additivity means that

$$\lambda(A \bigcup B) + \lambda(A \bigcap B) = \lambda(A) + \lambda(B) \tag{1}$$

whenever $A, B, A \bigcup B \in \mathcal{F}$.

By a *partition* (detaily, $(\mathcal{F}, \lambda)$-*partition*) of a set $A \in \mathcal{F}$ we mean a finite collection $\{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ such that

(i) $\mathcal{U}_1, \dots, \mathcal{U}_k \in \mathcal{F}$,

(ii) $\bigcup_{i=1}^{k} \mathcal{U}_i = A$,

(iii) $\lambda(\mathcal{U}_i \bigcap \mathcal{U}_j) = 0$ whenever $i \neq j$,

(iv) $t_i \in \mathcal{U}_i \, (i = 1, \dots, k)$.

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A finite collection $\{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ of subsets of $A \in \mathcal{F}$, satisfying conditions (i), (iii) and (iv), but not necessarily (ii), is said to be *decomposition* of A. We shall assume that $\mathcal{F}$ *separates points* in the following way: to any $A \in \mathcal{F}$ there exists a sequence $(\mathcal{A}_n)_n$ of partitions of A such that

(i) $\mathcal{A}_{n+1}$ is a refinement of $\mathcal{A}_n$,

(ii) to any $x, y \in A$, $x \neq y$, there exist $n \in \mathbb{N}$ and $B \in \mathcal{A}_n$ such that $x \in B$ and $y \notin B$.

We note that this assumption is fulfilled if the topological space X is metrizable or it satisfies the second axiom of countability (see [6]).

A *gauge* on a set $A \subset X$ is a mapping δ assigning to every point $x \in A$ a neighborhood $\delta(x)$ of x. If $\mathcal{D} = \{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ is a decomposition of A and δ is a gauge on A, then we say that $\mathcal{D}$ is δ*-fine* if $\mathcal{U}_i \subset \delta(t_i)$ for any $i \in \{1, 2, \dots, k\}$.

We obtain a simple example putting $X = [a, b] \subset \mathbb{R}$ with the usual topology, $\mathcal{F}$ =the family of all closed subintervals of X, $\lambda([\alpha, \beta]) = \beta - \alpha$, $a \le \alpha < \beta \le b$. Any gauge can be represented by a real function $d : [a, b] \to \mathbb{R}^+$, if we put $\delta(x) = (x - d(x), x + d(x))$.

Another example is the unbounded interval $[a, +\infty] = [a, +\infty) \bigcup \{+\infty\}$ considered as the one-point compactification of the locally compact space $[a, +\infty)$. The base of open sets consists of open subsets of $[a, +\infty)$ and the sets of the type $(b, +\infty) \bigcup \{+\infty\}$, $a \le b < +\infty$. Any gauge in $[a, +\infty]$ has the form $\delta(x) = (x - d(x), x + d(x))$, if $x \in [a, +\infty] \bigcap \mathbb{R}$, and $\delta(+\infty) = (b, +\infty] = (b, +\infty) \bigcup \{+\infty\}$, where d denotes a positive real-valued function defined on $[a, +\infty)$, and b denotes a real number.

Let us return to the definition of Kurzweil-Henstock integral (KH-integral) on X. If $\mathcal{D} = \{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ is a decomposition of a set A, and $f : X \to \mathbb{R}$, then we define the Riemann sum as follows:

$$S(f, \mathcal{D}) = \sum_{i=1}^{k} f(t_i)\lambda(\mathcal{U}_i),$$

if the sum exists in $\mathbb{R}$, with the convention $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$.

We note that the fact that $\mathcal{F}$ separates points guarantees the existence of at least one δ-fine partition $\mathcal{D}$ such that $S(f, \mathcal{D})$ is well-defined for any gauge δ (see [6], [8]).

Definition 2.1. A function $f : X \to \mathbb{R}$ is *integrable* on a set A if there exists $I \in \mathbb{R}$ such that $\forall \varepsilon > 0$ there exists a gauge δ on A such that

$$|S(f, \mathcal{D}) - I| \le \varepsilon \tag{2}$$

whenever $\mathcal{D}$ is a δ-fine partition of A such that $S(f, \mathcal{D})$ exists in $\mathbb{R}$. We denote

$$I = \int_A f$$

(see also [6], Definition 1.8., p. 154).

We now prove the following:

Theorem 3.1. *Let $X = X_0 \bigcup \{x_0\}$ be the one-point compactification of a locally compact space X_0. Let $f : X \to I\!R$ be a function such that $f(x_0) = 0$. Let $(A_n)_n$ be a sequence of sets, such that $A_n \in \mathcal{F}$, $A_n \subset int\, A_{n+1}$, $A_{n+1} \setminus int\, A_n \in \mathcal{F}$, $\lambda(A_n \setminus int\, A_n) = 0$ $(n = 1, 2, \dots)$, $\bigcup_{n=1}^{\infty} A_n = X_0$. Let f be integrable on A_n $(n = 1, 2, \dots)$ and let there exist in $I\!R$ an element I such that, $\forall\, \varepsilon > 0$, there exists an integer n_0 such that*

$$\left| \int_A f - I \right| \leq \varepsilon \quad \forall\, A \in \mathcal{F},\ A \supset A_{n_0}.$$

Then f is integrable on X and $\displaystyle\int_X f = I$.

Proof. Let ε be an arbitrary positive real number, and $n_0 \in I\!N$ be as in the hypotheses of the theorem. Put $A_0 = \emptyset$, $B_n = A_{n+1} \setminus int\, A_n$ $(n = 1, 2, \dots)$. Proceeding analogously as in [6], Lemma 1.10, and as in [2], we get that f is integrable on every subset of A_n belonging to $\mathcal{F}$ $(n = 1, 2, \dots)$ and thus, in particular, f is integrable on B_n $(n = 1, 2, \dots)$. Therefore, $\forall\, n \in I\!N$, there exists a gauge δ_n on B_n such that

$$\left| \int_{B_n} f - S(f, \mathcal{D}_n) \right| \leq \frac{\varepsilon}{2^{n+3}} \tag{3}$$

for any δ_n-fine partition $\mathcal{D}_n$ of B_n. From (3) and Henstock's Lemma (see also [6], Lemma 2.1., pp. 158-159; [5], Theorem 3.2.1., pp. 81-83), it follows that

$$\left| \int_{\bigcup_{i=1}^{h} \mathcal{V}_i} f - S(f, \mathcal{E}_n) \right| \leq \frac{\varepsilon}{2^{n+2}} \tag{4}$$

for each δ_n-fine decomposition $\mathcal{E}_n = \{(\mathcal{V}_1, t_1), \dots, (\mathcal{V}_h, t_h)\}$ of B_n. Evidently

$$B_n \bigcap B_{n+1} = A_n \setminus int\, A_n \quad \forall\, n \in I\!N.$$

Therefore

$$B_n = (B_n \bigcap B_{n-1}) \bigcup (int\, B_n) \bigcup (B_n \bigcap B_{n+1}) \quad \forall\, n.$$

Moreover, it is easy to check that

$$B_j \cap B_l = \emptyset \text{ whenever } |j - l| \geq 2 \tag{5}$$

and that

$$(int\, B_n) \cap (int\, B_{n+1}) = \emptyset \quad \forall\, n \in I\!N. \tag{6}$$

Now define a gauge δ on X by the following formula:

$$\delta(x) = \begin{cases} \delta_n(x) \bigcap (int\, B_n) \text{if}\, x \in int\, B_n, \\ \delta_n(x) \bigcap \delta_{n+1}(x) \bigcap (int\, A_{n+1}) \text{if}\, x \in B_n \bigcap B_{n+1}, \quad (n = 1, 2, \dots) \\ (X_0 \setminus A_{n_0}) \bigcup \{x_0\} \text{if}\, x = x_0. \end{cases} \tag{7}$$

Let $\mathcal{D} = \{(\mathcal{U}_1, t_1), \ldots, (\mathcal{U}_k, t_k)\}$ be a δ-fine partition of X. There exists $(\mathcal{U}_{i_0}, t_{i_0}) \in \mathcal{D}$, with $i_0 \in \{1, 2, \ldots, k\}$, such that $x_0 \in \mathcal{U}_{i_0}$. We shall prove that $t_{i_0} = x_0$. Namely, in the opposite case,

$$x_0 \in \mathcal{U}_{i_0} \subset \delta(t_{i_0}) \subset \delta_n(t_{i_0})$$

for some n. But $\delta_n(t) \subset X_0$ for $t \neq x_0$. We have obtained $x_0 \in X_0$, that is a contradiction.

Since $f(x_0) = 0$, the Riemann sum $S(f, \mathcal{D})$ has the form

$$\sum_{i=1,\ldots,k, i\neq i_0} f(t_i)\,\lambda(\mathcal{U}_i),$$

and $t_i \in X_0$ $(i = 1, \ldots, k, i \neq i_0)$. Let

$$A = \bigcup_{n\in T} B_n,$$

where

(8) $$T = \{n \in I\!N : \exists i \in \{1, \ldots, k\}, i \neq i_0 : B_n \cap \mathcal{U}_i \neq \emptyset\}.$$

By (7) , and since $\mathcal{D}$ is a δ-fine *partition* of X, we get that

(9) $$A \supset A_{n_0}$$

By hypothesis we have

(10) $$\left|\int_A f - I\right| \leq \varepsilon$$

We claim that, if $\mathcal{U}_i$, $i \neq i_0$, has nonempty intersection with at least two of the $int\, B_n$'s, then necessarily there exists $n \in I\!N$ such that the point t_i corresponding to $\mathcal{U}_i$ belongs to $B_n \cap B_{n+1}$. Indeed, if $t_i \in int\, B_n$ for some n, then, from (7) and the fact that $\mathcal{D}$ is a δ-fine partition of X, we'd have

$$\mathcal{U}_i \subset \delta(t_i) \subset int\, B_n :$$

this is impossible, by virtue of (5) and (6). From this and since

$$(B_{n-1} \bigcap B_n) \bigcap (B_n \bigcap B_{n+1}) = \emptyset \quad \forall n,$$

it follows that, for every $i = 1, 2, \ldots, k$, $i \neq i_0$, the B_n's having nonempty intersection with $\mathcal{U}_i$ are at most two, while the B_n's which have nonempty intersection with $\mathcal{U}_{i_0}$ can be infinitely many (even all the B_n's). Thus we proved that the set T in (8) is finite.

For $n \in T$ define a decomposition $\mathcal{E}_n$ of B_n in the following way:

$$\begin{aligned}\mathcal{E}_n =& \{(\mathcal{U}_i, t_i) : t_i \in int\, B_n\} \\ & \bigcup \{(\mathcal{U}_i \bigcap B_n, t_i) : t_i \in B_n \bigcap B_{n-1}\} \\ & \bigcup \{(\mathcal{U}_i \bigcap B_n, t_i) : t_i \in B_n \bigcap B_{n+1}\}.\end{aligned}$$

Then, by construction, we have:

$$S(f,\mathcal{D}) = \sum_{n\in T} S(f,\mathcal{E}_n) \tag{11}$$

by additivity of λ and since $A_n \setminus int\, A_n = B_n \bigcap B_{n+1} \subset int\, A_{n+1}$ and $\lambda(A_n \setminus int\, A_n) = 0 \;\forall\, n \in \mathbb{N}$.

Similarly,

$$\sum_{n\in T} \int_{\bigcup_{\mathcal{U}_i \subset\, int\, B_n, i\neq i_0} \mathcal{U}_i} f = \int_A f \tag{12}$$

Since $\mathcal{D}_n$ is δ_n-fine, we have (3). From (3), (10), (11), (12), and (9) we obtain:

$$|S(f,\mathcal{D}) - I| = \left| \sum_{n\in T} S(f,\mathcal{E}_n) - I \right| =$$

$$\left| \sum_{n\in T} \left(S(f,\mathcal{E}_n) - \int_{\bigcup_{\mathcal{U}_i \subset\, int\, B_n, i\neq i_0} \mathcal{U}_i} f \right) + \sum_{n\in T} \int_{\bigcup_{\mathcal{U}_i \subset\, int\, B_n, i\neq i_0} \mathcal{U}_i} f - I \right| \leq$$

$$\sum_{n\in T} \left| S(f,\mathcal{E}_n) - \int_{\bigcup_{\mathcal{U}_i \subset\, int\, B_n, i\neq i_0} \mathcal{U}_i} f \right| + \left| \int_A f - I \right| \leq \sum_{n\in T} \frac{\varepsilon}{2^{n+2}} + \varepsilon < 2\,\varepsilon.$$

From this the assertion follows. □

Applications

The following results are consequences of Theorem 3.1:

Proposition 4.1. ([5], Theorem 2.9.3., pp. 61-63) *Let $f : [a, +\infty] \to \mathbb{R}$ be such that $f(+\infty) = 0$, f be integrable on $[a, b]$ for any $b > a$, and let there exist in $\mathbb{R}$ the limit*

$$\lim_{b\to+\infty} \int_{[a,b]} f.$$

Then f is integrable on $[a, +\infty]$, and

$$\int_{[a,+\infty]} f = \lim_{b\to+\infty} \int_{[a,b]} f.$$

Proposition 4.2. (see also [5], Theorem 2.8.3., pp. 57-59 and Remark 2.8.4, p.57) *Let $a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \to \mathbb{R}$, f be integrable on $[a, x]$ for any $a \leq x < b$, and let there exist in $\mathbb{R}$ the limit*

$$\lim_{x\to b^-} \int_{[a,x]} f.$$

Then f is integrable on $[a, b]$, and

$$\int_{[a,b]} f = \lim_{x\to b^-} \int_{[a,x]} f.$$

Proof. We observe that $[a,b] = [a,b) \bigcup \{b\}$ can be considered as the one-point compactification of $[a,b)$. The only difference is that we did not assume $f(b) = 0$. Of course, one can put $g(x) = f(x) - f(b)$, and use Theorem 3.1 with respect to the function g. Then we have

$$\int_{[a,b]} g = \lim_{x \to b^-} \int_{[a,x]} g,$$

and hence

$$\begin{aligned}\int_{[a,b]} f &= f(b)(b-a) + \int_{[a,b]} g = \\ &= \lim_{x \to b^-} f(b)(x-a) + \lim_{x \to b^-} \int_{[a,x]} g = \\ &= \lim_{x \to b^-} \int_{[a,x]} (g + f(b)) = \lim_{x \to b^-} \int_{[a,x]} f.\end{aligned}$$

This concludes the proof. □

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JOIN AND INTERSECTION OF HYPERMAPS

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ABSTRACT. Hypermaps are generalisations of maps - 2-cell decompositions of closed surfaces. The correspondence between hypermaps and quotients of the group Δ freely generated by three involutions is well-known. In this correspondence hypermaps correspond to conjugacy classes of subgroups of Δ, and hypermap coverings to the subgroup containment.

Let $\mathcal{H}$ and $\mathcal{K}$ be two hypermaps. We shall introduce and study two binary operations - join and intersection, defined on hypermaps. The corresponding operations in the subgroup representation is the intersection of two subgroups of Δ and the subgroup closure in Δ. We investigate basic properties of the join and intersection, particular attention is paid to the study of orthogonal hypermaps, final sections are devoted to the study of the relationship of some algebraic and topological properties of hypermaps and the join and intersection. As a byproduct we get a method of comparison of two hypermaps which led us to the definition of the shared cover index. This transpired to be a generalisation of the chirality index defined in [3]. In fact, the chirality index of an oriented regular hypermap $\mathcal{H}$ is just the shared cover index of $\mathcal{H}$ with its mirror image.

1. INTRODUCTION

A topological map is a 2-cell decomposition of a compact connected surface. A hypermap is a certain abstraction of a topological map linking different fields of mathematics including combinatorics, group theory, geometry of Riemann surfaces, algebraic geometry and Galois theory. For a survey explaining these relations we refer the reader to [9,10]. Formally, a hypermap is a 4-tuple $(F; r_0, r_1, r_2)$, where F is a set of flags and r_i, $i = 0, 1, 2$ are fixed point free involutory permutations acting on F such that $\langle r_0, r_1, r_2\rangle$ is transitive on F.

It is known that any hypermap can be viewed as a quotient of the universal hypermap given by the action of the group $\Delta = \langle r_0, r_1, r_2; r_0^2 = r_1^2 = r_2^2 = 1\rangle$ on itself by left multiplication. This gives rise to a correspondence between subgroups of Δ, called hypermap subgroups in this context, and hypermaps. In particular, normal subgroups of finite index in Δ determine hypermaps which automorphism group acts regularly on the set of flags. Using the representation of hypermaps via hypermap subgroups it is easy to see that for any two regular hypermaps $\mathcal{H}$, $\mathcal{K}$ there is a least regular common cover $\mathcal{H} \vee \mathcal{K}$, called the join of $\mathcal{H}$ and $\mathcal{K}$, satisfying the following property: if a regular hypermap $\mathcal{X}$ covers both $\mathcal{H}$ and $\mathcal{K}$ then it covers

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$\mathcal{H} \vee \mathcal{K}$. Similarly, we define $\mathcal{H} \wedge \mathcal{K}$ to be the largest regular hypermap covered by $\mathcal{H}$ and $\mathcal{K}$, called here the intersection of hypermaps. Although both constructions are known [18,2,3], no systematic study of their properties (from the point of view of theory of hypermaps) was done, except the paper of S. Wilson [18] where the investigation is restricted to joins of maps.

The introduction is followed by a section where we develop some necessary definitions, notations and mention some basic facts on hypermaps and their representations. In Section 3 we introduce the join and intersection of two hypermaps and prove some fundamental results about them. In Section 4 we study the structure of the monodromy group of the join and intersection of two regular hypermaps, this is equivalent with the study of the corresponding automorphism groups. In Section 5 we study the orthogonality of two hypermaps, an interesting phenomenon related with the join and intersection of them. Final Sections are devoted to an investigation of orientability, reflexibility and self-duality of regular hypermaps in relation to the join and intersection. Several ideas and results from [3] and [18] are generalised there.

2. Hypermaps and subgroups of Δ

A *topological hypermap* $\mathcal{H}$ is a cellular embedding of a connected 3-valent graph X into a closed surface S such that the cells are 3-coloured (say by black, grey and white colours) with adjacent cells having different colours. Numbering the colours 0, 1 and 2, and labelling the edges of X with the missing adjacent cell number, we can define 3 fixed points free involutory permutations r_i, $i = 0, 1, 2$, on the set F of vertices of X; each r_i switches the pairs of vertices connected by i-edges (edges labelled i). The elements of F are called *flags* of $\mathcal{H}$ and the group G generated by r_0, r_1 and r_2 is called the *monodromy group* $\mathrm{Mon}(\mathcal{H})$ of the hypermap $\mathcal{H}$. The cells of $\mathcal{H}$ coloured 0, 1 and 2 are called the *hypervertices*, *hyperedges* and *hyperfaces*, respectively. Since the graph X is connected, the monodromy group acts transitively on F and the orbits of $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$ or $\langle r_0, r_2\rangle$ on F determine hyperfaces, hypervertices and hyperedges, respectively. Let $k = ord(r_0r_1)$, $m = ord(r_1r_2)$ and $n = ord(r_2r_0)$ be the orders of the respective elements in the monodromy group. The triple (k, m, n) is called the *type* of the hypermap. Maps are hypermaps satisfying condition $(r_0r_2)^2 = 1$. In other words, maps are hypermaps of type $(p, q, 2)$ or of type $(p, p, 1)$.

It is known that all information on the topological hypermap $\mathcal{H}$ is coded in the three associated fixed points free permutations acting on F (see for instance [4, 6, 11, 12, 14, 15]) . Thus we define a *hypermap* to be a 4-tuple $(F; r_0, r_1, r_2)$, where r_i, $i = 0, 1, 2$ are fixed point free involutions acting on F such that the action of $\mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$ is transitive. Let $\mathcal{H} = (F; r_0, r_1, r_2)$ and $\mathcal{K} = (F'; t_0, t_1, t_2)$. *A homomorphism* $\mathcal{H} \to \mathcal{K}$ is a mapping $\pi : F \to F'$ such that $t_i\pi = \pi r_i$, for each $i = 0, 1, 2$. Due to the transitivity of the action of $\mathrm{Mon}(\mathcal{K})$ a hypermap homomorphism is necessarily surjective, thus homomorphisms between hypermaps are alternatively called *coverings*. An easy but fundamental observation establishes that given covering $\pi : \mathcal{H} \to \mathcal{K}$ there is an induced group epimorphism $\pi^* : \mathrm{Mon}(\mathcal{H}) \to \mathrm{Mon}(\mathcal{K})$ taking $r_i \mapsto t_i$ for $i = 0, 1, 2$. Homomorphisms of hypermaps correspond to branched coverings of topological hypermaps mapping i-cells onto i-cells for $i = 0, 1, 2$. A bijective homomorphism $\mathcal{H} \to \mathcal{K}$ is an *isomorphism*, we write $\mathcal{H} \cong \mathcal{K}$ in this case. An *automorphism* $\mathcal{H} \to \mathcal{H}$ is a permutation of flags of $\mathcal{H}$ commuting with the involutions r_i, for each $i = 0, 1, 2$. In what follows, we shall always let the elements of the monodromy group of a hypermap $\mathcal{H}$ having left action on the flags, while the automorphisms of $\mathcal{H}$ will act from 'right'.

It is well-known (and easy to see) that the action of the automorphism group of a hypermap on its flags is semi-regular (i. e. the stabiliser of a flag is trivial). In the case the automorphism group Aut $(\mathcal{H})$ acts regularly on the flag-set of a hypermap $\mathcal{H}$, the hypermap $\mathcal{H}$ is called *regular*.

Besides the monodromy group $G = \mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2 \rangle$ of a hypermap we consider its even word subgroup generated by $G^+ = \langle \rho, \lambda \rangle$, where $R = r_1 r_2$ and $L = r_0 r_1$. Obviously, it is a subgroup of index at most two. If $[G : G^+] = 2$ the hypermap $\mathcal{H}$ is *orientable*. The category of *oriented hypermaps* is formed by triples $(D; R, L)$ where R, L are permutations generating a group (the *oriented monodromy group*) acting transitively on the set of darts D. The notions of homomorphism, of isomorphism and of automorphism are defined in the obvious way. An oriented map is *regular* if its automorphism group acts regularly on the set of darts.

Let us denote by

$$\Delta = \langle \rho_0, \rho_1, \rho_2; \;\; r_0^2 = r_1^2 = r_2^2 = 1 \rangle$$

the free product of three two-element groups.

The associated (infinite) hypermap $\mathcal{U} = (\Delta; \rho_0, \rho_1, \rho_2)$, with ρ_i $(i = 0, 1, 2)$ acting by left multiplication, will be called the *universal hypermap*. It follows that the monodromy group of any hypermap $\mathcal{H}$ is an epimorphic image of Δ and this epimorphism induces an action of Δ on flags of $\mathcal{H}$. Hence $\mathcal{H}$ can be represented as a hypermap $(\Delta/H; r_0', r_1', r_2')$, where H is a stabiliser of a flag in the action of Δ, Δ/H is the set of left cosets of H and the action of r_i' is defined by the rule $r_i'(xH) = r_i xH$ for $i = 0, 1, 2$. The group H of finite index is called the *hypermap subgroup* of $\mathcal{H}$. The above defined hypermap corresponding to a hypermap subgroup H will be denoted by $\mathcal{U}/H$ and will be called an *algebraic hypermap*. A routine calculation shows that two subgroups H_1 and H_2 of Δ determine isomorphic hypermaps if and only if they are conjugate. Hence, the representation of a hypermap by a hypermap subgroup is not unique, this is because in an irregular hypermap two flag stabilisers may be different although they are always conjugate. More generally, $\mathcal{H}$ covers $\mathcal{K}$ if and only if there exist $g \in \Delta$ such that $H^g \leq K$. As concerns properties of algebraic hypermaps the following (well-known) statement is worth to mention explicitly. Recall that given groups $H \leq G$ the normaliser $N_G(H)$ is a subgroup of G consisting of $g \in G$ such that $H^g = H$.

Proposition 2.1. *Let $\mathcal{H}$ be an algebraic map with a hypermap subgroup $H \leq \Delta$. Then $Aut(\mathcal{H}) = N_\Delta(H)/H$.*

Proof. Let φ be an automorphism of $\mathcal{H}$ taking H onto gH. We show that the assignment $A : \varphi \mapsto gH$ defines the required isomorphism. Since φ is an automorphism of $\mathcal{H}$, we have $hgH = h(H\varphi) = (hH)\varphi = ghH = gH$ for every $h \in H$. Thus g normalises H. By its definition A is a homomorphism. The semi-regularity of the action of the automorphism group implies that A is injective. To see that it is surjective, let us denote by φ_g the mapping $xH \mapsto xgH$. This is a well-defined automorphism if and only if for every $h \in H$, we have $hgH = gH$. But the latter statement means $g \in N_\Delta(H)$. □

It follows that a hypermap $\mathcal{H}$ is regular if and only if the associated hypermap subgroup is normal (see also [7,6]). Hence H is uniquely determined in this case. In what follows we (as a rule) denote by $H \trianglelefteq \Delta$ the hypermap subgroup associated with a regular hypermap $\mathcal{H}$. To establish a one-to-one correspondence between the normal subgroups of finite index of Δ and regular hypermaps we need to extend the family of all regular hypermaps by considering a *trivial hypermap* being the one-flag hypermap with the trivial action of the three defining involutory permutations. We shall use 1 to denote the trivial hypermap. The hypermap subgroup of the trivial hypermap is Δ. Let $\mathcal{H}$ and $\mathcal{K}$ are regular hypermaps. Then $\mathcal{H} \to \mathcal{K}$ if and only if $K \geq H$. Hence there is an isomorphism between the set of regular hypermaps partially ordered by the relation "to be a cover", and the set of normal (torsion free) subgroups of finite index ordered by the subgroup relation. In what follows this correspondence will be extensively employed. In fact, the whole paper is devoted to a detailed investigation of this fundamental correspondence. Let us remark that coverings between regular hypermaps are necessarily regular (see [13]), i. e. the group of covering transformations acts regularly on each flag-fiber. If $\mathcal{H} \to \mathcal{K}$ are regular hypermaps with the hypermap subgroups $H \leq K$ then the covering is defined by mapping $\pi : xH \mapsto xK$ and the *covering transformation group* is isomorphic to the kernel $\operatorname{Ker}\pi$ of the above group epimorphism $\pi : \Delta/H \to \Delta/K$.

Similar statements about the correspondence between oriented hypermaps and conjugacy classes of subgroups of finite index of the free 2-generator group $\Delta^+ < \Delta$ can be established. In particular, there is one-to-one correspondence between the isomorphism classes of oriented regular hypermaps and normal subgroups of finite index in Δ^+.

The reader interested to get more information on maps, hypermaps and related topics is referred to [4, 5, 6, 9, 10, 11, 15, 16]. As concerns the related parts of theory of permutation groups an old but popular monograph is [17].

3. Join and intersection of two hypermaps

Let $\mathcal{H} = \mathcal{U}/H$ and $\mathcal{K} = \mathcal{U}/K$ be algebraic hypermaps. Set $\mathcal{H} \vee \mathcal{K} = \mathcal{U}/(H \cap K)$ and $\mathcal{H} \wedge \mathcal{K} = \mathcal{U}/\langle H, K\rangle$. The hypermaps $\mathcal{H} \vee \mathcal{K}$, $\mathcal{H} \wedge \mathcal{K}$ will be called *join* and *intersection* of $\mathcal{H}$ and $\mathcal{K}$, respectively.

The following two propositions are direct consequences of definitions.

Proposition 3.1. *Let $\mathcal{H} = \mathcal{U}/H$ and $\mathcal{K} = \mathcal{U}/K$ be algebraic hypermaps. Then*

if both $\mathcal{H}$ and $\mathcal{K}$ are finite then $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are finite,
if both $\mathcal{H}$ and $\mathcal{K}$ are regular then $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are regular as well,
if $\mathcal{H} \to \mathcal{K}$ is a covering then $\mathcal{H} \vee \mathcal{K} = \mathcal{H}$ and $\mathcal{H} \wedge \mathcal{K} = \mathcal{K}$.
if a hypermap $\mathcal{X} = \mathcal{U}/X$ covers both $\mathcal{H}$ and $\mathcal{K}$ then it covers $\mathcal{H} \vee \mathcal{K}$,
if a hypermap $\mathcal{X} = \mathcal{U}/X$ is covered by both $\mathcal{H}$ and $\mathcal{K}$ then it is covered by $\mathcal{H} \vee \mathcal{K}$,

Proposition 3.2. *If $\mathcal{H}$ and $\mathcal{K}$ are regular hypermaps then $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are well-defined binary operations on isomorphism classes of hypermaps.*

Proof. The respective hypermap subgroups are unique. □

It follows from the above propositions that for any two regular hypermaps $\mathcal{H}$ and $\mathcal{K}$ there is a unique regular hypermap $\mathcal{Y} = \mathcal{H} \vee \mathcal{K}$ satisfying the following property:

if $\mathcal{X} \to \mathcal{H}$ and $\mathcal{X} \to \mathcal{K}$ then $\mathcal{X}$ covers the join $\mathcal{Y}$. Thus it make sense to speak on the *least common cover* of two regular maps $\mathcal{H}$ and $\mathcal{K}$. Similarly, any hypermap covered by two regular hypermaps is covered by $\mathcal{H} \wedge \mathcal{K}$ and so we can view the intersection as the *largest common quotient* of $\mathcal{H}$ and $\mathcal{K}$.

An irregular hypermap can be represented by two different hypermap subgroups H and H^g for some $g \in \Delta$. Thus the join and the intersection does not preserve isomorphism classes of hypermaps. Hence, they are binary operations on algebraic representations of hypermaps and not on the isomorphism classes. Since the intersection of normal subgroups as well as their product $HK = \langle H, K \rangle$ is a normal subgroup, we have not such a problem provided we restrict ourselves to the family of regular hypermaps, so we can speak on a join and intersection of (abstract) regular hypermaps. In a general case we shall always assume that with a given hypermap $\mathcal{H}$ a particular representative $H \leq \Delta$ of the respective conjugacy class of hypermap subgroups is associated. The latter is equivalent with considering a rooted hypermap, meaning a hypermap with a specified flag (the root of it). This approach is taken in [18].

The following lemma lists the properties of the join and intersection which are trivial consequences of the definitions. In particular, it follows that algebraic hypermaps form a lattice isomorphic to the lattice of all subgroups of Δ and regular hypermaps form a lattice isomorphic to the lattice of all normal subgroups of Δ. The ordering on regular hypermaps is given by hypermap coverings.

Lemma 3.3. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be algebraic hypermaps (regular hypermaps). Let $\mathcal{U}$ and 1 be the universal and trivial hypermaps. Then*

$\mathcal{X} \vee (\mathcal{Y} \vee \mathcal{Z}) = (\mathcal{X} \vee \mathcal{Y}) \vee \mathcal{Z}$,
$\mathcal{X} \vee \mathcal{Y} = \mathcal{Y} \vee \mathcal{X}$,
$\mathcal{X} \vee \mathcal{U} = \mathcal{U}$ *and* $\mathcal{X} \vee 1 = \mathcal{X}$,
$\mathcal{X} \wedge (\mathcal{Y} \vee \mathcal{Z}) \to (\mathcal{X} \wedge \mathcal{Y}) \vee (\mathcal{X} \wedge \mathcal{Z})$.

Interchanging joins and intersections in the above statements we get a dual version of the above lemma. In particular, we have

$$\mathcal{X} \vee (\mathcal{Y} \wedge \mathcal{Z}) \leftarrow (\mathcal{X} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Z}).$$

Let $\mathcal{H}$ be a hypermap. Denote by $|\mathcal{H}|$ the number of its flags. Of course if $\mathcal{H}$ is a regular hypermap we have $|\mathcal{H}| = |\mathrm{Mon}\,(\mathcal{H})| = |\Delta/H|$. The following statement relates the monodromy groups of the join and intersection of hypermaps with the monodromy groups of the original hypermaps.

Proposition 3.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. Then the monodromy group of $\mathcal{H} \vee \mathcal{K}$ is a subgroup of the direct product $Mon\,(\mathcal{H}) \times Mon\,(\mathcal{K})$ and we have*

$$Mon\,(\mathcal{H} \wedge \mathcal{K}) = Mon\,(\mathcal{H} \vee \mathcal{K}) / \big(H/H \cap K \times K/H \cap K\big),$$

where $H/H \cap K \times K/H \cap K$ is an internal direct product. Moreover,

$$|\mathcal{H} \vee \mathcal{K}| \cdot |\mathcal{H} \wedge \mathcal{K}| = |\mathcal{H}| \cdot |\mathcal{K}|.$$

Proof. We show that the mapping $\psi : g(H \cap K) \mapsto (gH, gK)$ is a monomorphism $\Delta/(H \cap K) \to \Delta/H \times \Delta/K$. Indeed, for any $x, y \in \Delta$

$$\psi((xH \cap K)(yH \cap K)) = \psi(xyH \cap K) = (xyH, xyK) =$$

$$(xH, xK)(yH, yK) = \psi(xH \cap K)\psi(yH \cap K).$$

Now let $\psi(xH \cap K) = 1 = (H, K)$ for some $x \in \Delta$. Then $(xH, xK) = (H, K)$, and consequently $x = 1$. Hence, ψ is a monomorphism.

By the third isomorphism theorem

$$HK/H \cap K = H/H \cap K \times K/H \cap K \cong HK/K \times HK/H.$$

Using this we get

$$|\mathcal{H} \vee \mathcal{K}||\mathcal{H} \wedge \mathcal{K}| = |\Delta/H \cap K||\Delta/HK| = |\Delta/HK||HK/K \times HK/H||\Delta/HK| = |\Delta/HK||HK/K||\Delta/HK||HK/H| = |\Delta/K||\Delta/H| = |\mathcal{K}||\mathcal{H}|.$$

By the second isomorphism theorem we obtain

$$\mathrm{Mon}\,(\mathcal{H} \wedge \mathcal{K}) = \Delta/HK \cong (\Delta/H \cap K)/(HK/H \cap K) = \mathrm{Mon}\,(\mathcal{H} \vee \mathcal{K})/\big(H/H \cap K \times K/H \cap K\big).$$

□

The equality $|\mathcal{H} \vee \mathcal{K}| \cdot |\mathcal{H} \wedge \mathcal{K}| = |\mathcal{H}| \cdot |\mathcal{K}|$, combined with the well-known statement in elementary number theory establishing

$$|\mathcal{H}| \cdot |\mathcal{K}| = gcd(|\mathcal{H}|, |\mathcal{K}|) \cdot lcm(|\mathcal{H}|, |\mathcal{K}|),$$

may suggest that $|\mathcal{H} \vee \mathcal{K}| = lcm(|\mathcal{H}|, |\mathcal{K}|)$, or equivalently $|\mathcal{H} \wedge \mathcal{K}| = gcd(|\mathcal{H}|, |\mathcal{K}|)$. However, this is not true in general. In general, we can only claim that $lcm(|\mathcal{H}|, |\mathcal{K}|)$ divides $|\mathcal{H} \vee \mathcal{K}|$, and $|\mathcal{H} \wedge \mathcal{K}|$ divides $gcd(|\mathcal{H}|, |\mathcal{K}|)$. The above two equalities imply

$$\frac{|\mathcal{H} \vee \mathcal{K}|}{lcm(|\mathcal{H}|, |\mathcal{K}|)} = \frac{gcd(|\mathcal{H}|, |\mathcal{K}|)}{|\mathcal{H} \wedge \mathcal{K}|}.$$

This observation led us to a new concept allowing us to relate two hypermaps. Given two regular hypermaps $\mathcal{H}$ and $\mathcal{K}$ the integer

$$s(\mathcal{H}, \mathcal{K}) = \frac{|\mathcal{H} \vee \mathcal{K}|}{lcm(|\mathcal{H}|, |\mathcal{K}|)} = \frac{gcd(|\mathcal{H}|, |\mathcal{K}|)}{|\mathcal{H} \wedge \mathcal{K}|}$$

will be called the *shared cover index* of $\mathcal{H}$ and $\mathcal{K}$. Clearly, if one of $\mathcal{H}$, $\mathcal{K}$ covers the other then $s(\mathcal{H}, \mathcal{K}) = 1$. Generally, it can be equal to any divisor of $gcd(|\mathcal{H}|, |\mathcal{K}|)$.

Replacing hypermaps by oriented hypermaps one can see that the concept of the shared cover index applies in the category of oriented regular maps as well. Here it can be viewed as a generalisation of the chirality index studied in [3]. Recall that by the *mirror image* of an oriented hypermap $\mathcal{H} = (R, L)$ we mean the hypermap $\mathcal{H}^r = (R^{-1}, L^{-1})$. The integer $\kappa(\mathcal{H}) = H/(H \cap H^r)$ is called the *chirality index* of $\mathcal{H}$, see [3].

Proposition 3.5. *Let $\mathcal{H}$ be an oriented regular hypermaps and $\mathcal{H}^r$ is the mirror image of it. Then $s(\mathcal{H}, \mathcal{H}^r) = \kappa(\mathcal{H})$, where $\kappa(\mathcal{H})$ is the chirality index of $\mathcal{H}$.*

Proof.

$$s(\mathcal{H}, \mathcal{H}^r) = \frac{|\mathcal{H} \vee \mathcal{H}^r|}{lcm(|\mathcal{H}|, |\mathcal{H}^r|)} = \frac{|\mathcal{H} \vee \mathcal{H}^r|}{|\mathcal{H}|} = \frac{|\Delta/H \cap H^r|}{|\Delta/H|} = |H/H \cap H^r| = \kappa(\mathcal{H}).$$

□

4. Monodromy groups of the join and intersection of two hypermaps

Throughout this section all the considered hypermaps will be regular. In the above section we have derived some information on the structure of the monodromy groups of $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$. In what follows we shall consider the problem how to calculate the above monodromy groups by using the action of monodromy groups of $\mathcal{H}$ and $\mathcal{K}$. Let $A = \langle r_0, \ldots, r_k \rangle$ and $B = \langle s_0, \ldots, s_k \rangle$ be two k-generated groups. Let us define their *monodromy product* $A \times_m B$ to be the subgroup of the direct product generated by (r_i, s_i), where $i = 0, 1, \ldots, k$. Note that S. Wilson calls it the parallel product in [18]. Further, denote by $\pi_1 : A \times_m B \to A$, $\pi_2 : A \times_m B \to B$ the natural projections erasing the second and first coordinate, respectively.

Theorem 4.1. *Let $\mathcal{H} = (A; r_0, r_1, r_2)$ and $\mathcal{K} = (B; s_0, s_1, s_2)$ be regular hypermaps. Then $Mon(\mathcal{H} \vee \mathcal{K}) = Mon(\mathcal{H}) \times_m Mon(\mathcal{K})$ and $Mon(\mathcal{H} \wedge \mathcal{K}) = Mon(\mathcal{H}) \times_m Mon(\mathcal{K})/Ker\, \pi_2 Ker\, \pi_1$.*

Proof. Let $\Delta = \langle R_0, R_1, R_2; R_0^2 = R_1^2 = R_2^2 = 1 \rangle$. Recall that the hypermap subgroup of $\mathcal{H}$ can be reconstructed as a stabiliser $H = STAB_\Delta(x_0)$ of a flag x_0 and similarly for $\mathcal{K}$, $K = STAB_\Delta(y_0)$. Denote by $\psi_1 : \mathcal{H} \to (\Delta/H; R_0H, R_1H, R_2H)$ the isomorphism of hypermaps and by $\psi_1^* : \mathrm{Mon}\,\mathcal{H} \to \Delta/H$ the induced group epimorphism sending $r_i \mapsto R_iH$, for $i = 0, 1, 2$. Similarly, denote by ψ_2 the isomorphism $\mathcal{K} \to (\Delta/K; R_0K, R_1K, R_2K)$ of hypermaps and by ψ_2^* the respective group epimorphism taking $s_i \mapsto R_iK$. Then we have an isomorphism $\Psi : \Delta/H \times_m \Delta/K \to \mathrm{Mon}\,(\mathcal{H}) \times_m \mathrm{Mon}\,(\mathcal{K})$ taking $(R_iH, R_iK) \to ((\psi_1^*)^{-1}(R_iH), (\psi_2^*)^{-1}(R_iK)) = (r_i, s_i)$.

In the proof of Proposition 3.4 we have already verified that the mapping $\Phi : \Delta/H \cap K \to \Delta/H \times_m \Delta/K$, taking $g(H \cap K)$ onto (gH, gK), is an isomorphism of groups. Now the composition $\Psi\Phi$ establishes an isomorphism $\mathrm{Mon}\,(\mathcal{H} \vee \mathcal{K}) \to \mathrm{Mon}\,(\mathcal{H}) \times_m \mathrm{Mon}\,(\mathcal{K})$.

Regarding the intersection of $\mathcal{H}$ and $\mathcal{K}$, by Proposition 3.4 we have

$$\mathrm{Mon}\,(\mathcal{H} \wedge \mathcal{K}) = \mathrm{Mon}\,(\mathcal{H} \vee \mathcal{K})/(H/H \cap K \times K/H \cap K).$$

In view of what we have proved it is enough to see that $\Psi\Phi$ sends $K/H \cap K$ onto $\mathrm{Ker}\,\pi_2$, and $H/H \cap K$ onto $\mathrm{Ker}\,\pi_1$. Indeed,

$$\Psi\Phi(K/H \cap K) = \Psi(\{(gH, K) | g \in K\}) =$$

$$\{(w, 1) | (w, 1) \in \mathrm{Mon}\,(\mathcal{H}) \times_m \mathrm{Mon}\,(\mathcal{K})\} = \mathrm{Ker}\,\pi_2.$$

Similar calculation verifies the statement $\Psi\Phi(H/H \cap K) = \mathrm{Ker}\,\pi_1$. □

We say that a covering $\mathcal{H} \to \mathcal{K}$ of regular hypermaps is *smooth* if both hypermaps are of the same type. Smooth covers of hypermaps correspond to unbranched covers of their topological equivalents.

Proposition 4.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. Then*

(a) the hypermap $\mathcal{H} \vee \mathcal{K}$ smoothly covers both $\mathcal{H}$, $\mathcal{K}$ if and only if the types of $\mathcal{H}$ and $\mathcal{K}$ are equal.

(b) if both $\mathcal{H}$ and $\mathcal{K}$ smoothly cover the intersection $\mathcal{H} \wedge \mathcal{K}$ then they have the same type.

We shall see later that the above implication (b) cannot be reversed.

5. Orthogonal hypermaps

Two regular hypermaps $\mathcal{H}$, $\mathcal{K}$ will be called *orthogonal* if $HK = \Delta$. We shall use $\mathcal{H} \perp \mathcal{K}$ to denote the orthogonality of $\mathcal{H}$ and $\mathcal{K}$. Let G, H be two groups. A *common epimorphic* image of G and H is a group Q such that there are epimorphisms $G \to Q$ and $H \to Q$. Let $H = \langle r_0, r_1, r_2 \rangle$, $K = \langle s_0, s_1, s_2 \rangle$ and $Q = \langle t_0, t_1, t_2 \rangle$ be groups. We say that Q is a *monodromic common epimorphic image* of H and K if both the assignments $r_i \mapsto t_i$ and $s_i \mapsto t_i$ (for $i = 0, 1, 2$) extend to group epimorphisms $H \to Q$ and $K \to Q$.

The following theorem gives several characterisations of the orthogonality.

Theorem 5.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. Then the following conditions are equivalent:*

(i) $\mathcal{H} \perp \mathcal{K}$,
(ii) $\mathcal{H} \wedge \mathcal{K}$ *is a trivial hypermap,*
(iii) $\mathcal{H}$ *and* $\mathcal{K}$ *have no nontrivial common quotients,*
(iv) *the monodromy groups $Mon(\mathcal{H})$ and $Mon(\mathcal{K})$ have no common monodromic epimorphic images,*
(v) $Mon(\mathcal{H} \vee \mathcal{K}) = Mon(\mathcal{H}) \times Mon(\mathcal{K})$.

Proof. *(i)* $\Leftrightarrow$ *(ii)* Since the flags of the intersection are the elements of Δ/HK, the intersection is a trivial hypermap if and only if $HK = \Delta$.

(ii) $\Leftrightarrow$ *(iii)*. If $\mathcal{H} \wedge \mathcal{K}$ is nontrivial then it forms a non-trivial common quotient. Vice-versa if there is a non-trivial common (possibly irregular) quotient $\mathcal{Q}$ then there are $g, h \in \Delta$ such that $\Delta > Q^g \geq K$ and $\Delta > Q^h \geq H$. By normality of both H and K we get $\Delta > Q^x \geq K$, $\Delta > Q^x \geq H$ for any $x \in \Delta$. Hence $\mathcal{Q}_\Delta = \mathcal{U}/\bigcap_{x \in \Delta} Q^x \to \mathcal{Q}$ is a non-trivial regular common quotient. However, since $\mathcal{Q}_\Delta$ is covered by $\mathcal{H} \wedge \mathcal{K}$, thus the intersection is a non-trivial hypermap.

(ii) $\Leftrightarrow$ *(v)*. By Proposition 3.4 $\mathrm{Mon}(\mathcal{H} \vee \mathcal{K}) \leq \mathrm{Mon}(\mathcal{H}) \times \mathrm{Mon}(\mathcal{K})$. The second part of Proposition 3.4 implies that the equality holds if and only if $\mathcal{H} \perp \mathcal{K}$.

(iii) $\Leftrightarrow$ *(iv)* If there is a common quotient $\mathcal{Q}$ for $\mathcal{H}$ and $\mathcal{K}$ then the coverings $\mathcal{H} \to \mathcal{Q}$ and $\mathcal{K} \to \mathcal{Q}$ induce, respectively, monodromy epimorphisms $\mathrm{Mon}(\mathcal{H}) \to \mathrm{Mon}(\mathcal{Q})$ and $\mathrm{Mon}(\mathcal{K}) \to \mathrm{Mon}(\mathcal{Q})$. Vice-versa, if Q is a monodromic common epimorphic image, then representing the hypermaps via hypermap subgroups we get that the assignments $gH \mapsto gQ$, $gK \mapsto gQ$, where g ranges in Δ extend to group epimorphisms. However, the same mappings establish coverings $\mathcal{U}/H \to \mathcal{U}/Q$ and $\mathcal{U}/K \to \mathcal{U}/Q$. The statement follows. $\square$

Denote by $\mathcal{O}$ the two-flag hypermap with $r_0 = r_1 = r_2$ being equal to the non-trivial involution interchanging the two flags. It is easy to see that the hypermap subgroup of $\mathcal{O}$ is Δ^+.

Proposition 5.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be (regular) hypermaps. If $\mathcal{H}$ and $\mathcal{K}$ are orthogonal then at least one of the hypermaps $\mathcal{H}$ and $\mathcal{K}$ is nonorientable.*

Proof. Assume both $\mathcal{H}$ and $\mathcal{K}$ are orientable. The orientability implies that both $H \leq \Delta^+$, $K \leq \Delta^+$ are subgroups of the even-word subgroup Δ^+ of Δ. Then $\mathcal{O} \cong \mathcal{U}/\Delta^+$ is a common non-trivial quotient, a contradiction. $\square$

In general, it can be difficult to see the orthogonality of hypermaps. In what follows we give some sufficient conditions implying the orthogonality of hypermaps. The following proposition is a straightforward consequence of Theorem 5.1.

Proposition 5.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. If the monodromy groups of $\mathcal{H}$ and $\mathcal{K}$ have no nontrivial common epimorphic images then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.*

Thus a regular hypermap with a non-abelian simple monodromy group is orthogonal to any other hypermap.

Numerical conditions implying the orthogonality may be useful in constructions. We shall present a sample of them.

Proposition 5.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps of types (m_0, m_1, m_2) and (n_0, n_1, n_2). Let one of them, say $\mathcal{H}$, be non-orientable.*

If for any two $i, j \in \{0, 1, 2\}$ the integers m_i, n_i and m_j, n_j are respectively coprimes then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.

Proof. Let the monodromy groups be generated by the triples of involutions $\mathrm{Mon}\,(\mathcal{H}) = \langle r_0, r_1, r_2 \rangle$ and $\mathrm{Mon}\,(\mathcal{K}) = \langle s_0, s_1, s_2 \rangle$. Denote by $R_i = r_i r_{i+1}$ and $S_i = s_i s_{i+1}$, $i = 0, 1, 2$. By the assumption, two of $gcd(R_i, S_i)$, $i = 0, 1, 2$ are equal to 1. Without loss of generality we assume $gcd(R_0, S_0) = 1$ and $gcd(R_2, S_2) = 1$. Then the following equality for the even word subgroup of the monodromy product holds true:

$$(\mathrm{Mon}\,(\mathcal{H}) \times_m \mathrm{Mon}\,(\mathcal{K}))^+ = \langle R_0, R_2 \rangle \times_m \langle S_0, S_2 \rangle = \mathrm{Mon}^+(\mathcal{H}) \times_m \mathrm{Mon}^+(\mathcal{K}).$$

To prove the orthogonality of $\mathcal{H}$ and $\mathcal{K}$ we show that the projections of the latter group into the coordinate factors contain isomorphic copies of the even-word subgroups of the original hypermaps. Since the orders of R_0 and S_0 are coprime, $(R_0, 1)$ and $(1, S_0)$ are elements of the cyclic group $\langle (R_0, S_0) \rangle$. For the same reason we see that $(R_2, 1)$ and $(1, S_2)$ belong to $\langle (R_2, S_2) \rangle$. Now observe $\mathrm{Mon}^+(\mathcal{H}) \cong \langle (R_0, 1), (R_2, 1) \rangle$, and similarly we get $\mathrm{Mon}^+(\mathcal{K}) \cong \langle (1, S_0), (1, S_2) \rangle$. Hence we have that $\mathrm{Mon}\,(\mathcal{H} \vee \mathcal{K})$ contains a subgroup $G = \mathrm{Mon}^+(\mathcal{H}) \times \mathrm{Mon}^+(\mathcal{K})$. Since $\mathcal{H}$ is non-orientable, $\mathrm{Mon}^+(\mathcal{H}) = \mathrm{Mon}\,(\mathcal{H})$. By Theorem 4.1 the monodromy group of the intersection

$$\mathrm{Mon}\,(\mathcal{H} \wedge \mathcal{K}) = \mathrm{Mon}\,(\mathcal{H}) \times_m \mathrm{Mon}\,(\mathcal{K}) / Ker\,\pi_2 Ker\,\pi_1.$$

Since $\mathrm{Mon}\,(\mathcal{H}) \times \mathrm{Mon}^+(\mathcal{K}) \le Ker\,\pi_2 Ker\,\pi_1$ the intersection is either trivial or a 2-flag hypermap. However, the only (regular) 2-flag hypermap is $\mathcal{O}$ which is obviously not covered by $\mathcal{H}$. Hence, the intersection is the trivial hypermap and we are done. □

A cellular embedding of a graph into a surface is called a *regular embedding* if the corresponding map is regular.

Proposition 5.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular maps determined by regular embeddings of two non-bipartite graphs with coprime valency. Then $\mathcal{H} \perp \mathcal{K}$ if and only if at least one of $\mathcal{H}$, $\mathcal{K}$ is non-orientable.*

Proof. If both embeddings define orientable maps then they both cover $\mathcal{O}$, and consequently, they are not orthogonal.

Let one of the maps associated with the embeddings of graphs is non-orientable. With the same notation as above we have $R_2^2 = 1 = S_2^2$, because the hypermaps are maps now. Since the valences of the maps are coprime we have that $(R_1, 1)$

and $(1, S_1)$ belong to the monodromy product of the even word subgroups. Since the graphs are non-bipartite there are identities of the form $\prod_{i=1}^{k} R_1^{m_i} R_2 = 1$, $\prod_{i=1}^{n} S_1^{p_i} S_2 = 1$, where k and n are some odd integers. Replacing above R_1 by $(R_1, 1)$, R_2 by (R_2, S_2), S_1 by $(1, S_1)$ and S_2 by (R_2, S_2) we get that the involutions $(R_2, 1)$ and $(1, S_2)$ are elements of the monodromy product. Hence the even-word subgroup of the monodromy product is the direct product of the even-word subgroups of the original maps. Now we can complete the proof as above. □

There are oriented versions of the above propositions. We shall state them without proofs.

Proposition 5.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be oriented regular hypermaps of types (m_0, m_1, m_2) and (n_0, n_1, n_2).*

If for any two $i, j \in \{0, 1, 2\}$ the integers m_i, n_i and m_j, n_j are respectively coprimes then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.

A cellular embedding of a graph into an orientable surface is called *orientably regular* if the corresponding oriented map is regular.

Proposition 5.7. *Orientably regular embeddings of non-bipartite graphs with coprime valency determine a couple of orthogonal oriented maps.*

6. Orientability, reflexibility and self-duality

Topological and algebraic properties of maps, as for instance, the orientability, the reflexibility and the self-duality have their counterparts in the associated algebraic representations. The aim of this section is to discuss the above properties and concepts in a relation with the join and with the intersection of two hypermaps.

6.1 Orientability.

An algebraic hypermap $\mathcal{H} = \mathcal{U}/H$ is *orientable* if $H \leq \Delta^+$, where $\Delta^+ = \langle r_1 r_2, r_2 r_0 \rangle$ is the even word subgroup of Δ (which is an index two subgroup).

The following statements are direct consequences of the definitions so we shall omit the proofs of them.

Proposition 6.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps with the respective hypermap subgroups H and K. Then*

if both $\mathcal{H}$ and $\mathcal{K}$ are orientable then both $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are orientable as well,

if one of $\mathcal{H}$, $\mathcal{K}$ is orientable and the other not then $\mathcal{H} \vee \mathcal{K}$ is orientable while $\mathcal{H} \wedge \mathcal{K}$ is nonorientable,

if both $\mathcal{H}$ and $\mathcal{K}$ are nonorientable then $\mathcal{H} \wedge \mathcal{K}$ is nonorientable as well.

Proposition 6.2. *Let $\mathcal{H}$ be a regular hypermap. The following statements are equivalent:*

$\mathcal{H}$ is orientable,

$\mathcal{H}$ covers $\mathcal{O}$,

$\mathcal{H} = \mathcal{H} \vee \mathcal{O}$,

$\mathcal{O} = \mathcal{H} \wedge \mathcal{O}$.

It follows that $\mathcal{H}$ is nonorientable if and only if $\mathcal{H} \perp \mathcal{O}$ and the algebraic counterpart to the well-known construction of the antipodal double cover over a nonorientable hypermap $\mathcal{H}$ is the construction of the join $\mathcal{H} \vee \mathcal{O}$ (cf. [18]). Let us remark that in the case of maps the first three items of Proposition 6.2 are covered by [18].

6.2 Reflexibility.

A regular oriented hypermap $\mathcal{K} = \mathcal{U}^+/K$ is *reflexible* if $K^{r_0} = K$. Note that since $\mathcal{K}$ is a normal subgroup of Δ^+, we have $K^{r_0} = K^{r_1} = K^{r_2}$. The hypermap $\mathcal{K}^r = \mathcal{U}^+/K^{r_0}$ will be called the *mirror image* of $\mathcal{K}$. Clearly, the join $\mathcal{K} \vee \mathcal{K}^r$ and the intersection $\mathcal{K} \wedge \mathcal{K}^r$ are reflexible hypermaps. In general, we have the following

Proposition 6.3. *Let $\mathcal{K}$ be an oriented regular hypermap. Then*

the join $\mathcal{K} \vee \mathcal{K}^r$ is the least reflexible regular oriented hypermap covering $\mathcal{K}$,
the intersection $\mathcal{K} \wedge \mathcal{K}^r$ is the largest reflexible regular oriented hypermap covered by $\mathcal{K}$.

As it was already noted, see Proposition 3.5, the integer $\kappa(\mathcal{K}) = s(\mathcal{K}, \mathcal{K}^r) = \frac{|\mathcal{K}|}{|\mathcal{K}\wedge\mathcal{K}^r|}$, called the chirality index in [3], can be used to measure of how much a given hypermap is far from being mirror symmetric. Moreover, the way how two hypermaps with the same chirality index are chiral can be of different quality. More precisely, for any oriented regular hypermap we have coverings $\mathcal{H} \vee \mathcal{H}^r \to \mathcal{H} \to \mathcal{H}\wedge\mathcal{H}^r$. It is proved in [3] that the two associated groups of covering transformations are isomorphic and their size is equal to the about mentioned chirality index which coincides with the shared cover index $s(\mathcal{H}, \mathcal{H}^r)$. The associated group is called the *chirality group* of $\mathcal{H}$. It is proved in [3] that any finite abelian group can be isomorphic to the chirality group of a regular hypermap. Members of several infinite families of non-abelian groups are proved to appear as chirality groups as well (see [3]).

6.3 Self-duality.

Let σ be a permutation of $\{0, 1, 2\}$. Clearly, σ induces an outer automorphism $\bar{\sigma}$ of Δ mapping $r_i \mapsto r_{i\sigma}$. A σ-dual of $\mathcal{H}$ is the hypermap $\mathcal{U}/\bar{\sigma}(H)$ with the hypermap subgroup $\bar{\sigma}(H)$. It may happen that $H = \bar{\sigma}(H)$, in this case $\mathcal{H}$ is called *σ-selfdual.* If $\mathcal{H}$ is σ-self-dual for all 6 possible permutations σ of the index-set we shall say that $\mathcal{H}$ is *totally selfdual.* Similarly as for the reflexibility we have the following statement.

Proposition 6.4. *Let $\mathcal{K}$ be a regular hypermap and let σ is a permutation of $\{0, 1, 2\}$. Then*

(1) *the join $\mathcal{K} \vee \mathcal{K}^{\bar{\sigma}}$ is the least σ-selfdual regular hypermap covering $\mathcal{K}$,*
(2) *the intersection $\mathcal{K} \wedge \mathcal{K}^{\bar{\sigma}}$ is the largest σ-selfdual regular hypermap covered by $\mathcal{K}$.*

In particular, if S_3 denotes the group of all permutations of $\{0, 1, 2\}$ then $\mathcal{U}/\bigcap_{\sigma\in S_3} K^{\bar{\sigma}}$ is the least totally selfdual hypermap covering $\mathcal{K}$. Similarly, $\mathcal{U}/\prod_{\sigma\in S_3} K^{\bar{\sigma}}$ is the largest totally selfdual hypermap covered by $\mathcal{K}$.

7 G-symmetric maps and hypermaps

The results of the previous section are just particular instances of a more general approach. Let Out (Δ) be the outer automorphism group of Δ. Recall that Out (Δ) = Aut (Δ)/Inn (Δ), where Inn (Δ) denotes the group of inner automorphisms of Δ acting by conjugation on Δ. The outer automorphism group Out (Δ) was described by L. James in [8]. It follows that Out $(\Delta) \cong PSL(2, Z)$ and it is generated by the 6 permutations permuting the three generators r_0, r_1 and r_2 and one twisting automorphism taking $r_0 \mapsto r_2r_0r_2$, $r_1 \mapsto r_1$, $r_2 \mapsto r_2$. The orbit of the action of Out(Δ) on a hypermap $\mathcal{H}$ is finite and can be constructed by making σ-duals and applying the twisting operator repeatedly.

Let $G \leq$ Out(Δ) be a subgroup. If $\mathcal{H}$ is a regular hypermap then for each $\phi \in G$ the hypermap $\mathcal{H}^\phi = \mathcal{U}/H^\phi$ is also regular. We say that $\mathcal{H}$ is *G-symmetric* if it is invariant with respect to G, i. e. $\mathcal{H} = \mathcal{H}^\phi$ for every $\phi \in G$, or equivalently, $H = H^\phi$ for every $\phi \in G$. The join $\bigvee_{\phi\in G}\mathcal{H}^\phi$ and the intersection $\bigwedge_{\phi\in G}\mathcal{H}^\phi$ are clearly G-symmetric hypermaps for any regular hypermap $\mathcal{H}$. By the definition $\mathcal{H}$ is G-symmetric if and only if $\mathcal{H} \cong \bigvee_{\phi\in G}\mathcal{H}^\phi$ or equivalently, $\mathcal{H} \cong \bigwedge_{\phi\in G}\mathcal{H}^\phi$. We have the following statement.

Proposition 7.1. *Let $\mathcal{H}$ be a regular hypermap and $G \leq Out(\Delta)$. Then*

the join $\bigvee_{\phi\in G}\mathcal{H}^\phi$ is the least G-symmetric regular hypermap covering $\mathcal{H}$,

the intersection $\bigwedge_{\phi\in G}\mathcal{H}^\phi$ is the largest G-symmetric regular hypermap covered by $\mathcal{H}$.

We can use the covering transformation groups of the coverings $\bigvee_{\phi\in G}\mathcal{H}^\phi \to \mathcal{H}$ and $\mathcal{H} \to \bigwedge_{\phi\in G}\mathcal{H}^\phi$ to measure of how much a given regular hypermap is far from being G-symmetric. Clearly, $\mathcal{H}$ is G-symmetric if and only if these coverings are trivial. In the case $|G| = 2$ we can say something more.

Proposition 7.2. *Let $\phi \in Out(\Delta)$ ($\phi \in Out(\Delta^+)$). Let $\mathcal{H}$ be a regular hypermap (an oriented regular hypermap). Then the groups of covering transformations of coverings $\mathcal{H} \vee \mathcal{H}^\phi \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{H} \wedge \mathcal{H}^\phi$ are isomorphic. Let this common group be denoted by $\mathcal{C}(\mathcal{H}, \mathcal{H}^\phi)$. The order of $\mathcal{C}(\mathcal{H}, \mathcal{H}^\phi)$ is the shared cover index $s(\mathcal{H}, \mathcal{H}^\phi)$.*

Proof. The first covering is defined by the epimorphism $\pi : \Delta/H \cap H^\phi \to \Delta/H$ taking $x(H \cap H^\phi) \mapsto xH$ for any $x \in \Delta$. Clearly, the kernel $\operatorname{Ker}\pi \cong H/H \cap H^\phi$.

The second covering is defined by the epimorphism $\sigma : \Delta/H \to \Delta/HH^\phi$ taking $xH \mapsto xHH^\phi$. Now the kernel is $\operatorname{Ker}\sigma \cong HH^\phi/H$.

By the third isomorphism theorem we have

$$\operatorname{Ker}\sigma \cong HH^\phi/H \cong H/H \cap H^\phi \cong \operatorname{Ker}\pi.$$

To complete the proof of the statement we proceed similarly as in the proof of Proposition 3.5

$$s(\mathcal{H}, \mathcal{H}^\phi) = \frac{gcd(|\mathcal{H}|, |\mathcal{H}^\phi|)}{|\mathcal{H} \wedge \mathcal{H}^\phi|} = \frac{|\mathcal{H}|}{|\mathcal{H} \wedge \mathcal{H}^\phi|} = \frac{|\Delta/H|}{|\Delta/HH^\phi|} = |HH^\phi/H| = |\operatorname{Ker}\sigma|.$$

□

It follows that if $G \leq$ Out(Δ) is of order two, the two covering transformation groups of the covering $\mathcal{H} \vee \mathcal{H}^\phi \to \mathcal{H} \to \mathcal{H} \wedge \mathcal{H}^\phi$ are isomorphic. In the special case, when $G \leq$ Out(Δ^+) is the group acting on an oriented map $\mathcal{H}$ by taking its mirror image $\mathcal{H}^r$ we get, as a corollary, Theorem 3 of [3]. In this case the chirality group mentioned in the previous section coincide with the group $\mathcal{C}(\mathcal{H}, \mathcal{H}^\phi)$.

If a regular hypermap $\mathcal{H}$ is G-symmetric for $G = Out(\Delta)$ we say that $\mathcal{H}$ is *characteristic.* In this case any automorphism of Δ leaves the hypermap subgroup H invariant. Hence H is a characteristic subgroup of Δ. Vice-versa, a characteristic subgroup $H \trianglelefteq \Delta$ of finite index determines an Out(Δ)-symmetric regular hypermap. Consequently, drawings of hypermaps sharing this property can be viewed as pictures of the characteristic subgroups of finite index of Δ. Since the structure of Out(Δ) was described by L. James in [8] we have the following:

Theorem 7.3. *Let $\mathcal{H} = (F; r_0, r_1, r_2)$ be a hypermap. The following statements are equivalent:*

$\mathcal{H}$ is regular and $Out(\Delta)$-symmetric (that is, $\mathcal{H}$ is characteristic),
$\mathcal{H} \cong \mathcal{U}/K$, where K is a characteristic subgroup of finite index,
$\mathcal{H}$ is regular and isomorphic to each of the following three hypermaps $(F; r_2, r_1, r_0)$, $(F; r_0, r_2, r_1)$ and $(F; r_2 r_0 r_2, r_1, r_2)$.

A similar statement can be formulated in the case of oriented hypermaps by using the fact $\mathrm{Out}(\Delta^+) = \langle \mathrm{Out}(\Delta), \rho \rangle$ where ρ is the automorphism mapping an oriented hypermap onto its mirror image $\rho : (D; R, L) \mapsto (D; R^{-1}, L^{-1})$, see [8].

Finally we stretch that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two G-symmetric hypermaps for some $G \leq \mathrm{Out}(\Delta)$ then the join $\mathcal{H}_1 \vee \mathcal{H}_2$ and the intersection $\mathcal{H}_1 \wedge \mathcal{H}_2$ are also G-symmetric hypermaps.

Example. Let us examine characteristic subgroups H of Δ of small index via the corresponding Δ-symmetric regular hypermaps $\mathcal{H}$. Since $\mathcal{H}$ is totally selfdual such a hypermap is of type (n, n, n) for some integer $n \geq 1$. There is just one non-trivial regular hypermap of type $(1, 1, 1)$ and it is $\mathcal{O} \cong \Delta/\Delta^+$. Obviously Δ^+ is characteristic. Also it is easy to see that we have only one Δ-symmetric regular hypermap $\mathcal{H}$ of type $(2, 2, 2)$ - this is actually the only characteristic map. Its topological hypermap arises by colouring the opposite faces of the cube by the same colour (see Fig. 1). Consequently, $\Delta/H = C_2^3$ is elementary abelian and $\mathcal{H} = \mathcal{D}$ [2].

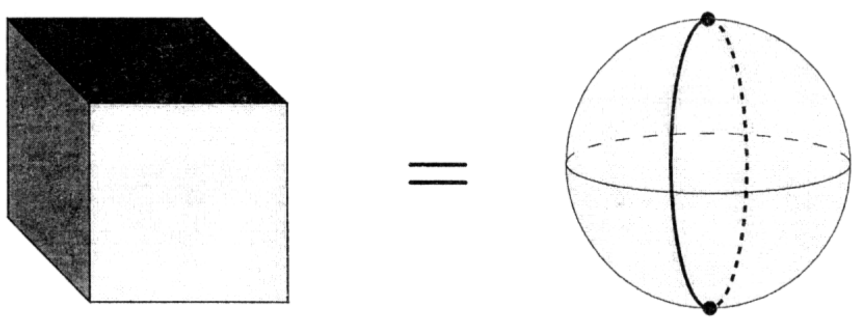

FIGURE 1

As concerns type $(3, 3, 3)$ we shall argue as follows. Clearly, $\mathcal{H}$ is one of the toroidal hypermaps classified by Corn and Singerman in [6]. The smallest representative of the family is the hypermap $\mathcal{H}_2$ drawn on Fig. 2.

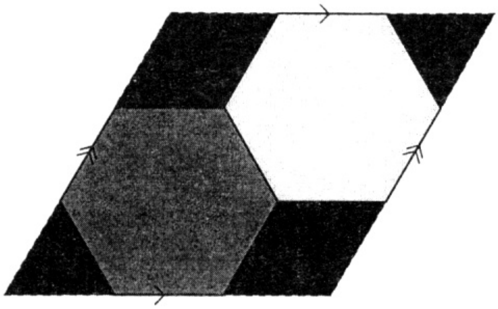

FIGURE 2

It has 6 flags and one hypervertex, hyperedge and hyperface, respectively. As all the regular toroidal hypermaps of type (3,3,3) it is totally selfdual. However, it is easy to see that the three involutory generators satisfy the relation $r_2r_0r_2 = r_1$. Hence, the twisting operator takes $\mathcal{H}_2 = (F; r_0, r_1, r_2)$ onto $(F; r_1, r_1, r_2)$. The latter hypermap is clearly not isomorphic to $\mathcal{H}_1$ since it has type $(1, 3, 3)$. Take two hypermaps $\mathcal{A}$ and $\mathcal{B}$ from the orbit of Δ with the respective types $(1, 3, 3)$ and $(3, 1, 3)$. By Proposition 5.6 the corresponding oriented hypermaps are orthogonal and so $\mathcal{A} \wedge \mathcal{B} = \mathcal{O}$. Hence $\mathcal{O}$ is the largest Δ-symmetric hypermap covered by both $\mathcal{A}$ and $\mathcal{B}$, and consequently, by $\mathcal{H}_2$ as well. The covering $\mathcal{H}_2 \to \mathcal{O} = \mathcal{H}_2 \wedge \mathcal{A}$ is a 3-fold covering. Hence, $\mathcal{H}_2 \vee \mathcal{A} = \mathcal{K}$ is a Δ-symmetric regular hypermap of type $(3, 3, 3)$, and the covering $\mathcal{K} \to \mathcal{H}_2$ is a 3-fold covering. Consequently, $\mathcal{K}$ has 18 flags. By [6] there is precisely one such hypermap $\mathcal{H}_3$ of type $(3, 3, 3)$ depicted on Fig.3. Since the oriented hypermaps $\mathcal{A}$ and $\mathcal{B}$ are orthogonal, the even word subgroup is isomorphic to the direct product $C_3 \times C_3$. The monodromy group of $\mathcal{H}_3$ is then a semidirect product of $(C_3 \times C_3)$ by C_2.

We can prove that this is a unique Δ-symmetric regular hypermap of type $(3, 3, 3)$. As a curiosity let us mention that the underlying 3-valent graph is known as the Pappus graph, which is related to the well-known Pappus configuration, a popular example of a finite geometry.

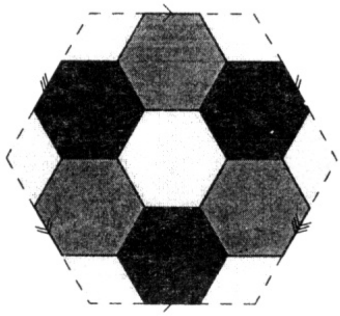

FIGURE 3

Since the join of two characteristic hypermaps is again characteristic the hypermap $\mathcal{H}_4 = \mathcal{D} \vee \mathcal{H}_3$ of type $(6, 6, 6)$ is also a characteristic hypermap. By Proposition 5.6 the oriended hypermaps corresponding to $\mathcal{D}$ and $\mathcal{H}_3$ are orthogonal. Hence the even word subgroup of the monodromy group of $\mathcal{H}_4$ is the direct product $C_2^2 \times C_3^2$ and it is of size 36. Consequently, $\mathcal{H}_4$ is a characteristic hypermap of genus 10.

To find a Δ-symmetric regular hypermap of type $(4, 4, 4)$ we checked the list of regular hypermaps of genus 2 in [1]. There is precisely one regular hypermap $\mathcal{H}_5$ of type $(4, 4, 4)$ and genus 2. The hypermap is totally selfdual (see Fig. 4). A direct computation verifies the relation $(r_2 r_0 r_2 r_1)^4 = 1$, hence the twisting operator applied on $\mathcal{H}_5$ gives a regular hypermap of type $(4, 4, 4)$ with the same number of flags. Since the orientability is preserved, it is a hypermap on an orientable surface of genus 2. Since there is just one regular hypermap of type $(4, 4, 4)$ on the surface of genus 2, it must be $\mathcal{H}_5$. Consequently, $\mathcal{H}_5$ is Δ-symmetric.

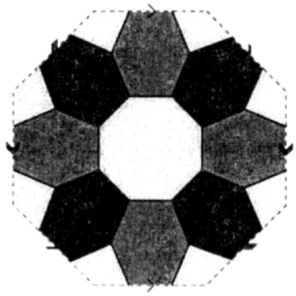

FIGURE 4

The join $\mathcal{H}_6 = \mathcal{H}_3 \vee \mathcal{H}_4$ is a characteristic hypermap of type $(12, 12, 12)$. By Proposition 5.6 the corresponding oriented hypermaps are orthogonal, hence the even word subgroup is the direct product of the even word subgroups of factors. Consequently, the size of the (full) monodromy group $|\mathrm{Mon}(\mathcal{H}_6)| = 144$ and the genus is 28.

If we restrict ourselves to maps then the characterisation of the outer automorphism group $\mathrm{Out}(\Delta(\infty, \infty, 2))$ done by Jones and Thornton [12] can be useful. Recall that $\Delta(\infty, \infty, 2) = \langle r_0, r_1, r_2; r_0^2 = r_1^2 = r_2^2 = (r_0 r_2)^2 = 1 \rangle$ is a monodromy group of the universal map covering any map. The outer automorphism group is isomorphic to S_3 and is generated by two operations (see [12]), first one defined by $(F; r_0, r_1, r_2) \to (F; r_2, r_1, r_0)$ and second one defined by $(F; r_0, r_1, r_2) \to (F; r_0 r_2, r_1, r_2)$. First one is known as the duality operation while the second one coincide with the Petrie operation. A more detailed discussion on $\mathrm{Out}(\Delta(\infty, \infty, 2))$-symmetric regular maps can be found in Section 5 of [12]. Theorem 3 in [12] is similar to our Theorem 7.3.

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APPLICATIONS OF LINE OBJECTS IN ROBOTICS

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ABSTRACT. In this paper the Lie algebra of the Lie group of Euclidean motions in E_3 is explained as the vector space A_6 of couples of vectors in E_3. All subalgebras and all 3-dimensional subspaces of A_6 which are orthogonal to themselves according to the Klein form and their kinematic interpretations are described. Vector fields in E_3 determined by elements of A_6 and their kinematic and dynamic interpretations are investigated

1 INTRODUCTION

Line Plücker's coordinates inspire applications of couples of vectors in robotics. First of all in this paper the Plücker's coordinates, basic structure properties such as the Klein and Killing forms, the Lie bracket in the algebra A_6 of all couples of vectors in Euclidean space E_3 are recalled. The algebra A_6 is isomorphic with the Lie algebra of the Lie group of all isometries preserving orientation in E_3. All subalgebras of A_6 and all 3-dimensional subspaces which are orthogonal to themselves according to the Klein form are described. Mechanical engineers use the notion of screws as a useful tool for solving of robotic problems. The roots of this notion are in $19^{\underline{th}}$ century, Ball [1]. We describe the set of screws as a projective 5-dimensional space P_5^σ of all 1-dimensional subspaces in A_6, so the sum of two screws has not sense. We show that the Lie bracket in A_6 induces both a map $P_5^\sigma \times P_5^\sigma \to P_5^\sigma$ and a map $\beta \times \beta \to \beta$ defined on couples of nonparallel lines in E_3, where β is the manifold of proper lines in E_3. Inspired by [2] and [4] we introduced vector fields in E_3 induced by elements of A_6 and give their kinematic and dynamic interpretations. This work does not give quite new original results except the description of subalgebras of A_6 and of their kinematic interpretations. Perhaps it will be useful from the point of view of explanation which is close to the papers [5] and [3]. We prefer the algebra of vector couples to the dual number and dual quaternion technique because of the cleaner geometrical and mechanical interpretation.

2 PLÜCKER LINE COORDINATES, VECTOR COUPLES, SCREWS

Let V_3 be the vector space associated to the Euclidean space E_3. The scalar or vector or mixed product of vectors in V_3 will be denoted by $\overline{a}.\overline{b}$ or $\overline{a} \times \overline{b}$ or

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$(\overline{a} \times \overline{b}).\overline{c}$ respectively. Let $(0, \overline{e}_1, \overline{e}_2, \overline{e}_3)$ be a Cartesian coordinate system in E_3. Let $p = AB$ be a line determined by its two different points A, B. The couple of vectors $\overline{s} = \overline{AB}, \overline{m} = \overline{OA} \times \overline{s} = \overline{OA} \times \overline{OB}$ is called Plücker line coordinates. In cartesian coordinates, $\overline{s} = (s_1 = b_1 - a_1, s_2 = b_2 - a_2, s_3 = b_3 - a_3)$, $\overline{m} = (m_1 = a_2b_3 - b_2a_3, m_2 = a_3b_1 - a_1b_3, m_3 = a_1b_2 - b_1a_2)$. Plücker line coordinates will be called canonical if $\|\overline{s}\| = \sqrt{\overline{s}.\overline{s}} = 1$. Let us note that Plücker coordinates $(\overline{s}, \overline{m})$ satisfy the equality $\overline{s}.\overline{m} = 0$.

Remark 1. Let (x_0, x_1, x_2, x_3) be homogeneous coordinates in E_3, where the equality $x_0 = 0$ means improper points (points in infinity). Let $[A, B]^T$ denote the matrix $\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}$. Then $p_{ik} = a_ib_k - a_kb_i$ $i, k = 0, 1, 2, 3$ are the homogeneous Plücker coordinates of a line $p = AB$. They satisfy the equality $\det[A, B; A, B]^T = p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$ which corresponds with the condition $\overline{s}.\overline{m} = 0$ in Plücker coordinates $(\overline{s}, \overline{m})$.

Plücker coordinates of a line p is a couple of two vectors $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$, $\overline{s}.\overline{m} = 0$. It is easy to see that the point $C, \overline{OC} = (\overline{s} \times \overline{m})/\overline{s}^2$, is the orthogonal projection of the coordinate origin 0 into p. If we change determining points of a line p then we get a couple $(k\overline{s}, k\overline{m})$. Changing the origin 0 we obtain a couple $(\overline{s}, \overline{m}' = \overline{m} + \overline{0'0} \times \overline{s})$. It means that the vector $\overline{m}$ of the Plücker coordinates $(\overline{s}, \overline{m})$ depends on the origin 0 but the scalar product $\overline{s}.\overline{m}$ does not depend on 0.

Vice versa, an ordered couple of vectors $(\overline{s}, \overline{m})$, $0 \neq \overline{s}$, $\overline{m} \in V_3$, determines the line p in the direction $\overline{s}$ and passing through the point $C, \overline{OC} = \overline{s} \times \overline{m}/\overline{s}^2$. This line will be called the line of the couple $(\overline{s}, \overline{m})$. The line of a couple $(0, \overline{m})$ is the unproper line p of all parallel plains the normal vector of which is $\overline{m}$. There is not any line of the couple $(0, 0)$. We use $p = \pi((\overline{s}, \overline{m}))$ for $(\overline{s}, \overline{m}) \neq (\overline{0}, \overline{0})$.

Let us remind that the set of all ordered couples $(\overline{s}, \overline{m}) \in V_3 \times V_3$ has a real vector space structure where

$$k_1(\overline{s}_1, \overline{m}_1) + k_2(\overline{s}_2, \overline{m}_2) = (k_1\overline{s}_1 + k_2\overline{s}_2, k_1\overline{m}_1 + k_2\overline{m}_2)$$

Lema 1. *Let p be the line of couple $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$. Then every couple $(\overline{s}', \overline{m}')$ with the line p is of the form $\overline{s}' = k\overline{s}$, $\overline{m}' = k\overline{m} + u\overline{s}$, $0 \neq k$, $u \in R$.*

Proof. The line p is the line of a couple $(\overline{s}', \overline{m}')$ if and only if $\overline{s}' = k\overline{s}$ and $\overline{OC}' = \overline{OC}$. Comparing $\overline{OC}' = \frac{\overline{s}' \times \overline{m}'}{(\overline{s}')^2} = \frac{k\overline{s} \times \overline{m}'}{k^2\overline{s}^2} = \frac{\overline{s} \times \overline{m}'}{k\overline{s}^2}$ with $\overline{OC} = \frac{\overline{s} \times \overline{m}}{s^2}$ we get $\overline{m}' = k\overline{m} + u\overline{s}$. □

The set β_p of all couples $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$ with the same proper line p is two-parametric. If $X_i = (\overline{s}_i, \overline{m}_i) = (k_i\overline{s}, k_i\overline{m} + u_i\overline{s}) \in \beta_p$, $i = 1, 2$, then for $k \neq 0$ also $kX_1 \in \beta_p$ and for $k_1 + k_2 \neq 0$ also $X_1 + X_2 \in \beta_p$. Denote $R(V_3 \times V_3) := \{(\overline{s}, \overline{m}) \in V_3 \times V_3; \overline{s}^2 \neq 0\}$. We say that $X_i = (\overline{s}_i, \overline{m}_i) \in R(V_3 \times V_3)$, $i = 1, 2$, are L-equivalent iff there are $0 \neq k$, $u \in R$ such that $\overline{s}_2 = k\overline{s}_1$, $\overline{m}_2 = k\overline{m}_1 + u\overline{s}_1$,i.e. iff there is a line p, that $X_1, X_2 \in \beta_p$. Denote β the space of all L-equivalence classes in $R(V_3 \times V_3)$. There is a one-to-one correspondence between the set of all proper lines in E_3 and the set β. Then β is a 4-dimensional manifold. Let $\pi_1 : R(V_3 \times V_3) \to \beta$ be the map where $\pi_1(\overline{s}, \overline{m})$ is the class of L-equivalent elements determined by $(\overline{s}, \overline{m})$. Certainly $\pi_1 : R(V_3 \times V_3) \to \beta$ is a fibre manifold, fibre $\pi_1^{-1}(p) = \beta_p$ of which have an almost vector space structure, i. e. under the conditions introduced above kX_1 and $X_1 + X_2$ belong to the same fibre as X_1 and X_2.

An ordered couple $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$ is called the Plücker's couple if $\overline{s}.\overline{m} = 0$. It is said to be canonical if also $\overline{s}^2 = 1$.

Lema 2. *Let $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$ be a Plücker's couple. Let p be the line of $(\overline{s}, \overline{m})$. Then $(\overline{s}, \overline{m})$ is the Plücker's coordinate of p.*

Proof. Let us remind two well known equalities

$$\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a}.\overline{c})\overline{b} - (\overline{a}.\overline{b})\overline{c} \tag{1}$$

$$(\overline{a} \times \overline{b}).(\overline{c} \times \overline{d}) = (\overline{a}.\overline{c})(\overline{b}.\overline{d}) - (\overline{b}.\overline{c})(\overline{a}.\overline{d}) \tag{2}$$

Using the equality (1) and $\overline{s}.\overline{m} = 0$ we get

$$\overline{OC} \times \overline{s} = \frac{\overline{s} \times \overline{m}}{\overline{s}^2} \times \overline{s} = \frac{\overline{s}^2\overline{m} - (\overline{s}.\overline{m})\overline{s}}{\overline{s}^2} = \overline{m}.$$

□

Definition 1. *Every 1-dimensional subspace in $V_3 \times V_3$ will be called a screw. Every couple $X = (\overline{s}, \overline{m}) \neq (0, 0)$ determines the screw $\langle X \rangle$ where $\langle M \rangle$ denotes the vector space spanned on a set $M \subset V_3 \times V_3$. The couple X is called a representative of the screw $\langle X \rangle$. A screw $\langle X \rangle$ is called proper or improper if $\overline{s} \neq \overline{0}$ or $\overline{s} = \overline{0}$ respectively.*

It is clear that if X is a representative of $\langle X \rangle$ then every representative of $\langle X \rangle$ is of the form kX, $k \neq 0$, and then all representatives have the same line of couple which will be called the line of $\langle X \rangle$.

It immediately follows from the definition of screws that the set P_5^σ of all screws is a projective 5-dimensional space. Let $\pi_2 : V_3 \times V_3 \to P_5^\sigma$ be a such map that $\pi_2(X) = \langle X \rangle$. It means that π_2 is a 1-dimensional vector fibration.

Let $X = (\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$, be a couple of vectors. Denote $h := (\overline{s}.\overline{m})/\overline{s}^2$. It is easy to prove the following property.

Lemma 3. *The number h does not depend on a choice of a representative of the screw $\langle X \rangle$.*

Definition 2. *The number $h = (\overline{s}.\overline{m})/\overline{s}^2$ will be called pitch of the screw $\langle X \rangle, X = (\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$. If $\overline{s} = \overline{0}$ we put $h = \infty$.*

Let us recall that $h\overline{s}$ is the orthogonal projection of $\overline{m}$ into $\overline{s}$.

Corollary of Lemma 2. *Two proper screws which have the same screw line are both of the form $\langle(\overline{s}, \overline{m})\rangle$ and $\langle(\overline{s}, \overline{m} + u\overline{s})\rangle$, $\overline{s} \neq \overline{0}$, $u \neq 0$. It means that the set of all screws with the same screw line form one-parametric family. If h is the pitch of the first screw then the pitch of the second one is $h + u$.*

Remark 2. It is conspicuous that a proper screw is determined by its line and by its pitch h. This property is often taken as the definition of screws, see for example [4], [5].

Remark 3. Let us emphasize that the sum of two screws has not any sense because sums of different representatives have not to belong to the same screw.

A proper screw $\langle(\overline{s}, \overline{m})\rangle$, $\overline{s} \neq \overline{0}$, is called the Plücker's screw if $h = \overline{s}.\overline{m} = 0$.

Lemma 4. *There is a unique Plücker's screw in the set of all proper screws with the same screw line.*

Proof. Let $\langle(\overline{s},\overline{m})\rangle$, $\overline{s}\neq\overline{0}$ be a screw with the screw line p. Then every screw with the screw line p is of the form $\langle(\overline{s},\overline{m}+u\overline{s})\rangle$. But this screw is the Plücker's one iff $\overline{s}.(\overline{m}+u\overline{s})=0$, i. e. iff $u=-(\overline{s}.\overline{m})/s^2=-h$. It completes our proof.

It is clear that there is a one-to-one correspondence between the set P^σ of Plücker's screws and the space of all lines in E_3, i. e. between the spaces P^σ and β.

Remark 4. It is clear that the line p of a couple $(\overline{s},\overline{m})$, $\overline{s}\neq\overline{0}$ depends on the choice of origin O. If O' is another origin then the line p' of the couple $(\overline{s},\overline{m})$ is the image of p in the translation determined by the vector $\overline{00'}$.

3. Lie Algebra Of Vector Couples

Vector space $V_3\times V_3$ of all couples $(\overline{s},\overline{m})$ is closely connected with geometry of lines in E_3. Remind that $\overline{s}\in V_3$ is the direction of the line p of a couple $(\overline{s},\overline{m})$ and does not depend on coordinate systems. In contrary $\overline{b}$ depends on the choice of the origin 0, but $\overline{s}.\overline{m}$ is independent on 0.

So in the space $V_3\times V_3$ there are natural scalar and vector bilinear forms which gives useful information about geometrical and physical objects connected with lines in E_3. Remind them.

a)Klein scalar bilinear form KL:
Let $X_i=(\overline{s}_i,\overline{m}_i)\in V_3\times V_3$, $i=1,2$. Then

$$KL(X_1,X_2)=\overline{s}_1.\overline{m}_2+\overline{s}_2.\overline{m}_1$$

It is a symmetric regular bilinear scalar form on $V_3\times V_3$ of the signature $(+,+,+,-,-,-)$. Its quadratic form will be written in the form $KL(X):=\frac{1}{2}KL(X,X)=\overline{s}.\overline{m}$. Vectors $X_1,X_2\in V_3\times V_3$ will be called KL-orthogonal if $KL(X_1,X_2)=0$.

A subspace $B\subset V_3\times V_3$ is called KL-orthogonal to a subspace $A\subset V_3\times V_3$ if $KL(X,Y)=0$ for every $X\in A$ and every $Y\in B$. There is a unique subspace A^K which is totally KL-orthogonal to a subspace $A\subset V_3\times V_3$, i. e. if any vector subspace B is KL-orthogonal to A then $B\subset A^k$.

From the definition of KL-orthogonality it follows

1) A couple $X=(\overline{s},\overline{m})$, $\overline{s}\neq\overline{0}$ is KL-orthogonal to itself if and only if is a Plücker's couple.

2) If couples X, Y are KL-orthogonal then kX, uY are also KL-orthogonal.

So we can introduce KL-orthogonality in the case of screws. We say that two screws $\langle X\rangle$, $\langle Y\rangle$ are KL-orthogonal if X, Y are KL-orthogonal. Then $\langle X\rangle$ is KL-orthogonal to itself iff is a Plücker's screw.

Lemma 5. *Let p_1, p_2 be two non-parallel lines in E_3. Then p_1 and p_2 are crossing if and only if their Plücker's screws are KL-orthogonal.*

Proof. Let $X_i=(\overline{s}_i,\overline{m}_i)$, $\overline{s}_i^2=1$, $\overline{s}_i.\overline{m}_i=0$, $\overline{s}_2\neq k\overline{s}_1$, $i=1,2$, be a representative of the Plücker's screw the line of which is p_i. The line p_i is passing cross the point C_i, $\overline{OC}_i=\overline{s}_i\times\overline{m}_i$. Then the lines p_1, p_2 are crossing if and only if $0=\overline{C_1C_2}.(\overline{s}_1\times\overline{s}_2)=(\overline{s}_2\times\overline{m}_2-\overline{s}_1\times\overline{m}_1).(\overline{s}_1\times\overline{s}_2)=$ (*use the equality* (2)) $=(\overline{s}_2.\overline{s}_1)(\overline{m}_2.\overline{s}_2)-$

$(\overline{m}_2.\overline{s}_1)\overline{s}_2^2 - \overline{s}_1^2(\overline{m}_1.\overline{s}_2) + (\overline{m}_1.\overline{s}_1)(\overline{s}_1.\overline{s}_2) = -(\overline{s}_1.\overline{m}_2 + \overline{s}_2.\overline{m}_1) = -KL(X_1, X_2)$. It completes our proof.

b) Killing scalar bilinear form K:
Let $X_i = (\overline{s}_i, \overline{m}_i) \in V_3 \times V_3$. Put

$$K(X_1, X_2) = \overline{s}_1.\overline{s}_2$$

It means that K is a symmetric singular bilinear form on $V_3 \times V_3$. Its corresponding quadratic form will be written in the form $K(X) := K(X, X) = \overline{s}^2$. Then the line of X is improper iff $K(X) = 0$.

c) Lie bracket - vector bilinear form on $V_3 \times V_3$:
The vector product $\overline{a} \times \overline{b}$ of $\overline{a}, \overline{b} \in V_3$ is an example of the Lie bracket of two vectors. It is a skew-symmetric vector bilinear form on V_3. The well known and useful Lie bracket in $V_3 \times V_3$ is defined as follows.

If $X_i = (\overline{s}_i, \overline{m}_i) \in V_3 \times V_3$, $i = 1, 2$, then we put

$$[X_1, X_2] := (\overline{s}_1 \times \overline{s}_2, \overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1)$$

It is easy to show that the Jacobian identity

$$[X_1, [X_2, X_3]] + [X_3, [X_1, X_2]] + [X_2, [X_3, X_1]] = 0$$

is satisfied. Thus the vector space $V_3 \times V_3$ becomes a Lie algebra.

The vector space $V_3 \times V_3$ endowed with the Klein form KL, Killing form K and by the Lie bracket will be rewritten by A_6 instead $V_3 \times V_3$. It is well known that this Lie algebra A_6 is isomorphic with the Lie algebra of the Lie group of all orientation preserving isometries in E_3.

The following properties immediately follow from the definition of Lie bracket.

(1) If the line p_2 of $X_2 = (\overline{s}_2, \overline{m}_2)$, $\overline{s}_2 \neq \overline{0}$, is parallel with the line p_1 of $X_1 = (\overline{s}_1, \overline{m}_1)$, $\overline{s}_1 \neq \overline{0}$, i. e. if $\overline{s}_2 = k\overline{s}_1$, $k \neq 0$, then $[X_1, X_2] = (0, \overline{s}_1 \times (k\overline{m}_2 - \overline{m}_1)$ and thus the line of $[X_1, X_2]$ is improper.

(2) If $X_3 \in \langle X_1 \rangle$, i. e. $X_3 = kX_1$ and $X_4 \in \langle X_2 \rangle$, $X_4 = uX_2$, then $[X_3, X_4] = ku[X_1, X_2]$. It means that $[X_3, X_4] \in \langle [X_1, X_2] \rangle$. Thus we get the map $P_5^\sigma \times P_5^\sigma \to P_5^\sigma$, $(\langle X_1 \rangle, \langle X_2 \rangle) \mapsto \langle [X_1, X_2] \rangle$. Let us recall the representation $ad : A_6 \to L(A_6)$ of the Lie algebra A_6 in the vector space $L(A_6)$ of all linear maps on A_6 defined by the rule $ad_X(Y) = [X, Y]$. So we have a representation ad^σ of A_6 in the set of maps on P_5^σ, $ad_X^\sigma(\langle Y \rangle) = \langle [X, Y] \rangle$.

(3) Quite analogously it is easy to see that the Lie bracket preserves the L-equivalence classes, i. e. if $X_i, Y_i \in \beta_{pi}$, $i = 1, 2$, and p is the line of $[X_1, X_2]$ then $[Y_1, Y_2] \in \beta_p$.

(4) By direct calculation we get $KL(X_1, [X_1, X_2]) = \overline{s}_1.(\overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1) + (\overline{s}_1 \times \overline{s}_2).\overline{m}_1 = 0$. Therefore the Lie bracket $[X_1, X_2]$ is KL-orthogonal to X_i, $i = 1, 2$ and thus also the screw $\langle [X_1, X_2] \rangle$ is KL-orthogonal to $\langle X_i \rangle$, $i = 1, 2$.

Lemma 5. *Let p_i be the line of $X_i = (\overline{s}_i, \overline{m}_i)$, $\overline{s}_i \neq \overline{0}$, $i = 1, 2$, $\overline{s}_2 \neq k\overline{s}_1$. Then the line p of $[X_1, X_2]$ is the axis of the lines p_1, p_2, i. e. p intersects p_1 and p_2 orthogonally.*

Scratch of proof. We can suppose that $\overline{s}_i^2 = 1$, $i = 1, 2$. Certainly p is orthogonal to p_i, $i = 1, 2$. The line p_i is passing through $C_i, \overline{OC}_i = \overline{s}_i \times \overline{m}_i$ and the line p goes through $C, \overline{OC} = (\overline{s}_1 \times \overline{s}_2) \times (\overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1)/(\overline{s}_1 \times \overline{s}_2)^2$. Using the equality (1), $\overline{s}_1.\overline{s}_2 = \cos\alpha$, $(\overline{s}_1 \times \overline{s}_2)^2 = \sin^2\alpha$ it is easy to see that $\overline{C_1C}.(\overline{s}_1 \times (\overline{s}_1 \times \overline{s}_2)) = 0$, i. e. that p and p_1 are crossing. Analogously p and p_2 are also crossing. □

Corollary 1. *Let L be the manifold of all proper lines in E_3. Then according to property 3 the Lie bracket in A_6 induces the map from $L \times L$ into L in which the image of two non-parallel lines p_1, p_2 is the line p which intersects p_1 and p_2 orthogonally.*

Corollary 2. *The line $\tilde{p}$ of a couple $k_1X_1 + k_2X_2$ orthogonally intersects the line p of $[X_1, X_2]$ because $[X_1, k_1X_1 + k_2X_2] = k_2[X_1, X_2]$ and thus (by Lemma 5) p orthogonally intersects $\tilde{p}$.*

Lemma 6. *Let $X_i = (\overline{s}_i, \overline{m}_i)$, $i = 1, 2$, $\overline{s}_1 \times \overline{s}_2 \neq \overline{0}$ be two Plücker's couples. Then $[X_1, X_2]$ is a Plücker's couple if either the lines p_1, p_2 of X_1, X_2 respectively are orthogonal or X_1, X_2 are KL-orthogonal.*

Proof. $[X_1, X_2] = (\overline{s}_1 \times \overline{s}_2, \overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1)$. Then $[X_1, X_2]$ is a Plücker's couple iff $0 = (\overline{s}_1 \times \overline{s}_2).(\overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1) = s_1^2(\overline{s}_2.\overline{m}_2) - (\overline{s}_2.\overline{s}_1)(\overline{s}_1.\overline{m}_2) - (\overline{s}_1.\overline{s}_2)(\overline{s}_2.\overline{m}_1) + \overline{s}_2^2(\overline{s}_1.\overline{m}_1) = -(\overline{s}_1.\overline{s}_2)(\overline{s}_1.\overline{m}_2 + \overline{s}_2.\overline{m}_1)$. □

Recall that the Lie algebra A_6 has two basic subalgebras:
$V_3^\rho = \{(\overline{s}, 0), \overline{s} \in V_3\}$, $V_3^\tau = \{(0, \overline{m}), \overline{m} \in V_3\}$, $A_6 = V_3^\rho \oplus V_3^\tau$.
The line of $(\overline{s}, 0)$ goes through origin 0 and the line of a couple $(0, \overline{m})$ is improper. If $X_1, X_2 \in V_3^\tau$ then $[X_1, X_2] = (0, 0)$. If $X_1 \in V_3^\rho$, $X_2 \in V_3^\tau$ then $[X_1, X_2] \in V_3^\tau$. It means that V_3^ρ acts on V_3^τ by the Lie bracket; in detail, $ad_{X_1}(X_2) = [X_1, X_2] \in V_3^\tau$.

In the next part of this chapter we will try to describe all subalgebras in A_6, i. e. all vector subspaces A in A_6 for which $[A, A] \subset A$.

1. Every 1-dimensional subspace $A_1 \subset A_6$ is a subalgebra because $[X_1, X_2] = 0$ for $X_2 = kX_1$.
2. Let $A_2 \subset A_6$ be a 2-dimensional subspace. Let $X_i = (\overline{s}_i, \overline{m}_i)$, $i = 1, 2$, is a base in A_2. Then $[X_1, X_2] = (\overline{s}_1 \times \overline{s}_2, \overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1)$ belongs to A_2 if and only if $\overline{s}_1 \times \overline{s}_2 = k_1\overline{s}_1 + k_2\overline{s}_2$, $\overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1 = k_1\overline{m}_1 + k_2\overline{m}_2$. The former equality is satisfied iff $\overline{s}_1 \times \overline{s}_2 = 0$, i. e. iff $\overline{s}_2 = k\overline{s}_1$. There are two cases:
   a) if $\overline{s}_1 = \overline{0}$, then $\overline{s}_2 = \overline{0}$, i. e. $A_2 \subset V_3^\tau$.
   b) Let $\overline{s}_1 \neq \overline{0}$. As $k_1 = 0 = k_2$ then $\overline{s}_1 \times (\overline{m}_2 - k\overline{m}_1) = \overline{0}$, i. e. $\overline{m}_2 = k\overline{m}_1 + u\overline{s}_1$, i. e. there is a proper line p in E_3 such that $A_2 = \langle \beta_p \rangle$ is the vector space spanned on β_p. We get

Lemma 7. *A two-dimensional subspace $A_2 \subset A_6$ is a subalgebra if and only if either $A_2 \subset V_3^\tau$ or if $A_2 = \langle \beta_p \rangle$ for a proper line p.*

Remark 5. If A_2 is not a subalgebra, (i. e. if X_1, X_2 is a base in A_2 and $[X_1, X_2] \notin A_2$), then the proper line of all couples $X \in A_2$ form two-parametric family $\pi(A_2)$ of lines which orthogonally intersect the line of $[X_1, X_2]$. This line can be called axis of A_2. Recall that in differential geometry of lines a two-parametric family of lines is called a congruence of lines.

3. In this part we will investigate two problems: Under what conditions a 3-dimensional subspace $A_3 \subset A_6$ is a subalgebra and under what conditions A_3 is totaly KL-orthogonal to itself, i. e. $A_3 = A_3^K$.

Let $p_i : A_6 = V_3 \times V_3 \to V_3$ be the projection on the i-th factor, $i = 1, 2$, i. e. $p_1(\overline{s}, \overline{m}) = \overline{s}$, $p_2(\overline{s}, \overline{m}) = \overline{m}$.

There are cases for A_3:

a) $p_1(A_3) = \overline{0} \in V_3$. Then $A_3 = V_3^\tau$ is a subalgebra. As $KL(V_3^\tau, V_3^\tau) = 0$ then $A_3 = A_3^K$, i. e. A_3 is totaly KL-orthogonal to itself.

b) dim $p_1(A_3) = 1$, dim $p_2(A_3) = 3$. Always we can choose a base $X_1 = (\overline{s}_1, \overline{m}_1)$, $X_2 = (\overline{0}, \overline{m}_2)$, $X_3 = (\overline{0}, \overline{m}_3)$ in A_3, where $V_3 = \langle \overline{m}_1, \overline{m}_2, \overline{m}_3 \rangle$, $\overline{s}_1^2 \neq 0$. Then $[X_1, X_2] = (\overline{0}, \overline{s}_1 \times \overline{m}_2)$, $[X_1, X_3] = (\overline{0}, \overline{s}_1 \times \overline{m}_3)$, $[X_2, X_3] = \overline{0}$, $KL(X_1) = \overline{s}_1.\overline{m}_1$, $KL(X_3) = 0$, $KL(X_1, X_2) = \overline{s}_1.\overline{m}_2$, $KL(X_1, X_3) = \overline{s}_1.\overline{m}_3$, $KL(X_2, X_3) = 0$.

It gives

Lemma 8. *A 3-dimensional subspace A_3, $dimp_1 A_3 = 1$, $dimp_2 A_3 = 3$ is a subalgebra if $p_1(A_3)$ is orthogonal to $p_2(A_3 \cap V_3^\tau)$ in V_3. The equality $A_3^K = A_3$ cannot be satisfied.*

c) $dimp_1(A_3) = 1$, $dimp_2(A_3) = 2$. There is in A_3 a base $X_1(\overline{s}_1, \overline{0})$, $X_2 = (\overline{0}, \overline{m}_2)$, $X_3 = (\overline{0}, \overline{m}_3)$. Then for X_i, X_j and $KL(X_i, X_j)$ we obtain the same equalities as in b) except $KL(X_1) = 0$.

So we have

Lemma 9. *A 3-dimensional subspace A_3, $dimp_1(A_3) = 1$, $dimp_2(A_3) = 2$, is a subalgebra iff is KL-orthogonal to itself, i. e. iff $p_1(A_3)$ is orthogonal to $p_2(A_3)$ in V_3.*

d) $dimp_1(A_3) = 2$, $dimp_2(A_3) \geq 1$. Always we can choose a base $X_1 = (\overline{s}_1, \overline{m}_1)$, $X_2 = (\overline{s}_2, \overline{m}_2)$, $X_3 = (\overline{0}, \overline{m}_3)$, where $\overline{s}_1$ $\overline{s}_2$ are independent. Then $[X_1, X_2] = (\overline{s}_1 \times \overline{s}_2, \cdot)$. If A_3 is a subalgebra then $\overline{s}_1 \times \overline{s}_2 = k_1\overline{s}_1 + k_2\overline{s}_2$. It is impossible. We get $KL(X_1) = \overline{s}_1.\overline{m}_1$, $KL(\overline{s}_2) = \overline{s}_2.\overline{m}_2$, $KL(X_3) = 0$, $KL(X_1, X_2) = \overline{s}_1.\overline{m}_2 + \overline{m}_1.\overline{s}_2$, $KL(X_1, X_3) = \overline{s}_1.\overline{m}_3$, $KL(X_2, X_3) = \overline{s}_2.\overline{m}_3$. If $dimp_2(A_3) = 1$ then we can choose $\overline{m}_1 = \overline{0} = \overline{m}_2$. Then $A_3^K = A_3$ iff $\overline{m}_3 = k\overline{s}_1 \times \overline{s}_2$. If $dimp_2(A_3) = 2$ we can put $\overline{m}_1 = \overline{0}$. Then $A_3^K = A_3$ iff $\overline{m}_2 = k_2\overline{s}_1 \times \overline{s}_2$, $\overline{m}_3 = k_3\overline{s}_1 \times \overline{s}_2$. It is impossible. If $dimp_2(A_2) = 3$ then $\overline{m}_1$, $\overline{m}_2$, $\overline{m}_3$ we can chose as an orthonormal base in V_3. Then $A_3^k = A_3$ iff $\overline{s}_1 = \overline{m}_1 \times \overline{m}_3$, $\overline{s}_2 = \overline{m}_2 \times \overline{m}_3$, $\overline{m}_3 = \overline{s}_1 \times \overline{s}_2$, i. e. iff $\overline{s}_1 = -\overline{m}_2$, $\overline{s}_2 = \overline{m}_1$. We get

Lemma 10. *A 3-dimensional subspace A_3, $dimp_1(A_3) = 2$, $dimp_2(A_3) \geq 1$, is not a subalgebra. If $dimp_2(A_3) = 1$ then $A_3^K = A_3$ iff $p_2(A_3)$ is orthogonal to $p_1(A_3)$ in V_3. If $dimp_2(A_3) = 2$ then $A_3^K \neq A_3$. If $dimp_2(A_3) = 3$ then $A_3^K = A_3$ iff $A_3 = \langle (-\overline{m}_2, \overline{m}_1), (\overline{m}_1, \overline{m}_2), (\overline{0}, \overline{m}_3) \rangle$ where $\overline{m}_1$, $\overline{m}_2$, $\overline{m}_3$ is an orthonormal base in V_3. To every 2-dimensional subspace $V_2 = \langle \overline{s}_1, \overline{s}_2 \rangle \subset V_3$ there is a unique $A_3 = \langle (\overline{s}_1, \overline{s}_2), (\overline{s}_2, -\overline{s}_1), (0, \overline{s}_1 \times \overline{s}_2) \rangle$ such that $A_3^K = A_3$. A_3 does not depend on choice of orthonormal base $\overline{s}_1$, $\overline{s}_2$.*

e) $\dim p_1(A_3) = 3$, $\dim p_2(A_3) = 1$. Choosing a base $X_1 = (\overline{s}_1, \overline{0})$, $X_2 = (\overline{s}_2, 0)$, $X_3 = (\overline{s}_3, \overline{m}_3)$ it is easy to show that A_3 is not subalgebra and $A_3^K \neq A_3$.

f) $\dim(A_3) = 3$, $\dim p_2(A_3) \geq 2$. We can chose a base $X_1 = (\overline{s}_1, \overline{m}_1)$, $X_2 = (\overline{s}_2, \overline{m}_2)$, $X_3 = (\overline{s}_3, \overline{m}_3)$ where $\overline{s}_1$, $\overline{s}_2$, $\overline{s}_3$ are orthonormal: $\overline{s}_1 \times \overline{s}_2 = \overline{s}_3$,

$\overline{s}_1 \times \overline{s}_3 = -\overline{s}_2$, $\overline{s}_2 \times \overline{s}_3 = \overline{s}_1$. Let $\overline{m_i} = \sum_{j=1}^{3} m_i^j \overline{s}_j$, $i = 1, 2, 3$. Calculating $[X_i, X_k]$ we obtain. A subspace A_3 is a subalgebra if and only if

$$(3) \qquad m_1^1 = m_2^2 = m_3^3 = 0, m_2^3 + m_3^2 = 0, m_1^3 + m_3^1 = 0, m_1^2 + m_2^1 = 0$$

As $KL(X_i) = m_i^i$, $i = 1, 2, 3$, $KL(X_1, X_2) = m_2^1 + m_1^2$, $KL(X_1, X_3) = m_3^1 + m_1^3$, $KL(X_2, X_3) = m_3^2 + m_2^3$ therefore the equality $A_3^K = A_3$ is satisfied iff (3) is true.

The equalities (3) give: $\overline{m}_1 = -m_1^2\overline{s}_2 + m_1^3\overline{s}_3$, $\overline{m}_2 = m_1^2\overline{s}_1 - m_2^3\overline{s_3}$, $\overline{m}_3 = -m_1^3\overline{s}_1 + m_2^3\overline{s}_2$.

Put $\overline{m} := m_2^3\overline{s}_1 + m_1^3\overline{s}_2 + m_1^2\overline{s}_3$. Then $\overline{m}_1 = \overline{s}_1 \times \overline{m}$, $\overline{m}_2 = \overline{s}_2 \times \overline{m}$, $\overline{m}_3 = \overline{s}_3 \times \overline{m}$. The rank r of the system $(\overline{m}_1, \overline{m}_2, \overline{m}_3)$ is 2. We have proved.

Lemma 11.. *A vector subspace A_3, $\dim p_1(A_3) = 3$, $\dim p_2(A_3) = 2$ is a subalgebra if a one of the following equivalent conditions is satisfied:*

1. $A_3^K = A_3$
2. *A_3 is the subspace of couples $(\overline{s}, \overline{s} \times \overline{m})$, $\overline{s} \in V_3$ and $\overline{m} \neq \overline{0}$ is a given vector.*

If $\dim p_1(A_3) = 3$, $\dim p_2(A_3) = 3$ then A_3 is not subalgebra and $A_3^K \neq A_3$.

4. Let A_4 be a 4-dimensional vector subspace in A_6. There are cases:

a) $\dim p_1 A_4 = 1$, $\dim p_2 A_4 = 3$. Choosing a base $X_1 = (\overline{s}_1, \overline{0})$, $X_i = (\overline{0}, \overline{m}_i)$, $i = 2, 3, 4$ and calculating $[X_i, X_j]$ we get

Lemma 12. *A 4-dimensional vector subspace A_4, $\dim p_1(A_4) = 1$, $\dim p_2 A_4 = 3$ is always a subalgebra.*

Let us remark that $A_4 \cap V_3^\tau = V_3^\tau$ in this case.

b)$\dim p_1 A_4 \geq 2$, $\dim p_2 A_4 \geq 2$. Always we can choose a suitable base and show that A_4 cannot be a subalgebra.

5. In the case when A_5 is a vector subspace always there are bases by which can be shown that A_5 cannot be a subalgebra.

Let us introduce survey of all subalgebras in A_6:

1. All 1-dimensional vector subspaces have the subalgebra structure.
2. A 2-dimensional vector subspace A_2 is a subalgebra if either $A_2 = \langle \beta_p \rangle$ for some line p or $A_2 \subset V_3^\tau$.
3. A 3-dimensional vector subspace A_3 is a subalgebra in the cases
   a) $A_3 = V_3^\rho$, $A_3 = V_3^\tau$
   b) $\dim p_1(A_3) = 1$, $\dim p_2(A_3) = 3$ and $p_1(A_3)$ is orthogonal to $p_2(A_3 \cap V_3^\tau)$ in V_3
   c) $\dim p_1(A_3) = 1$, $\dim p_2(A_3) = 2$ and $p_1(A_3)$ is orthogonal to $p_2(A_3)$ in V_3
   d) $A_3 = \{(\overline{s}, \overline{s} \times \overline{m}) \in A_6, \overline{s} \in V_3, \overline{m} \neq \overline{0}$ *is a given vector*$\}$
4. A 4-dimensional vector subspace A_4 is a subalgebra iff $\dim p_1 A_4 = 1$, $\dim p_2 A_4 = 3$.

Remark 6. Let $\pi(A)_f$ denote the set of all proper lines of couples $(\overline{s}, \overline{m}) \in A$, $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$. It is easy to see that in the cases 3b, 3c, 4 $\pi(A)_f$ is a set of all lines parallel with the direction $p_1(A)$. A line of $\pi(A_3)_f$ from the case 3d goes through the point $C, \overline{OC} = \frac{\overline{s} \times (\overline{s} \times \overline{m})}{s^2} = \frac{(\overline{s}.\overline{m})}{s^2}\overline{s} - \overline{m}$. Therefore $\pi(A_3)_f$ is the set of lines p

going through points C on the sphere S^2 with the center $M, \overline{OM} = -\frac{1}{2}\overline{m}$ and with radius $r = \frac{1}{2}\sqrt{\overline{m}^2}$. If $\overline{s}.\overline{m} \neq 0$, i. e. $\overline{OC} \neq -\overline{m}$ then p is in the direction $\overline{s}$ and so it is unique. If $\overline{s}.\overline{m} = 0$, i. e. $\overline{OC} = -\overline{m}$ then every line p going through C and orthogonal to $\overline{m}$ belongs to $\pi(A_3)_f$. So $\pi(A_3)_f$ is a two-parametric family of lines in E_3, i. e. it is a congruence of lines. Recall that in the case of a general vector subspace A_3, $\pi(A_3)_f$ is a 3-parametric family of lines in E_3 that is called a complex of lines.

4. Canonical Vector Fields In E_3 Induced By A_6

Recall that a vector field on a differentiable manifold M is a rule ξ by which a tangent vector $\xi(x)$ at $x \in M$ is determined for every $x \in M$. In the case of $M = E_3$ $\xi(x) \in V_3$.

Definition 3. *Let $X = (\overline{s}, \overline{m}) \in A_6$ and 0 be a given point in E_3. This couple X and 0 determine a vector field $\xi_{(X,0)}$ by the following rule:*

a) *If $\overline{s} = \overline{0}$ then $\xi_{(X,0)}(Y) = \overline{m}$ for any $Y \in E_3$,*

b) *Let $\overline{s} \neq \overline{0}$. Let $h\overline{s}$ be the orthogonal projection $\overline{m}$ into $\overline{s}$, i. e. $h = (\overline{s}.\overline{m})/\overline{s}^2$. Let $\overline{OC} = (\overline{s} \times \overline{m})/\overline{s}^2$. Then*

$$\xi_{(X,0)}(Y) = \overline{s} \times \overline{CY} + h\overline{s}, Y \in E_3. \tag{4}$$

This vector field will be called the field of X.

Lemma 13. *The value of the field of X at 0 is $\overline{m}$, $\xi_{(X,0)}(0) = \overline{m}$.*

Proof. If $X = (0, \overline{m})$, i. e. $\overline{s} = 0$, then assertion is true. If $\overline{s} \neq \overline{0}$ then using (1) we get

$$\xi_{(X,0)}(0) = \overline{s} \times \overline{CO} + h\overline{s} = -\overline{s} \times (\overline{s} \times \overline{m})/\overline{s}^2 + h\overline{s} = -[(\overline{s}.\overline{m})\overline{s} - \overline{s}^2\overline{m}]/\overline{s}^2 + h\overline{s} = \overline{m}.$$

□

Corollary 3. *For the value of the vector field $\xi_{(X,0)}$ at $Y \in E_3$ we get $\xi_{(X,0)}(Y) = \overline{s} \times \overline{CY} + h\overline{s} = \overline{s} \times (\overline{C0} + \overline{0Y}) + h\overline{s} = \overline{s} \times \overline{C0} + \overline{s} \times \overline{0Y} + h\overline{s}$, i. e.*

$$\xi_{(X,0)}(Y) = \overline{s} \times \overline{0Y} + \overline{m} \tag{5}$$

It immediately gives:

a)

$$\xi_{(kX,0)}(Y) = k\xi_{(X,0)}(Y) \tag{6}$$

b) If two couples $X_i = (\overline{s}_i, \overline{m}_i) = 1, 2$, have the same line of couple, i. e. if $\overline{s}_2 = k\overline{s}_1$, $\overline{m}_2 = k\overline{m}_1 + u\overline{s}_1$ then

$$\xi_{(X_2,0)} = k\xi_{(X_1,0)} + u\overline{s}_1$$

Remark 7. If we change origin, if we choose 0' instead of 0 then from (4) or from (5) we get

$\xi_{(X,0')}(Y) = \overline{s} \times \overline{C'Y} + hs = \overline{s} \times (\overline{C'C} + \overline{CY}) + hs = \xi_{(X,0)}(Y) + \overline{s} \times \overline{C'C}$ or

$\xi_{(X,0')}(Y) = \overline{s} \times \overline{0'Y} + \overline{m} = \overline{s} \times (\overline{0'0} + 0Y) + \overline{m} = \xi_{(X,0)}(Y) + \overline{s} \times \overline{0'0}$ respectively.

Let CE_3 denote a set of all vector fields on E_3. It is a real vector space. Let $\xi : V_6 \to CE_3$ be a map defined by the rule $\xi(X) = \xi_{(X,0)}$.

Proposition 1. *The map $\xi : A_6 \to \xi(A_6) \subset CE_3$ is an isomorphism of vector spaces.*

Proof. By the equality (6) $\xi(kX) = k\xi(X)$.

Let $X_i = (\overline{s}_i, \overline{m}_i)$, $i = 1, 2$. According to the definition of ξ we consider the following cases:

a) $\overline{s}_1 = \overline{0} = \overline{s}_2 : \xi(X_1 + X_2)(Y) = \overline{m}_1 + \overline{m}_2 = \xi(X_1)(Y) + \xi(X_2)(Y)$.

b) $\overline{s}_1 = \overline{0}, \overline{s}_2 \neq \overline{0} : \xi(X_1 + X_2)(Y) = \overline{s}_2 \times \overline{0Y} + \overline{m}_1 + \overline{m}_2 = \xi(X_1)(Y) + \xi(X_2)(Y)$.

c) $\overline{s}_1 \neq \overline{0}$, $\overline{s}_2 = \overline{0}$: analogously $\xi(X_1 + X_2)(Y) = \xi(X_1)(Y) + \xi(X_2)(Y))$.

d) $\overline{s}_1 \neq \overline{0}, \overline{s}_2 \neq \overline{0} : \xi(X_1 + X_2)(Y) = (\overline{s}_1 + \overline{s}_2) \times \overline{0Y} + (\overline{m}_1 + \overline{m}_2) = \xi(X_1)(Y) + \xi(X_2)(Y)$.

We have proved that ξ is a linear map. We will show that $\ker \xi = \overline{0}$. Let $\xi(X) = \overline{0} \in CE_3$. If $\overline{s} = \overline{0}$ then $\overline{0} = \xi(X)(Y) = \overline{m}$, i. e. $\xi = (\overline{0}, \overline{0})$. If $\overline{s} \neq \overline{0}$ then $\overline{0} = \xi(X)(Y) = \overline{s} \times \overline{0Y} + \overline{m}$ for all $Y \in E_3$. It is possible only if $\overline{s} = \overline{0}$, $\overline{m} = \overline{0}$. It completes our proof. □

The vector subspace $\xi(A_6)$ will be denoted as $SCE_3 := \xi(A_6)$. On the vector space SCE_3 by the isomorphism ξ the following bilinear forms are induced:

a) Klein form $SKL(\xi(X_1), \xi(X_2)) = KL(X_1, X_2)$,

b) Killing form $SK(\xi(X_1), \xi(X_2)) = K(X_1, X_2)$,

c) Lie bracket $[\xi(X_1), \xi(X_2)] = \xi[X_1, X_2]$.

Let $X_i = (\overline{s}_i, \overline{m}_i)$, $i = 1, 2$, be two couples. Then the isomorphism ξ inspires the following shapes for the above introduced forms.

Proposition 2.

a)$SKL(\xi(X_1), \xi(X_2)) = \overline{s}_1.\xi(X_2) + \overline{s}_2.\xi(X_1)$

b)$SK(\xi(X_1), \xi(X_2)) = \overline{s}_1.\overline{s}_2$

c)$[\xi(X_1), \xi(X_2)] = \overline{s}_1 \times \xi(X_2) - \overline{s}_2 \times \xi(X_1)$

Proof. Using the equalities (1) and (6) we get successively

a) $\overline{s}_1.\xi(X_2) + \overline{s}_2.\xi(X_1) = \overline{s}_1.(\overline{s}_2 \times \overline{0Y} + \overline{m}_2) + \overline{s}_2.(\overline{s}_1 \times \overline{0Y} + \overline{m}_1) = \overline{s}_1.\overline{m}_2 + \overline{s}_2.\overline{m}_1 = KL(X_1, X_2)$.

b) $\overline{s}_1.\overline{s}_2 = K(X_1, X_2)$

c) $\overline{s}_1 \times \xi(X_2) - \overline{s}_2 \times \xi(X_1) = \overline{s}_1 \times (\overline{s}_2 \times \overline{OY} + \overline{m}_2) - \overline{s}_2 \times (\overline{s}_1 \times \overline{0Y} + \overline{m}_1) = (\overline{s}_1.\overline{0Y})\overline{s}_2 - (\overline{s}_1.\overline{s}_2)\overline{0Y} + \overline{s}_1 \times \overline{m}_2 - (\overline{s}_2.\overline{0Y})\overline{s}_1 + (\overline{s}_1.\overline{s}_2)\overline{0Y} - \overline{s}_2 \times \overline{m}_1 = \xi([X_1, X_2])$. □

Remark about trajectories of the vector field $\xi(X)$. Let us remind that a trajectory of a vector field is a curve the tangent vectors of which are values of the vector field in points of this curve. So if $Y = Y(t)$ is the equation of a trajectory of the field $\xi(X)$ then

$$\dot{Y} = \xi_{(X,0)}(Y(t))$$

Using the equality (5) we can this equation to rewrite as follows

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}.$$

It is a system of differential equations the solution of which are trajectories of the vector field $\xi(X)$.

5. Kinematic Interpretation Of The Field $\xi(X)$ Of A Couple X

In this chapter we use a notation $X = (\overline{w}, \overline{b})$ instead $(\overline{s}, \overline{m})$. We will distinguish two cases.

a)If $\overline{w} = \overline{0}$ then the value of the vector field $\xi(X)$ is $\overline{b}$ at any $Y \in E_3$. We can interpret this values as the instants velocities of equable straightforward motions (translation motions). The trajectories of this motions are lines in the direction $\overline{b}$.

2. Let $\overline{w} \neq \overline{0}$. Recall that the line p of the couple $(\overline{w}, \overline{b}) \in A_6$ is going through the point $C, \overline{0C} = \overline{w} \times \overline{b}/\overline{w}^2$ in the direction $\overline{w}$. Let us consider the equable screw motion in E_3 which is composition of two motions: the first part is the rotation around the axis p with the constant angle velocity $\overline{w}$ and the second one is the translation motion in E_3 in the direction $\overline{w}$ with the constant velocity $h\overline{w}, h = (\overline{w}.\overline{b})/\overline{w}^2$. (We will say that the line p is the axis of this equable screw motion). The velocity $\overline{v}$ of this motion at a point Y satisfies the equality

$$\overline{v} = \overline{w} \times \overline{CY} + h\overline{w}.$$

According to (5) $\overline{v}$ is the value of the vector field $\xi(X)$ at $Y \in E_3$. We have proved.

Theorem. *Let $X = (\overline{w}, \overline{b}) \in A_6$. Then the vector field $\xi(X)$ is the velocity field of the following motions:*
If $\overline{w} = \overline{0}$ then it is a translation motion with the velocity $\overline{b}$.
If $\overline{w} \neq \overline{0}$ then this motion is the equable screw motion around the line p of the couple X with constant angle velocity $\overline{w}$ and with translation constant velocity $h\overline{w}, h = (\overline{w}.\overline{b})/\overline{w}^2$.

Remark 8. If $\overline{w}.\overline{b} = 0$, $\overline{w} \neq \overline{0}$, i. e. if $KL(X) = 0$, $K(X) \neq 0$, i. e. if $X = (\overline{w}, \overline{b})$ is a Plücker's couple then $\xi(X)$ is a field of velocities of the clean rotation around the line of X with constant angle velocity $\overline{w}$. Couples X belonging to the same Plücker's screw $\langle X \rangle$ determine rotations around the line of $\langle X \rangle$ with different angle velocities. When $KL(X) \neq 0$, $K(X) \neq 0$ then the trajectories of the field $\xi(X)$ are screw curves the axis of which is the line of X. The motions determined by the couples of a screw $\langle X \rangle$, $KL(X) \neq 0$, $K(X) \neq 0$, are equable screw motions around the line of $\langle X \rangle$ with the same pitch h. In general two couples $X_1, X_2, K(X_2) \neq 0$, with the same line p of couple, i. e. $X_i \in \beta_p$, determined equable screw motions around p with different angular velocities and pitches.

Remark 9 (about pitch h). By definition $h = (\overline{w}.\overline{b})/\overline{w}^2$ and then $v = |h|\|\overline{w}\|$, $\|\overline{w}\|^2 = \overline{w}.\overline{w}$, is the translation velocity of the motion determined by $X = (\overline{w}, \overline{b})$, $\overline{w} \neq \overline{0}$. Then $|h| = \frac{v}{w}$, $w = \|\overline{w}\|$. So $|h|$ is the translation length according to revolution with angle of radian around the line p of the couple X. It will be called specific lift. If $h > 0$ then we say that both motion parts, rotation and translation, have positive orientation, (If the rotation is in the direction of fingers of the right hand then the translation is in the direction of the thumb.), and in the opposite case $h < 0$ we say about negative orientation.

Remark 10 (about influence of a choice of a origin 0). If we use a point 0' instead 0 then the line of a couple $X = (\overline{w}, \overline{b})$ is the line $p = \tau(p)$ where τ is the translation determined by the vector $\overline{00'}$. Now $C' = \tau(C)$, $\overline{O'C'} = (\overline{w} \times \overline{b})/\overline{w}^2 = \overline{0C}$. The velocity of the equable screw motion around p' with angular velocity $\overline{w}$ and translation velocity $\overline{w}$ in a point Y is

$$\xi_{(X,0')}(Y) = \overline{w} \times C'Y + h\overline{w} = \overline{w} \times (\overline{C'C} + \overline{C}Y) + h\overline{w} = \xi_{(X,0)} + \overline{w} \times \overline{C'C}.$$

Remark 11 (about subgroups of motions induced by couples of the Lie subalgebras of A_6). It is well known that to every Lie algebra A there is a Lie group $G(A)$ the Lie algebra of which is just A. By our investigations in the 3-th chapter there are 9 types of subalgebras.

a) $A_1 = \langle X \rangle$, $X = (\overline{s}, \overline{m}) \neq (\overline{0}, \overline{0})$. If $\overline{s} = 0$ then the corresponding group $G(A_1)$ is the group of all translations with constant velocities $k\overline{m}$. If $\overline{s} \neq \overline{0}$ then $G(A_1)$ is the group of all equable screw motions around the line of X with the same pitch h.

b_1) $A_2 = \langle \beta_p \rangle$ for a line p, i. e. $A_2 = \{(k\overline{w}, k\overline{b} + u\overline{w}), \overline{w} \neq 0, k, u \in R\}$. The group $G(A_2)$ induced by A_2 is the group of all equable screw motions around p including all translations in the direction of p and rotations around p.

b_2) If $A_2 \subset V_3^\tau$, $A_2 = \langle (0, \overline{m}_1), (0, \overline{m}_2) \rangle$ then corresponding group is the group of all translations with the velocities $\overline{v} \in A_2$.

c_1) If $A_3 = V_3^\tau$ or $A_3 = V_3^\rho$ then the corresponding group is the group of all translations in E_3 or of all rotations about origin 0.

c_2) $A_3 \subset A_6$ with properties: $\dim(p_1 A_3) = 1$, $\dim(p_2 A_3) = 3$, $p_1(A_3)$ is orthogonal to $p_2(A_3 \cap V_3^\tau)$. Then $(\overline{w}, \overline{0}) \notin A_3$ and there is $(\overline{w}, \overline{b}) \in A_3$, $\overline{w} \neq \overline{0} \neq \overline{b}$. Let p be the line of $(\overline{w}, \overline{b})$. Then $G(A_3)$ is generated by all equable screw motions around lines parallel with p except the one going through origin 0 and by all translations with velocities $\overline{v}$ orthogonal to p.

c_3) $A_3 \subset A_6$ with properties: $\dim(p_1 A_3) = 1$, $\dim(p_2 A_3) = 2$, $p_1(A_3)$ is orthogonal to $p_2(A_3)$ in V_3. Then $(\overline{w}, \overline{0}) \in A_3$ and $G(A_3)$ is generated as in the case c_2 including equable screw motions around the line going through origin 0 in the direction $\overline{w}$.

c_4) $A_3 = \{(\overline{w}, \overline{w} \times \overline{m}) \in A_6, \overline{w} \in V_3, \overline{m} \neq \overline{0}$ *is a given vector*$\}$. Then $G(A_3)$ is generated by all equable rotations around the lines going through points $C, \overline{0C} \neq -\overline{m}$, of the sphere S_2 (describing in the Remark in the end of the 3-d chapter) and around all lines orthogonal to $\overline{m}$ going through $C, \overline{0C} = -\overline{m}$.

d) $A_4 \subset A_6$ with properties: $\dim(p_1 A_4) = 1$, $\dim(p_2 A_4) = 3$. Then $V_3^\tau \subset A_4$ and $G(A_4)$ is generated as in the case C_3 including all translations in E_3.

5. Dynamic Interpretation Of A Vector Fields $\xi(X)$

Firstly we recall effects of a force on a rigid body. Let a force $\overline{f}$ affects on a rigid body Ω at a point $C \in \Omega$. The line $p = (C, \overline{f})$ going through C in the direction $\overline{f}$ is called the line of $\overline{f}$. The result of effect of $\overline{f}$ at a point $Y \in \Omega$ does not depend on a choice of a point C on the line p of $\overline{f}$. A measure of this effect is moment of the force $\overline{f}$ at Y, i. e. the vector $\overline{YC} \times \overline{f}$. Denote $\overline{m} := \overline{OC} \times \overline{f}$ the moment of $\overline{f}$ at origin 0. We get a Plücker's couple $(\overline{f}, \overline{m} = \overline{OC} \times \overline{f})$ the line of which is just the line of $\overline{f}$.

Remind further, that the effect of a couple of forces $(\overline{f}, -\overline{f}, \overline{r})$ with its arm $\overline{r}$ is the same at every point $Y \in \Omega$. A measure of this effect is moment $\overline{r} \times \overline{f}$ of the couple of forces.

Let $X = (\overline{f}, \overline{m}) \in A_6$ be a couple of vectors. Let $\xi(X)$ be the vector fields on E_3 determined by X. The above considerations inspire the following dynamic interpretation of the vector field $\xi(X)$.

a) If $X = (0, \overline{m})$ then $\xi(X)$ is the vector field the value of which in every point $Y \in \Omega$ is the moment $\overline{m}$ of some couple of forces.
b) Let $X = (\overline{f}, \overline{m})$ be a Plücker's couple, i.e. $\overline{f}.\overline{m} = 0$. The line p of X we interpret as the line of a force $\overline{f}$. Then $\xi(X)$ is the vector field the value of which in a point Y is the moment $\overline{YC} \times \overline{f}$ of the force $\overline{f}$ at Y, where $\overline{OC} = (\overline{f} \times \overline{m})/\overline{f}^2$, i. e. C is the orthogonal projection of the origin 0 into p. The value of this field in 0 is $\overline{m} = \overline{OC} \times \overline{f}$.
c) Let $X = (\overline{f}, \overline{m})$ is not Plücker's couple, i. e. $\overline{f}.\overline{m} \neq 0$. Recall that $h\overline{f}, h = (\overline{f}.\overline{m})/\overline{f}^2$, is the orthogonal projection $\overline{m}$ into $\overline{f}$. Then the vectors $\overline{f}, \overline{m} - h\overline{f}$ are orthogonal in V_3 and $(\overline{f}, \overline{m}) = (\overline{f}, \overline{m} - h\overline{f}) + (0, h\overline{f})$ where $(\overline{f}, \overline{m} - h\overline{f})$ is a Plücker's couple. So the vector field $\xi(X)$ is the sum of the vector fields $\xi((\overline{f}, \overline{m} - h\overline{f}))$ and $\xi((0, h\overline{f}))$, i. e.

$$\xi(X)(Y) = \overline{YC} \times \overline{f} + h\overline{f} = \xi((\overline{f}, \overline{m} - h\overline{f})) + \xi((0, h\overline{f})).$$

This means that the value of the field $\xi(x)$ in a point Y is the sum of the moment of the force $\overline{f}$ at Y and the moment $h\overline{f}$ of some couple of forces.

Values of the vector field $\xi(X)$ interpreted by moments of forces can be called dynamic effects of a couple X.

Recall that in literature the following notions are used. Elements of the Lie algebra A_6, i. e. couples $X = (\overline{s}, \overline{m})$, are called motors. If the vector field $\xi(X)$ of a motor X is interpreted as a vector field of velocities then X is called twist.

If $\xi(X)$ is interpreted as a vector field of moments then X is called wrench. If two wrenches $X_1, X_2 \in A_6$ belong to the same screw, i. e. if $X_2 = kX_1$ then $\xi(X_2) = k\xi(X_1)$, i. e. the dynamic effect of X_2 is a multiple of the dynamic effect of X_1. In general if wrenches X_1, X_2 have the same line of couple, i. e. if $X_2 = (k\overline{f}_1, k\overline{m}_1 + uf_1)$ then the dynamic effect of X_2 is the sum of a multiple of the dynamic effect of X_1 and of a moment of some couple of forces.

Remark 12. (about a twist-wrench interpretation of $KL(X_1, X_2)$):
Let a twist $X_1 = (\overline{w}, \overline{b}) \in A_6$ determined an equable screw motion of a body Ω around the line p_1 of X_1 with angle velocity $\overline{w}$ and with translation velocity $h\overline{w}$. Then $\overline{b}$ is the velocity of origin 0. Let $X_2 = (\overline{f}, \overline{m}) \in A_6$ is a wrench, i. e. $\overline{f}$ is a force the line of which is the line of X_2 and $\xi(X_2)$ is a such vector field that $\xi(X_2)(Y) = \overline{YC}_2 \times \overline{f} + h\overline{f}$ is the sum of the moment of $\overline{f}$ at Y and of the moment $h\overline{f}$ of some souple of forces. Recall that $\xi(X_2)(0) = \overline{m}$. The value $KL(X_1, X_2) = \overline{f}.\overline{b} + \overline{w}.\overline{m}$ can be interpreted as follows. We can say that $\overline{f}.\overline{b}$ is a translation effect of $\overline{f}$ and $\overline{w}.\overline{m}$ is a rotation effect of $\overline{f}$ at the origin 0 of the body Ω moving by a equable screw motion. Then $KL(X_1, X_2)$ can be called a power given to the solid Ω, moving under the twist X_1, by the wrench X_2 per unit of time.

Remark on a motion of the effector of a robot.. We consider the effector of a robot as a rigid solid Ω. The moving effector determines in Ω the vector field of velocities of points $Y \in \Omega$ at any time t. This vector field is the vector field of velocities of a equable screw motion around an instantaneous axis and thus it is determined by a couple $X(t) = (\overline{w}(t), \overline{b}(t)) \in A_6$. So a moving effector determines a curve $t \mapsto X(t)$

in A_6. Vice versa, a curve $X(t)$ in A_6 states a movement of a effector the trajectories of which are solutions of the non-autonomous differential system

$$\dot{Y} = \xi_{X(t),0} Y, \dot{\overline{y}} = \overline{\omega}(t) \times \overline{y} + \overline{b}(t).$$

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THE LATTICE OF VARIETIES OF ORGRAPHS

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Abstract. In [5] we investigated varieties of orgraphs (i.e. oriented graphs) as classes of orgraphs closed under isomorphic images, suborgraph identifications and induced suborgraphs, and we studied the lattice of varieties of orgraphs. We paid particular attention to varieties containing no nontrivial tournament. In this paper we pay attention to the part of the lattice of varieties of orgraphs which consists of varieties generated by sets of notrivial tournaments.

1. Introduction

A useful tool for investigations of some properties of graphs is a choice of suitable closure operators and examinations classes of graphs closed under these operators. For example, classes of graphs closed under induced subgraphs are called hereditary in [12] and induced hereditary in [3], and were considered in several papers. Classes of graphs closed under other operators are considered, for example, in [2] and [6]. In the paper [5] were considered classes of orgraphs closed under isomorphic images, suborgraph identification and induced suborgraphs.

By an *orgraph* we mean directed graph $\mathcal{G}(V,E)$ without loops with the following property:

> for every two distinct vertices $u, v \in V$, at most one of the edges uv and vu is an arc from E.

We briefly write uv instead of $[u,v]$ for vertices $u, v \in V$.

We can associate to every orgraph $\mathcal{G}(V,E)$ the graph $\mathcal{G}^\star(V^\star, E^\star)$ by omitting the orientation of all edges, i.e.

$V^\star = V$ and $\{u,v\} \in E^\star$ iff $uv \in E$ or $vu \in E$.

An orgraph $\mathcal{G}(V,E)$ is called

- weakly connected if $\mathcal{G}^\star(V^\star, E^\star)$ is connected,
- a weak cycle if $\mathcal{G}^\star(V^\star, E^\star)$ is a cycle,
- a tournament if $\mathcal{G}^\star(V^\star, E^\star)$ is a complete graph.

Let us recall that by a *suborgraph identification* of orgraphs $\mathcal{G}_1$, $\mathcal{G}_2$ we mean gluing of the orgraphs $\mathcal{G}_1$, $\mathcal{G}_2$ in their weakly connected induced suborgraphs $\mathcal{G}'_1$, $\mathcal{G}'_2$ which are isomorphic (we choose an isomorphism between $\mathcal{G}'_1$ and $\mathcal{G}'_2$ and identify the corresponding vertices of $\mathcal{G}'_1$ and $\mathcal{G}'_2$ [7]).

In this paper we follow the notation of [5]. If $\mathbb{K}$ is a set of orgraphs we denote

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- by $\Gamma(\mathbb{K})$ the smallest class of weakly connected orgraphs containing the set $\mathbb{K}$ and closed under suborgraph identifications ,

- by $S(\mathbb{K})$ the class of all weakly connected induced suborgraphs of orgraphs from $\mathbb{K}$,

- by $I(\mathbb{K})$ the class of all isomorphic images of orgraphs from $\mathbb{K}$.

Definition 1.1. A set $\mathbb{K}$ of orgraphs closed under isomorphic images, induced weakly connected suborgraphs and suborgraph identifications is called *a variety*; that is $\mathbb{K}$ is a variety if

$$I(\mathbb{K}) \subseteq \mathbb{K},\ S(\mathbb{K}) \subseteq \mathbb{K} \text{ and } \Gamma(\mathbb{K}) \subseteq \mathbb{K}.$$

Obviously, I, S, Γ are closure operators on the system of all sets of weakly connected orgraphs. By [4, Theorem 5.2] we obtain the next statement.

Theorem 1.1. *The set of all varieties of orgraphs with set inclusion as the partial ordering is a complete lattice (denoted by* $\mathbf{L}(I, S, \Gamma)$*).*

We denote by $V(\mathbb{K})$ the smallest variety of orgraphs containing a given set $\mathbb{K}$ of orgraphs. We will say that $V(\mathbb{K})$ is generated by the set $\mathbb{K}$.

The following lemma and corollary play an important role in investigations of varieties of orgraphs.

Lemma 1.2. *Let* $\mathcal{G}(V, E)$ *be a weakly connected orgraph which is neither a tournament nor a weak cycle. Then there exist two nonadjacent vertices* $u, v \in V$ *such that* $\mathcal{G} - \{u, v\}$ *is a weakly connected orgraph.*

Proof. The statement immediately follows from [8] or [10, page 208]. □

Corollary 1.3. *If* $\mathcal{G}(V, E)$ *is a weakly connected orgraph which is neither a weak cycle nor a tournament, then* $\mathcal{G}$ *is isomorphic to a suborgraph identification of two proper weakly connected suborgraphs of* $\mathcal{G}$.

Proof. By Lemma 1.2 there are two nonadjacent vertices $u, v \in V$ such that $\mathcal{G} - \{u, v\}$ is weakly connected. Let f be the identity on the suborgraph $\mathcal{G} - \{u, v\}$. The orgraphs $\mathcal{G}_1 = \mathcal{G} - \{u\}$ and $\mathcal{G}_2 = \mathcal{G} - \{v\}$ are proper weakly connected induced suborgraphs of $\mathcal{G}$, and obviously $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2$. □

Whenever uv is an arc of an orgraph $\mathcal{G}(V, E)$, the vertex u is called an *adjacent vertex to* v and v is called an *adjacent vertex from* u. An *outdegree* (an *indegree*) of a vertex $v \in V$ in the orgraph $\mathcal{G}(V, E)$ is the number of vertices adjacent from v (to v). When outdegree of a vertex v is i and indegree of v is j, we will say that v is of type $v_{(j)}^{(i)}$ and write simply v_j^i, when no confusion can arise.

A tournament will be denoted by $\mathcal{T}_n(V, E)$ or briefly by $\mathcal{T}_n$. We say that a tournament $\mathcal{T}_n(V, E)$ is of type $\mathcal{T}^{(o_1, o_2, \ldots, o_k)}$, $o_i \leq o_{i+1}$ for each $i = 1, \ldots, k-1$, if $V = \{v_1, \ldots, v_k\}$ and o_1, o_2, ... , o_k are the outdegres of the vertices v_1, v_2, ... , v_k, respectively. When the tournament $\mathcal{T}_n$ is of the type $\mathcal{T}^{(o_1, o_2, \ldots, o_k)}$, we more precisely write $\mathcal{T}_n = \langle v_1^{(o_1)}, v_2^{(o_2)}, \ldots, v_k^{(o_k)} \rangle$. Let us note that the notation $\mathcal{T}^{(o_1, o_2, \ldots, o_k)}$of k-vertex tournament is ambiguous for $k \geq 5$. We identify a tournament with its type if $k \leq 4$. The tournament $\mathcal{T}^{(1,1,1)}$ was denoted (as the weak cycle) by $\mathcal{C}_{(3,0)}$ and the tournament $\mathcal{T}^{(0,1,2)}$ was denoted by $\mathcal{C}_{(2,1)}$ in [5]. We say that a tournament $\mathcal{T}_n(V, E)$ is *nontrivial* if $|V| \geq 3$.

According to [5] we denote

- by $\mathcal{C}_{(4,1)}$ the weak cycle with two adjacent vertices of the types v_0^2, v_2^0 and three vertices of the type v_1^1 (see Figure 1a),
- by $\mathcal{C}_{(3,2)}$ the weak cycle with two nonadjacent vertices of the types v_0^2, v_2^0 and three vertices of the type v_1^1 (see Figure 1b).

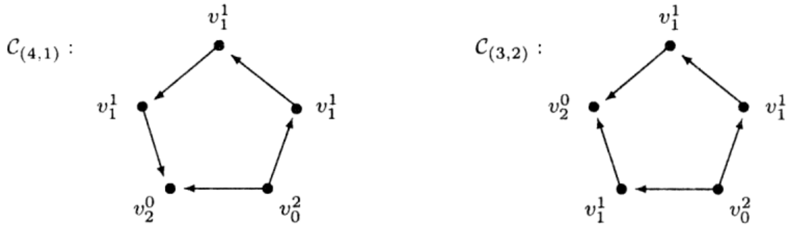

Figure 1a-b

We denote by $\mathbf{0}$ the smallest element of the lattice $\mathbf{L}(I, S, \Gamma)$ and by $\mathbf{1}$ the greatest element of the lattice $\mathbf{L}(I, S, \Gamma)$.

In the paper [5] we showed that the interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ of the lattice $\mathbf{L}(I, S, \Gamma)$ is isomorphic to the lattice $\mathbf{3} \oplus \mathbf{D}^d$ where $\oplus$ is the linear (ordinal) sum of the 3-element chain and the lattice $\mathbf{D}^d$, where $\mathbf{D}^d$ is the dual lattice of the lattice $\mathbf{D}$ of all nonnegative integers with the divisibility relation as the partial ordering. A variety of orgraphs belongs to the interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ iff it contains no nontrivial tournament.

In the next section we pay attention to the interval $[\mathbf{V}(\mathcal{C}_{(4,1)}, \mathbf{1}]$ of the lattice $\mathbf{L}(I, S, \Gamma)$.

In [5] we used a characteristic of a weak cycle. Let $\mathcal{C}(V, E)$ be a weak cycle of the length n. If all arcs of $\mathcal{C}$ have the same orientation, we say that the characteristic of $\mathcal{C}$ is n. On the other hand, if arcs of $\mathcal{C}$ have not the same orientation, we choose an arc $vw \in E$, and we call all arcs of $\mathcal{C}$ having the same orientation as vw *positive*; the other arcs are *negative*. The *characteristic* $ch(\mathcal{C})$ of the weak cycle $\mathcal{C}$ is $|p - n|$, where p is the number of all positive arcs of $\mathcal{C}$ and n is the number of all negative arcs of $\mathcal{C}$.

The next lemmas were proved in [5] and will be used in this paper. First, we denote analogously as in [5]
- by $\mathcal{C}_{(1,1,\ldots,1)}$ a weak cycle containing no vertex of the type v_1^1,
- by $\mathcal{C}_{(n,0)}$, $n \geq 3$, an n-vertex weak cycle containing only vertices of the type v_1^1,
- by $\mathcal{C}_{(3,1)}$ the weak cycle with two adjacent vertices of the type v_1^1, one vertex of the type v_0^2 and one vertex of the type v_2^0.

Lemma 1.4. *Let* $\mathbf{V}$ *be a variety of orgraphs. Let* $\mathcal{C}$ *be a weak cycle different from weak cycles of the type* $\mathcal{C}_{(1,1,\ldots,1)}$. *If* $\mathcal{C} \in \mathbf{V}$ *and* $\mathcal{C}'$ *is a weak cycle for which*

$$ch(\mathcal{C}') = ch(\mathcal{C}), \quad \mathcal{C}' \neq \mathcal{C}_{(3,0)} \text{ and } \mathcal{C}' \neq \mathcal{C}_{(2,1)}$$

(i.e. the characteristics of the weak cycles $\mathcal{C}$, $\mathcal{C}'$ *are the same and* $\mathcal{C}'$ *is not a tournament) then* $\mathcal{C}' \in \mathbf{V}$, *too.*

Lemma 1.5. *Let* $\mathbf{V}$ *be a variety generated by a weak cycle* $\mathcal{C}_{(n,0)}$, $n \geq 3$, *or by* $\mathcal{C}_{(3,1)}$, *or by* $\mathcal{C}_{(3,2)}$. *Let* $\mathcal{G}$ *be an orgraph containing no nontrivial tournament as an induced suborgraph . Then* $\mathcal{G} \in \mathbf{V}$ *iff the characteristic of each weak cycle of the orgraph* $\mathcal{G}$ *is a multiple of the characteristic of the generating weak cycle.*

2. Varieties containing some nontrivial tournaments

By Corollary 1.3 every variety of orgraphs is generated by a set of weak cycles and tournaments. Therefore the next lemmas related to minimal nontrivial tournaments (minimal with respect the relation of being a subtournament) will prove useful.

Lemma 2.1. *A tournament $\mathcal{T}_n$ does not contain the subtournament $\mathcal{T}^{(1,1,1)}$ if and only if the tournament $\mathcal{T}_n$ is of the type $\mathcal{T}^{(0,1,\dots,k)}$.*

Proof. We prove the statement by induction on the number of vertices of tournaments.

The statement is evidently true for 3-vertex tournaments.

Let the statement be true for any k-vertex tournaments.

1. Let $\mathcal{T}_n = \langle v_0^{(0)}, v_1^{(1)}, \dots, v_k^{(k)} \rangle$ be a tournament of the type $\mathcal{T}^{(0,1,\dots,k)}$. We prove that $\mathcal{T}_n$ does not contain the tournament $\mathcal{T}^{(1,1,1)}$. Omitting of the vertex $v_k^{(k)}$ of $\mathcal{T}_n$ (the outdegree of the vertex v_k is k) we obtain k-vertex tournament $\mathcal{T}_n'$ of the type $\mathcal{T}^{(0,1,\dots,k-1)}$. Evidently any 3-vertex subtournament of the tournament $\mathcal{T}_n$ is either subtournament of the tournament $\mathcal{T}_n'$ or a subtournament containing the vertex v_k. The tournament $\mathcal{T}_n'$ contains no subtournament of the type $\mathcal{T}^{(1,1,1)}$ (by induction hypothesis) and the indegree of the vertex v_k is zero, therefore the statement follows.

2. Let $\mathcal{T}_n$ be a k+1-vertex tournament of type different from the type $\mathcal{T}^{(0,1,\dots,k)}$. We prove that $\mathcal{T}^{(1,1,1)}$ is its subtournament. Omitting a vertex v of $\mathcal{T}_n$ we obtain k-vertex tournament $\mathcal{T}_n'$.

a) If $\mathcal{T}_n'$ contains the subtournament $\mathcal{T}^{(1,1,1)}$ then $\mathcal{T}^{(1,1,1)}$ is the subtournament of the tournament $\mathcal{T}_n$, too.

b) If $\mathcal{T}_n'$ contains no subtournament of the type $\mathcal{T}^{(1,1,1)}$ then $\mathcal{T}_n'$ is a tournament of the type $\mathcal{T}^{(0,1,\dots,k-1)}$ by induction hypothesis. Let $\mathcal{T}_n' = \langle u_0^{(0)}, u_1^{(1)}, \dots, u_{k-1}{}^{(k-1)} \rangle$.

If there exist two vertices $u_i, u_j \in V(\mathcal{T}_n')$, $i < j$, such that $u_i v \in E(\mathcal{T}_n)$ and $v u_j \in E(\mathcal{T}_n)$ then the tournament $\langle v, u_j, u_i \rangle = \mathcal{T}^{(1,1,1)}$ is the subtournament of $\mathcal{T}_n$.

Otherwise, the tournament $\langle v, u_0, u_1, \dots, u_{k-1} \rangle$ or $\langle u_0, u_1, \dots, u_{k-1}, v \rangle$ or $\langle u_0, u_1, \dots, u_s, v, u_{(s+1)}, \dots, u_{k-1} \rangle$, $0 \le s \le k-1$, is of the type $\mathcal{T}^{(0,1,\dots,k)}$, a contradiction. □

It is easy to verify that the next statement is true.

Lemma 2.2. *The tournament $\mathcal{T}^{(0,1,2)}$ is a subtournament of every nontrivial tournament $\mathcal{T}_n \neq \mathcal{T}^{(1,1,1)}$.*

Now, we focus our attention to varieties containing at least one nontrivial tournament.

Lemma 2.3. *Let $\mathbb{M}$ be a set of orgraphs . If a nontrivial tournament $\mathcal{T}_n$ is not a suborgraph of any orgraph from $\mathbb{M}$ then $\mathcal{T}_n \notin \mathbf{V}(\mathbb{M})$ (i.e. $\mathcal{T}_n$ does not belong to the variety generated by the set $\mathbb{M}$).*

Proof. It immediately follows from the following fact. If $\mathcal{T}_n$ is neither a suborgraph of an orgraph $\mathcal{G}_1$ nor a suborgraph of an orgraph $\mathcal{G}_2$, then obviously $\mathcal{T}_n$ is not a suborgraph of any suborgraph identification of the orgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$. □

Lemma 2.4. *The variety $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ contains every weak cycle $\mathcal{C} \neq \mathcal{C}_{(3,0)}$, and it covers the variety $\mathbf{V}(\mathcal{C}_{(3,2)})$.*

Proof. First we recall that $\mathcal{T}^{(0,1,2)} = \mathcal{C}_{(2,1)}$ and $\mathcal{C}_{(3,0)} = \mathcal{T}^{(1,1,1)}$. The weak cycle $\mathcal{C}_{(3,2)}$ belongs to the variety $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ by Lemma 1.4, and therefore the variety $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ contains every weak cycle $\mathcal{C} \neq \mathcal{C}_{(3,0)}$ by Lemma 1.5. The variety $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ contains only one tournament (by Lemma 2.3) and the statemant follows. □

Corollary 2.5. *Every variety* $\mathbf{V} \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$ *is generated by a suitable set of tournaments.*

Proof. The variety $\mathbf{V} \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$ is generated by a set $\mathbb{M} = \mathbb{M}_1 \cup \mathbb{M}_2$, where $\mathbb{M}_1$ is a set of weak cycles and $\mathbb{M}_2$ is a set of tournaments (and we suppose $\mathcal{T}^{(1,1,1)} \in \mathbb{M}_2$ if $\mathcal{C}_{(3,0)} \in \mathbb{M}_1$). We can assume that the set $\mathbb{M}_2$ of tournaments is closed under subtournaments (and so $\mathcal{T}^{(0,1,2)} \in \mathbb{M}_2$). It follows $\mathbf{V}(\mathbb{M}_1 \cup \mathbb{M}_2) = \mathbf{V}(\mathbb{M}_2)$ by Lemma 2.4. □

Corollary 2.6. *Let* $\mathbb{M}_1$, $\mathbb{M}_2$ *be sets of nontrivial tournaments closed under nontrivial subtournaments and let* $\mathbf{V}(\mathbb{M}_1) \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$ *and* $\mathbf{V}(\mathbb{M}_2) \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$. *The variety* $\mathbf{V}(\mathbb{M}_1)$ *is covered by the variety* $\mathbf{V}(\mathbb{M}_2)$ *if and only if there exists a tournament* $\mathcal{T}_n^{\blacktriangledown}$ *such that* $\mathbb{M}_2 = \mathbb{M}_1 \cup \{\mathcal{T}_n^{\blacktriangledown}\}$ *and* $\mathcal{T}_n^{\blacktriangledown} \notin \mathbb{M}_1$.

Now we investigate relations between varieties which contain the tournament $\mathcal{T}^{(1,1,1)}$.

Lemma 2.7. *a) The variety* $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ *covers only one variety* $\mathbf{V}(\mathcal{C}_{(4,1)})$.
b) The variety $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)})$ *covers only two varieties* $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ *and* $\mathbf{V}(\mathcal{C}_{(3,2)})$.
c) The variety $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)})$ *is covered by the variety* $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{T}^{(0,1,2)})$.

Proof. a) The variety $\mathbf{V}(\mathcal{T}^{(1,1,1)}) = \mathbf{V}(\mathcal{C}_{(3,0)})$ does not contain any nontrivial tournament $\mathcal{T}_n \neq \mathcal{T}^{(1,1,1)}$ and the weak cycle $\mathcal{C}_{(4,1)}$ belongs to $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ by Lemma 1.4. On the other hand the tournament $\mathcal{T}^{(1,1,1)}$ does not belong to the variety $\mathcal{C}_{(4,1)}$ by Lemma 2.3. A weak cycle $\mathcal{C}$ belongs to the variety $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ if and only if the characteristic of $\mathcal{C}$ is a multiple of the number 3 (by Lemma 1.5) and the statement follows.
b) The variety $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)})$ does not contain nontrivial tournament $\mathcal{T}_n \neq \mathcal{T}^{(1,1,1)}$ and contains every weak cycle $\mathcal{C} \neq \mathcal{C}_{(2,1)}$ (the weak cycle $C_{(2,1)} = \mathcal{T}^{(0,1,2)}$ is the tournament). Let us recall again that the variety $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ contains a weak cycle only if and only if its characteristic is a multiple of the number 3 and the variety $\mathbf{V}(\mathcal{C}_{(3,2)})$ contains every weak cycle $\mathcal{C} \neq \mathcal{C}_{(3,0)}$, by Lemma 1.5.
c) The variety $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{T}^{(0,1,2)})$ contains only two nontrivial tournaments $\mathcal{T}^{(1,1,1)}$ and $\mathcal{T}^{(0,1,2)}$ and all weak cycles, and the above considerations yield the statement. □

Thus, we have proved the next statement.

Theorem 2.8. *The lattice* $\mathbf{L}(I, S, \Gamma)$ *consists of the interval* $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ *(containing all varieties without nontrivial tournaments) and order filter with two minimal elements* $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ *and* $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ *(see Figure 2). The pair of varieties* $\langle \mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)}), \mathbf{V}(\mathcal{T}^{(0,1,2)}) \rangle$ *is the splitting pair of the lattice* $\mathbf{L}(I, S, \Gamma)$ *(i.e. for every variety* $\mathbf{V}$ *either* $\mathbf{V} \leq \mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)})$ *or* $\mathbf{V} \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$*).*

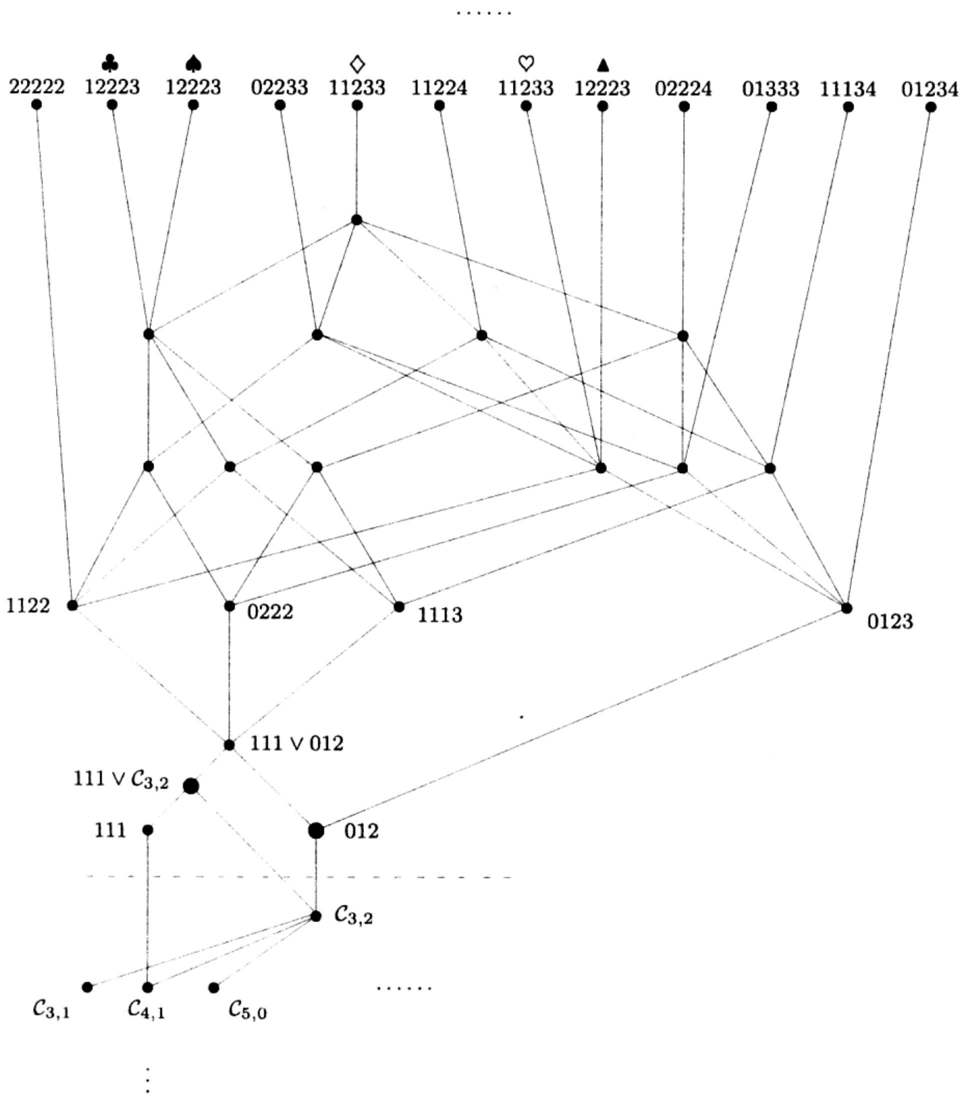

Figure 2

In Figure 2, the generators are used to denote the corresponding varieties, where tournaments are denoted by their types. Since some different tournaments with at least 5 vertices have the same type we depicted all 5-vertex tournaments in Figure 3 (by [11]).

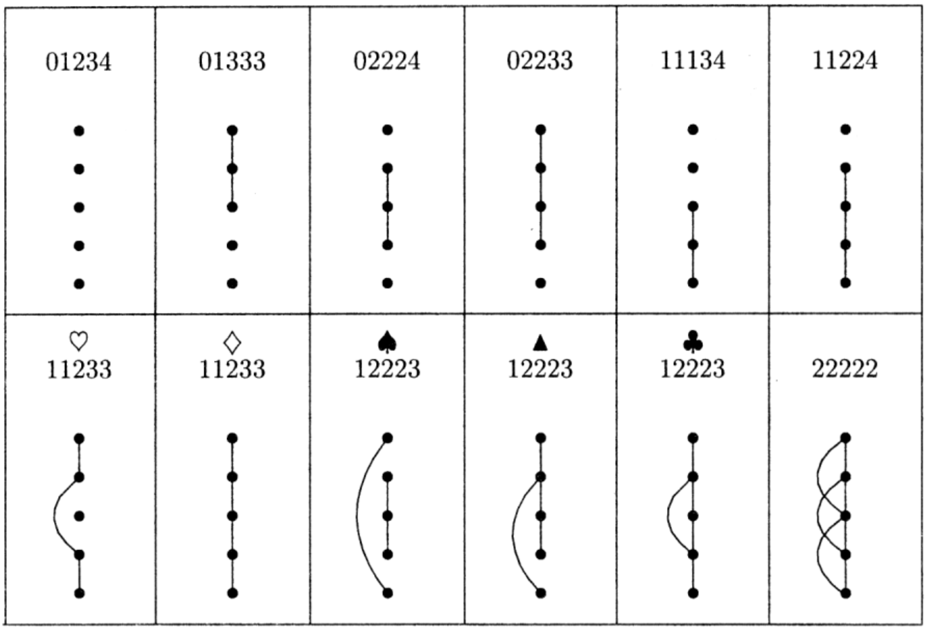

Figure 3
(Upward arcs are shown; downward arcs are implied)

In [5] we showed that the sublattice (the interval) $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ of the lattice $\mathbf{L}(I, S, \Gamma)$ is distributive. Now we will strengthen the statement.

Theorem 2.9. *The lattice* $\mathbf{L}(I, S, \Gamma)$ *is distributive.*

Proof.
a) First, we show that the sublattice (the interval) $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ is a distributive lattice. Let $\mathbb{M}_1$, $\mathbb{M}_2$ be sets of nontrivial tournaments closed under subtournaments and let $\mathbf{V}_1 = \mathbf{V}(\mathbb{M}_1)$, $\mathbf{V}_2 = \mathbf{V}(\mathbb{M}_2)$ be varieties generated by the sets $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively. By the above lemmas we have

$\mathbf{V}_1 \vee \mathbf{V}_2 = \mathbf{V}(\mathbb{M}_1 \cup \mathbb{M}_2)$ and $\mathbf{V}_1 \wedge \mathbf{V}_2 = \mathbf{V}(\mathbb{M}_1 \cap \mathbb{M}_2)$ if $\mathbb{M}_1 \cap \mathbb{M}_2 \neq \emptyset$.

Notice that $\mathbb{M}_1 \cap \mathbb{M}_2 = \emptyset$ if one of these sets is $\{\mathcal{T}^{(1,1,1)}\}$ and the other contains only tournaments of the type $\mathcal{T}^{(0,\dots,k)}$. It implies that the sublattice $[\mathbf{V}(\mathcal{T}^{(0,1,2)}), \mathbf{1}]$ of the lattice $\mathbf{L}(I, S, \Gamma)$ is distributive. Therefore the sublattice $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ is also distributive as is easy to check.

b) We show that the lattice $\mathbf{L}(I, S, \Gamma)$ contains neither the pentagon $\mathbf{N}_5$ nor the diamant $\mathbf{M}_3$.

Suppose, on the contrary, that the diamant $\mathbf{M}_3$ is a sublattice of the lattice $\mathbf{L}(I, S, \Gamma)$. At least two noncomparable elements of $\mathbf{M}_3$ belong to the interval $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ or to the interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)}]$. It follows that the sublattice $\mathbf{M}_3$ is a sublattice of the interval $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ or of the interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)}]$, and both mentioned intervals are distributive lattices, a contradiction.

Suppose, on the contrary, that the pentagon $\mathbf{N}_5$ is a sublattice of the lattice $\mathbf{L}(I, S, \Gamma)$. Let a, b, c be elements of $\mathbf{N}_5$, $a < c$, $a \parallel b$, $c \parallel b$, c covers a. Then both elements a, c belong either to interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ or to interval $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$. Since the intervals $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ and $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ are distributive the element b belongs to the other interval. There are only two possibilities: $a = \mathbf{V}(\mathcal{C}_{(4,1)})$ and

$c = \mathbf{V}(\mathcal{T}^{(1,1,1)})$ and $b \in [\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ (for example $b = \mathbf{V}(\mathcal{C}_{(5,0)})$) or $b = \mathbf{V}(\mathcal{C}_{(3,0)}) \in [\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ and $a, c \in [\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ (for example $c = \mathbf{V}(\mathcal{C}_{(5,0)})$, $a = \mathbf{V}(\mathcal{C}_{(10,0)})$). In this case we have $b \vee a < b \vee c$ or $b \wedge a < b \wedge c$ (see Lemma 1.5), a contradiction. □

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ON α-PSEUDODIMENSION OF MONOUNARY ALGEBRAS

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ABSTRACT. In this paper the notion of α-realizer is defined. There are found necessary and sufficient conditions under which an α-realizer of a connected monounary algebra exists. Next we deal with α-pseudodimension of a product of some special types of monounary algebras.

1 INTRODUCTION

Let $\mathcal{U}$ be the class of all monounary algebras and let $\alpha = (L, f)$ be a fixed element of $\mathcal{U}$. To each $(A, f) \in \mathcal{U}$ we assign a cardinal which will be denoted by α-pdim(A, f); we say that this cardinal is the α-pseudodimension of (A, f).

Our definition is in accordance with that used by V. Novák and M. Novotný [6] (cf. especially Example 6.4 of [6]).

The most of results concern the case when both (A, f) and (L, f) are finite connected monounary algebras.

First we study α-realizers of $(A, f) \in \mathcal{U}$. There are found necessary and sufficient conditions under which an α-realizer of a connected monounary algebra exists. Next some special types α are dealt with and we determine α-pdim(A, f) in the case when (A, f) is a direct product of sticks.

After the World War II, O.Borůvka formulated a problem concerning matrices commuting with a given matrix, that led to study homomorphisms of monounary algebras. His problem stimulated the investigation of these algebras; monounary algebras were investigated e.g. by M.Novotný [7],[8], O.Kopeček [3], E.Nelson [4], D.Jakubíková–Studenovská [1],[2]. The concept of pseudodimension was introduced in [5] for ordered sets. Later it was extended by Novák and Novotný [6] to the concept of α-pseudodimension of arbitrary relational structures.

For the terminology and definitions cf. Section 2.

2 α-REALIZER

In this section we start with defining of the notions we will use below. Then we investigate α-realizers of (A, f).

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Definition 2.1. Let $n \in N, k \in N \cup \{0\}$. The algebra of the type (n, k) is the monounary algebra (B, f), where $B = Z_n \cup \{m \in N : m \leq k\}$ (where $Z_n = \{0_n, \ldots, (n-1)_n\}$ is the set of all integers mod n),

$$f(i_n) = (i+1)_n \text{ for each } i \in Z, f(1) = 0_n,$$

$$f(m) = m - 1 \text{ for each } m \in N, 1 < m \leq k.$$

In the case when $n = 1$, the algebra (B, f) is called a stick or a stick of type k.

Notation 2.2. Let (B, f) be a connected monounary algebra. We denote by $C(B)$ the set of all cyclic elements of (B, f) and $R(B) = |C(B)|$.

The degree $s(x)$ of an element $x \in B$ was defined in [7] (cf. also [1]) as follows:

Let us denote by $B^{(\infty)}$ the set of all elements $x \in B$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of elements belonging to B with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $B^{(0)} = \{x \in B : f^{-1}(x) = \emptyset\}$. Now we define a set $B^{(\lambda)} \subseteq B$ for each ordinal λ by induction. Let $\lambda > 0$ be an ordinal. Assume that we have defined $B^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$B^{(\lambda)} = \{x \in B - \bigcup_{\alpha < \lambda} B^{(\alpha)} : f^{-1}(x) \subseteq \bigcup_{\alpha < \lambda} B^{(\alpha)}\}.$$

The sets $B^{(\lambda)}$ (where λ is an ordinal or $\lambda \doteq \infty$) are pairwise disjoint. For each $x \in B$, either $x \in B^{(\infty)}$ or there is an ordinal λ with $x \in B^{(\lambda)}$. In the former case we put $s(x) = \infty$, in the latter we set $s(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

Suppose that $R(B) \neq 0$. If $B = C(B)$, then we put $h(B) = 0$. If $B \neq C(B)$, then we define $h(B) = 1 + \sup \{s(x) : x \in B - C(B)\}$.

Notice that the definition of $s(x)$ implies that if $B \neq C(B)$, then

$$h(B) = 1 + \sup\{s(x) : x \in B - C(B), f(x) \in C(B)\}.$$

Remark. Let us remark that we considerably apply results of M.Novotný [7],[8] concerning homomorphisms of monounary algebras. E.g., without further reference we will use that if (A, f) and (B, f) are monounary algebras, then

(1) if φ is a homomorphism of (A, f) into (B, f), then $s(\varphi(x)) \geq s(x)$ for each $x \in A$,
(2) if φ is a homomorphism of (A, f) into (B, f) and $x \in A$ belongs to a cycle C, then $\varphi(x)$ belongs to a cycle $D \subseteq B$ such that $|D|$ divides $|C|$.

Notation 2.3. We will denote by (Z, f) and (N, f) the monounary algebra such that $f(i) = i + 1$ for each $i \in Z$ or $i \in N$, respectively.

Further, for a cardinal k let (N_k, f) be a fixed monounary algebra such that

$$N_k = N \cup D, \ N \cap D = \emptyset, \ |D| = k,$$

$$f(a) = \begin{cases} a + 1 & \text{if } a \in N, \\ 1 & \text{if } a \in D. \end{cases}$$

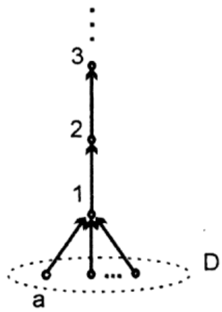


FIG. 1

Definition 2.4. Let $(A, f) \in \mathcal{U}$ and let $\{\varphi_j : j \in J\}$ be a nonempty system of mappings of A into L such that for any $x, y \in A$ we have

$$y = f(x) \Longleftrightarrow (\forall j \in J)(\varphi_j(y) = f(\varphi_j(x))).$$

Then $\{\varphi_j : j \in J\}$ is said to be an α-realizer of (A, f).

If no α-realizer of (A, f) exists, then we set

$$\alpha\text{-pdim } (A, f) = 0.$$

Further, suppose that there exists some α-realizer of (A, f); then we put

$$\alpha \text{ -pdim } (A, f) = \min\{|J| : \{\varphi_j : j \in J\} \text{ is an } \alpha \text{ -realizer of } (A, f)\}.$$

This cardinal is called α-pseudodimension of (A, f).

This definition immediately yields the following two assertions:

Lemma 2.5. Let $(A, f) \in \mathcal{U}$ and suppose that $\{\varphi_j : j \in J\}$ is an α-realizer of (A, f). For $j \in J$, the mapping φ_j is a homomorphism of (A, f) into (L, f).

Lemma 2.6. Let $(A, f) \in \mathcal{U}$ and let Υ be a nonempty system of homomorphisms of (A, f) into (L, f). Then Υ is an α-realizer of (A, f) if and only if the following implication is valid for each $x, y \in A$

$$(*) \qquad ((\forall \varphi \in \Upsilon)(\varphi(y) = \varphi(f(x)))) \Rightarrow y = f(x).$$

Corollary 2.7. If $(A, f) \in \mathcal{U}$ and there exists an injective homomorphism of (A, f) into (L, f), then $\alpha\text{-pdim}(A, f) = 1$.

Lemma 2.8. Let (A, f) and (L, f) be connected monounary algebras. If there exists an α-realizer of (A, f), then $R(A) = R(L)$.

Proof. Suppose that Υ is an α-realizer of (A, f).

a) First assume that $R(L) = m \in N$. Then there exists $x \in A$ such that $\varphi(f(x)) \in C(L)$. Put $y = f^{m+1}(x)$. For $\varphi \in \Upsilon$ we get

$$\varphi(y) = \varphi(f^{m+1}(x)) = f^{m+1}(\varphi(x)) = f^m(\varphi(f(x))) = \varphi(f(x)).$$

Since Υ is an α-realizer, $(*)$ of 2.6 yields that $y = f(x)$, i.e., $f^m(f(x)) = f(x)$. Therefore $R(A)$ divides m. From 2.5 it follows that $R(L)$ divides $R(A)$ (because each φ is a homomorphism of (A, f) into (L, f)), thus we obtain $R(A) = R(L)$.

b) Now suppose that $R(L) = 0$. According to 2.5, $R(A) = 0$, too. □

Theorem 2.9. Let (A, f) and (L, f) be connected monounary algebras such that $R(A) = R(L) \neq 0$. An α-realizer of (A, f) exists if and only if $h(A) \leq h(L)$.

Proof. Let Υ be an α-realizer of (A, f). Let $a \in A - C(A)$ be such that $f(a) \in C(A)$. There exists $x \in C(A)$ with $f^2(x) = f(a)$. Put $c = f(x)$. Then $f(c) = f(a) \in C(A)$. First suppose that $\varphi(a) \in C(L)$ for each $\varphi \in \Upsilon$. This implies that for $\varphi \in \Upsilon$ we have

$$\varphi(a) = \varphi(c) = \varphi(f(x)),$$

thus by $(*)$, $a = f(x)$, a contradiction. Hence there exists $\psi \in \Upsilon$ such that $b = \psi(a) \notin C(L)$. Since ψ is a homomorphism, the element $f(b)$ is cyclic and

$$s(a) \leq s(\psi(a)) = s(b).$$

Therefore $h(A) \leq h(L)$.

Conversely, assume that $h(A) \leq h(L)$. If $h(A) = 0$, then let φ_0 be an arbitrary isomorphism of A onto $C(L)$. It is obvious that $\Upsilon = \{\varphi_0\}$ is an α-realizer of A. Now let $h(A) \neq 0$. Let $u \in A - C(A)$. Then there is $a \in A - C(A)$ with $f(a) \in C(A)$ and $u \in f^{-n}(a)$ for some $n \in N \cup \{0\}$. The relation $h(A) \leq h(L)$ implies that there is $b \in L - C(L)$ such that $f(b) \in C(L)$ and that $s(a) \leq s(b)$. Let b be a fixed element with this property. Obviously, $s(u) \leq s(b)$. By [8], Thm., p.157 there exists a homomorphism ψ_u of (A, f) into (L, f) having the following properties:

(1) $\psi_u(u) = b$,
(2) if $v \notin \bigcup_{m \in N \cup \{0\}} f^{-m}(u) \cup \{f^k(u) : k \in N\}$, then $\psi_u(v) \in C(L)$.

Denote $\Upsilon = \{\psi_u : u \in A - C(A)\}$. Let us verify that Υ is an α-realizer of (A, f) according to $(*)$. Assume that $x, y \in A$ and that $\psi_u(y) = \psi_u(f(x))$ for each $\psi_u \in \Upsilon$.

a) If $f(x) \notin C(A)$, then take $u = f(x)$. We get $\psi_u(y) = \psi_u(u)$. Since $\psi_u^{-1}(\psi_u(u))$ is a one-element set $\{u\}$ by (2), this implies that $y = u$, i.e., $y = f(x)$.
b) Let $f(x) \in C(A)$. If $y \notin C(A)$ then $\psi_y(y) \notin C(L)$, hence $\psi_y(y) \neq \psi_y(f(x))$, a contradiction. Thus $y \in C(A)$. Take an arbitrary $\varphi \in \Upsilon$. Then φ is an isomorphism of $C(A)$ onto $C(L)$, thus the relation $\varphi(f(x)) = \varphi(y)$ yields that $f(x) = y$.

Therefore Υ is an α-realizer of (A, f). □

Theorem 2.10. Let (A, f) and (L, f) be connected monounary algebras such that $R(A) = R(L) = 0$. Let $P = \{u \in A : |f^{-1}(u)| > 1,\ f^{-2}(u) \neq \emptyset\}$. An α-realizer of (A, f) exists if and only if one of the following conditions is satisfied:

(a) $(A, f) \cong (N, f)$ or $(A, f) \cong (N_k, f)$ for some $k \in \text{Card}$;
(b) $(A, f) \cong (Z, f)$ and there is a subalgebra of (L, f) isomorphic to (Z, f);
(c) $P \neq \emptyset$ and for each $u \in A$, $q_1, q_2 \in f^{-1}(u)$, $q_1 \neq q_2$ such that $f^{-1}(q_1) \neq \emptyset$ there are $v \in L$ and distinct elements $t_1, t_2 \in f^{-1}(v)$ such that $s(f^k(u)) \leq s(f^k(v))$, $s(q_i) \leq s(t_i)$ for each $k \in N \cup \{0\}$, $i \in \{1, 2\}$.

Proof. Let Υ be an α-realizer of (A, f). First suppose that $P \neq \emptyset$. Take $u \in P$, $x \in f^{-2}(u)$, $q_2 \in f^{-1}(u) - \{f(x)\}$. Let $q_1 = f(x)$. Since Υ is an α-realizer, we obtain that there is $\varphi \in \Upsilon$ such that $\varphi(q_2) \neq \varphi(q_1)$. Put $v = \varphi(u)$. Then $s(f^k(u)) \leq s(\varphi(f^k(u))) = s(f^k(v))$, $s(q_i) \leq s(\varphi(q_i))$ for each $k \in N \cup \{0\}$, $i \in \{1, 2\}$, hence (c) is valid. Now let $P = \emptyset$. Then (A, f) is isomorphic to one of the algebras $(Z, f), (N, f), (N_k, f)$ for some $k \in \text{Card}$, i.e., either (a) is valid or $(A, f) \cong (Z, f)$.

Each $\varphi \in \Upsilon$ is a homomorphism, thus if $(A, f) \cong (Z, f)$ then (Z, f) is isomorphic to some subalgebra of (L, f). Therefore one of the conditions (a) – (c) is satisfied.

Conversely, let one of the conditions (a) – (c) be valid. If (a) or (b) is valid, then there exists a homomorphism φ_0 of (A, f) into (L, f); put $\Upsilon = \{\varphi_0\}$. Let $x, y \in A$, $\varphi_0(y) = \varphi_0(f(x))$, $y \neq f(x)$. If (A, f) is isomorphic to (N, f) or to (Z, f), then each homomorphism of (A, f) into (L, f) is injective. Let (A, f) be (up to isomorphism) (N_k, f) for some $k \in$ Card. Further, the relation $\varphi_0(y) = \varphi_0(f(x))$ implies that $\{y, f(x)\} \subseteq D$, which is a contradiction, since $f(x) \in N - D$.

Let (c) hold. If $u \in P$, $p \in f^{-2}(u)$, $q \in f^{-1}(u) - \{f(p)\}$ then take $q_1 = f(p)$, $q_2 = q$; by (c) (according to [8], as in the proof of Thm.2.9) there exists a homomorphism ψ_{upq} of (A, f) into (L, f) such that

(1) $\psi_{upq}(u) = v$,

(2) $\psi_{upq}(q) \neq \psi_{upq}(f(p))$.

Let Υ the set of all homomorphisms of the form ψ_{upq}. We will show that Υ is an α-realizer of (A, f). Let $x, y \in A$ and suppose that $\varphi(y) = \varphi(f(x))$ for each $\varphi \in \Upsilon$. Put $z = f(x)$. From the connectedness we infer that $f^m(y) = f^n(z)$ for some $m, n \in N \cup \{0\}$; we can assume that m, n are the smallest nonnegative integers with this property. Since $\Upsilon \neq \emptyset$ and $\varphi(y) = \varphi(z)$ for $\varphi \in \Upsilon$, we get that $m \neq 0$ and $n \neq 0$. Denote $u = f^m(y)$, $p = f^{n-1}(x)$, $q = f^{m-1}(y)$. In view of the relation $\varphi_{upq} \in \Upsilon$ we have

(3) $\psi_{upq}(y) = \psi_{upq}(z)$.

This implies

$f^m(\psi_{upq}(z)) = f^m(\psi_{upq}(y)) = \psi_{upq}(f^m(y)) = \psi_{upq}(u) = \psi_{upq}(f^n(z)) = f^n(\psi_{upq}(z))$,
hence $m = n$. Assume that $y \neq z$. Next,
$\psi_{upq}(q) = \psi_{upq}(f^{n-1}(y)) = f^{n-1}(\psi_{upq}(y))$, i.e., $\psi_{upq}(y) \in f^{-(n-1)}(\psi_{upq}(q))$.
Similarly we obtain $\psi_{upq}(f(p)) = \psi_{upq}(f(f^{n-1}(x)) = f^{n-1}(\psi_{upq}(f(x)))$,
i.e., $\psi_{upq}(f(x)) \in f^{-(n-1)}(\psi_{upq}(f(p))$. In view of (3) we get

$$f^{-(n-1)}(\psi_{upq}(q)) \cap f^{-(n-1)}(\psi_{upq}(f(p))) \neq \emptyset,$$

which is a contradiction to (2). This concludes the proof. □

3 α-PDIMENSION AND A PRODUCT OF STICKS

In this section we deal with realizers of type (n, k), $n \in N, k \in N \cup \{0\}$. Further, we find the value of $(1, k)$-pseudodimension of a direct product of sticks.

Lemma 3.1. Let $n \in N, k \in N \cup \{0\}$. An (n, k)-realizer of a connected monounary algebra (A, f) exists if and only if $R(A) = n$ and $f^k(a) \in C(A)$ for each $a \in A$.

Proof. The assertion is a corollary of 2.9. □

Theorem 3.2. Let $n \in N, k \in N \cup \{0\}$.

a) If $k = 0$ or $n = 1, k = 1$, then (n, k)-pdim $(A, f) = 1$ for each monounary algebra such that an (n, k)-realizer exists.

b) Let $k = 1, n \geq 2$. If $m \in \{1, 2, \ldots, n\}$, then there exists $(A, f) \in \mathcal{U}$ such that (n, k)-pdim $(A, f) = m$. If $m \in N, m > n$, then (n, k)-pdim $(A, f) \neq m$ for each $(A, f) \in \mathcal{U}$.

c) If $k \geq 2, m \in N$, then there exists $(A, f) \in \mathcal{U}$ such that (n, k)-pdim $(A, f) = m$.

Proof.

a) Assume that an (n,k)-realizer of (A,f) exists. If $k=0$, then $|A|=|C(A)|=n$ by 3.1 and there is an isomorphism φ_0 of A onto Z_n. Then 2.7 implies that $(n,0)$-pdim $(A,f)=1$. If $n=1$, $k=1$, then 3.1 implies that there is $c \in A$ such that $f(a)=c$ for each $a \in A$. Put $\varphi(c)=0_1$, $\varphi(a)=1$ for each $a \in A-\{c\}$. Then $\{\varphi\}$ is a $(1,1)$-realizer of (A,f) and $(1,1)$-pdim $(A,f)=1$.

b) From the assumption it follows that $L=Z_n \cup \{1\}$.
Let $m \in \{1,\dots,n\}$. We put $A=Z_n \cup \{1,\dots,m\}$, $f(i_n)=(i+1)_n$ for each $i \in Z$, $f(l)=l_n$ for each $l \in \{1,\dots,m\}$. For $j \in \{1,\dots,m\}$ we define a mapping $\varphi_j : A \to L$ as follows: $\varphi_j(i_n)=(i-j)_n$ for each $i \in Z$, $\varphi_j(j)=1$, $\varphi_j(l)=(l-1-j)_n$ for each $l \in \{1,\dots,m\}-\{j\}$. (cf. Fig.2.)

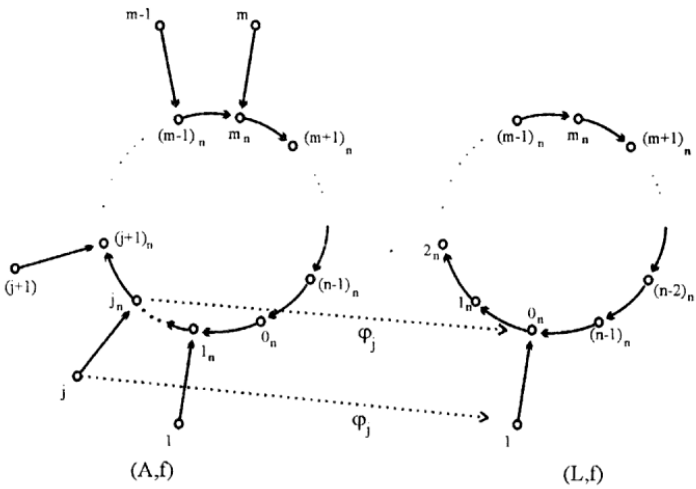


FIG. 2

It is easy to verify that φ_j is a homomorphism for each $j \in \{1,\dots,m\}$. Denote $\Upsilon=\{\varphi_j : j \in \{1,\dots,m\}\}$. Let $x,y \in A$, $\varphi_j(y)=\varphi_j(f(x))$ for each $j \in \{1,\dots,m\}$. We have $\varphi_j^{-1}(1)=\{j\}$ for each j, thus $y \neq j$ for each j. Thus $y \in Z_n$. Next, $f(x) \in Z_n$, hence in view of the fact that any φ_j is a bijection of $C(A)$ onto $C(L)$, the relation $\varphi_j(y)=\varphi_j(f(x))$ implies that $y=f(x)$. Therefore Υ is an $(n,1)$-realizer of (A,f) and $(n,1)$-pdim $(A,f) \le m$.

Suppose that Υ' is an $(n,1)$-realizer of (A,f). Let $j \in \{1,\dots,m\}$. If $\psi(j) \in Z_n$ for each $\psi \in \Upsilon'$, then

$$\psi(j)=\psi((j-1)_n)=\psi(f((j-2)_n)),$$

$$j=f((j-2)_n),$$

which is a contradiction. Thus there exists $\psi_j \in \Upsilon'$ such that $\psi_j(j)=1$. If $\psi_j=\psi_l$ for $j,l \in \{1,\dots,m\}$, then $\psi_j(l)=1=\psi_j(j)$, which implies

$$\psi_j(j_n)=\psi_j(f(j))=f(\psi_j(j))=f(1)=0_n.$$

Similarly, $\psi_j(l_n) = 0_n$. Further, we have

$$0_n = \psi_j(l_n) = \psi_j(f^{l-j}(j_n)) = f^{l-j}(\psi_j(j_n)) = f^{l-j}(0_n) = (l-j)_n,$$

thus $l = j$. Hence $|\Upsilon'| \geq m$, therefore $(n,1)$-pdim $(A,f) = m$.

Let $m > n$ and suppose that there is $(A,f) \in \mathcal{U}$ with $(n,1)$-pdim $(A,f) = m$. Thus there is an $(n,1)$-realizer Υ of (A,f) with $|\Upsilon| = m$. According to 3.1 we have $R(A) = n$ and $f(a) \in C(A)$ for each $a \in A$. Up to isomorphism,

$$A = Z_n \cup D_1 \cup D_2 \cup \cdots \cup D_n,$$

$$f(i_n) = (i+1)_n \text{ for each } i \in Z,$$

$$f(d) = l_n \text{ for each } d \in D_l,\ l \in \{1,\ldots,n\}.$$

Let $j \in \{1,\ldots,n\}$. Define a mapping $\varphi_j : A \to L$ as follows:

$$\varphi_j(i_n) = (i-j)_n \text{ each } i \in Z,$$

$$\varphi_j(d) = \begin{cases} 1 & \text{if } d \in D_j, \\ (l-1-j)_n & \text{if } d \in D_l,\ l \neq j. \end{cases}$$

It is easy to verify that $\{\varphi_j : j \in \{1,\ldots,n\}\}$ is an $(n,1)$-realizer of (A,f), hence $(n,1)$-pdim $(A,f) \leq n$, which is a contradiction.

c) Let $k \geq 2, m \in N$. There exists $t \in N$ such that $2^{m-1} < t+1 \leq 2^m$. We denote by (A,f) a monounary algebra such that $A = Z_n \cup \{a_1,\ldots,a_t\} \cup \{b_1,\ldots,b_t\}$ (suppose that all these elements are distinct and they do not belong to Z_n), where $f(i_n) = (i+1)_n$ for each $i \in Z$, $f(a_l) = 0_n$, $f(b_l) = a_l$ for each $l = 1,\ldots,t$.

There exist 2^m distinct m-tuples of the elements $(n-1)_n, 1$. Thus there exists a set $Q = \{q_1,\ldots,q_t\}$ of m-tuples of the elements $(n-1)_n$, 1 with $|Q| = t$, $q \neq ((n-1)_n,\ldots,(n-1)_n)$ for each $q \in Q$. For $j \in \{1,\ldots,m\}$, $l \in \{1,\ldots,t\}$ let $q_l(j)$ be the projection of q_l into the j-th coordinate. Let $j \in \{1,\ldots,m\}$; we will define a mapping φ_j as follows. For $l \in \{1,\ldots,t\}$ we put

$$\varphi_j(i_n) = i_n \text{ for each } i \in Z,$$

$$\varphi_j(a_l) = q_l(j),$$

$$\varphi_j(b_l) = \begin{cases} 2 & \text{if } q_l(j) = 1, \\ (n-2)_n & \text{otherwise.} \end{cases}$$

It is easy to verify that $\{\varphi_j : j \in \{1,\ldots,m\}\}$ is a set of homomorphisms and that it is an (n,k)-realizer of (A,f) (cf. Example 1).

Next suppose that Υ is an (n,k)-realizer of (A,f), $\Upsilon = \{\psi_1,\ldots,\psi_r\}$, $r = |\Upsilon| < m$. For $l \in \{1,\ldots,t\}$ consider an r-tuple $p^{(l)}$ such that for $j \in \{1,2,\ldots,r\}$

$$p^{(l)}(j) = \begin{cases} 0 & \text{if } \psi_j(a_l) \in Z_n, \\ 1 & \text{otherwise.} \end{cases}$$

Let $l \in \{1,\ldots,t\}$. If $p^{(l)}(j) = 0$ for each $j \in \{1,\ldots,r\}$, then $\psi_j(a_l) \in Z_n$ for each $j \in \{1,\ldots,r\}$; then $\psi_j(a_l) = \psi_j((n-1)_n) = \psi_j(f((n-2)_n))$ and the

definition of an (n,k)-realizer implies that $a_l = f((n-2)_n)$, a contradiction. Therefore each r-tuple $p^{(l)}$ does not consist of zeros only.
Since ψ_j is a homomorphism, $\psi_j(a_l) = 1$ or $\psi_j(a_l) \in Z_n$. If $l, l' \in \{1, \dots, t\}$ and $j \in \{1, \dots, r\}$, then either

(1) $$\psi_j(a_l) = \psi_j(a_{l'}) \text{ or}$$

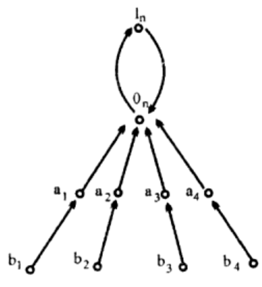
(2) $$\psi_j(a_l) = 1, \psi_j(a_{l'}) = (n-1)_n \text{ or}$$

(3) $$\psi_j(a_l) = (n-1)_n, \psi_j(a_{l'}) = 1.$$

By the assumption, $r \leq m-1$, $2^r - 1 \leq 2^{m-1} - 1 < t$, thus there exist $l, l' \in \{1, \dots, t\}$, $l \neq l'$ such that $p^{(l)} = p^{(l')}$. Then $p^{(l)}(j) = p^{(l')}(j)$ for each $j \in \{1, \dots, r\}$. Then we obtain that the cases (2) and (3) yield a contradiction, thus $\psi_j(a_l) = \psi_j(a_{l'})$ for each $j \in \{1, \dots, r\}$. This implies that for each $j \in \{1, \dots, r\}$, $\psi_j(f(b_l)) = \psi_j(a_l) = \psi_j(a_{l'})$. According the fact that Υ is an (n,k)-realizer we get $f(b_l) = a_{l'}$, which is a contradiction.

Thus we have shown that (n,k)-pdim$(A, f) = m$. □

Example 1. Let $n = 2$ and $k \geq 2$. For $m = 3$ we will define (A, f) such that $(2,k)$-pdim(A, f) is equal to m. Let us follow the proof of theorem 3.2c). The relation $2^2 < t + 1 \leq 2^3$ implies $t \in \{4, 5, 6, 7\}$. Let $t = 4$.



(A,f)

FIG. 3

There exists 2^3 of 3-tuples of elements $1, 1_2$: $(1,1,1)$, $(1,1,1_2)$, $(1,1_2,1)$, $(1_2,1,1)$, $(1,1_2,1_2)$, $(1_2,1,1_2)$, $(1_2,1_2,1)$, $(1_2,1_2,1_2)$. Next we choose four elements of them ($t = 4$), e.g., let $q_1 = (1,1,1)$, $q_2 = (1,1,1_2)$, $q_3 = (1,1_2,1)$ and $q_4 = (1_2,1,1)$. Put $Q = \{q_1, q_2, q_3, q_4\}$. We can define three ($m = 3$) mappings $\varphi_1, \varphi_2, \varphi_3$.

	0_2	1_2	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
φ_1	0_2	1_2	1	1	1	1_2	2	2	2	0_2
φ_2	0_2	1_2	1	1	1_2	1	2	2	0_2	2
φ_3	0_2	1_2	1	1_2	1	1	2	0_2	2	2

It can be verified that $\{\varphi_1, \varphi_2, \varphi_3\}$ is a $(2,k)$-realizer of algebra (A, f) and $(2,k)$-pdim$(A, f) = 3$.

Theorem 3.3. Let (A, f) be a direct product of sticks $(A_1, f), (A_2, f), \dots, (A_m, f)$ of types $k_1, k_2, \dots, k_m$ such that $|\{i \in \{1, \dots, m\} : k_i = 1\}| \le 1$. If $k \ge k_i$ for each $i \in \{1, \dots, m\}$, then $(1, k)$-pdim $(A, f) = m$.

Proof. Let (L, f) be a monounary algebra of type k, $k \ge k_i$ for each $i \in \{1, \dots, m\}$. We can suppose that $L = \{0, 1, \dots, k\}$, $A_i = \{0, 1, \dots, k_i\}$,

$$f(j) = \begin{cases} j-1 & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

in L and in A_i for $i \in \{1, \dots, m\}$. For $j \in \{1, \dots, m\}$ we define a mapping $\varphi_j : A_1 \times \dots \times A_m \to L$ such that $\varphi_j((a_1, \dots, a_m)) = a_j$. Put $\Upsilon = \{\varphi_j : j \in \{1, \dots, m\}\}$. If $x, y \in A$, $\varphi_j(y) = \varphi_j(f(x))$ for each $j \in \{1, \dots, m\}$, then $y = f(x)$. Thus Υ is a $(1, k)$-realizer of (A, f) and $(1, k)$-pdim $(A, f) \le m$.
Suppose that Υ' is a $(1, k)$-realizer of (A, f), $|\Upsilon'| < m$. Denote $\bar{0} = (0, 0, \dots, 0) \in A$. Obviously, $\psi(\bar{0}) = 0$ for each $\psi \in \Upsilon'$. We have $|f^{-1}(\bar{0})| = |\{a = (a_1, \dots, a_m) : a_i \in \{0, 1\} \text{ for each } i \in \{1, \dots, m\}\}| = 2^m$. Next, if $a \in f^{-1}(\bar{0})$, then $\psi(a) \in \{0, 1\}$ for each $\psi \in \Upsilon'$. Since $2^{|\Upsilon'|} < 2^m$, there are $a, b \in f^{-1}(\bar{0})$, $a \neq b$ such that $\psi(a) = \psi(b)$ for each $\psi \in \Upsilon'$. Without loss of generality, in view of the assumption that $|\{i \in \{1, \dots, m\} : k_i = 1\}| \le 1$ we get that $f^{-1}(b) \neq \emptyset$; let $x \in f^{-1}(b)$. Then $\psi(a) = \psi(f(x))$ for each $\psi \in \Upsilon'$. The set Υ' is a $(1, k)$-realizer, of (A, f), thus $(*)$ implies $a = f(x) = b$, which is a contradiction. Therefore $(1, k)$-pdim $(A, f) = m$. □

Lemma 3.4. Let (B, f) be a monounary algebra fulfilling the condition

(c) if $b \in B$, then there is $b' \in B$ with $f(b) = f(b')$, $f^{-1}(b') = \emptyset$.

Let (E, f) be a 1-stick. Then $(B, f) \times (E, f)$ fulfils (c) and if $(1, k)$-pdim$(B, f) = p$, then $(1, k)$-pdim$((B, f) \times (E, f)) = p$.

Proof. Without loss of generality, $E = \{0, 1\}$. First we show (c) for the algebra $(B, f) \times (E, f)$. Let $(b, e) \in B \times E$. By (c), there is $b' \in B$ with $f(b) = f(b')$, $f^{-1}(b') = \emptyset$. Take $(b', e) \in B \times E$. Then

$$f((b', e)) = (f(b'), f(e)) = (f(b), f(e)) = f((b, e)),$$

$$f^{-1}((b', e)) = \{(x_1, x_2) : x_1 \in f^{-1}(b'), \ x_2 \in f^{-1}(e)\} = \emptyset.$$

Further suppose that $(1, k)$-pdim$(B, f) = p$ and that Υ is a $(1, k)$-realizer of (B, f). For $\varphi \in \Upsilon$ we define a mapping $\bar{\varphi} : B \times E \to L$ as follows. Let $(b, e) \in B \times E$, b' be the element corresponding to b in view of the condition (c). We put

$$\bar{\varphi}((b, e)) = \begin{cases} \varphi(b) \text{ if } e = 0, \\ \varphi(b') \text{ if } e = 1; \end{cases}$$

$\bar{\Upsilon} = \{\bar{\varphi} : \varphi \in \Upsilon\}$. To prove that $\bar{\Upsilon}$ is a $(1, k)$-realizer of $(B, f) \times (E, f)$ assume that (x, e), $(y, j) \in B \times E$ and that $\bar{\varphi}(f((x, e))) = \bar{\varphi}((y, j))$ for each $\bar{\varphi} \in \bar{\Upsilon}$. For any $\varphi \in \Upsilon$ we have

$$\bar{\varphi}(f((x, e))) = \bar{\varphi}((f(x), 0)) = \varphi(f(x)).$$

If $j = 0$, then

$$\bar{\varphi}((y, j)) = \bar{\varphi}((y, 0)) = \varphi(y),$$

thus the fact that $x, y \in B$ and that Υ is a $(1,k)$-realizer of (B, f) implies that $y = f(x)$, hence

$$(y, j) = (f(x), 0) = f((x, 0)) = f((x, e)).$$

Let $j = 1$. To $y \in B$ there is $y' \in B$ with $f(y) = f(y')$, $f^{-1}(y') = \emptyset$. For any $\varphi \in \Upsilon$,

$$\bar{\varphi}((y, j)) = \bar{\varphi}((y, 1)) = \varphi(y'),$$

i.e., $\varphi(y') = \varphi(f(x))$ for each $\varphi \in \Upsilon$. Since Υ is a $(1,k)$-realizer of (B, f), this implies that $y' = f(x)$, which is a contradiction, because $f^{-1}(y') = \emptyset$. Thus j cannot be 1. Therefore

$$(1,k)\text{-pdim}((B, f) \times (E, f)) \leq (1,k)\text{-pdim}(B, f).$$

The converse relation is obvious, thus

$(1,k)\text{-pdim}((B, f) \times (E, f)) = (1,k)\text{-pdim}(B, f)$. □

Corollary 3.5. Let (A, f) be a direct product of sticks $(A_1, f), \ldots, (A_n, f)$ of types $k_1, k_2, \ldots, k_m$ and assume that $|\{i \in \{1, \ldots, m\} : k_i = 1\}| = t > 1$. If $k \geq k_i$ for each $i \in \{1, \ldots, m\}$, then $(1,k)$-pdim $(A, f) = m - t + 1$.

Proof. The assertion is a consequence of 3.3 and 3.4; we can proceed by induction.

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SOME ERROR ESTIMATES IN THE NEWTON METHOD

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Abstract. For the numerical solution of the equation $f(x) = 0$ by the Newton method the inequality $|x_{n-1} - x_n| \le \delta$ is often used as a stopping rule, where $\delta > 0$ is prescribed. We show that this inequality yields no information about $|x_n - x_0|$, where x_0 is a root, because the inequality $|x_n - x_0| \le |x_{n-1} - x_n|$ is not true in a general case. We give several simple estimates for $|x_n - x_0|$. Particularly, we give a sufficient condition under which $|x_n - x_0| \le |x_{n-1} - x_n|$.

We consider the equation $f(x) = 0$, where f is a convex or concave strictly monotone function of the class C^1 on the interval $\langle a, b\rangle$ such that $f(a)f(b) < 0$ and $\min(|f'(a)|, |f'(b)|) > 0$. To find a numerical solution x_0 of this equation the Newton method is often used. Namely, put

$$x_1 = \begin{cases} b \text{ if } f \text{ is convex and increasing or concave and decreasing} \\ a \text{ if } f \text{ is convex and decreasing or concave and increasing} \end{cases}$$

and

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \qquad \text{for } n > 1 .$$

Then we obtain a monotone sequence $(x_n)_{n=1}^{\infty}$ which converges to the root x_0. The inequality $|x_{n-1} - x_n| \le \delta$ is often used as a stopping rule, see [2, p. 87]. This rule is only formal, i.e. it does not imply $|x_n - x_0| \le \delta$, because the estimate $|x_n - x_0| \le |x_{n-1} - x_n|$ is false in a general case. This fact is illustrated on Figure 1.

To disprove this estimate we also consider the equation $\tan x = x$, or equivalently $\tan x - x = 0$, on the interval $\langle \pi, \frac{3\pi}{2})$. Put

$$f(x) = \tan x - x .$$

Since

$$f(\pi) < 0 \text{ and } \lim_{x \nearrow \frac{3\pi}{2}} f(x) = +\infty ,$$

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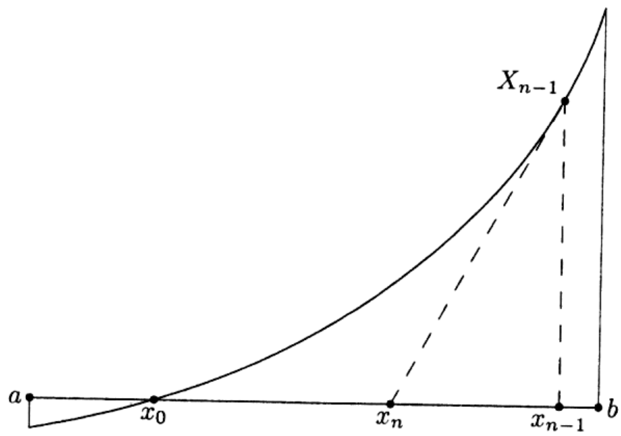

FIGURE 1 *Newton method*

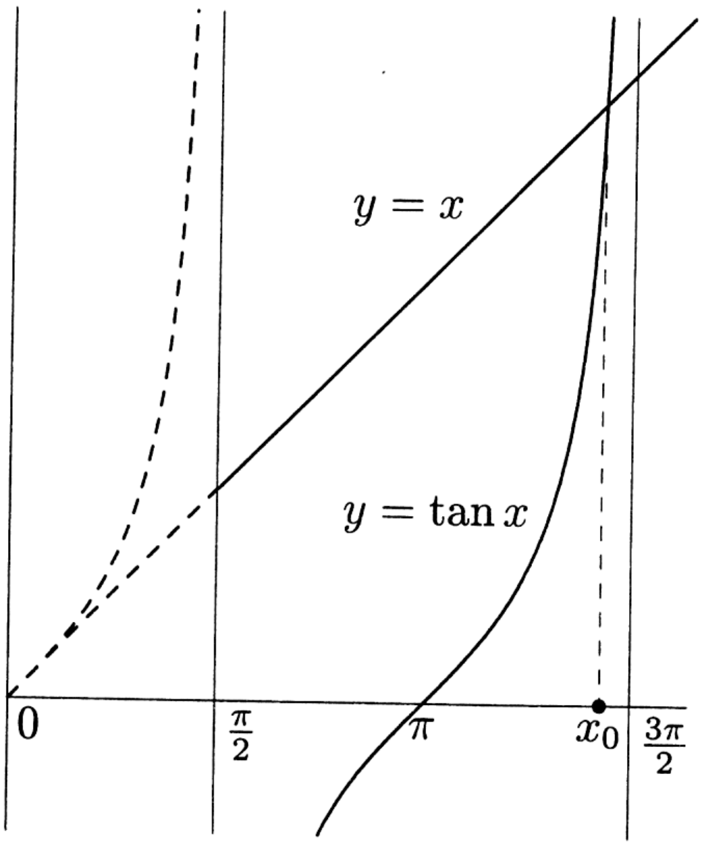

FIGURE 2 *Equation* $\tan x = x$

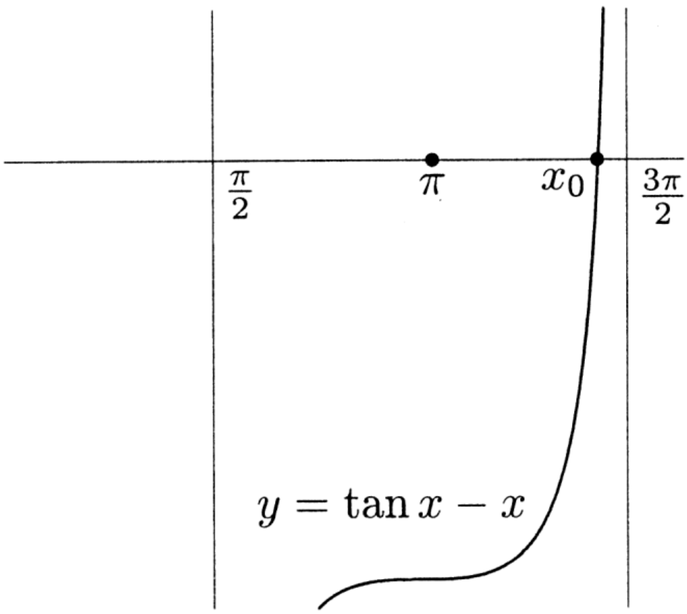

FIGURE 3 *Equation* $\tan x - x = 0$

our equation has a root in the interval $(\pi, \frac{3\pi}{2})$.
The derivative
$$f'(x) = \frac{1}{\cos^2 x} - 1 = \tan^2 x$$
is positive and increasing on $(\pi, \frac{3\pi}{2})$. Therefore, f is an increasing and convex function on this interval. We solve this equation by the Newton method starting with $x_1 = \frac{3\pi}{2} - 10^{-4}$. The results are presented in Table 1.

We see that the estimate $|x_n - x_0| \leqslant |x_n - x_{n-1}|$ is false. For the behaviour of the sequence $(x_n)_{n=1}^{\infty}$ note that
$$x_n = g(x_{n-1}), \text{ where } g(x) = \frac{x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} \ .$$
The point $\frac{3\pi}{2}$ is not a root of our equation, but it is a repulsive fixed point of the function g, because
$$g'\left(\frac{3\pi}{2}\right) = 2 > 1 \ .$$
For any $x_1 \in \left(x_0, \frac{3\pi}{2}\right)$ we obtain a sequence converging to x_0, but if the initial point x_1 is closed to $\frac{3\pi}{2}$ then also x_n is closed to $\frac{3\pi}{2}$ for many n. Particularly, if we take $x_1 = \frac{3\pi}{2} - 10^{-9}$, then we obtain an example which shows that the convergence of the Newton method may be slower than the convergence of the bisection method.

Remark. All values in this paper were evaluated onto 18-20 significant digits and then all outputs were rounded onto 10 significant digits.

For the evaluation of x_0 with a given accuracy δ there are several possibilities. The simplest way is the evaluation of x_n until the inequality $f(x_n - \delta)f(x_n + \delta) < 0$

n	x_n	$x_{n-1} - x_n$	$x_n - x_0$
1	4.712288980		0.218879522
2	4.712189028	0.000099953	0.218779570
3	4.711989263	0.000199764	0.218579805
4	4.711590298	0.000398965	0.218180841
5	4.710794622	0.000795677	0.217385164
6	4.709212237	0.001582385	0.215802779
7	4.706083007	0.003129230	0.212673549
8	4.699964094	0.006118913	0.206554636
9	4.688264213	0.011699881	0.194854755
10	4.666864413	0.021399800	0.173454955
11	4.630993761	0.035870652	0.137584303
12	4.580235510	0.050758252	0.086826052
13	4.528239646	0.051995864	0.034830188
14	4.499076575	0.029163071	0.005667117
15	4.493560666	0.005515909	0.000151208
16	4.493409566	0.000151100	0.000000108
17	4.493409458	0.000000108	0.000000000
18	4.493409458	0.000000000	0.000000000

TABLE 1

is satisfied. The second possibility is a combination of the Newton method with a modified method of false position, [1, p.183], i.e. together with x_n we evaluate also ξ_n by

$$\xi_1 = a \quad \text{and}$$

$$\xi_n = x_n - \frac{f(x_n)(x_n - \xi_{n-1})}{f(x_n) - f(\xi_{n-1})} \qquad \text{for } n > 1 .$$

Then we have $\xi_n < x_0 < x_n$, (when the function f is increasing and convex). It means that the inequality $x_n - \xi_n < \delta$ guarantees $x_n - x_0 < \delta$. Table 2 contains the results of evaluation of ξ_n and x_n for the equation $\tan x = x$ starting with $\xi_1 = 4.3$ and $x_1 = 4.7$.

n	ξ_n	x_n	$x_n - \xi_n$
1	4.300000000	4.700000000	0.400000000
2	4.320114416	4.688331848	0.368217432
3	4.354413674	4.666984472	0.312570798
4	4.404248369	4.631183287	0.226934918
5	4.456982727	4.580473096	0.123490370
6	4.487397534	4.528429052	0.041031518
7	4.493247036	4.499138109	0.005891073
8	4.493409340	4.493563964	0.000154625
9	4.493409458	4.493409570	0.000000113
10	4.493409458	4.493409458	0.000000000

TABLE 2 *Left and right estimates of the root*

For the comparison we evaluate ξ_n by the (simple) method of false position, i.e.

$$\xi_n = b - \frac{f(b)(b-\xi_{n-1})}{f(b)-f(\xi_{n-1})} \ , \text{ where}$$
$$b = 4.7 \qquad \text{and} \qquad \xi_1 = 4.3 \ .$$

n	ξ_n
1	4.300000000
2	4.310325422
3	4.320114062
4	4.329392330
5	4.338185494
6	4.346517706
7	4.354412045
8	4.361890542
9	4.368974227
10	4.375683153

TABLE 3 *Simple method of false position*

Tables 2 and 3 show that the convergence of the modified method of false position convergence is faster then the convergence of the simple method. Roughly speaking, the Newton method accelerates the convergence of the modified method of false position.

Another left estimates ξ_n of the root may be evaluated by the following scheme, see [4, p.180]. Put

$$\xi_1 = a \qquad \text{and}$$
$$\xi_n = \xi_{n-1} - \frac{f(\xi_{n-1})}{f'(x_{n-1})} \qquad \text{for } n > 1 \ .$$

The results are contained in Table 4.

n	ξ_n	x_n	$x_n - \xi_n$
1	4.300000000	4.700000000	0.400000000
2	4.301166132	4.688331848	0.387165716
3	4.305311541	4.666984472	0.361672931
4	4.318465687	4.631183287	0.312717600
5	4.352138102	4.580473096	0.228334994
6	4.410902541	4.528429052	0.117526511
7	4.466942647	4.499138109	0.032195462
8	4.490428002	4.493563964	0.003135962
9	4.493368097	4.493409570	0.000041473
10	4.493409450	4.493409458	0.000000008
11	4.493409458	4.493409458	0.000000000

TABLE 4 *Another left estimates of the root*

The another possibility of the estimate of $|x_n - x_0|$ is given by the following theorem, cf. [1, p.163] and [4, p.183].

Theorem 1. *Let f be a convex or concave strictly monotone function of the class C^1 on the interval $\langle a, b\rangle$ such that $f(a)f(b) < 0$, $|f'(a)| > 0$ and $|f'(b)| > 0$. Let $x_0 \in \langle a, b\rangle$ be the root of f and the sequence $(x_n)_{n=1}^{\infty}$ be of the Newton method. Then*

$$|x_n - x_0| \leq \frac{|f(x_n)|}{A}, \quad \text{where} \quad A = \min(|f'(a)|, |f'(b)|) .$$

Proof. Since f is strictly monotone, $f'(a)$ and $f'(b)$ have the same sign. Moreover,

$$\min_{x\in\langle a,b\rangle} f'(x) = \min(|f'(a)|, |f'(b)|) ,$$

because f is convex or concave. Therefore,

$$|f(x_n)| = |f(x_n) - f(x_0)| = |f'(\xi)||x_n - x_0| \geqslant A(|x_n - x_0| ,$$

where ξ is between x_n and x_0.

Example. Take values from Table 1 and $a = 4.45$. Then $a < x_0$, because $f(a) < 0$. Since $A = f'(a)$, we obtain

$$0 \leq x_{18} - x_0 \leq \frac{f(x_{18})}{f'(4.45)} \doteq 1.3 \cdot 10^{-10}$$

Finally, we prove the following result.

Theorem 2. *Let f be a convex or concave strictly monotone function of the class C^1 on the interval $\langle a, b\rangle$ such that $f(a)f(b) < 0$, $|f'(a)| > 0$ and $|f'(b)| > 0$. Let $x_0 \in \langle a, b\rangle$ be the root of f and the sequence $(x_n)_{n=1}^{\infty}$ be of the Newton method. Then*

$$0 \leq |x_n - x_0| \leqslant \left(\frac{|f'(x_{n-1})|}{A} - 1\right)|x_{n-1} - x_n| \leqslant \left(\frac{B}{A} - 1\right)|x_{n-1} - x_n| ,$$

where

$$A = \min(|f'(a)|, |f'(b)|) \text{ and } B = \max(|f'(a)|, |f'(b)|) .$$

Particularly,

$$|x_n - x_0| \leqslant |x_{n-1} - x_n| \text{ whenever } B \leqslant 2A \text{ or } f'(x_{n-1}) \leqslant 2A .$$

Proof. Without loss of generality we may assume that the function f is increasing and convex. We have

$f(x_{n-1}) = f'(x_{n-1})(x_{n-1} - x_n)$
$f(x_{n-1}) = f(x_{n-1}) - f(x_0) = f'(\xi)(x_{n-1} - x_0) = f'(\xi)(x_{n-1} - x_n) + f'(\xi)(x_n - x_0)$,
where $x_0 < \xi < x_{n-1}$.

Therefore,

$$f'(\xi)(x_n - x_0) = f'(x_{n-1})(x_{n-1} - x_n) - f'(\xi)(x_{n-1} - x_n)$$

and

$$x_n - x_0 = \frac{f'(x_{n-1})}{f'(\xi)}(x_{n-1} - x_n) - (x_{n-1} - x_n) = \left(\frac{f'(x_{n-1})}{f'(\xi)} - 1\right)(x_{n-1} - x_n)\,.$$

Since f is convex and $a < x_0 < \xi < x_{n-1} < b$, we have $f'(a) \leqslant f'(\xi)$ and $f'(x_{n-1}) \leqslant f'(b)$. So, we obtain the desired inequality.

Example. Take values from Table 1 and $a = 4.45$. We obtain

$$0 \le x_{18} - x_0 \le \left(\frac{f'(x_{17})}{f'(4.45)} - 1\right)(x_{17} - x_{18}) \le 4.5 \cdot 10^{-10}\,.$$

(By Table 1 we have $x_{17} - x_{18} = 0$. However, both values x_{17} and x_{18} are rounded. Therefore, we have used $x_{17} - x_{18} \le 10^{-9}$.)

Remark. If the inequality $B \leqslant 2A$ is not satisfied, then the interval $\langle a, b\rangle$ is too wide. We may use the bisection method until this inequality is satisfied and then the Newton method may be used with a true estimate $x_n - x_0 \le x_{n-1} - x_n$.

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