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LINE DIGRAPHS OF COMPLETE BIPARTITE SYMMETRIC DIGRAPHS ARE RATIONAL

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ABSTRACT. A digraph D is divisible by t if its arc set can be partitioned into t subsets, such that the sub–digraphs (called factors) induced by the subsets are all isomorphic. If D has q arcs, then it is t-rational if it is divisible by t or t does not divide q. D is rational if it is t-rational for all $t \geqq 2$. In this note, we show that graphs $L(K^*_{n,n})$ are rational.

1. INTRODUCTION

An isomorphic factorization of a digraph D is a partition of its arc set into subsets such that the sub–digraphs (called factors) induced by the subsets are mutually isomorphic. If there exists an isomorphic factorization D into t factors, we say that D is divisible by t. For given t and a given digraph D having precisely q arcs, an obvious necessary condition for the divisibility of D by t is that t divides q. This is called the divisibility condition for D and t. D is t-rational if D is divisible by t or the divisibility condition for D and t is not satisfied, otherwise D is t-irrational; D is rational if it is t-rational for all $t \geqq 2$, otherwise D is irrational, in which case D is t-irrational for some $t \geqq 2$.

The problem which concerns us is to find values of r and t for which all r-regular digraphs are t-rational. Wormald [6] has shown that for fixed t and r such that $2 \leqq t \leqq r$, almost all r-regular digraphs are not divisible by t, and for fixed $t \geqq 2$ almost all regular tournaments also are not divisible by t. Further, in [6] it was proved that all 1–regular digraphs are rational. For r-regular graphs, some results in the direction were achieved in [1, 2, 3, 4, 5, 6].

The line digraph $L(D)$ of a digraph $D(V, A)$ has the arc set of D as its vertex set, and there is an arc from xy to zw in $L(D)$ if $y = z$. The aim of this paper is to prove the divisibility of digraphs $L(K^*_{n,n})$ by t for any t dividing the number of its arcs.

2. RESULT

Let $K^*_{n,n}$ be a complete bipartite symmetric digraph with partite sets V_1 and V_2, where $|V_1| = |V_2| = n$. Then the line graph digraph $L(K^*_{n,n})$ is a digraph with $2n^2$ vertices, $2n^3$ arcs and regular of degree n. We shall prove the following theorem:

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Theorem. *Let $n > 1$ be any positive integer. Then the line graph $L(K^*_{n,n})$ is rational.*

Proof. Assume $t|2n^3$ for any positive integer $t > 1$. We show that then $L(K^*_{n,n})$ is divisible by t.

Let $K^*_{n,n}$ denote the complete bipartite symmetric digraph with partite sets $V_1 = \{x_1, x_2, \ldots, x_n\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$, and let

$$\begin{aligned}\alpha_1 =&(x_1y_1x_2y_2\ldots x_ny_n)(x_1y_2x_2y_3\ldots x_ny_1)\ldots(x_1y_nx_2y_1\ldots x_ny_{n-1})\\ &(y_1x_1y_2x_2\ldots y_nx_n)(y_2x_1y_3x_2\ldots y_1x_n)\ldots(y_nx_1y_1x_2\ldots y_{n-1}x_n) =\\ =&\varepsilon_1\varepsilon_2\ldots\varepsilon_n\gamma_1\gamma_2\ldots\gamma_n\end{aligned}$$

be the vertex permutation of $L(K^*_{n,n})$. Let α_2 denote a permutation of arcs of $L(K^*_{n,n})$ that is induced by the permutation α_1. The induced arc permutation α_2 is seen to have the property that the length of every cycle is n, and that the number of these cycles is equal to $2n^2$. Thus induced permutation α_2 has the expression of the form of a product of cycles

$$\alpha_2 = \prod_{i=1}^{n}\prod_{j=1}^{n}\varepsilon_i\gamma_j \cdot \prod_{j=1}^{n}\prod_{i=1}^{n}\gamma_j\varepsilon_i.$$

Define now a new digraph $K^*(A, B)$ with partite sets $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_n\}$. Let every vertex $u_i(v_i)$ correspond to the cycle $\varepsilon_i(\gamma_i)$, $i = 1, 2, \ldots, n$, and let the vertex $u_i(v_j)$ be connected by an arc with the vertex $v_j(u_i)$ if and only if a cycle $\varepsilon_i\gamma_j(\gamma_i\varepsilon_i)$ belongs to α_2. It is evident that the digraph $K^*(A, B)$ is isomorphic to $K^*_{n,n}$. Next, let $\overrightarrow{K}(X, Y)$ denote a complete bipartite digraph which contains all arcs of which start–vertex is from X and end–vertex is from Y.

The exact construction of t isomorphic factors of $L(K^*_{n,n})$ depends on the parity of t.

Case 1. Let t be even and let $t|2n^3$. Then $t = 2r$ for some positive integer r, and therefore r divides n^3. Let $\gcd(r, n^2) = r_1$. Consequently, there exist positive integers b and c, such that $b|n$, $c|n$, and $r_1 = bc$. Next, let $r|r_1 = d$. Then obviously $d|n$.

Divide $K^*(A, B)$ into two isomorphic digraphs $\overrightarrow{K}(A, B)$ and $\overrightarrow{K}(B, A)$. Owing to this it is sufficient to construct a decomposition of $\overrightarrow{K}(A, B)$ into r isomorphic sub–digraphs.

Firstly, construct the decomposition of $\overrightarrow{K}(A, B)$ into r_1 isomorphic sub–digraphs. Let $A = \bigcup_{k=1}^{b} A_k$, $B = \bigcup_{s=1}^{c} B_s$, $|A_k| = n/b$, and $|B_s| = n/c$, where the sets A_k and B_s are mutually disjoint. Define r_1 sub–digraphs of $\overrightarrow{K}(A, B)$ in the following way: $G_{ks} = \overrightarrow{K}(A_k, B_s)$ for every ordered couple $(k, s) \in \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$. Sub–digraphs G_{ks} are all isomorphic because there exists an isomorphism between $G_{k_1s_1}$ and $G_{k_2s_2}$ induced by mapping $(k_1, s_1) \to (k_2, s_2)$. By the backward application of the previous correspondence on the digraphs G_{ks}, we get the decomposition of $L(K^*_{n,n})$ into r_1 isomorphic sub–digraphs, denote them by F_{ks}. To complete the proof it now suffices, without loss of generality, to decompose F_{11} into d isomorphic sub–digraphs. Let one part of the complete bipartite digraph F_{11} contains vertices of cycles $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n/b}$ and second part contains vertices of cycles $\gamma_1, \gamma_2, \ldots, \gamma_{n/c}$. Denote by α_2/F_{11} the reduced induced arc permutation which contains only those cycles in α_2 which correspond to arcs belonging to F_{11}. Then

$$\alpha_2/F_{11} = \prod_{i=1}^{n/b} \prod_{j=1}^{n/c} \varepsilon_i \gamma_j$$

is the product cycles having lengths which are multiples of d as $d|n$. Choose now from each cycle of the permutation α_2/F_{11} an arc h_{ij} that occupies the first place in given cycle and put

$$E = E(F_{111}) = \{(\alpha_2/F_{11})^{ud}(h_{ij}); u \geqq 0\}.$$

Then $\{E, (\alpha_2/F_{11})(E), \ldots, (\alpha_2/F_{11})^{d-1}(E)\}$ is a partition of the arcs of F_{11}. This constitutes an isomorphic factorization of F_{11}, as the sub–digraph F_{111} induced by E is isomorphic to the sub–digraphs of F_{11} induced by each of $(\alpha_2/F_{11})(E), \ldots$ Isomorphisms between F_{111} and these sub–digraphs are provided by the corresponding powers of α_1. Hence F_{11} is divisible by d. In consequence of preceding follows that the digraph $L(K^*_{n,n})$ is divisible by t.

Case 2. Let t be odd and let $t|2n^3$. Then t divides n^3. Let $(t, n^2) = t_1$ and let $t/t_1 = d$. Then obviously d must divide n. Since $t_1|n^2$, then there exist positive integers b and c such that $b|n$, $c|n$, and $t_1 = bc$. Consider subsets A_k and B_s that have the same meaning as in the Case 1.

Suppose $b > 1$ and $c > 1$. Define for every ordered couple $(k, s) \in \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$ digraphs $G_{ks} = \overrightarrow{K}(A_k, B_s) \cup \overrightarrow{K}(B_{s+1}, A_k)$ where the addition $s + 1$ is taken modulo c with residues $1, 2, \ldots, c$. It is seen that every digraph G_{ks} is isomorphic to the directed "path" $\overrightarrow{P}_3$ with "vertices" B_{s+1}, A_k and B_s, whereupon these paths are arc–disjoint. Then digraphs F_{ks} obtained from G_{ks} by analogous fashion as stated above provide an isomorphic factorization of $L(K^*_{n,n})$ into t_1 factors. Take now the digraph F_{11} and decompose it into d isomorphic sub–digraphs. Let one part of the bipartite digraph F_{11} contains vertices that are elements of cycles $\varepsilon_1, \varepsilon_2, \ldots, e_{n/b}$ and the second part contains vertices that are elements of cycles $\gamma_1, \gamma_2, \ldots, \gamma_{n/c}, \gamma_{n/c+1}, \ldots, \gamma_{2n/c}$, and let the induced reduced permutation α_2/F_{11} contains only those cycles from α_2 for which there exists a corresponding arc in F_{11}. Then

$$\alpha_2/F_{11} = \prod_{i=1}^{n/b} \prod_{j=1}^{n/c} \varepsilon_i \gamma_j \cdot \prod_{j=n/c+1}^{2n/c} \prod_{i=1}^{n/b} \gamma_j \varepsilon_i,$$

where all cycles of α_2/F_{11} have lengths which are multiples of d. From each cycle $\varepsilon_i \gamma_j$ and $\gamma_j \varepsilon_i$ in α_2/F_{11} choose the first arc h_{ij} and e_{ji}, respectively, and put

$$E = E(F_{111}) = \{(\alpha_2/F_{11})^{ud}(h_{ij}), (\alpha_2/F_{11})^{ud}(e_{ji}); u \geqq 0\}.$$

The system $\{E, (\alpha_2/F_{11}(E), \ldots, (\alpha_2/F_{11})^{d-1}(E)\}$ is a partition of the arc set of F_{11} and sub–digraphs induced by these subsets provides an isomorphic factorization of F_{11}. Thus F_{11} is divisible by d und in consequence of the preceding the digraph $L(K^*_{n,n})$ is divisible by t.

Suppose now $b > 1$ and $c = 1$. Then $d = 1$ and $t = b$. Define in this case b sub–digraphs of $K^*(A, B)$ in this way: $G_k = \overrightarrow{K}(A_k, B) \cup \overrightarrow{K}(B, A_{k+1})$ for $k = 1, 2, \ldots, b$. The addition $k + 1$ is taken modulo b with residues $1, 2, \ldots, b$. Note, that subsets A_k have the same meaning as in Case 1. As above, the gained sub–digraphs F_k are all isomorphic because each of them is isomorphic to the directed "path" $\overrightarrow{P}_3$ with "vertices" A_k, B and A_{k+1}, and these directed "paths" are arc–disjoint. Hence $L(K^*_{n,n})$ is divisible by t. Since for every t dividing $2n^3$ the digraph $L(K^*_{n,n})$ is divisible by t, then $L(K^*_{n,n})$ is rational, which completes the proof.

In conclusion, we note that analogous theorem for non-oriented graphs $L(K_{n,n})$ will be proved in the forthcoming paper.

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