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A NOTE ON THE IMPROPER KURZWEIL-HENSTOCK INTEGRAL

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Abstract. A connection is studied between the improper Kurzweil-Henstock integral on the real line and the integral over a compact space.

Introduction

In [5] two possibilities are mentioned of defining the improper Kurzweil-Henstock integral on the real line (see also [2] for a more general range). In [1] and [6] the Kurzweil-Henstock construction has been examined for a general compact range. It is natural to consider one-point compactification of the real line. Therefore we work with the compactification and we prove a convergence theorem in compact spaces describing the situation from the real case.

Kurzweil-Henstock integral in compact topological spaces

Let $\mathbb{N}$ be the set of all strictly positive integers, $\mathbb{R}$ the set of the real numbers, $\mathbb{R}^+$ be the set of all strictly positive real numbers. Let X be a Hausdorff compact topological space. If $A \subset X$, then the interior of the set A is denoted by $int\, A$.

We shall work with a family $\mathcal{F}$ of compact subsets of X closed under the intersection and a monotone and additive mapping $\lambda : \mathcal{F} \to [0, +\infty]$. The additivity means that

$$\lambda(A \bigcup B) + \lambda(A \bigcap B) = \lambda(A) + \lambda(B) \tag{1}$$

whenever $A, B, A \bigcup B \in \mathcal{F}$.

By a *partition* (detaily, $(\mathcal{F}, \lambda)$-*partition*) of a set $A \in \mathcal{F}$ we mean a finite collection $\{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ such that

(i) $\mathcal{U}_1, \dots, \mathcal{U}_k \in \mathcal{F}$,

(ii) $\bigcup_{i=1}^{k} \mathcal{U}_i = A$,

(iii) $\lambda(\mathcal{U}_i \bigcap \mathcal{U}_j) = 0$ whenever $i \neq j$,

(iv) $t_i \in \mathcal{U}_i \, (i = 1, \dots, k)$.

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A finite collection $\{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ of subsets of $A \in \mathcal{F}$, satisfying conditions (i), (iii) and (iv), but not necessarily (ii), is said to be *decomposition* of A. We shall assume that $\mathcal{F}$ *separates points* in the following way: to any $A \in \mathcal{F}$ there exists a sequence $(\mathcal{A}_n)_n$ of partitions of A such that

(i) $\mathcal{A}_{n+1}$ is a refinement of $\mathcal{A}_n$,
(ii) to any $x, y \in A$, $x \neq y$, there exist $n \in I\!N$ and $B \in \mathcal{A}_n$ such that $x \in B$ and $y \notin B$.

We note that this assumption is fulfilled if the topological space X is metrizable or it satisfies the second axiom of countability (see [6]).

A *gauge* on a set $A \subset X$ is a mapping δ assigning to every point $x \in A$ a neighborhood $\delta(x)$ of x. If $\mathcal{D} = \{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ is a decomposition of A and δ is a gauge on A, then we say that $\mathcal{D}$ is δ*-fine* if $\mathcal{U}_i \subset \delta(t_i)$ for any $i \in \{1, 2, \dots, k\}$.

We obtain a simple example putting $X = [a, b] \subset I\!R$ with the usual topology, $\mathcal{F}$ =the family of all closed subintervals of X, $\lambda([\alpha, \beta]) = \beta - \alpha$, $a \le \alpha < \beta \le b$. Any gauge can be represented by a real function $d : [a, b] \to I\!R^+$, if we put $\delta(x) = (x - d(x), x + d(x))$.

Another example is the unbounded interval $[a, +\infty] = [a, +\infty) \bigcup \{+\infty\}$ considered as the one-point compactification of the locally compact space $[a, +\infty)$. The base of open sets consists of open subsets of $[a, +\infty)$ and the sets of the type $(b, +\infty) \bigcup \{+\infty\}$, $a \le b < +\infty$. Any gauge in $[a, +\infty]$ has the form $\delta(x) = (x - d(x), x + d(x))$, if $x \in [a, +\infty] \bigcap I\!R$, and $\delta(+\infty) = (b, +\infty] = (b, +\infty) \bigcup \{+\infty\}$, where d denotes a positive real-valued function defined on $[a, +\infty)$, and b denotes a real number.

Let us return to the definition of Kurzweil-Henstock integral (KH-integral) on X. If $\mathcal{D} = \{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ is a decomposition of a set A, and $f : X \to I\!R$, then we define the Riemann sum as follows:

$$S(f, \mathcal{D}) = \sum_{i=1}^{k} f(t_i)\lambda(\mathcal{U}_i),$$

if the sum exists in $I\!R$, with the convention $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$.

We note that the fact that $\mathcal{F}$ separates points guarantees the existence of at least one δ-fine partition $\mathcal{D}$ such that $S(f, \mathcal{D})$ is well-defined for any gauge δ (see [6], [8]).

Definition 2.1. A function $f : X \to I\!R$ is *integrable* on a set A if there exists $I \in I\!R$ such that $\forall\, \varepsilon > 0$ there exists a gauge δ on A such that

$$|S(f, \mathcal{D}) - I| \le \varepsilon \tag{2}$$

whenever $\mathcal{D}$ is a δ-fine partition of A such that $S(f, \mathcal{D})$ exists in $I\!R$. We denote

$$I = \int_A f$$

(see also [6], Definition 1.8., p. 154).

We now prove the following:

Theorem 3.1. *Let $X = X_0 \bigcup \{x_0\}$ be the one-point compactification of a locally compact space X_0. Let $f : X \to I\!R$ be a function such that $f(x_0) = 0$. Let $(A_n)_n$ be a sequence of sets, such that $A_n \in \mathcal{F}$, $A_n \subset int\, A_{n+1}$, $A_{n+1} \setminus int\, A_n \in \mathcal{F}$, $\lambda(A_n \setminus int\, A_n) = 0$ $(n = 1, 2, \dots)$, $\bigcup_{n=1}^{\infty} A_n = X_0$. Let f be integrable on A_n $(n = 1, 2, \dots)$ and let there exist in $I\!R$ an element I such that, $\forall\, \varepsilon > 0$, there exists an integer n_0 such that*

$$\left| \int_A f - I \right| \leq \varepsilon \quad \forall\, A \in \mathcal{F},\, A \supset A_{n_0}.$$

Then f is integrable on X and $\int_X f = I$.

Proof. Let ε be an arbitrary positive real number, and $n_0 \in I\!N$ be as in the hypotheses of the theorem. Put $A_0 = \emptyset$, $B_n = A_{n+1} \setminus int\, A_n$ $(n = 1, 2, \dots)$. Proceeding analogously as in [6], Lemma 1.10, and as in [2], we get that f is integrable on every subset of A_n belonging to $\mathcal{F}$ $(n = 1, 2, \dots)$ and thus, in particular, f is integrable on B_n $(n = 1, 2, \dots)$. Therefore, $\forall\, n \in I\!N$, there exists a gauge δ_n on B_n such that

$$\left| \int_{B_n} f - S(f, \mathcal{D}_n) \right| \leq \frac{\varepsilon}{2^{n+3}} \tag{3}$$

for any δ_n-fine partition $\mathcal{D}_n$ of B_n. From (3) and Henstock's Lemma (see also [6], Lemma 2.1., pp. 158-159; [5], Theorem 3.2.1., pp. 81-83), it follows that

$$\left| \int_{\bigcup_{i=1}^{h} \mathcal{V}_i} f - S(f, \mathcal{E}_n) \right| \leq \frac{\varepsilon}{2^{n+2}} \tag{4}$$

for each δ_n-fine decomposition $\mathcal{E}_n = \{(\mathcal{V}_1, t_1), \dots, (\mathcal{V}_h, t_h)\}$ of B_n. Evidently

$$B_n \bigcap B_{n+1} = A_n \setminus int\, A_n \quad \forall\, n \in I\!N.$$

Therefore

$$B_n = (B_n \bigcap B_{n-1}) \bigcup (int\, B_n) \bigcup (B_n \bigcap B_{n+1}) \quad \forall\, n.$$

Moreover, it is easy to check that

$$B_j \cap B_l = \emptyset \text{ whenever } |j - l| \geq 2 \tag{5}$$

and that

$$(int\, B_n) \cap (int\, B_{n+1}) = \emptyset \quad \forall\, n \in I\!N. \tag{6}$$

Now define a gauge δ on X by the following formula:

$$\delta(x) = \begin{cases} \delta_n(x) \bigcap (int\, B_n) \text{if } x \in int\, B_n, \\ \delta_n(x) \bigcap \delta_{n+1}(x) \bigcap (int\, A_{n+1}) \text{if } x \in B_n \bigcap B_{n+1}, \quad (n = 1, 2, \dots) \\ (X_0 \setminus A_{n_0}) \bigcup \{x_0\} \text{if } x = x_0. \end{cases} \tag{7}$$

Let $\mathcal{D} = \{(\mathcal{U}_1, t_1), \dots, (\mathcal{U}_k, t_k)\}$ be a δ-fine partition of X. There exists $(\mathcal{U}_{i_0}, t_{i_0}) \in \mathcal{D}$, with $i_0 \in \{1, 2, \dots, k\}$, such that $x_0 \in \mathcal{U}_{i_0}$. We shall prove that $t_{i_0} = x_0$. Namely, in the opposite case,

$$x_0 \in \mathcal{U}_{i_0} \subset \delta(t_{i_0}) \subset \delta_n(t_{i_0})$$

for some n. But $\delta_n(t) \subset X_0$ for $t \neq x_0$. We have obtained $x_0 \in X_0$, that is a contradiction.

Since $f(x_0) = 0$, the Riemann sum $S(f, \mathcal{D})$ has the form

$$\sum_{i=1,\dots,k, i \neq i_0} f(t_i)\, \lambda(\mathcal{U}_i),$$

and $t_i \in X_0$ $(i = 1, \dots, k, i \neq i_0)$. Let

$$A = \bigcup_{n \in T} B_n,$$

where

$$T = \{n \in I\!N : \exists i \in \{1, \dots, k\}, i \neq i_0 : B_n \cap \mathcal{U}_i \neq \emptyset\}. \tag{8}$$

By (7) , and since $\mathcal{D}$ is a δ-fine *partition* of X, we get that

$$A \supset A_{n_0} \tag{9}$$

By hypothesis we have

$$\left| \int_A f - I \right| \leq \varepsilon \tag{10}$$

We claim that, if $\mathcal{U}_i$, $i \neq i_0$, has nonempty intersection with at least two of the $int\, B_n$'s, then necessarily there exists $n \in I\!N$ such that the point t_i corresponding to $\mathcal{U}_i$ belongs to $B_n \cap B_{n+1}$. Indeed, if $t_i \in int\, B_n$ for some n, then, from (7) and the fact that $\mathcal{D}$ is a δ-fine partition of X, we'd have

$$\mathcal{U}_i \subset \delta(t_i) \subset int\, B_n :$$

this is impossible, by virtue of (5) and (6). From this and since

$$(B_{n-1} \bigcap B_n) \bigcap (B_n \bigcap B_{n+1}) = \emptyset \quad \forall n,$$

it follows that, for every $i = 1, 2, \dots, k$, $i \neq i_0$, the B_n's having nonempty intersection with $\mathcal{U}_i$ are at most two, while the B_n's which have nonempty intersection with $\mathcal{U}_{i_0}$ can be infinitely many (even all the B_n's). Thus we proved that the set T in (8) is finite.

For $n \in T$ define a decomposition $\mathcal{E}_n$ of B_n in the following way:

$$\begin{aligned} \mathcal{E}_n =& \{(\mathcal{U}_i, t_i) : t_i \in int\, B_n\} \\ & \bigcup \{(\mathcal{U}_i \bigcap B_n, t_i) : t_i \in B_n \bigcap B_{n-1}\} \\ & \bigcup \{(\mathcal{U}_i \bigcap B_n, t_i) : t_i \in B_n \bigcap B_{n+1}\}. \end{aligned}$$

Then, by construction, we have:

$$S(f,\mathcal{D}) = \sum_{n\in T} S(f,\mathcal{E}_n) \tag{11}$$

by additivity of λ and since $A_n \setminus int\, A_n = B_n \bigcap B_{n+1} \subset int\, A_{n+1}$ and $\lambda(A_n \setminus int\, A_n) = 0\ \forall\, n \in I\!N$.

Similarly,

$$\sum_{n\in T} \int_{\bigcup_{\mathcal{U}_i \subset int\, B_n, i\neq i_0} \mathcal{U}_i} f = \int_A f \tag{12}$$

Since $\mathcal{D}_n$ is δ_n-fine, we have (3). From (3), (10), (11), (12), and (9) we obtain:

$$|S(f,\mathcal{D}) - I| = \left|\sum_{n\in T} S(f,\mathcal{E}_n) - I\right| =$$

$$\left|\sum_{n\in T}\left(S(f,\mathcal{E}_n) - \int_{\bigcup_{\mathcal{U}_i \subset int\, B_n, i\neq i_0}\mathcal{U}_i} f\right) + \sum_{n\in T}\int_{\bigcup_{\mathcal{U}_i \subset int\, B_n, i\neq i_0}\mathcal{U}_i} f - I\right| \leq$$

$$\sum_{n\in T}\left|S(f,\mathcal{E}_n) - \int_{\bigcup_{\mathcal{U}_i \subset int\, B_n, i\neq i_0}\mathcal{U}_i} f\right| + \left|\int_A f - I\right| \leq \sum_{n\in T}\frac{\varepsilon}{2^{n+2}} + \varepsilon < 2\,\varepsilon.$$

From this the assertion follows. □

Applications

The following results are consequences of Theorem 3.1:

Proposition 4.1. ([5], Theorem 2.9.3., pp. 61-63) *Let $f : [a,+\infty] \to I\!R$ be such that $f(+\infty) = 0$, f be integrable on $[a,b]$ for any $b > a$, and let there exist in $I\!R$ the limit*

$$\lim_{b\to+\infty}\int_{[a,b]} f.$$

Then f is integrable on $[a,+\infty]$, and

$$\int_{[a,+\infty]} f = \lim_{b\to+\infty}\int_{[a,b]} f.$$

Proposition 4.2. (see also [5], Theorem 2.8.3., pp. 57-59 and Remark 2.8.4, p.57) *Let $a,b \in I\!R$, $a < b$, $f : [a,b] \to I\!R$, f be integrable on $[a,x]$ for any $a \leq x < b$, and let there exist in $I\!R$ the limit*

$$\lim_{x\to b^-}\int_{[a,x]} f.$$

Then f is integrable on $[a,b]$, and

$$\int_{[a,b]} f = \lim_{x\to b^-}\int_{[a,x]} f.$$

Proof. We observe that $[a,b] = [a,b) \bigcup \{b\}$ can be considered as the one-point compactification of $[a,b)$. The only difference is that we did not assume $f(b) = 0$. Of course, one can put $g(x) = f(x) - f(b)$, and use Theorem 3.1 with respect to the function g. Then we have

$$\int_{[a,b]} g = \lim_{x \to b^-} \int_{[a,x]} g,$$

and hence

$$\int_{[a,b]} f = f(b)(b-a) + \int_{[a,b]} g =$$
$$= \lim_{x \to b^-} f(b)(x-a) + \lim_{x \to b^-} \int_{[a,x]} g =$$
$$= \lim_{x \to b^-} \int_{[a,x]} (g + f(b)) = \lim_{x \to b^-} \int_{[a,x]} f.$$

This concludes the proof. □

References

[1] S. I. Ahmed and W. F. Pfeffer, *A Riemann integral in a locally compact Hausdorff space*, J. Australian Math. Soc. **41** (1986), 115-137.

[2] A. Boccuto and B. Riečan, *On the Henstock-Kurzweil integral for Riesz-space-valued functions defined on unbounded intervals*, Czechoslovak Math. J. (2001) (to appear).

[3] R. Henstock, *The general theory of integration*, Clarendon Press, Oxford, 1991.

[4] J. Kurzweil, *Nicht absolut-konvergente Integrale*, Teubner, Leipzig, 1980.

[5] L. P. Lee and R. Výborný, *The integral: An easy approach after Kurzweil and Henstock*, Cambridge Univ. Press, Cambridge, 2000.

[6] B. Riečan, *On the Kurzweil integral in compact topological spaces*, Rad. Mat **2** (1986), 151-163.

[7] B. Riečan and T. Neubrunn, *Integral, Measure and Ordering*, Kluwer, Dordrecht, and Ister, Bratislava, 1997.

[8] B. Riečan and M. Vrábelová, *The Kurzweil construction of an integral in ordered spaces*, Czechoslovak Math. J. **48 (123)** (1998), 565-574.

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