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JOIN AND INTERSECTION OF HYPERMAPS

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ABSTRACT. Hypermaps are generalisations of maps - 2-cell decompositions of closed surfaces. The correspondence between hypermaps and quotients of the group Δ freely generated by three involutions is well-known. In this correspondence hypermaps correspond to conjugacy classes of subgroups of Δ, and hypermap coverings to the subgroup containment.

Let $\mathcal{H}$ and $\mathcal{K}$ be two hypermaps. We shall introduce and study two binary operations - join and intersection, defined on hypermaps. The corresponding operations in the subgroup representation is the intersection of two subgroups of Δ and the subgroup closure in Δ. We investigate basic properties of the join and intersection, particular attention is paid to the study of orthogonal hypermaps, final sections are devoted to the study of the relationship of some algebraic and topological properties of hypermaps and the join and intersection. As a byproduct we get a method of comparison of two hypermaps which led us to the definition of the shared cover index. This transpired to be a generalisation of the chirality index defined in [3]. In fact, the chirality index of an oriented regular hypermap $\mathcal{H}$ is just the shared cover index of $\mathcal{H}$ with its mirror image.

1. INTRODUCTION

A topological map is a 2-cell decomposition of a compact connected surface. A hypermap is a certain abstraction of a topological map linking different fields of mathematics including combinatorics, group theory, geometry of Riemann surfaces, algebraic geometry and Galois theory. For a survey explaining these relations we refer the reader to [9,10]. Formally, a hypermap is a 4-tuple $(F; r_0, r_1, r_2)$, where F is a set of flags and r_i, $i = 0, 1, 2$ are fixed point free involutory permutations acting on F such that $\langle r_0, r_1, r_2\rangle$ is transitive on F.

It is known that any hypermap can be viewed as a quotient of the universal hypermap given by the action of the group $\Delta = \langle r_0, r_1, r_2; r_0^2 = r_1^2 = r_2^2 = 1\rangle$ on itself by left multiplication. This gives rise to a correspondence between subgroups of Δ, called hypermap subgroups in this context, and hypermaps. In particular, normal subgroups of finite index in Δ determine hypermaps which automorphism group acts regularly on the set of flags. Using the representation of hypermaps via hypermap subgroups it is easy to see that for any two regular hypermaps $\mathcal{H}$, $\mathcal{K}$ there is a least regular common cover $\mathcal{H} \vee \mathcal{K}$, called the join of $\mathcal{H}$ and $\mathcal{K}$, satisfying the following property: if a regular hypermap $\mathcal{X}$ covers both $\mathcal{H}$ and $\mathcal{K}$ then it covers

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$\mathcal{H} \vee \mathcal{K}$. Similarly, we define $\mathcal{H} \wedge \mathcal{K}$ to be the largest regular hypermap covered by $\mathcal{H}$ and $\mathcal{K}$, called here the intersection of hypermaps. Although both constructions are known [18,2,3], no systematic study of their properties (from the point of view of theory of hypermaps) was done, except the paper of S. Wilson [18] where the investigation is restricted to joins of maps.

The introduction is followed by a section where we develop some necessary definitions, notations and mention some basic facts on hypermaps and their representations. In Section 3 we introduce the join and intersection of two hypermaps and prove some fundamental results about them. In Section 4 we study the structure of the monodromy group of the join and intersection of two regular hypermaps, this is equivalent with the study of the corresponding automorphism groups. In Section 5 we study the orthogonality of two hypermaps, an interesting phenomenon related with the join and intersection of them. Final Sections are devoted to an investigation of orientability, reflexibility and self-duality of regular hypermaps in relation to the join and intersection. Several ideas and results from [3] and [18] are generalised there.

2. Hypermaps and subgroups of Δ

A *topological hypermap* $\mathcal{H}$ is a cellular embedding of a connected 3-valent graph X into a closed surface S such that the cells are 3-coloured (say by black, grey and white colours) with adjacent cells having different colours. Numbering the colours 0, 1 and 2, and labelling the edges of X with the missing adjacent cell number, we can define 3 fixed points free involutory permutations r_i, $i = 0$, 1, 2, on the set F of vertices of X; each r_i switches the pairs of vertices connected by i-edges (edges labelled i). The elements of F are called *flags* of $\mathcal{H}$ and the group G generated by r_0, r_1 and r_2 is called the *monodromy group* $\mathrm{Mon}\,(\mathcal{H})$ of the hypermap $\mathcal{H}$. The cells of $\mathcal{H}$ coloured 0, 1 and 2 are called the *hypervertices*, *hyperedges* and *hyperfaces*, respectively. Since the graph X is connected, the monodromy group acts transitively on F and the orbits of $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$ or $\langle r_0, r_2\rangle$ on F determine hyperfaces, hypervertices and hyperedges, respectively. Let $k = ord(r_0r_1)$, $m = ord(r_1r_2)$ and $n = ord(r_2r_0)$ be the orders of the respective elements in the monodromy group. The triple (k, m, n) is called the *type* of the hypermap. Maps are hypermaps satisfying condition $(r_0r_2)^2 = 1$. In other words, maps are hypermaps of type $(p, q, 2)$ or of type $(p, p, 1)$.

It is known that all information on the topological hypermap $\mathcal{H}$ is coded in the three associated fixed points free permutations acting on F (see for instance [4, 6, 11, 12, 14, 15]) . Thus we define a *hypermap* to be a 4-tuple $(F; r_0, r_1, r_2)$, where r_i, $i = 0, 1, 2$ are fixed point free involutions acting on F such that the action of $\mathrm{Mon}\,(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$ is transitive. Let $\mathcal{H} = (F; r_0, r_1, r_2)$ and $\mathcal{K} = (F'; t_0, t_1, t_2)$. *A homomorphism* $\mathcal{H} \to \mathcal{K}$ is a mapping $\pi : F \to F'$ such that $t_i\pi = \pi r_i$, for each $i = 0, 1, 2$. Due to the transitivity of the action of $\mathrm{Mon}\,(\mathcal{K})$ a hypermap homomorphism is necessarily surjective, thus homomorphisms between hypermaps are alternatively called *coverings*. An easy but fundamental observation establishes that given covering $\pi : \mathcal{H} \to \mathcal{K}$ there is an induced group epimorphism $\pi^* : \mathrm{Mon}\,(\mathcal{H}) \to \mathrm{Mon}\,(\mathcal{K})$ taking $r_i \mapsto t_i$ for $i = 0, 1, 2$. Homomorphisms of hypermaps correspond to branched coverings of topological hypermaps mapping i-cells onto i-cells for $i = 0, 1, 2$. A bijective homomorphism $\mathcal{H} \to \mathcal{K}$ is an *isomorphism*, we write $\mathcal{H} \cong \mathcal{K}$ in this case. An *automorphism* $\mathcal{H} \to \mathcal{H}$ is a permutation of flags of $\mathcal{H}$ commuting with the involutions r_i, for each $i = 0, 1, 2$. In what follows, we shall always let the elements of the monodromy group of a hypermap $\mathcal{H}$ having left action on the flags, while the automorphisms of $\mathcal{H}$ will act from 'right'.

It is well-known (and easy to see) that the action of the automorphism group of a hypermap on its flags is semi-regular (i. e. the stabiliser of a flag is trivial). In the case the automorphism group Aut $(\mathcal{H})$ acts regularly on the flag-set of a hypermap $\mathcal{H}$, the hypermap $\mathcal{H}$ is called *regular.*

Besides the monodromy group $G = \text{Mon}\,(\mathcal{H}) = \langle r_0, r_1, r_2 \rangle$ of a hypermap we consider its even word subgroup generated by $G^+ = \langle \rho, \lambda \rangle$, where $R = r_1 r_2$ and $L = r_0 r_1$. Obviously, it is a subgroup of index at most two. If $[G : G^+] = 2$ the hypermap $\mathcal{H}$ is *orientable.* The category of *oriented hypermaps* is formed by triples $(D; R, L)$ where R, L are permutations generating a group (the *oriented monodromy group*) acting transitively on the set of darts D. The notions of homomorphism, of isomorphism and of automorphism are defined in the obvious way. An oriented map is *regular* if its automorphism group acts regularly on the set of darts.

Let us denote by

$$\Delta = \langle \rho_0, \rho_1, \rho_2; \;\; r_0^2 = r_1^2 = r_2^2 = 1 \rangle$$

the free product of three two-element groups.

The associated (infinite) hypermap $\mathcal{U} = (\Delta; \rho_0, \rho_1, \rho_2)$, with ρ_i $(i = 0, 1, 2)$ acting by left multiplication, will be called the *universal hypermap.* It follows that the monodromy group of any hypermap $\mathcal{H}$ is an epimorphic image of Δ and this epimorphism induces an action of Δ on flags of $\mathcal{H}$. Hence $\mathcal{H}$ can be represented as a hypermap $(\Delta/H; r_0', r_1', r_2')$, where H is a stabiliser of a flag in the action of Δ, Δ/H is the set of left cosets of H and the action of r_i' is defined by the rule $r_i'(xH) = r_i xH$ for $i = 0, 1, 2$. The group H of finite index is called the *hypermap subgroup* of $\mathcal{H}$. The above defined hypermap corresponding to a hypermap subgroup H will be denoted by $\mathcal{U}/H$ and will be called an *algebraic hypermap.* A routine calculation shows that two subgroups H_1 and H_2 of Δ determine isomorphic hypermaps if and only if they are conjugate. Hence, the representation of a hypermap by a hypermap subgroup is not unique, this is because in an irregular hypermap two flag stabilisers may be different although they are always conjugate. More generally, $\mathcal{H}$ covers $\mathcal{K}$ if and only if there exist $g \in \Delta$ such that $H^g \leq K$. As concerns properties of algebraic hypermaps the following (well-known) statement is worth to mention explicitly. Recall that given groups $H \leq G$ the normaliser $N_G(H)$ is a subgroup of G consisting of $g \in G$ such that $H^g = H$.

Proposition 2.1. *Let $\mathcal{H}$ be an algebraic map with a hypermap subgroup $H \leq \Delta$. Then $Aut(\mathcal{H}) = N_\Delta(H)/H$.*

Proof. Let φ be an automorphism of $\mathcal{H}$ taking H onto gH. We show that the assignment $A : \varphi \mapsto gH$ defines the required isomorphism. Since φ is an automorphism of $\mathcal{H}$, we have $hgH = h(H\varphi) = (hH)\varphi = ghH = gH$ for every $h \in H$. Thus g normalises H. By its definition A is a homomorphism. The semi-regularity of the action of the automorphism group implies that A is injective. To see that it is surjective, let us denote by φ_g the mapping $xH \mapsto xgH$. This is a well-defined automorphism if and only if for every $h \in H$, we have $hgH = gH$. But the latter statement means $g \in N_\Delta(H)$. $\square$

It follows that a hypermap $\mathcal{H}$ is regular if and only if the associated hypermap subgroup is normal (see also [7,6]). Hence H is uniquely determined in this case. In what follows we (as a rule) denote by $H \trianglelefteq \Delta$ the hypermap subgroup associated with a regular hypermap $\mathcal{H}$. To establish a one-to-one correspondence between the normal subgroups of finite index of Δ and regular hypermaps we need to extend the family of all regular hypermaps by considering a *trivial hypermap* being the one-flag hypermap with the trivial action of the three defining involutory permutations. We shall use 1 to denote the trivial hypermap. The hypermap subgroup of the trivial hypermap is Δ. Let $\mathcal{H}$ and $\mathcal{K}$ are regular hypermaps. Then $\mathcal{H} \to \mathcal{K}$ if and only if $K \geq H$. Hence there is an isomorphism between the set of regular hypermaps partially ordered by the relation "to be a cover", and the set of normal (torsion free) subgroups of finite index ordered by the subgroup relation. In what follows this correspondence will be extensively employed. In fact, the whole paper is devoted to a detailed investigation of this fundamental correspondence. Let us remark that coverings between regular hypermaps are necessarily regular (see [13]), i. e. the group of covering transformations acts regularly on each flag-fiber. If $\mathcal{H} \to \mathcal{K}$ are regular hypermaps with the hypermap subgroups $H \leq K$ then the covering is defined by mapping $\pi : xH \mapsto xK$ and the *covering transformation group* is isomorphic to the kernel $\operatorname{Ker}\pi$ of the above group epimorphism $\pi : \Delta/H \to \Delta/K$.

Similar statements about the correspondence between oriented hypermaps and conjugacy classes of subgroups of finite index of the free 2-generator group $\Delta^+ < \Delta$ can be established. In particular, there is one-to-one correspondence between the isomorphism classes of oriented regular hypermaps and normal subgroups of finite index in Δ^+.

The reader interested to get more information on maps, hypermaps and related topics is referred to [4, 5, 6, 9, 10, 11, 15, 16]. As concerns the related parts of theory of permutation groups an old but popular monograph is [17].

3. Join and intersection of two hypermaps

Let $\mathcal{H} = \mathcal{U}/H$ and $\mathcal{K} = \mathcal{U}/K$ be algebraic hypermaps. Set $\mathcal{H} \vee \mathcal{K} = \mathcal{U}/(H \cap K)$ and $\mathcal{H} \wedge \mathcal{K} = \mathcal{U}/\langle H, K\rangle$. The hypermaps $\mathcal{H} \vee \mathcal{K}$, $\mathcal{H} \wedge \mathcal{K}$ will be called *join* and *intersection* of $\mathcal{H}$ and $\mathcal{K}$, respectively.

The following two propositions are direct consequences of definitions.

Proposition 3.1. *Let $\mathcal{H} = \mathcal{U}/H$ and $\mathcal{K} = \mathcal{U}/K$ be algebraic hypermaps. Then*

if both $\mathcal{H}$ and $\mathcal{K}$ are finite then $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are finite,
if both $\mathcal{H}$ and $\mathcal{K}$ are regular then $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are regular as well,
if $\mathcal{H} \to \mathcal{K}$ is a covering then $\mathcal{H} \vee \mathcal{K} = \mathcal{H}$ and $\mathcal{H} \wedge \mathcal{K} = \mathcal{K}$.
if a hypermap $\mathcal{X} = \mathcal{U}/X$ covers both $\mathcal{H}$ and $\mathcal{K}$ then it covers $\mathcal{H} \vee \mathcal{K}$,
if a hypermap $\mathcal{X} = \mathcal{U}/X$ is covered by both $\mathcal{H}$ and $\mathcal{K}$ then it is covered by $\mathcal{H} \vee \mathcal{K}$,

Proposition 3.2. *If $\mathcal{H}$ and $\mathcal{K}$ are regular hypermaps then $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are well-defined binary operations on isomorphism classes of hypermaps.*

Proof. The respective hypermap subgroups are unique. □

It follows from the above propositions that for any two regular hypermaps $\mathcal{H}$ and $\mathcal{K}$ there is a unique regular hypermap $\mathcal{Y} = \mathcal{H} \vee \mathcal{K}$ satisfying the following property:

if $\mathcal{X} \to \mathcal{H}$ and $\mathcal{X} \to \mathcal{K}$ then $\mathcal{X}$ covers the join $\mathcal{Y}$. Thus it make sense to speak on the *least common cover* of two regular maps $\mathcal{H}$ and $\mathcal{K}$. Similarly, any hypermap covered by two regular hypermaps is covered by $\mathcal{H} \wedge \mathcal{K}$ and so we can view the intersection as the *largest common quotient* of $\mathcal{H}$ and $\mathcal{K}$.

An irregular hypermap can be represented by two different hypermap subgroups H and H^g for some $g \in \Delta$. Thus the join and the intersection does not preserve isomorphism classes of hypermaps. Hence, they are binary operations on algebraic representations of hypermaps and not on the isomorphism classes. Since the intersection of normal subgroups as well as their product $HK = \langle H, K\rangle$ is a normal subgroup, we have not such a problem provided we restrict ourselves to the family of regular hypermaps, so we can speak on a join and intersection of (abstract) regular hypermaps. In a general case we shall always assume that with a given hypermap $\mathcal{H}$ a particular representative $H \leq \Delta$ of the respective conjugacy class of hypermap subgroups is associated. The latter is equivalent with considering a rooted hypermap, meaning a hypermap with a specified flag (the root of it). This approach is taken in [18].

The following lemma lists the properties of the join and intersection which are trivial consequences of the definitions. In particular, it follows that algebraic hypermaps form a lattice isomorphic to the lattice of all subgroups of Δ and regular hypermaps form a lattice isomorphic to the lattice of all normal subgroups of Δ. The ordering on regular hypermaps is given by hypermap coverings.

Lemma 3.3. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be algebraic hypermaps (regular hypermaps). Let $\mathcal{U}$ and* 1 *be the universal and trivial hypermaps. Then*

$\mathcal{X} \vee (\mathcal{Y} \vee \mathcal{Z}) = (\mathcal{X} \vee \mathcal{Y}) \vee \mathcal{Z}$,
$\mathcal{X} \vee \mathcal{Y} = \mathcal{Y} \vee \mathcal{X}$,
$\mathcal{X} \vee \mathcal{U} = \mathcal{U}$ *and* $\mathcal{X} \vee 1 = \mathcal{X}$,
$\mathcal{X} \wedge (\mathcal{Y} \vee \mathcal{Z}) \to (\mathcal{X} \wedge \mathcal{Y}) \vee (\mathcal{X} \wedge \mathcal{Z})$.

Interchanging joins and intersections in the above statements we get a dual version of the above lemma. In particular, we have

$$\mathcal{X} \vee (\mathcal{Y} \wedge \mathcal{Z}) \leftarrow (\mathcal{X} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Z}).$$

Let $\mathcal{H}$ be a hypermap. Denote by $|\mathcal{H}|$ the number of its flags. Of course if $\mathcal{H}$ is a regular hypermap we have $|\mathcal{H}| = |\mathrm{Mon}\,(\mathcal{H})| = |\Delta/H|$. The following statement relates the monodromy groups of the join and intersection of hypermaps with the monodromy groups of the original hypermaps.

Proposition 3.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. Then the monodromy group of $\mathcal{H} \vee \mathcal{K}$ is a subgroup of the direct product $Mon\,(\mathcal{H}) \times Mon\,(\mathcal{K})$ and we have*

$$Mon\,(\mathcal{H} \wedge \mathcal{K}) = Mon\,(\mathcal{H} \vee \mathcal{K})/\big(H/H \cap K \times K/H \cap K\big),$$

where $H/H \cap K \times K/H \cap K$ is an internal direct product. Moreover,

$$|\mathcal{H} \vee \mathcal{K}| \cdot |\mathcal{H} \wedge \mathcal{K}| = |\mathcal{H}| \cdot |\mathcal{K}|.$$

Proof. We show that the mapping $\psi : g(H \cap K) \mapsto (gH, gK)$ is a monomorphism $\Delta/(H \cap K) \to \Delta/H \times \Delta/K$. Indeed, for any $x, y \in \Delta$

$$\psi((xH \cap K)(yH \cap K)) = \psi(xyH \cap K) = (xyH, xyK) =$$

$$(xH, xK)(yH, yK) = \psi(xH \cap K)\psi(yH \cap K).$$

Now let $\psi(xH \cap K) = 1 = (H, K)$ for some $x \in \Delta$. Then $(xH, xK) = (H, K)$, and consequently $x = 1$. Hence, ψ is a monomorphism.

By the third isomorphism theorem

$$HK/H \cap K = H/H \cap K \times K/H \cap K \cong HK/K \times HK/H.$$

Using this we get

$$|\mathcal{H} \vee \mathcal{K}||\mathcal{H} \wedge \mathcal{K}| = |\Delta/H \cap K||\Delta/HK| = |\Delta/HK||HK/K \times HK/H||\Delta/HK| =$$
$$|\Delta/HK||HK/K||\Delta/HK||HK/H| = |\Delta/K||\Delta/H| = |\mathcal{K}||\mathcal{H}|.$$

By the second isomorphism theorem we obtain

$$\text{Mon}\,(\mathcal{H} \wedge \mathcal{K}) = \Delta/HK \cong (\Delta/H \cap K)/(HK/H \cap K) =$$
$$\text{Mon}\,(\mathcal{H} \vee \mathcal{K})/\big(H/H \cap K \times K/H \cap K\big).$$

□

The equality $|\mathcal{H}\vee\mathcal{K}|\cdot|\mathcal{H}\wedge\mathcal{K}| = |\mathcal{H}|\cdot|\mathcal{K}|$, combined with the well-known statement in elementary number theory establishing

$$|\mathcal{H}| \cdot |\mathcal{K}| = gcd(|\mathcal{H}|, |\mathcal{K}|) \cdot lcm(|\mathcal{H}|, |\mathcal{K}|),$$

may suggest that $|\mathcal{H} \vee \mathcal{K}| = lcm(|\mathcal{H}|, |\mathcal{K}|)$, or equivalently $|\mathcal{H} \wedge \mathcal{K}| = gcd(|\mathcal{H}|, |\mathcal{K}|)$. However, this is not true in general. In general, we can only claim that $lcm(|\mathcal{H}|, |\mathcal{K}|)$ divides $|\mathcal{H}\vee\mathcal{K}|$, and $|\mathcal{H}\wedge\mathcal{K}|$ divides $gcd(|\mathcal{H}|, |\mathcal{K}|)$. The above two equalities imply

$$\frac{|\mathcal{H} \vee \mathcal{K}|}{lcm(|\mathcal{H}|, |\mathcal{K}|)} = \frac{gcd(|\mathcal{H}|, |\mathcal{K}|)}{|\mathcal{H} \wedge \mathcal{K}|}.$$

This observation led us to a new concept allowing us to relate two hypermaps. Given two regular hypermaps $\mathcal{H}$ and $\mathcal{K}$ the integer

$$s(\mathcal{H}, \mathcal{K}) = \frac{|\mathcal{H} \vee \mathcal{K}|}{lcm(|\mathcal{H}|, |\mathcal{K}|)} = \frac{gcd(|\mathcal{H}|, |\mathcal{K}|)}{|\mathcal{H} \wedge \mathcal{K}|}$$

will be called the *shared cover index* of $\mathcal{H}$ and $\mathcal{K}$. Clearly, if one of $\mathcal{H}$, $\mathcal{K}$ covers the other then $s(\mathcal{H}, \mathcal{K}) = 1$. Generally, it can be equal to any divisor of $gcd(|\mathcal{H}|, |\mathcal{K}|)$.

Replacing hypermaps by oriented hypermaps one can see that the concept of the shared cover index applies in the category of oriented regular maps as well. Here it can be viewed as a generalisation of the chirality index studied in [3]. Recall that by the *mirror image* of an oriented hypermap $\mathcal{H} = (R, L)$ we mean the hypermap $\mathcal{H}^r = (R^{-1}, L^{-1})$. The integer $\kappa(\mathcal{H}) = H/(H \cap H^r)$ is called the *chirality index* of $\mathcal{H}$, see [3].

Proposition 3.5. *Let $\mathcal{H}$ be an oriented regular hypermaps and $\mathcal{H}^r$ is the mirror image of it. Then $s(\mathcal{H}, \mathcal{H}^r) = \kappa(\mathcal{H})$, where $\kappa(\mathcal{H})$ is the chirality index of $\mathcal{H}$.*

Proof.

$$s(\mathcal{H}, \mathcal{H}^r) = \frac{|\mathcal{H} \vee \mathcal{H}^r|}{lcm(|\mathcal{H}|, |\mathcal{H}^r|)} = \frac{|\mathcal{H} \vee \mathcal{H}^r|}{|\mathcal{H}|} = \frac{|\Delta/H \cap H^r|}{|\Delta/H|} = |H/H \cap H^r| = \kappa(\mathcal{H}).$$

□

4. Monodromy groups of the join and intersection of two hypermaps

Throughout this section all the considered hypermaps will be regular. In the above section we have derived some information on the structure of the monodromy groups of $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$. In what follows we shall consider the problem how to calculate the above monodromy groups by using the action of monodromy groups of $\mathcal{H}$ and $\mathcal{K}$. Let $A = \langle r_0, \ldots, r_k \rangle$ and $B = \langle s_0, \ldots, s_k \rangle$ be two k-generated groups. Let us define their *monodromy product* $A \times_m B$ to be the subgroup of the direct product generated by (r_i, s_i), where $i = 0, 1, \ldots, k$. Note that S. Wilson calls it the parallel product in [18]. Further, denote by $\pi_1 : A \times_m B \to A$, $\pi_2 : A \times_m B \to B$ the natural projections erasing the second and first coordinate, respectively.

Theorem 4.1. *Let $\mathcal{H} = (A; r_0, r_1, r_2)$ and $\mathcal{K} = (B; s_0, s_1, s_2)$ be regular hypermaps. Then $Mon(\mathcal{H} \vee \mathcal{K}) = Mon(\mathcal{H}) \times_m Mon(\mathcal{K})$ and $Mon(\mathcal{H} \wedge \mathcal{K}) = Mon(\mathcal{H}) \times_m Mon(\mathcal{K})/Ker\,\pi_2 Ker\,\pi_1$.*

Proof. Let $\Delta = \langle R_0, R_1, R_2; R_0^2 = R_1^2 = R_2^2 = 1 \rangle$. Recall that the hypermap subgroup of $\mathcal{H}$ can be reconstructed as a stabiliser $H = STAB_\Delta(x_0)$ of a flag x_0 and similarly for $\mathcal{K}$, $K = STAB_\Delta(y_0)$. Denote by $\psi_1 : \mathcal{H} \to (\Delta/H; R_0H, R_1H, R_2H)$ the isomorphism of hypermaps and by $\psi_1^* : \mathrm{Mon}\,\mathcal{H} \to \Delta/H$ the induced group epimorphism sending $r_i \mapsto R_iH$, for $i = 0, 1, 2$. Similarly, denote by ψ_2 the isomorphism $\mathcal{K} \to (\Delta/K; R_0K, R_1K, R_2K)$ of hypermaps and by ψ_2^* the respective group epimorphism taking $s_i \mapsto R_iK$. Then we have an isomorphism $\Psi : \Delta/H \times_m \Delta/K \to \mathrm{Mon}\,(\mathcal{H}) \times_m \mathrm{Mon}\,(\mathcal{K})$ taking $(R_iH, R_iK) \to ((\psi_1^*)^{-1}(R_iH), (\psi_2^*)^{-1}(R_iK)) = (r_i, s_i)$.

In the proof of Proposition 3.4 we have already verified that the mapping $\Phi : \Delta/H \cap K \to \Delta/H \times_m \Delta/K$, taking $g(H \cap K)$ onto (gH, gK), is an isomorphism of groups. Now the composition $\Psi\Phi$ establishes an isomorphism $\mathrm{Mon}\,(\mathcal{H} \vee \mathcal{K}) \to \mathrm{Mon}\,(\mathcal{H}) \times_m \mathrm{Mon}\,(\mathcal{K})$.

Regarding the intersection of $\mathcal{H}$ and $\mathcal{K}$, by Proposition 3.4 we have

$$\mathrm{Mon}\,(\mathcal{H} \wedge \mathcal{K}) = \mathrm{Mon}\,(\mathcal{H} \vee \mathcal{K})/(H/H \cap K \times K/H \cap K).$$

In view of what we have proved it is enough to see that $\Psi\Phi$ sends $K/H \cap K$ onto $\mathrm{Ker}\ \pi_2$, and $H/H \cap K$ onto $\mathrm{Ker}\ \pi_1$. Indeed,

$$\Psi\Phi(K/H \cap K) = \Psi(\{(gH, K) | g \in K\}) =$$

$$\{(w, 1) | (w, 1) \in \mathrm{Mon}\,(\mathcal{H}) \times_m \mathrm{Mon}\,(\mathcal{K})\} = \mathrm{Ker}\,\pi_2.$$

Similar calculation verifies the statement $\Psi\Phi(H/H \cap K) = \mathrm{Ker}\,\pi_1$. □

We say that a covering $\mathcal{H} \to \mathcal{K}$ of regular hypermaps is *smooth* if both hypermaps are of the same type. Smooth covers of hypermaps correspond to unbranched covers of their topological equivalents.

Proposition 4.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. Then*

(a) the hypermap $\mathcal{H} \vee \mathcal{K}$ smoothly covers both $\mathcal{H}$, $\mathcal{K}$ if and only if the types of $\mathcal{H}$ and $\mathcal{K}$ are equal.

(b) if both $\mathcal{H}$ and $\mathcal{K}$ smoothly cover the intersection $\mathcal{H} \wedge \mathcal{K}$ then they have the same type.

We shall see later that the above implication (b) cannot be reversed.

Two regular hypermaps $\mathcal{H}$, $\mathcal{K}$ will be called *orthogonal* if $HK = \Delta$. We shall use $\mathcal{H} \perp \mathcal{K}$ to denote the orthogonality of $\mathcal{H}$ and $\mathcal{K}$. Let G, H be two groups. A *common epimorphic* image of G and H is a group Q such that there are epimorphisms $G \to Q$ and $H \to Q$. Let $H = \langle r_0, r_1, r_2 \rangle$, $K = \langle s_0, s_1, s_2 \rangle$ and $Q = \langle t_0, t_1, t_2 \rangle$ be groups. We say that Q is a *monodromic common epimorphic image* of H and K if both the assignments $r_i \mapsto t_i$ and $s_i \mapsto t_i$ (for $i = 0, 1, 2$) extend to group epimorphisms $H \to Q$ and $K \to Q$.

The following theorem gives several characterisations of the orthogonality.

Theorem 5.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. Then the following conditions are equivalent:*

(i) $\mathcal{H} \perp \mathcal{K}$,
(ii) $\mathcal{H} \wedge \mathcal{K}$ *is a trivial hypermap,*
(iii) $\mathcal{H}$ *and* $\mathcal{K}$ *have no nontrivial common quotients,*
(iv) *the monodromy groups* $Mon(\mathcal{H})$ *and* $Mon(\mathcal{K})$ *have no common monodromic epimorphic images,*
(v) $Mon(\mathcal{H} \vee \mathcal{K}) = Mon(\mathcal{H}) \times Mon(\mathcal{K})$.

Proof. *(i)* $\Leftrightarrow$ *(ii)* Since the flags of the intersection are the elements of Δ / HK, the intersection is a trivial hypermap if and only if $HK = \Delta$.

(ii) $\Leftrightarrow$ *(iii)*. If $\mathcal{H} \wedge \mathcal{K}$ is nontrivial then it forms a non-trivial common quotient. Vice-versa if there is a non-trivial common (possibly irregular) quotient $\mathcal{Q}$ then there are $g, h \in \Delta$ such that $\Delta > Q^g \geq K$ and $\Delta > Q^h \geq H$. By normality of both H and K we get $\Delta > Q^x \geq K$, $\Delta > Q^x \geq H$ for any $x \in \Delta$. Hence $\mathcal{Q}_\Delta = \mathcal{U} / \bigcap_{x \in \Delta} Q^x \to \mathcal{Q}$ is a non-trivial regular common quotient. However, since $\mathcal{Q}_\Delta$ is covered by $\mathcal{H} \wedge \mathcal{K}$, thus the intersection is a non-trivial hypermap.

(ii) $\Leftrightarrow$ *(v)*. By Proposition 3.4 $\mathrm{Mon}(\mathcal{H} \vee \mathcal{K}) \leq \mathrm{Mon}(\mathcal{H}) \times \mathrm{Mon}(\mathcal{K})$. The second part of Proposition 3.4 implies that the equality holds if and only if $\mathcal{H} \perp \mathcal{K}$.

(iii) $\Leftrightarrow$ *(iv)* If there is a common quotient $\mathcal{Q}$ for $\mathcal{H}$ and $\mathcal{K}$ then the coverings $\mathcal{H} \to \mathcal{Q}$ and $\mathcal{K} \to \mathcal{Q}$ induce, respectively, monodromy epimorphisms $\mathrm{Mon}(\mathcal{H}) \to \mathrm{Mon}(\mathcal{Q})$ and $\mathrm{Mon}(\mathcal{K}) \to \mathrm{Mon}(\mathcal{Q})$. Vice-versa, if Q is a monodromic common epimorphic image, then representing the hypermaps via hypermap subgroups we get that the assignments $gH \mapsto gQ$, $gK \mapsto gQ$, where g ranges in Δ extend to group epimorphisms. However, the same mappings establish coverings $\mathcal{U}/H \to \mathcal{U}/Q$ and $\mathcal{U}/K \to \mathcal{U}/Q$. The statement follows. $\square$

Denote by $\mathcal{O}$ the two-flag hypermap with $r_0 = r_1 = r_2$ being equal to the non-trivial involution interchanging the two flags. It is easy to see that the hypermap subgroup of $\mathcal{O}$ is Δ^+.

Proposition 5.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be (regular) hypermaps. If $\mathcal{H}$ and $\mathcal{K}$ are orthogonal then at least one of the hypermaps $\mathcal{H}$ and $\mathcal{K}$ is nonorientable.*

Proof. Assume both $\mathcal{H}$ and $\mathcal{K}$ are orientable. The orientability implies that both $H \leq \Delta^+$, $K \leq \Delta^+$ are subgroups of the even-word subgroup Δ^+ of Δ. Then $\mathcal{O} \cong \mathcal{U}/\Delta^+$ is a common non-trivial quotient, a contradiction. $\square$

In general, it can be difficult to see the orthogonality of hypermaps. In what follows we give some sufficient conditions implying the orthogonality of hypermaps. The following proposition is a straightforward consequence of Theorem 5.1.

Proposition 5.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps. If the monodromy groups of $\mathcal{H}$ and $\mathcal{K}$ have no nontrivial common epimorphic images then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.*

Thus a regular hypermap with a non-abelian simple monodromy group is orthogonal to any other hypermap.

Numerical conditions implying the orthogonality may be useful in constructions. We shall present a sample of them.

Proposition 5.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps of types (m_0, m_1, m_2) and (n_0, n_1, n_2). Let one of them, say $\mathcal{H}$, be non-orientable.*

If for any two $i, j \in \{0, 1, 2\}$ the integers m_i, n_i and m_j, n_j are respectively coprimes then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.

Proof. Let the monodromy groups be generated by the triples of involutions $\mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$ and $\mathrm{Mon}(\mathcal{K}) = \langle s_0, s_1, s_2\rangle$. Denote by $R_i = r_i r_{i+1}$ and $S_i = s_i s_{i+1}$, $i = 0, 1, 2$. By the assumption, two of $gcd(R_i, S_i)$, $i = 0, 1, 2$ are equal to 1. Without loss of generality we assume $gcd(R_0, S_0) = 1$ and $gcd(R_2, S_2) = 1$. Then the following equality for the even word subgroup of the monodromy product holds true:

$$(\mathrm{Mon}(\mathcal{H}) \times_m \mathrm{Mon}(\mathcal{K}))^+ = \langle R_0, R_2\rangle \times_m \langle S_0, S_2\rangle = \mathrm{Mon}^+(\mathcal{H}) \times_m \mathrm{Mon}^+(\mathcal{K}).$$

To prove the orthogonality of $\mathcal{H}$ and $\mathcal{K}$ we show that the projections of the latter group into the coordinate factors contain isomorphic copies of the even-word subgroups of the original hypermaps. Since the orders of R_0 and S_0 are coprime, $(R_0, 1)$ and $(1, S_0)$ are elements of the cyclic group $\langle(R_0, S_0)\rangle$. For the same reason we see that $(R_2, 1)$ and $(1, S_2)$ belong to $\langle(R_2, S_2)\rangle$. Now observe $\mathrm{Mon}^+(\mathcal{H}) \cong \langle(R_0, 1), (R_2, 1)\rangle$, and similarly we get $\mathrm{Mon}^+(\mathcal{K}) \cong \langle(1, S_0), (1, S_2)\rangle$. Hence we have that $\mathrm{Mon}(\mathcal{H} \vee \mathcal{K})$ contains a subgroup $G = \mathrm{Mon}^+(\mathcal{H}) \times \mathrm{Mon}^+(\mathcal{K})$. Since $\mathcal{H}$ is non-orientable, $\mathrm{Mon}^+(\mathcal{H}) = \mathrm{Mon}(\mathcal{H})$. By Theorem 4.1 the monodromy group of the intersection

$$\mathrm{Mon}(\mathcal{H} \wedge \mathcal{K}) = \mathrm{Mon}(\mathcal{H}) \times_m \mathrm{Mon}(\mathcal{K}) / Ker\,\pi_2 Ker\,\pi_1.$$

Since $\mathrm{Mon}(\mathcal{H}) \times \mathrm{Mon}^+(\mathcal{K}) \leq Ker\,\pi_2 Ker\,\pi_1$ the intersection is either trivial or a 2-flag hypermap. However, the only (regular) 2-flag hypermap is $\mathcal{O}$ which is obviously not covered by $\mathcal{H}$. Hence, the intersection is the trivial hypermap and we are done. □

A cellular embedding of a graph into a surface is called a *regular embedding* if the corresponding map is regular.

Proposition 5.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular maps determined by regular embeddings of two non-bipartite graphs with coprime valency. Then $\mathcal{H} \perp \mathcal{K}$ if and only if at least one of $\mathcal{H}$, $\mathcal{K}$ is non-orientable.*

Proof. If both embeddings define orientable maps then they both cover $\mathcal{O}$, and consequently, they are not orthogonal.

Let one of the maps associated with the embeddings of graphs is non-orientable. With the same notation as above we have $R_2^2 = 1 = S_2^2$, because the hypermaps are maps now. Since the valences of the maps are coprime we have that $(R_1, 1)$

and $(1, S_1)$ belong to the monodromy product of the even word subgroups. Since the graphs are non-bipartite there are identities of the form $\prod_{i=1}^{k} R_1^{m_i} R_2 = 1$, $\prod_{i=1}^{n} S_1^{p_i} S_2 = 1$, where k and n are some odd integers. Replacing above R_1 by $(R_1, 1)$, R_2 by (R_2, S_2), S_1 by $(1, S_1)$ and S_2 by (R_2, S_2) we get that the involutions $(R_2, 1)$ and $(1, S_2)$ are elements of the monodromy product. Hence the even-word subgroup of the monodromy product is the direct product of the even-word subgroups of the original maps. Now we can complete the proof as above. □

There are oriented versions of the above propositions. We shall state them without proofs.

Proposition 5.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be oriented regular hypermaps of types (m_0, m_1, m_2) and (n_0, n_1, n_2).*

If for any two $i, j \in \{0, 1, 2\}$ the integers m_i, n_i and m_j, n_j are respectively coprimes then the hypermaps $\mathcal{H}$ and $\mathcal{K}$ are orthogonal.

A cellular embedding of a graph into an orientable surface is called *orientably regular* if the corresponding oriented map is regular.

Proposition 5.7. *Orientably regular embeddings of non-bipartite graphs with coprime valency determine a couple of orthogonal oriented maps.*

6. Orientability, reflexibility and self-duality

Topological and algebraic properties of maps, as for instance, the orientability, the reflexibility and the self-duality have their counterparts in the associated algebraic representations. The aim of this section is to discuss the above properties and concepts in a relation with the join and with the intersection of two hypermaps.

6.1 Orientability.

An algebraic hypermap $\mathcal{H} = \mathcal{U}/H$ is *orientable* if $H \leq \Delta^+$, where $\Delta^+ = \langle r_1 r_2, r_2 r_0 \rangle$ is the even word subgroup of Δ (which is an index two subgroup).

The following statements are direct consequences of the definitions so we shall omit the proofs of them.

Proposition 6.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be regular hypermaps with the respective hypermap subgroups H and K. Then*

if both $\mathcal{H}$ and $\mathcal{K}$ are orientable then both $\mathcal{H} \vee \mathcal{K}$ and $\mathcal{H} \wedge \mathcal{K}$ are orientable as well,

if one of $\mathcal{H}$, $\mathcal{K}$ is orientable and the other not then $\mathcal{H} \vee \mathcal{K}$ is orientable while $\mathcal{H} \wedge \mathcal{K}$ is nonorientable,

if both $\mathcal{H}$ and $\mathcal{K}$ are nonorientable then $\mathcal{H} \wedge \mathcal{K}$ is nonorientable as well.

Proposition 6.2. *Let $\mathcal{H}$ be a regular hypermap. The following statements are equivalent:*

$\mathcal{H}$ is orientable,

$\mathcal{H}$ covers $\mathcal{O}$,

$\mathcal{H} = \mathcal{H} \vee \mathcal{O}$,

$\mathcal{O} = \mathcal{H} \wedge \mathcal{O}$.

It follows that $\mathcal{H}$ is nonorientable if and only if $\mathcal{H} \perp \mathcal{O}$ and the algebraic counterpart to the well-known construction of the antipodal double cover over a nonorientable hypermap $\mathcal{H}$ is the construction of the join $\mathcal{H} \vee \mathcal{O}$ (cf. [18]). Let us remark that in the case of maps the first three items of Proposition 6.2 are covered by [18].

6.2 Reflexibility.

A regular oriented hypermap $\mathcal{K} = \mathcal{U}^+/K$ is *reflexible* if $K^{r_0} = K$. Note that since $\mathcal{K}$ is a normal subgroup of Δ^+, we have $K^{r_0} = K^{r_1} = K^{r_2}$. The hypermap $\mathcal{K}^r = \mathcal{U}^+/K^{r_0}$ will be called the *mirror image* of $\mathcal{K}$. Clearly, the join $\mathcal{K} \vee \mathcal{K}^r$ and the intersection $\mathcal{K} \wedge \mathcal{K}^r$ are reflexible hypermaps. In general, we have the following

Proposition 6.3. *Let $\mathcal{K}$ be an oriented regular hypermap. Then*

the join $\mathcal{K} \vee \mathcal{K}^r$ is the least reflexible regular oriented hypermap covering $\mathcal{K}$, the intersection $\mathcal{K} \wedge \mathcal{K}^r$ is the largest reflexible regular oriented hypermap covered by $\mathcal{K}$.

As it was already noted, see Proposition 3.5, the integer $\kappa(\mathcal{K}) = s(\mathcal{K}, \mathcal{K}^r) = \frac{|\mathcal{K}|}{|\mathcal{K} \wedge \mathcal{K}^r|}$, called the chirality index in [3], can be used to measure of how much a given hypermap is far from being mirror symmetric. Moreover, the way how two hypermaps with the same chirality index are chiral can be of different quality. More precisely, for any oriented regular hypermap we have coverings $\mathcal{H} \vee \mathcal{H}^r \to \mathcal{H} \to \mathcal{H} \wedge \mathcal{H}^r$. It is proved in [3] that the two associated groups of covering transformations are isomorphic and their size is equal to the about mentioned chirality index which coincides with the shared cover index $s(\mathcal{H}, \mathcal{H}^r)$. The associated group is called the *chirality group* of $\mathcal{H}$. It is proved in [3] that any finite abelian group can be isomorphic to the chirality group of a regular hypermap. Members of several infinite families of non-abelian groups are proved to appear as chirality groups as well (see [3]).

6.3 Self-duality.

Let σ be a permutation of $\{0, 1, 2\}$. Clearly, σ induces an outer automorphism $\bar{\sigma}$ of Δ mapping $r_i \mapsto r_{i\sigma}$. A σ-dual of $\mathcal{H}$ is the hypermap $\mathcal{U}/\bar{\sigma}(H)$ with the hypermap subgroup $\bar{\sigma}(H)$. It may happen that $H = \bar{\sigma}(H)$, in this case $\mathcal{H}$ is called *σ-selfdual.* If $\mathcal{H}$ is σ-self-dual for all 6 possible permutations σ of the index-set we shall say that $\mathcal{H}$ is *totally selfdual.* Similarly as for the reflexibility we have the following statement.

Proposition 6.4. *Let $\mathcal{K}$ be a regular hypermap and let σ is a permutation of $\{0, 1, 2\}$. Then*

(1) *the join $\mathcal{K} \vee \mathcal{K}^{\bar{\sigma}}$ is the least σ-selfdual regular hypermap covering $\mathcal{K}$,*
(2) *the intersection $\mathcal{K} \wedge \mathcal{K}^{\bar{\sigma}}$ is the largest σ-selfdual regular hypermap covered by $\mathcal{K}$.*

In particular, if S_3 denotes the group of all permutations of $\{0, 1, 2\}$ then $\mathcal{U}/\bigcap_{\sigma \in S_3} K^{\bar{\sigma}}$ is the least totally selfdual hypermap covering $\mathcal{K}$. Similarly, $\mathcal{U}/\prod_{\sigma \in S_3} K^{\bar{\sigma}}$ is the largest totally selfdual hypermap covered by $\mathcal{K}$.

7 G-symmetric maps and hypermaps

The results of the previous section are just particular instances of a more general approach. Let Out (Δ) be the outer automorphism group of Δ. Recall that Out (Δ) = Aut (Δ)/Inn (Δ), where Inn (Δ) denotes the group of inner automorphisms of Δ acting by conjugation on Δ. The outer automorphism group Out (Δ) was described by L. James in [8]. It follows that Out $(\Delta) \cong PSL(2, Z)$ and it is generated by the 6 permutations permuting the three generators r_0, r_1 and r_2 and one twisting automorphism taking $r_0 \mapsto r_2 r_0 r_2$, $r_1 \mapsto r_1$, $r_2 \mapsto r_2$. The orbit of the action of Out (Δ) on a hypermap $\mathcal{H}$ is finite and can be constructed by making σ-duals and applying the twisting operator repeatedly.

Let $G \leq \text{Out}(\Delta)$ be a subgroup. If $\mathcal{H}$ is a regular hypermap then for each $\phi \in G$ the hypermap $\mathcal{H}^\phi = \mathcal{U}/H^\phi$ is also regular. We say that $\mathcal{H}$ is *G-symmetric* if it is invariant with respect to G, i. e. $\mathcal{H} = \mathcal{H}^\phi$ for every $\phi \in G$, or equivalently, $H = H^\phi$ for every $\phi \in G$. The join $\bigvee_{\phi\in G} \mathcal{H}^\phi$ and the intersection $\bigwedge_{\phi\in G} \mathcal{H}^\phi$ are clearly G-symmetric hypermaps for any regular hypermap $\mathcal{H}$. By the definition $\mathcal{H}$ is G-symmetric if and only if $\mathcal{H} \cong \bigvee_{\phi\in G} \mathcal{H}^\phi$ or equivalently, $\mathcal{H} \cong \bigwedge_{\phi\in G} \mathcal{H}^\phi$. We have the following statement.

Proposition 7.1. *Let $\mathcal{H}$ be a regular hypermap and $G \leq Out(\Delta)$. Then*

the join $\bigvee_{\phi\in G} \mathcal{H}^\phi$ is the least G-symmetric regular hypermap covering $\mathcal{H}$,

the intersection $\bigwedge_{\phi\in G} \mathcal{H}^\phi$ is the largest G-symmetric regular hypermap covered by $\mathcal{H}$.

We can use the covering transformation groups of the coverings $\bigvee_{\phi\in G} \mathcal{H}^\phi \to \mathcal{H}$ and $\mathcal{H} \to \bigwedge_{\phi\in G} \mathcal{H}^\phi$ to measure of how much a given regular hypermap is far from being G-symmetric. Clearly, $\mathcal{H}$ is G-symmetric if and only if these coverings are trivial. In the case $|G| = 2$ we can say something more.

Proposition 7.2. *Let $\phi \in Out(\Delta)$ ($\phi \in Out(\Delta^+)$). Let $\mathcal{H}$ be a regular hypermap (an oriented regular hypermap). Then the groups of covering transformations of coverings $\mathcal{H} \vee \mathcal{H}^\phi \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{H} \wedge \mathcal{H}^\phi$ are isomorphic. Let this common group be denoted by $\mathcal{C}(\mathcal{H},\mathcal{H}^\phi)$. The order of $\mathcal{C}(\mathcal{H},\mathcal{H}^\phi)$ is the shared cover index $s(\mathcal{H},\mathcal{H}^\phi)$.*

Proof. The first covering is defined by the epimorphism $\pi : \Delta/H \cap H^\phi \to \Delta/H$ taking $x(H \cap H^\phi) \mapsto xH$ for any $x \in \Delta$. Clearly, the kernel $\text{Ker}\,\pi \cong H/H \cap H^\phi$.

The second covering is defined by the epimorphism $\sigma : \Delta/H \to \Delta/HH^\phi$ taking $xH \mapsto xHH^\phi$. Now the kernel is $\text{Ker}\,\sigma \cong HH^\phi/H$.

By the third isomorphism theorem we have

$$\text{Ker}\,\sigma \cong HH^\phi/H \cong H/H \cap H^\phi \cong \text{Ker}\,\pi.$$

To complete the proof of the statement we proceed similarly as in the proof of Proposition 3.5

$$s(\mathcal{H},\mathcal{H}^\phi) = \frac{gcd(|\mathcal{H}|,|\mathcal{H}^\phi|)}{|\mathcal{H}\wedge\mathcal{H}^\phi|} = \frac{|\mathcal{H}|}{|\mathcal{H}\wedge\mathcal{H}^\phi|} = \frac{|\Delta/H|}{|\Delta/HH^\phi|} = |HH^\phi/H| = |\text{Ker}\,\sigma|.$$

□

It follows that if $G \leq \text{Out}(\Delta)$ is of order two, the two covering transformation groups of the covering $\mathcal{H} \vee \mathcal{H}^\phi \to \mathcal{H} \to \mathcal{H} \wedge \mathcal{H}^\phi$ are isomorphic. In the special case, when $G \leq \text{Out}(\Delta^+)$ is the group acting on an oriented map $\mathcal{H}$ by taking its mirror image $\mathcal{H}^r$ we get, as a corollary, Theorem 3 of [3]. In this case the chirality group mentioned in the previous section coincide with the group $\mathcal{C}(\mathcal{H},\mathcal{H}^\phi)$.

If a regular hypermap $\mathcal{H}$ is G-symmetric for $G = Out(\Delta)$ we say that $\mathcal{H}$ is *characteristic*. In this case any automorphism of Δ leaves the hypermap subgroup H invariant. Hence H is a characteristic subgroup of Δ. Vice-versa, a characteristic subgroup $H \trianglelefteq \Delta$ of finite index determines an Out (Δ)-symmetric regular hypermap. Consequently, drawings of hypermaps sharing this property can be viewed as pictures of the characteristic subgroups of finite index of Δ. Since the structure of Out (Δ) was described by L. James in [8] we have the following:

Theorem 7.3. *Let $\mathcal{H} = (F; r_0, r_1, r_2)$ be a hypermap. The following statements are equivalent:*

$\mathcal{H}$ is regular and $Out(\Delta)$-symmetric (that is, $\mathcal{H}$ is characteristic),
$\mathcal{H} \cong \mathcal{U}/K$, where K is a characteristic subgroup of finite index,
$\mathcal{H}$ is regular and isomorphic to each of the following three hypermaps $(F; r_2, r_1, r_0)$, $(F; r_0, r_2, r_1)$ and $(F; r_2r_0r_2, r_1, r_2)$.

A similar statement can be formulated in the case of oriented hypermaps by using the fact $\mathrm{Out}(\Delta^+) = \langle \mathrm{Out}(\Delta), \rho \rangle$ where ρ is the automorphism mapping an oriented hypermap onto its mirror image $\rho : (D; R, L) \mapsto (D; R^{-1}, L^{-1})$, see [8].

Finally we stretch that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two G-symmetric hypermaps for some $G \leq \mathrm{Out}(\Delta)$ then the join $\mathcal{H}_1 \vee \mathcal{H}_2$ and the intersection $\mathcal{H}_1 \wedge \mathcal{H}_2$ are also G-symmetric hypermaps.

Example. Let us examine characteristic subgroups H of Δ of small index via the corresponding Δ-symmetric regular hypermaps $\mathcal{H}$. Since $\mathcal{H}$ is totally selfdual such a hypermap is of type (n, n, n) for some integer $n \geq 1$. There is just one non-trivial regular hypermap of type $(1, 1, 1)$ and it is $\mathcal{O} \cong \Delta/\Delta^+$. Obviously Δ^+ is characteristic. Also it is easy to see that we have only one Δ-symmetric regular hypermap $\mathcal{H}$ of type $(2, 2, 2)$ - this is actually the only characteristic map. Its topological hypermap arises by colouring the opposite faces of the cube by the same colour (see Fig. 1). Consequently, $\Delta/H = C_2^3$ is elementary abelian and $\mathcal{H} = \mathcal{D}$ [2].

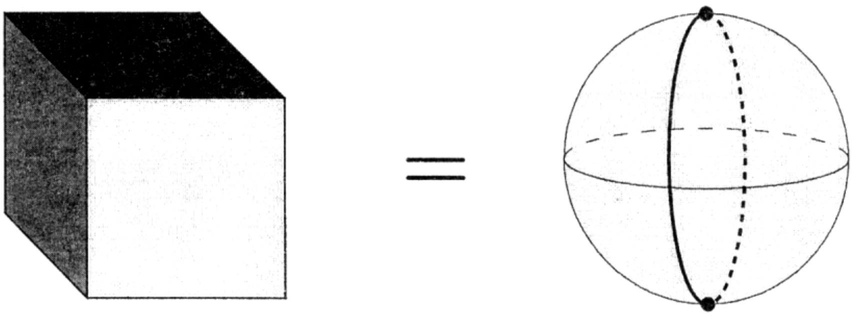

FIGURE 1

As concerns type $(3, 3, 3)$ we shall argue as follows. Clearly, $\mathcal{H}$ is one of the toroidal hypermaps classified by Corn and Singerman in [6]. The smallest representative of the family is the hypermap $\mathcal{H}_2$ drawn on Fig. 2.

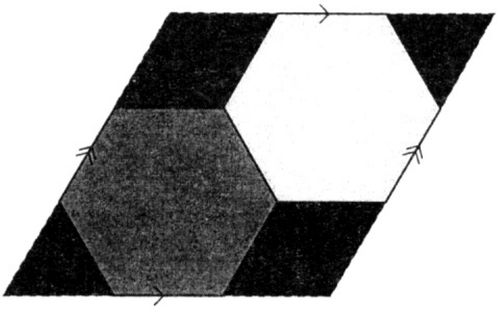

FIGURE 2

It has 6 flags and one hypervertex, hyperedge and hyperface, respectively. As all the regular toroidal hypermaps of type (3,3,3) it is totally selfdual. However, it is easy to see that the three involutory generators satisfy the relation $r_2r_0r_2 = r_1$. Hence, the twisting operator takes $\mathcal{H}_2 = (F; r_0, r_1, r_2)$ onto $(F; r_1, r_1, r_2)$. The latter hypermap is clearly not isomorphic to $\mathcal{H}_1$ since it has type $(1,3,3)$. Take two hypermaps $\mathcal{A}$ and $\mathcal{B}$ from the orbit of Δ with the respective types $(1,3,3)$ and $(3,1,3)$. By Proposition 5.6 the corresponding oriented hypermaps are orthogonal and so $\mathcal{A} \wedge \mathcal{B} = \mathcal{O}$. Hence $\mathcal{O}$ is the largest Δ-symmetric hypermap covered by both $\mathcal{A}$ and $\mathcal{B}$, and consequently, by $\mathcal{H}_2$ as well. The covering $\mathcal{H}_2 \to \mathcal{O} = \mathcal{H}_2 \wedge \mathcal{A}$ is a 3-fold covering. Hence, $\mathcal{H}_2 \vee \mathcal{A} = \mathcal{K}$ is a Δ-symmetric regular hypermap of type $(3,3,3)$, and the covering $\mathcal{K} \to \mathcal{H}_2$ is a 3-fold covering. Consequently, $\mathcal{K}$ has 18 flags. By [6] there is precisely one such hypermap $\mathcal{H}_3$ of type $(3,3,3)$ depicted on Fig.3. Since the oriented hypermaps $\mathcal{A}$ and $\mathcal{B}$ are orthogonal, the even word subgroup is isomorphic to the direct product $C_3 \times C_3$. The monodromy group of $\mathcal{H}_3$ is then a semidirect product of $(C_3 \times C_3)$ by C_2.

We can prove that this is a unique Δ-symmetric regular hypermap of type $(3,3,3)$. As a curiosity let us mention that the underlying 3-valent graph is known as the Pappus graph, which is related to the well-known Pappus configuration, a popular example of a finite geometry.

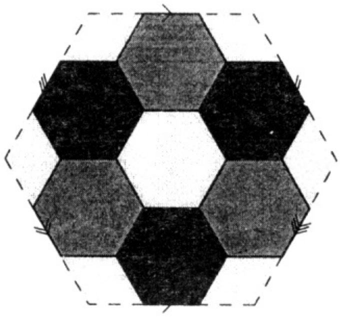

FIGURE 3

Since the join of two characteristic hypermaps is again characteristic the hypermap $\mathcal{H}_4 = \mathcal{D} \vee \mathcal{H}_3$ of type $(6,6,6)$ is also a characteristic hypermap. By Proposition 5.6 the oriended hypermaps corresponding to $\mathcal{D}$ and $\mathcal{H}_3$ are orthogonal. Hence the even word subgroup of the monodromy group of $\mathcal{H}_4$ is the direct product $C_2^2 \times C_3^2$ and it is of size 36. Consequently, $\mathcal{H}_4$ is a characteristic hypermap of genus 10.

To find a Δ-symmetric regular hypermap of type $(4,4,4)$ we checked the list of regular hypermaps of genus 2 in [1]. There is precisely one regular hypermap $\mathcal{H}_5$ of type $(4,4,4)$ and genus 2. The hypermap is totally selfdual (see Fig. 4). A direct computation verifies the relation $(r_2r_0r_2r_1)^4 = 1$, hence the twisting operator applied on $\mathcal{H}_5$ gives a regular hypermap of type $(4,4,4)$ with the same number of flags. Since the orientability is preserved, it is a hypermap on an orientable surface of genus 2. Since there is just one regular hypermap of type $(4,4,4)$ on the surface of genus 2, it must be $\mathcal{H}_5$. Consequently, $\mathcal{H}_5$ is Δ-symmetric.

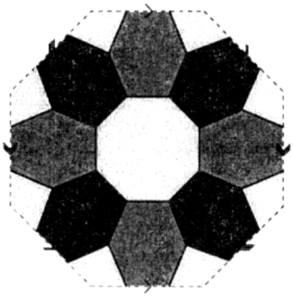

FIGURE 4

The join $\mathcal{H}_6 = \mathcal{H}_3 \vee \mathcal{H}_4$ is a characteristic hypermap of type $(12,12,12)$. By Proposition 5.6 the corresponding oriented hypermaps are orthogonal, hence the even word subgroup is the direct product of the even word subgroups of factors. Consequently, the size of the (full) monodromy group $|\mathrm{Mon}(\mathcal{H}_6)| = 144$ and the genus is 28.

If we restrict ourselves to maps then the characterisation of the outer automorphism group $\mathrm{Out}(\Delta(\infty,\infty,2))$ done by Jones and Thornton [12] can be useful. Recall that $\Delta(\infty,\infty,2) = \langle r_0, r_1, r_2; r_0^2 = r_1^2 = r_2^2 = (r_0r_2)^2 = 1\rangle$ is a monodromy group of the universal map covering any map. The outer automorphism group is isomorphic to S_3 and is generated by two operations (see [12]), first one defined by $(F; r_0, r_1, r_2) \to (F; r_2, r_1, r_0)$ and second one defined by $(F; r_0, r_1, r_2) \to (F; r_0r_2, r_1, r_2)$. First one is known as the duality operation while the second one coincide with the Petrie operation. A more detailed discussion on $\mathrm{Out}(\Delta(\infty,\infty,2))$-symmetric regular maps can be found in Section 5 of [12]. Theorem 3 in [12] is similar to our Theorem 7.3.

8. ACKNOWLEDGEMENT

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