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APPLICATIONS OF LINE OBJECTS IN ROBOTICS

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ABSTRACT. In this paper the Lie algebra of the Lie group of Euclidean motions in E_3 is explained as the vector space A_6 of couples of vectors in E_3. All subalgebras and all 3-dimensional subspaces of A_6 which are orthogonal to themselves according to the Klein form and their kinematic interpretations are described. Vector fields in E_3 determined by elements of A_6 and their kinematic and dynamic interpretations are investigated

1 INTRODUCTION

Line Plücker's coordinates inspire applications of couples of vectors in robotics. First of all in this paper the Plücker's coordinates, basic structure properties such as the Klein and Killing forms, the Lie bracket in the algebra A_6 of all couples of vectors in Euclidean space E_3 are recalled. The algebra A_6 is isomorphic with the Lie algebra of the Lie group of all isometries preserving orientation in E_3. All subalgebras of A_6 and all 3-dimensional subspaces which are orthogonal to themselves according to the Klein form are described. Mechanical engineers use the notion of screws as a useful tool for solving of robotic problems. The roots of this notion are in $19^{\underline{th}}$ century, Ball [1]. We describe the set of screws as a projective 5-dimensional space P_5^σ of all 1-dimensional subspaces in A_6, so the sum of two screws has not sense. We show that the Lie bracket in A_6 induces both a map $P_5^\sigma \times P_5^\sigma \to P_5^\sigma$ and a map $\beta \times \beta \to \beta$ defined on couples of nonparallel lines in E_3, where β is the manifold of proper lines in E_3. Inspired by [2] and [4] we introduced vector fields in E_3 induced by elements of A_6 and give their kinematic and dynamic interpretations. This work does not give quite new original results except the description of subalgebras of A_6 and of their kinematic interpretations. Perhaps it will be useful from the point of view of explanation which is close to the papers [5] and [3]. We prefer the algebra of vector couples to the dual number and dual quaternion technique because of the cleaner geometrical and mechanical interpretation.

2 PLÜCKER LINE COORDINATES, VECTOR COUPLES, SCREWS

Let V_3 be the vector space associated to the Euclidean space E_3. The scalar or vector or mixed product of vectors in V_3 will be denoted by $\overline{a}.\overline{b}$ or $\overline{a} \times \overline{b}$ or

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$(\overline{a} \times \overline{b}).\overline{c}$ respectively. Let $(0, \overline{e}_1, \overline{e}_2, \overline{e}_3)$ be a Cartesian coordinate system in E_3. Let $p = AB$ be a line determined by its two different points A, B. The couple of vectors $\overline{s} = \overline{AB}, \overline{m} = \overline{OA} \times \overline{s} = \overline{OA} \times \overline{OB}$ is called Plücker line coordinates. In cartesian coordinates, $\overline{s} = (s_1 = b_1 - a_1, s_2 = b_2 - a_2, s_3 = b_3 - a_3)$, $\overline{m} = (m_1 = a_2b_3 - b_2a_3, m_2 = a_3b_1 - a_1b_3, m_3 = a_1b_2 - b_1a_2)$. Plücker line coordinates will be called canonical if $\|\overline{s}\| = \sqrt{\overline{s}.\overline{s}} = 1$. Let us note that Plücker coordinates $(\overline{s}, \overline{m})$ satisfy the equality $\overline{s}.\overline{m} = 0$.

Remark 1. Let (x_0, x_1, x_2, x_3) be homogeneous coordinates in E_3, where the equality $x_0 = 0$ means improper points (points in infinity). Let $[A, B]^T$ denote the matrix $\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}$. Then $p_{ik} = a_ib_k - a_kb_i$ $i, k = 0, 1, 2, 3$ are the homogeneous Plücker coordinates of a line $p = AB$. They satisfy the equality $\det[A, B; A, B]^T = p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$ which corresponds with the condition $\overline{s}.\overline{m} = 0$ in Plücker coordinates $(\overline{s}, \overline{m})$.

Plücker coordinates of a line p is a couple of two vectors $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$, $\overline{s}.\overline{m} = 0$. It is easy to see that the point $C, \overline{OC} = (\overline{s} \times \overline{m})/\overline{s}^2$, is the orthogonal projection of the coordinate origin 0 into p. If we change determining points of a line p then we get a couple $(k\overline{s}, k\overline{m})$. Changing the origin 0 we obtain a couple $(\overline{s}, \overline{m}' = \overline{m} + \overline{0'0} \times \overline{s})$. It means that the vector $\overline{m}$ of the Plücker coordinates $(\overline{s}, \overline{m})$ depends on the origin 0 but the scalar product $\overline{s}.\overline{m}$ does not depend on 0.

Vice versa, an ordered couple of vectors $(\overline{s}, \overline{m})$, $0 \neq \overline{s}$, $\overline{m} \in V_3$, determines the line p in the direction $\overline{s}$ and passing through the point $C, \overline{OC} = \overline{s} \times \overline{m}/\overline{s}^2$. This line will be called the line of the couple $(\overline{s}, \overline{m})$. The line of a couple $(0, \overline{m})$ is the unproper line p of all parallel plains the normal vector of which is $\overline{m}$. There is not any line of the couple $(0, 0)$. We use $p = \pi((\overline{s}, \overline{m}))$ for $(\overline{s}, \overline{m}) \neq (\overline{0}, \overline{0})$.

Let us remind that the set of all ordered couples $(\overline{s}, \overline{m}) \in V_3 \times V_3$ has a real vector space structure where

$$k_1(\overline{s}_1, \overline{m}_1) + k_2(\overline{s}_2, \overline{m}_2) = (k_1\overline{s}_1 + k_2\overline{s}_2, k_1\overline{m}_1 + k_2\overline{m}_2)$$

Lema 1. *Let p be the line of couple $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$. Then every couple $(\overline{s}', \overline{m}')$ with the line p is of the form $\overline{s}' = k\overline{s}$, $\overline{m}' = k\overline{m} + u\overline{s}$, $0 \neq k$, $u \in R$.*

Proof. The line p is the line of a couple $(\overline{s}', \overline{m}')$ if and only if $\overline{s}' = k\overline{s}$ and $\overline{OC}' = \overline{OC}$. Comparing $\overline{OC}' = \frac{\overline{s}' \times \overline{m}'}{(\overline{s}')^2} = \frac{k\overline{s} \times \overline{m}'}{k^2\overline{s}^2} = \frac{\overline{s} \times \overline{m}'}{k\overline{s}^2}$ with $\overline{OC} = \frac{\overline{s} \times \overline{m}}{s^2}$ we get $\overline{m}' = k\overline{m} + u\overline{s}$. □

The set β_p of all couples $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$ with the same proper line p is two-parametric. If $X_i = (\overline{s}_i, \overline{m}_i) = (k_i\overline{s}, k_i\overline{m} + u_i\overline{s}) \in \beta_p$, $i = 1, 2$, then for $k \neq 0$ also $kX_1 \in \beta_p$ and for $k_1 + k_2 \neq 0$ also $X_1 + X_2 \in \beta_p$. Denote $R(V_3 \times V_3) := \{(\overline{s}, \overline{m}) \in V_3 \times V_3; \overline{s}^2 \neq 0\}$. We say that $X_i = (\overline{s}_i, \overline{m}_i) \in R(V_3 \times V_3)$, $i = 1, 2$, are L-equivalent iff there are $0 \neq k$, $u \in R$ such that $\overline{s}_2 = k\overline{s}_1$, $\overline{m}_2 = k\overline{m}_1 + u\overline{s}_1$,i.e. iff there is a line p, that $X_1, X_2 \in \beta_p$. Denote β the space of all L-equivalence classes in $R(V_3 \times V_3)$. There is a one-to-one correspondence between the set of all proper lines in E_3 and the set β. Then β is a 4-dimensional manifold. Let $\pi_1 : R(V_3 \times V_3) \to \beta$ be the map where $\pi_1(\overline{s}, \overline{m})$ is the class of L-equivalent elements determined by $(\overline{s}, \overline{m})$. Certainly $\pi_1 : R(V_3 \times V_3) \to \beta$ is a fibre manifold, fibre $\pi_1^{-1}(p) = \beta_p$ of which have an almost vector space structure, i. e. under the conditions introduced above kX_1 and $X_1 + X_2$ belong to the same fibre as X_1 and X_2.

An ordered couple $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$ is called the Plücker's couple if $\overline{s}.\overline{m} = 0$. It is said to be canonical if also $\overline{s}^2 = 1$.

Lema 2. *Let $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$ be a Plücker's couple. Let p be the line of $(\overline{s}, \overline{m})$. Then $(\overline{s}, \overline{m})$ is the Plücker's coordinate of p.*

Proof. Let us remind two well known equalities

$$\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a}.\overline{c})\overline{b} - (\overline{a}.\overline{b})\overline{c} \tag{1}$$

$$(\overline{a} \times \overline{b}).(\overline{c} \times \overline{d}) = (\overline{a}.\overline{c})(\overline{b}.\overline{d}) - (\overline{b}.\overline{c})(\overline{a}.\overline{d}) \tag{2}$$

Using the equality (1) and $\overline{s}.\overline{m} = 0$ we get

$$\overline{OC} \times \overline{s} = \frac{\overline{s} \times \overline{m}}{\overline{s}^2} \times \overline{s} = \frac{\overline{s}^2\overline{m} - (\overline{s}.\overline{m})\overline{s}}{\overline{s}^2} = \overline{m}.$$

□

Definition 1. *Every 1-dimensional subspace in $V_3 \times V_3$ will be called a screw. Every couple $X = (\overline{s}, \overline{m}) \neq (0, 0)$ determines the screw $\langle X \rangle$ where $\langle M \rangle$ denotes the vector space spanned on a set $M \subset V_3 \times V_3$. The couple X is called a representative of the screw $\langle X \rangle$. A screw $\langle X \rangle$ is called proper or improper if $\overline{s} \neq \overline{0}$ or $\overline{s} = \overline{0}$ respectively.*

It is clear that if X is a representative of $\langle X \rangle$ then every representative of $\langle X \rangle$ is of the form kX, $k \neq 0$, and then all representatives have the same line of couple which will be called the line of $\langle X \rangle$.

It immediately follows from the definition of screws that the set P_5^σ of all screws is a projective 5-dimensional space. Let $\pi_2 : V_3 \times V_3 \to P_5^\sigma$ be a such map that $\pi_2(X) = \langle X \rangle$. It means that π_2 is a 1-dimensional vector fibration.

Let $X = (\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$, be a couple of vectors. Denote $h := (\overline{s}.\overline{m})/\overline{s}^2$. It is easy to prove the following property.

Lemma 3. *The number h does not depend on a choice of a representative of the screw $\langle X \rangle$.*

Definition 2. *The number $h = (\overline{s}.\overline{m})/\overline{s}^2$ will be called pitch of the screw $\langle X \rangle, X = (\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$. If $\overline{s} = \overline{0}$ we put $h = \infty$.*

Let us recall that $h\overline{s}$ is the orthogonal projection of $\overline{m}$ into $\overline{s}$.

Corollary of Lemma 2. *Two proper screws which have the same screw line are both of the form $\langle(\overline{s}, \overline{m})\rangle$ and $\langle(\overline{s}, \overline{m} + u\overline{s})\rangle$, $\overline{s} \neq \overline{0}$, $u \neq 0$. It means that the set of all screws with the same screw line form one-parametric family. If h is the pitch of the first screw then the pitch of the second one is $h + u$.*

Remark 2. It is conspicuous that a proper screw is determined by its line and by its pitch h. This property is often taken as the definition of screws, see for example [4], [5].

Remark 3. Let us emphasize that the sum of two screws has not any sense because sums of different representatives have not to belong to the same screw.

A proper screw $\langle(\overline{s}, \overline{m})\rangle$, $\overline{s} \neq \overline{0}$, is called the Plücker's screw if $h = \overline{s}.\overline{m} = 0$.

Lemma 4. *There is a unique Plücker's screw in the set of all proper screws with the same screw line.*

Proof. Let $\langle(\overline{s}, \overline{m})\rangle$, $\overline{s} \neq \overline{0}$ be a screw with the screw line p. Then every screw with the screw line p is of the form $\langle(\overline{s}, \overline{m} + u\overline{s})\rangle$. But this screw is the Plücker's one iff $\overline{s}.(\overline{m} + u\overline{s}) = 0$, i. e. iff $u = -(\overline{s}.\overline{m})/s^2 = -h$. It completes our proof.

It is clear that there is a one-to-one correspondence between the set P^σ of Plücker's screws and the space of all lines in E_3, i. e. between the spaces P^σ and β.

Remark 4. It is clear that the line p of a couple $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$ depends on the choice of origin O. If O' is another origin then the line p' of the couple $(\overline{s}, \overline{m})$ is the image of p in the translation determined by the vector $\overline{00'}$.

3. Lie Algebra Of Vector Couples

Vector space $V_3 \times V_3$ of all couples $(\overline{s}, \overline{m})$ is closely connected with geometry of lines in E_3. Remind that $\overline{s} \in V_3$ is the direction of the line p of a couple $(\overline{s}, \overline{m})$ and does not depend on coordinate systems. In contrary $\overline{b}$ depends on the choice of the origin 0, but $\overline{s}.\overline{m}$ is independent on 0.

So in the space $V_3 \times V_3$ there are natural scalar and vector bilinear forms which gives useful information about geometrical and physical objects connected with lines in E_3. Remind them.

a)Klein scalar bilinear form KL:
Let $X_i = (\overline{s}_i, \overline{m}_i) \in V_3 \times V_3$, $i = 1, 2$. Then

$$KL(X_1, X_2) = \overline{s}_1.\overline{m}_2 + \overline{s}_2.\overline{m}_1$$

It is a symmetric regular bilinear scalar form on $V_3 \times V_3$ of the signature $(+, +, +, -, -, -)$. Its quadratic form will be written in the form $KL(X) := \frac{1}{2}KL(X, X) = \overline{s}.\overline{m}$. Vectors $X_1, X_2 \in V_3 \times V_3$ will be called KL-orthogonal if $KL(X_1, X_2) = 0$.

A subspace $B \subset V_3 \times V_3$ is called KL-orthogonal to a subspace $A \subset V_3 \times V_3$ if $KL(X, Y) = 0$ for every $X \in A$ and every $Y \in B$. There is a unique subspace A^K which is totally KL-orthogonal to a subspace $A \subset V_3 \times V_3$, i. e. if any vector subspace B is KL-orthogonal to A then $B \subset A^k$.

From the definition of KL-orthogonality it follows

1) A couple $X = (\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$ is KL-orthogonal to itself if and only if is a Plücker's couple.

2) If couples X, Y are KL-orthogonal then kX, uY are also KL-orthogonal.

So we can introduce KL-orthogonality in the case of screws. We say that two screws $\langle X\rangle$, $\langle Y\rangle$ are KL-orthogonal if X, Y are KL-orthogonal. Then $\langle X\rangle$ is KL-orthogonal to itself iff is a Plücker's screw.

Lemma 5. *Let p_1, p_2 be two non-parallel lines in E_3. Then p_1 and p_2 are crossing if and only if their Plücker's screws are KL-orthogonal.*

Proof. Let $X_i = (\overline{s}_i, \overline{m}_i)$, $\overline{s}_i^2 = 1$, $\overline{s}_i.\overline{m}_i = 0$, $\overline{s}_2 \neq k\overline{s}_1$, $i = 1, 2$, be a representative of the Plücker's screw the line of which is p_i. The line p_i is passing cross the point C_i, $\overline{OC}_i = \overline{s}_i \times \overline{m}_i$. Then the lines p_1, p_2 are crossing if and only if $0 = \overline{C_1C_2}.(\overline{s}_1 \times \overline{s}_2) = (\overline{s}_2 \times \overline{m}_2 - \overline{s}_1 \times \overline{m}_1).(\overline{s}_1 \times \overline{s}_2) =$ (*use the equality* (2)) $= (\overline{s}_2.\overline{s}_1)(\overline{m}_2.\overline{s}_2) -$

$(\overline{m}_2.\overline{s}_1)\overline{s}_2^2 - \overline{s}_1^2(\overline{m}_1.\overline{s}_2) + (\overline{m}_1.\overline{s}_1)(\overline{s}_1.\overline{s}_2) = -(\overline{s}_1.\overline{m}_2 + \overline{s}_2.\overline{m}_1) = -KL(X_1, X_2)$. It completes our proof.

b) Killing scalar bilinear form K:
Let $X_i = (\overline{s}_i, \overline{m}_i) \in V_3 \times V_3$. Put

$$K(X_1, X_2) = \overline{s}_1.\overline{s}_2$$

It means that K is a symmetric singular bilinear form on $V_3 \times V_3$. Its corresponding quadratic form will be written in the form $K(X) := K(X, X) = \overline{s}^2$. Then the line of X is improper iff $K(X) = 0$.

c) Lie bracket - vector bilinear form on $V_3 \times V_3$:
The vector product $\overline{a} \times \overline{b}$ of $\overline{a}, \overline{b} \in V_3$ is an example of the Lie bracket of two vectors. It is a skew-symmetric vector bilinear form on V_3. The well known and useful Lie bracket in $V_3 \times V_3$ is defined as follows.

If $X_i = (\overline{s}_i, \overline{m}_i) \in V_3 \times V_3$, $i = 1, 2$, then we put

$$[X_1, X_2] := (\overline{s}_1 \times \overline{s}_2, \overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1)$$

It is easy to show that the Jacobian identity

$$[X_1, [X_2, X_3]] + [X_3, [X_1, X_2]] + [X_2, [X_3, X_1]] = 0$$

is satisfied. Thus the vector space $V_3 \times V_3$ becomes a Lie algebra.

The vector space $V_3 \times V_3$ endowed with the Klein form KL, Killing form K and by the Lie bracket will be rewritten by A_6 instead $V_3 \times V_3$. It is well known that this Lie algebra A_6 is isomorphic with the Lie algebra of the Lie group of all orientation preserving isometries in E_3.

The following properties immediately follow from the definition of Lie bracket.

(1) If the line p_2 of $X_2 = (\overline{s}_2, \overline{m}_2)$, $\overline{s}_2 \neq \overline{0}$, is parallel with the line p_1 of $X_1 = (\overline{s}_1, \overline{m}_1)$, $\overline{s}_1 \neq \overline{0}$, i. e. if $\overline{s}_2 = k\overline{s}_1$, $k \neq 0$, then $[X_1, X_2] = (0, \overline{s}_1 \times (k\overline{m}_2 - \overline{m}_1))$ and thus the line of $[X_1, X_2]$ is improper.

(2) If $X_3 \in \langle X_1 \rangle$, i. e. $X_3 = kX_1$ and $X_4 \in \langle X_2 \rangle$, $X_4 = uX_2$, then $[X_3, X_4] = ku[X_1, X_2]$. It means that $[X_3, X_4] \in \langle [X_1, X_2] \rangle$. Thus we get the map $P_5^\sigma \times P_5^\sigma \to P_5^\sigma$, $(\langle X_1 \rangle, \langle X_2 \rangle) \mapsto \langle [X_1, X_2] \rangle$. Let us recall the representation $ad : A_6 \to L(A_6)$ of the Lie algebra A_6 in the vector space $L(A_6)$ of all linear maps on A_6 defined by the rule $ad_X(Y) = [X, Y]$. So we have a representation ad^σ of A_6 in the set of maps on P_5^σ, $ad_X^\sigma(\langle Y \rangle) = \langle [X, Y] \rangle$.

(3) Quite analogously it is easy to see that the Lie bracket preserves the L-equivalence classes, i. e. if $X_i, Y_i \in \beta_{pi}$, $i = 1, 2$, and p is the line of $[X_1, X_2]$ then $[Y_1, Y_2] \in \beta_p$.

(4) By direct calculation we get $KL(X_1, [X_1, X_2]) = \overline{s}_1.(\overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1) + (\overline{s}_1 \times \overline{s}_2).\overline{m}_1 = 0$. Therefore the Lie bracket $[X_1, X_2]$ is KL-orthogonal to X_i, $i = 1, 2$ and thus also the screw $\langle [X_1, X_2] \rangle$ is KL-orthogonal to $\langle X_i \rangle$, $i = 1, 2$.

Lemma 5. *Let p_i be the line of $X_i = (\overline{s}_i, \overline{m}_i)$, $\overline{s}_i \neq \overline{0}$, $i = 1, 2$, $\overline{s}_2 \neq k\overline{s}_1$. Then the line p of $[X_1, X_2]$ is the axis of the lines p_1, p_2, i. e. p intersects p_1 and p_2 orthogonally.*

Scratch of proof. We can suppose that $\overline{s}_i^2 = 1$, $i = 1, 2$. Certainly p is orthogonal to p_i, $i = 1, 2$. The line p_i is passing through $C_i, \overline{OC}_i = \overline{s}_i \times \overline{m}_i$ and the line p goes through $C, \overline{OC} = (\overline{s}_1 \times \overline{s}_2) \times (\overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1)/(\overline{s}_1 \times \overline{s}_2)^2$. Using the equality (1), $\overline{s}_1.\overline{s}_2 = \cos\alpha$, $(\overline{s}_1 \times \overline{s}_2)^2 = \sin^2\alpha$ it is easy to see that $\overline{C_1C}.(\overline{s}_1 \times (\overline{s}_1 \times \overline{s}_2)) = 0$, i. e. that p and p_1 are crossing. Analogously p and p_2 are also crossing. □

Corollary 1. *Let L be the manifold of all proper lines in E_3. Then according to property 3 the Lie bracket in A_6 induces the map from $L \times L$ into L in which the image of two non-parallel lines p_1, p_2 is the line p which intersects p_1 and p_2 orthogonally.*

Corollary 2. *The line $\tilde{p}$ of a couple $k_1X_1 + k_2X_2$ orthogonally intersects the line p of $[X_1, X_2]$ because $[X_1, k_1X_1 + k_2X_2] = k_2[X_1, X_2]$ and thus (by Lemma 5) p orthogonally intersects $\tilde{p}$.*

Lemma 6. *Let $X_i = (\overline{s}_i, \overline{m}_i)$, $i = 1, 2$, $\overline{s}_1 \times \overline{s}_2 \neq \overline{0}$ be two Plücker's couples. Then $[X_1, X_2]$ is a Plücker's couple if either the lines p_1, p_2 of X_1, X_2 respectively are orthogonal or X_1, X_2 are KL-orthogonal.*

Proof. $[X_1, X_2] = (\overline{s}_1 \times \overline{s}_2, \overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1)$. Then $[X_1, X_2]$ is a Plücker's couple iff $0 = (\overline{s}_1 \times \overline{s}_2).(\overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1) = s_1^2(\overline{s}_2.\overline{m}_2) - (\overline{s}_2.\overline{s}_1)(\overline{s}_1.\overline{m}_2) - (\overline{s}_1.\overline{s}_2)(\overline{s}_2.\overline{m}_1) + \overline{s}_2^2(\overline{s}_1.\overline{m}_1) = -(\overline{s}_1.\overline{s}_2)(\overline{s}_1.\overline{m}_2 + \overline{s}_2.\overline{m}_1)$. □

Recall that the Lie algebra A_6 has two basic subalgebras:
$V_3^\rho = \{(\overline{s}, 0), \overline{s} \in V_3\}$, $V_3^\tau = \{(0, \overline{m}), \overline{m} \in V_3\}$, $A_6 = V_3^\rho \oplus V_3^\tau$.
The line of $(\overline{s}, 0)$ goes through origin 0 and the line of a couple $(0, \overline{m})$ is improper. If $X_1, X_2 \in V_3^\tau$ then $[X_1, X_2] = (0, 0)$. If $X_1 \dot{\in} V_3^\rho$, $X_2 \in V_3^\tau$ then $[X_1, X_2] \in V_3^\tau$. It means that V_3^ρ acts on V_3^τ by the Lie bracket; in detail, $ad_{X_1}(X_2) = [X_1, X_2] \in V_3^\tau$.

In the next part of this chapter we will try to describe all subalgebras in A_6, i. e. all vector subspaces A in A_6 for which $[A, A] \subset A$.

1. Every 1-dimensional subspace $A_1 \subset A_6$ is a subalgebra because $[X_1, X_2] = 0$ for $X_2 = kX_1$.
2. Let $A_2 \subset A_6$ be a 2-dimensional subspace. Let $X_i = (\overline{s}_i, \overline{m}_i)$, $i = 1, 2$, is a base in A_2. Then $[X_1, X_2] = (\overline{s}_1 \times \overline{s}_2, \overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1)$ belongs to A_2 if and only if $\overline{s}_1 \times \overline{s}_2 = k_1\overline{s}_1 + k_2\overline{s}_2$, $\overline{s}_1 \times \overline{m}_2 - \overline{s}_2 \times \overline{m}_1 = k_1\overline{m}_1 + k_2\overline{m}_2$. The former equality is satisfied iff $\overline{s}_1 \times \overline{s}_2 = 0$, i. e. iff $\overline{s}_2 = k\overline{s}_1$. There are two cases:
   a) if $\overline{s}_1 = \overline{0}$, then $\overline{s}_2 = \overline{0}$, i. e. $A_2 \subset V_3^\tau$.
   b) Let $\overline{s}_1 \neq \overline{0}$. As $k_1 = 0 = k_2$ then $\overline{s}_1 \times (\overline{m}_2 - k\overline{m}_1) = \overline{0}$, i. e. $\overline{m}_2 = k\overline{m}_1 + u\overline{s}_1$, i. e. there is a proper line p in E_3 such that $A_2 = \langle \beta_p \rangle$ is the vector space spanned on β_p. We get

Lemma 7. *A two-dimensional subspace $A_2 \subset A_6$ is a subalgebra if and only if either $A_2 \subset V_3^\tau$ or if $A_2 = \langle \beta_p \rangle$ for a proper line p.*

Remark 5. If A_2 is not a subalgebra, (i. e. if X_1, X_2 is a base in A_2 and $[X_1, X_2] \notin A_2$), then the proper line of all couples $X \in A_2$ form two-parametric family $\pi(A_2)$ of lines which orthogonally intersect the line of $[X_1, X_2]$. This line can be called axis of A_2. Recall that in differential geometry of lines a two-parametric family of lines is called a congruence of lines.

3. In this part we will investigate two problems: Under what conditions a 3-dimensional subspace $A_3 \subset A_6$ is a subalgebra and under what conditions A_3 is totaly KL-orthogonal to itself, i. e. $A_3 = A_3^K$.

Let $p_i : A_6 = V_3 \times V_3 \to V_3$ be the projection on the i-th factor, $i = 1, 2$, i. e. $p_1(\overline{s}, \overline{m}) = \overline{s}$, $p_2(\overline{s}, \overline{m}) = \overline{m}$.

There are cases for A_3:

a) $p_1(A_3) = \overline{0} \in V_3$. Then $A_3 = V_3^{\tau}$ is a subalgebra. As $KL(V_3^{\tau}, V_3^{\tau}) = 0$ then $A_3 = A_3^K$, i. e. A_3 is totaly KL-orthogonal to itself.

b) dim $p_1(A_3) = 1$, dim $p_2(A_3) = 3$. Always we can choose a base $X_1 = (\overline{s}_1, \overline{m}_1)$, $X_2 = (\overline{0}, \overline{m}_2)$, $X_3 = (\overline{0}, \overline{m}_3)$ in A_3, where $V_3 = \langle \overline{m}_1, \overline{m}_2, \overline{m}_3 \rangle$, $\overline{s}_1^2 \neq 0$. Then $[X_1, X_2] = (\overline{0}, \overline{s}_1 \times \overline{m}_2)$, $[X_1, X_3] = (\overline{0}, \overline{s}_1 \times \overline{m}_3)$, $[X_2, X_3] = \overline{0}$, $KL(X_1) = \overline{s}_1.\overline{m}_1$, $KL(X_3) = 0$, $KL(X_1, X_2) = \overline{s}_1.\overline{m}_2$, $KL(X_1, X_3) = \overline{s}_1.\overline{m}_3$, $KL(X_2, X_3) = 0$.

It gives

Lemma 8. *A 3-dimensional subspace A_3, $dimp_1 A_3 = 1$, $dimp_2 A_3 = 3$ is a subalgebra if $p_1(A_3)$ is orthogonal to $p_2(A_3 \cap V_3^{\tau})$ in V_3. The equality $A_3^K = A_3$ cannot be satisfied.*

c) $dimp_1(A_3) = 1$, $dimp_2(A_3) = 2$. There is in A_3 a base $X_1(\overline{s}_1, \overline{0})$, $X_2 = (\overline{0}, \overline{m}_2)$, $X_3 = (\overline{0}, \overline{m}_3)$. Then for X_i, X_j and $KL(X_i, X_j)$ we obtain the same equalities as in b) except $KL(X_1) = 0$.

So we have

Lemma 9. *A 3-dimensional subspace A_3, $dimp_1(A_3) = 1$, $dimp_2(A_3) = 2$, is a subalgebra iff is KL-orthogonal to itself, i. e. iff $p_1(A_3)$ is orthogonal to $p_2(A_3)$ in V_3.*

d) $dimp_1(A_3) = 2$, $dimp_2(A_3) \geq 1$. Always we can choose a base $X_1 = (\overline{s}_1, \overline{m}_1)$, $X_2 = (\overline{s}_2, \overline{m}_2)$, $X_3 = (\overline{0}, \overline{m}_3)$, where $\overline{s}_1$ $\overline{s}_2$ are independent. Then $[X_1, X_2] = (\overline{s}_1 \times \overline{s}_2, \cdot)$. If A_3 is a subalgebra then $\overline{s}_1 \times \overline{s}_2 = k_1\overline{s}_1 + k_2\overline{s}_2$. It is impossible. We get $KL(X_1) = \overline{s}_1.\overline{m}_1$, $KL(\overline{s}_2) = \overline{s}_2.\overline{m}_2$, $KL(X_3) = 0$, $KL(X_1, X_2) = \overline{s}_1.\overline{m}_2 + \overline{m}_1.\overline{s}_2$, $KL(X_1, X_3) = \overline{s}_1.\overline{m}_3$, $KL(X_2, X_3) = \overline{s}_2.\overline{m}_3$. If $dimp_2(A_3) = 1$ then we can choose $\overline{m}_1 = \overline{0} = \overline{m}_2$. Then $A_3^K = A_3$ iff $\overline{m}_3 = k\overline{s}_1 \times \overline{s}_2$. If $dimp_2(A_3) = 2$ we can put $\overline{m}_1 = \overline{0}$. Then $A_3^K = A_3$ iff $\overline{m}_2 = k_2\overline{s}_1 \times \overline{s}_2$, $\overline{m}_3 = k_3\overline{s}_1 \times \overline{s}_2$. It is impossible. If $dimp_2(A_2) = 3$ then $\overline{m}_1$, $\overline{m}_2$, $\overline{m}_3$ we can chose as an orthonormal base in V_3. Then $A_3^k = A_3$ iff $\overline{s}_1 = \overline{m}_1 \times \overline{m}_3$, $\overline{s}_2 = \overline{m}_2 \times \overline{m}_3$, $\overline{m}_3 = \overline{s}_1 \times \overline{s}_2$, i. e. iff $\overline{s}_1 = -\overline{m}_2$, $\overline{s}_2 = \overline{m}_1$. We get

Lemma 10. *A 3-dimensional subspace A_3, $dimp_1(A_3) = 2$, $dimp_2(A_3) \geq 1$, is not a subalgebra. If $dimp_2(A_3) = 1$ then $A_3^K = A_3$ iff $p_2(A_3)$ is orthogonal to $p_1(A_3)$ in V_3. If $dimp_2(A_3) = 2$ then $A_3^K \neq A_3$. If $dimp_2(A_3) = 3$ then $A_3^K = A_3$ iff $A_3 = \langle(-\overline{m}_2, \overline{m}_1), (\overline{m}_1, \overline{m}_2), (\overline{0}, \overline{m}_3)\rangle$ where $\overline{m}_1$, $\overline{m}_2$, $\overline{m}_3$ is an orthonormal base in V_3. To every 2-dimensional subspace $V_2 = \langle \overline{s}_1, \overline{s}_2 \rangle \subset V_3$ there is a unique $A_3 = \langle(\overline{s}_1, \overline{s}_2), (\overline{s}_2, -\overline{s}_1), (0, \overline{s}_1 \times \overline{s}_2)\rangle$ such that $A_3^K = A_3$. A_3 does not depend on choice of orthonormal base $\overline{s}_1$, $\overline{s}_2$.*

e) $\dim p_1(A_3) = 3$, $\dim p_2(A_3) = 1$. Choosing a base $X_1 = (\overline{s}_1, \overline{0})$, $X_2 = (\overline{s}_2, 0)$, $X_3 = (\overline{s}_3, \overline{m}_3)$ it is easy to show that A_3 is not subalgebra and $A_3^K \neq A_3$.

f) $\dim(A_3) = 3$, $\dim p_2(A_3) \geq 2$. We can chose a base $X_1 = (\overline{s}_1, \overline{m}_1), X_2 = (\overline{s}_2, \overline{m}_2)$, $X_3 = (\overline{s}_3, \overline{m}_3)$ where $\overline{s}_1$, $\overline{s}_2$, $\overline{s}_3$ are orthonormal: $\overline{s}_1 \times \overline{s}_2 = \overline{s}_3$,

$\overline{s}_1 \times \overline{s}_3 = -\overline{s}_2$, $\overline{s}_2 \times \overline{s}_3 = \overline{s}_1$. Let $\overline{m_i} = \sum_{j=1}^{3} m_i^j \overline{s}_j$, $i = 1,2,3$. Calculating $[X_i, X_k]$ we obtain. A subspace A_3 is a subalgebra if and only if

$$(3) \qquad m_1^1 = m_2^2 = m_3^3 = 0, m_2^3 + m_3^2 = 0, m_1^3 + m_3^1 = 0, m_1^2 + m_2^1 = 0$$

As $KL(X_i) = m_i^i$, $i = 1,2,3$, $KL(X_1, X_2) = m_2^1 + m_1^2$, $KL(X_1, X_3) = m_3^1 + m_1^3$, $KL(X_2, X_3) = m_3^2 + m_2^3$ therefore the equality $A_3^K = A_3$ is satisfied iff (3) is true.
The equalities (3) give: $\overline{m}_1 = -m_1^2\overline{s}_2 + m_1^3\overline{s}_3$, $\overline{m}_2 = m_1^2\overline{s}_1 - m_2^3\overline{s_3}$, $\overline{m}_3 = -m_1^3\overline{s}_1 + m_2^3\overline{s}_2$.
Put $\overline{m} := m_2^3\overline{s}_1 + m_1^3\overline{s}_2 + m_1^2\overline{s}_3$. Then $\overline{m}_1 = \overline{s}_1 \times \overline{m}$, $\overline{m}_2 = \overline{s}_2 \times \overline{m}$, $\overline{m}_3 = \overline{s}_3 \times \overline{m}$.
The rank r of the system $(\overline{m}_1, \overline{m}_2, \overline{m}_3)$ is 2. We have proved.

Lemma 11.. *A vector subspace A_3, $\dim p_1(A_3) = 3$, $\dim p_2(A_3) = 2$ is a subalgebra if a one of the following equivalent conditions is satisfied:*

1. $A_3^K = A_3$
2. *A_3 is the subspace of couples $(\overline{s}, \overline{s} \times \overline{m})$, $\overline{s} \in V_3$ and $\overline{m} \neq \overline{0}$ is a given vector.*

If $\dim p_1(A_3) = 3$, $\dim p_2(A_3) = 3$ then A_3 is not subalgebra and $A_3^K \neq A_3$.
4. Let A_4 be a 4-dimensional vector subspace in A_6. There are cases:

a) $\dim p_1 A_4 = 1$, $\dim p_2 A_4 = 3$. Choosing a base $X_1 = (\overline{s}_1, \overline{0})$, $X_i = (\overline{0}, \overline{m}_i)$, $i = 2,3,4$ and calculating $[X_i, X_j]$ we get

Lemma 12. *A 4-dimensional vector subspace A_4, $\dim p_1(A_4) = 1$, $\dim p_2 A_4 = 3$ is always a subalgebra.*

Let us remark that $A_4 \cap V_3^\tau = V_3^\tau$ in this case.
b)$\dim p_1 A_4 \geq 2$, $\dim p_2 A_4 \geq 2$. Always we can choose a suitable base and show that A_4 cannot be a subalgebra.
5. In the case when A_5 is a vector subspace always there are bases by which can be shown that A_5 cannot be a subalgebra.
Let us introduce survey of all subalgebras in A_6:

1. All 1-dimensional vector subspaces have the subalgebra structure.
2. A 2-dimensional vector subspace A_2 is a subalgebra if either $A_2 = \langle \beta_p \rangle$ for some line p or $A_2 \subset V_3^\tau$.
3. A 3-dimensional vector subspace A_3 is a subalgebra in the cases
   a) $A_3 = V_3^\rho$, $A_3 = V_3^\tau$
   b) $\dim p_1(A_3) = 1$, $\dim p_2(A_3) = 3$ and $p_1(A_3)$ is orthogonal to $p_2(A_3 \cap V_3^\tau)$ in V_3
   c) $\dim p_1(A_3) = 1$, $\dim p_2(A_3) = 2$ and $p_1(A_3)$ is orthogonal to $p_2(A_3)$ in V_3
   d) $A_3 = \{(\overline{s}, \overline{s} \times \overline{m}) \in A_6, \overline{s} \in V_3, \overline{m} \neq \overline{0}$ *is a given vector*$\}$
4. A 4-dimensional vector subspace A_4 is a subalgebra iff $\dim p_1 A_4 = 1$, $\dim p_2 A_4 = 3$.

Remark 6. Let $\pi(A)_f$ denote the set of all proper lines of couples $(\overline{s}, \overline{m}) \in A$, $(\overline{s}, \overline{m})$, $\overline{s} \neq \overline{0}$. It is easy to see that in the cases 3b, 3c, 4 $\pi(A)_f$ is a set of all lines parallel with the direction $p_1(A)$. A line of $\pi(A_3)_f$ from the case 3d goes through the point $C, \overline{OC} = \frac{\overline{s} \times (\overline{s} \times \overline{m})}{s^2} = \frac{(\overline{s}.\overline{m})}{s^2}\overline{s} - \overline{m}$. Therefore $\pi(A_3)_f$ is the set of lines p

going through points C on the sphere S^2 with the center $M, \overline{OM} = -\frac{1}{2}\overline{m}$ and with radius $r = \frac{1}{2}\sqrt{\overline{m}^2}$. If $\overline{s}.\overline{m} \neq 0$, i. e. $\overline{OC} \neq -\overline{m}$ then p is in the direction $\overline{s}$ and so it is unique. If $\overline{s}.\overline{m} = 0$, i. e. $\overline{OC} = -\overline{m}$ then every line p going through C and orthogonal to $\overline{m}$ belongs to $\pi(A_3)_f$. So $\pi(A_3)_f$ is a two-parametric family of lines in E_3, i. e. it is a congruence of lines. Recall that in the case of a general vector subspace A_3, $\pi(A_3)_f$ is a 3-parametric family of lines in E_3 that is called a complex of lines.

4. Canonical Vector Fields In E_3 Induced By A_6

Recall that a vector field on a differentiable manifold M is a rule ξ by which a tangent vector $\xi(x)$ at $x \in M$ is determined for every $x \in M$. In the case of $M = E_3$ $\xi(x) \in V_3$.

Definition 3. *Let $X = (\overline{s}, \overline{m}) \in A_6$ and 0 be a given point in E_3. This couple X and 0 determine a vector field $\xi_{(X,0)}$ by the following rule:*

a) *If $\overline{s} = \overline{0}$ then $\xi_{(X,0)}(Y) = \overline{m}$ for any $Y \in E_3$,*
b) *Let $\overline{s} \neq \overline{0}$. Let $h\overline{s}$ be the orthogonal projection $\overline{m}$ into $\overline{s}$, i. e. $h = (\overline{s}.\overline{m})/\overline{s}^2$. Let $\overline{OC} = (\overline{s} \times \overline{m})/\overline{s}^2$. Then*

$$\xi_{(X,0)}(Y) = \overline{s} \times \overline{CY} + h\overline{s}, Y \in E_3. \tag{4}$$

This vector field will be called the field of X.

Lemma 13. *The value of the field of X at 0 is $\overline{m}$, $\xi_{(X,0)}(0) = \overline{m}$.*

Proof. If $X = (0, \overline{m})$, i. e. $\overline{s} = 0$, then assertion is true. If $\overline{s} \neq \overline{0}$ then using (1) we get

$$\xi_{(X,0)}(0) = \overline{s} \times \overline{CO} + h\overline{s} = -\overline{s} \times (\overline{s} \times \overline{m})/\overline{s}^2 + h\overline{s} = -[(\overline{s}.\overline{m})\overline{s} - \overline{s}^2\overline{m}]/\overline{s}^2 + h\overline{s} = \overline{m}.$$

□

Corollary 3. *For the value of the vector field $\xi_{(X,0)}$ at $Y \in E_3$ we get $\xi_{(X,0)}(Y) = \overline{s} \times \overline{CY} + h\overline{s} = \overline{s} \times (\overline{C0} + \overline{0Y}) + h\overline{s} = \overline{s} \times \overline{C0} + \overline{s} \times \overline{0Y} + h\overline{s}$, i. e.*

$$\xi_{(X,0)}(Y) = \overline{s} \times \overline{0Y} + \overline{m} \tag{5}$$

It immediately gives:

a)

$$\xi_{(kX,0)}(Y) = k\xi_{(X,0)}(Y) \tag{6}$$

b) If two couples $X_i = (\overline{s}_i, \overline{m}_i) = 1, 2$, have the same line of couple, i. e. if $\overline{s}_2 = k\overline{s}_1$, $\overline{m}_2 = k\overline{m}_1 + u\overline{s}_1$ then

$$\xi_{(X_2,0)} = k\xi_{(X_1,0)} + u\overline{s}_1$$

Remark 7. If we change origin, if we choose $0'$ instead of 0 then from (4) or from (5) we get

$\xi_{(X,0')}(Y) = \overline{s} \times \overline{C'Y} + hs = \overline{s} \times (\overline{C'C} + \overline{CY}) + hs = \xi_{(X,0)}(Y) + \overline{s} \times \overline{C'C}$ or
$\xi_{(X,0')}(Y) = \overline{s} \times \overline{0'Y} + \overline{m} = \overline{s} \times (\overline{0'0} + 0Y) + \overline{m} = \xi_{(X,0)}(Y) + \overline{s} \times \overline{0'0}$ respectively.

Let CE_3 denote a set of all vector fields on E_3. It is a real vector space.
Let $\xi : V_6 \to CE_3$ be a map defined by the rule $\xi(X) = \xi_{(X,0)}$.

Proposition 1. *The map $\xi : A_6 \to \xi(A_6) \subset CE_3$ is an isomorphism of vector spaces.*

Proof. By the equality (6) $\xi(kX) = k\xi(X)$.

Let $X_i = (\overline{s}_i, \overline{m}_i)$, $i = 1, 2$. According to the definition of ξ we consider the following cases:

a) $\overline{s}_1 = \overline{0} = \overline{s}_2 : \xi(X_1 + X_2)(Y) = \overline{m}_1 + \overline{m}_2 = \xi(X_1)(Y) + \xi(X_2)(Y)$.

b) $\overline{s}_1 = \overline{0}, \overline{s}_2 \neq \overline{0} : \xi(X_1 + X_2)(Y) = \overline{s}_2 \times \overline{0Y} + \overline{m}_1 + \overline{m}_2 = \xi(X_1)(Y) + \xi(X_2)(Y)$.

c) $\overline{s}_1 \neq \overline{0}$, $\overline{s}_2 = \overline{0}$: analogously $\xi(X_1 + X_2)(Y) = \xi(X_1)(Y) + \xi(X_2)(Y))$.

d) $\overline{s}_1 \neq \overline{0}, \overline{s}_2 \neq \overline{0} : \xi(X_1 + X_2)(Y) = (\overline{s}_1 + \overline{s}_2) \times \overline{0Y} + (\overline{m}_1 + \overline{m}_2) = \xi(X_1)(Y) + \xi(X_2)(Y)$.

We have proved that ξ is a linear map. We will show that $\ker \xi = \overline{0}$. Let $\xi(X) = \overline{0} \in CE_3$. If $\overline{s} = \overline{0}$ then $\overline{0} = \xi(X)(Y) = \overline{m}$, i. e. $\xi = (\overline{0}, \overline{0})$. If $\overline{s} \neq \overline{0}$ then $\overline{0} = \xi(X)(Y) = \overline{s} \times \overline{0Y} + \overline{m}$ for all $Y \in E_3$. It is possible only if $\overline{s} = \overline{0}$, $\overline{m} = \overline{0}$. It completes our proof. □

The vector subspace $\xi(A_6)$ will be denoted as $SCE_3 := \xi(A_6)$. On the vector space SCE_3 by the isomorphism ξ the following bilinear forms are induced:

a) Klein form $SKL(\xi(X_1), \xi(X_2)) = KL(X_1, X_2)$,

b) Killing form $SK(\xi(X_1), \xi(X_2)) = K(X_1, X_2)$,

c) Lie bracket $[\xi(X_1), \xi(X_2)] = \xi[X_1, X_2]$.

Let $X_i = (\overline{s}_i, \overline{m}_i)$, $i = 1, 2$, be two couples. Then the isomorphism ξ inspires the following shapes for the above introduced forms.

Proposition 2.

a)$SKL(\xi(X_1), \xi(X_2)) = \overline{s}_1.\xi(X_2) + \overline{s}_2.\xi(X_1)$

b)$SK(\xi(X_1), \xi(X_2)) = \overline{s}_1.\overline{s}_2$

c)$[\xi(X_1), \xi(X_2)] = \overline{s}_1 \times \xi(X_2) - \overline{s}_2 \times \xi(X_1)$

Proof. Using the equalities (1) and (6) we get successively

a) $\overline{s}_1.\xi(X_2) + \overline{s}_2.\xi(X_1) = \overline{s}_1.(\overline{s}_2 \times \overline{0Y} + \overline{m}_2) + \overline{s}_2.(\overline{s}_1 \times \overline{0Y} + \overline{m}_1) = \overline{s}_1.\overline{m}_2 + \overline{s}_2.\overline{m}_1 = KL(X_1, X_2)$.

b) $\overline{s}_1.\overline{s}_2 = K(X_1, X_2)$

c) $\overline{s}_1 \times \xi(X_2) - \overline{s}_2 \times \xi(X_1) = \overline{s}_1 \times (\overline{s}_2 \times \overline{OY} + \overline{m}_2) - \overline{s}_2 \times (\overline{s}_1 \times \overline{0Y} + \overline{m}_1) = (\overline{s}_1.\overline{0Y})\overline{s}_2 - (\overline{s}_1.\overline{s}_2)\overline{0Y} + \overline{s}_1 \times \overline{m}_2 - (\overline{s}_2.\overline{0Y})\overline{s}_1 + (\overline{s}_1.\overline{s}_2)\overline{0Y} - \overline{s}_2 \times \overline{m}_1 = \xi([X_1, X_2])$. □

Remark about trajectories of the vector field $\xi(X)$. Let us remind that a trajectory of a vector field is a curve the tangent vectors of which are values of the vector field in points of this curve. So if $Y = Y(t)$ is the equation of a trajectory of the field $\xi(X)$ then

$$\dot{Y} = \xi_{(X,0)}(Y(t))$$

Using the equality (5) we can this equation to rewrite as follows

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}.$$

It is a system of differential equations the solution of which are trajectories of the vector field $\xi(X)$.

5. Kinematic Interpretation Of The Field $\xi(X)$ Of A Couple X

In this chapter we use a notation $X = (\overline{w}, \overline{b})$ instead $(\overline{s}, \overline{m})$. We will distinguish two cases.

a)If $\overline{w} = \overline{0}$ then the value of the vector field $\xi(X)$ is $\overline{b}$ at any $Y \in E_3$. We can interpret this values as the instants velocities of equable straightforward motions (translation motions). The trajectories of this motions are lines in the direction $\overline{b}$.

2. Let $\overline{w} \neq \overline{0}$. Recall that the line p of the couple $(\overline{w}, \overline{b}) \in A_6$ is going through the point $C, \overline{0C} = \overline{w} \times \overline{b}/\overline{w}^2$ in the direction $\overline{w}$. Let us consider the equable screw motion in E_3 which is composition of two motions: the first part is the rotation around the axis p with the constant angle velocity $\overline{w}$ and the second one is the translation motion in E_3 in the direction $\overline{w}$ with the constant velocity $h\overline{w}, h = (\overline{w}.\overline{b})/\overline{w}^2$. (We will say that the line p is the axis of this equable screw motion). The velocity $\overline{v}$ of this motion at a point Y satisfies the equality

$$\overline{v} = \overline{w} \times \overline{CY} + h\overline{w}.$$

According to (5) $\overline{v}$ is the value of the vector field $\xi(X)$ at $Y \in E_3$. We have proved.

Theorem. *Let $X = (\overline{w}, \overline{b}) \in A_6$. Then the vector field $\xi(X)$ is the velocity field of the following motions:*
If $\overline{w} = \overline{0}$ then it is a translation motion with the velocity $\overline{b}$.
If $\overline{w} \neq \overline{0}$ then this motion is the equable screw motion around the line p of the couple X with constant angle velocity $\overline{w}$ and with translation constant velocity $h\overline{w}, h = (\overline{w}.\overline{b})/\overline{w}^2$.

Remark 8. If $\overline{w}.\overline{b} = 0$, $\overline{w} \neq \overline{0}$, i. e. if $KL(X) = 0$, $K(X) \neq 0$, i. e. if $X = (\overline{w}, \overline{b})$ is a Plücker's couple then $\xi(X)$ is a field of velocities of the clean rotation around the line of X with constant angle velocity $\overline{w}$. Couples X belonging to the same Plücker's screw $\langle X \rangle$ determine rotations around the line of $\langle X \rangle$ with different angle velocities. When $KL(X) \neq 0$, $K(X) \neq 0$ then the trajectories of the field $\xi(X)$ are screw curves the axis of which is the line of X. The motions determined by the couples of a screw $\langle X \rangle$, $KL(X) \neq 0$, $K(X) \neq 0$, are equable screw motions around the line of $\langle X \rangle$ with the same pitch h. In general two couples $X_1, X_2, K(X_2) \neq 0$, with the same line p of couple, i. e. $X_i \in \beta_p$, determined equable screw motions around p with different angular velocities and pitches.

Remark 9 (about pitch h). By definition $h = (\overline{w}.\overline{b})/\overline{w}^2$ and then $v = |h|\|\overline{w}\|$, $\|\overline{w}\|^2 = \overline{w}.\overline{w}$, is the translation velocity of the motion determined by $X = (\overline{w}, \overline{b})$, $\overline{w} \neq \overline{0}$. Then $|h| = \frac{v}{w}$, $w = \|\overline{w}\|$. So $|h|$ is the translation length according to revolution with angle of radian around the line p of the couple X. It will be called specific lift. If $h > 0$ then we say that both motion parts, rotation and translation, have positive orientation, (If the rotation is in the direction of fingers of the right hand then the translation is in the direction of the thumb.), and in the opposite case $h < 0$ we say about negative orientation.

Remark 10 (about influence of a choice of a origin 0). If we use a point 0' instead 0 then the line of a couple $X = (\overline{w}, \overline{b})$ is the line $p = \tau(p)$ where τ is the translation determined by the vector $\overline{00'}$. Now $C' = \tau(C)$, $\overline{O'C'} = (\overline{w} \times \overline{b})/\overline{w}^2 = \overline{0C}$. The velocity of the equable screw motion around p' with angular velocity $\overline{w}$ and translation velocity $\overline{w}$ in a point Y is

$$\xi_{(X,0')}(Y) = \overline{w} \times C'Y + h\overline{w} = \overline{w} \times (\overline{C'C} + \overline{C}Y) + h\overline{w} = \xi_{(X,0)} + \overline{w} \times \overline{C'C}.$$

Remark 11 (about subgroups of motions induced by couples of the Lie subalgebras of A_6). It is well known that to every Lie algebra A there is a Lie group $G(A)$ the Lie algebra of which is just A. By our investigations in the 3-th chapter there are 9 types of subalgebras.

a) $A_1 = \langle X \rangle$, $X = (\overline{s}, \overline{m}) \neq (\overline{0}, \overline{0})$. If $\overline{s} = 0$ then the corresponding group $G(A_1)$ is the group of all translations with constant velocities $k\overline{m}$. If $\overline{s} \neq \overline{0}$ then $G(A_1)$ is the group of all equable screw motions around the line of X with the same pitch h.

b_1) $A_2 = \langle \beta_p \rangle$ for a line p, i. e. $A_2 = \{(k\overline{w}, k\overline{b} + u\overline{w}), \overline{w} \neq 0, k, u \in R\}$. The group $G(A_2)$ induced by A_2 is the group of all equable screw motions around p including all translations in the direction of p and rotations around p.

b_2) If $A_2 \subset V_3^\tau$, $A_2 = \langle (0, \overline{m}_1), (0, \overline{m}_2) \rangle$ then corresponding group is the group of all translations with the velocities $\overline{v} \in A_2$.

c_1) If $A_3 = V_3^\tau$ or $A_3 = V_3^\rho$ then the corresponding group is the group of all translations in E_3 or of all rotations about origin 0.

c_2) $A_3 \subset A_6$ with properties: $\dim(p_1 A_3) = 1$, $\dim(p_2 A_3) = 3$, $p_1(A_3)$ is orthogonal to $p_2(A_3 \cap V_3^\tau)$. Then $(\overline{w}, \overline{0}) \notin A_3$ and there is $(\overline{w}, \overline{b}) \in A_3$, $\overline{w} \neq \overline{0} \neq \overline{b}$. Let p be the line of $(\overline{w}, \overline{b})$. Then $G(A_3)$ is generated by all equable screw motions around lines parallel with p except the one going through origin 0 and by all translations with velocities $\overline{v}$ orthogonal to p.

c_3) $A_3 \subset A_6$ with properties: $\dim(p_1 A_3) = 1$, $\dim(p_2 A_3) = 2$, $p_1(A_3)$ is orthogonal to $p_2(A_3)$ in V_3. Then $(\overline{w}, \overline{0}) \in A_3$ and $G(A_3)$ is generated as in the case c_2 including equable screw motions around the line going through origin 0 in the direction $\overline{w}$.

c_4) $A_3 = \{(\overline{w}, \overline{w} \times \overline{m}) \in A_6, \overline{w} \in V_3, \overline{m} \neq \overline{0}$ *is a given vector*$\}$. Then $G(A_3)$ is generated by all equable rotations around the lines going through points $C, \overline{0C} \neq -\overline{m}$, of the sphere S_2 (describing in the Remark in the end of the 3-d chapter) and around all lines orthogonal to $\overline{m}$ going through $C, \overline{0C} = -\overline{m}$.

d) $A_4 \subset A_6$ with properties: $\dim(p_1 A_4) = 1$, $\dim(p_2 A_4) = 3$. Then $V_3^\tau \subset A_4$ and $G(A_4)$ is generated as in the case C_3 including all translations in E_3.

5. Dynamic Interpretation Of A Vector Fields $\xi(X)$

Firstly we recall effects of a force on a rigid body. Let a force $\overline{f}$ affects on a rigid body Ω at a point $C \in \Omega$. The line $p = (C, \overline{f})$ going through C in the direction $\overline{f}$ is called the line of $\overline{f}$. The result of effect of $\overline{f}$ at a point $Y \in \Omega$ does not depend on a choice of a point C on the line p of $\overline{f}$. A measure of this effect is moment of the force $\overline{f}$ at Y, i. e. the vector $\overline{YC} \times \overline{f}$. Denote $\overline{m} := \overline{OC} \times \overline{f}$ the moment of $\overline{f}$ at origin 0. We get a Plücker's couple $(\overline{f}, \overline{m} = \overline{OC} \times \overline{f})$ the line of which is just the line of $\overline{f}$.

Remind further, that the effect of a couple of forces $(\overline{f}, -\overline{f}, \overline{r})$ with its arm $\overline{r}$ is the same at every point $Y \in \Omega$. A measure of this effect is moment $\overline{r} \times \overline{f}$ of the couple of forces.

Let $X = (\overline{f}, \overline{m}) \in A_6$ be a couple of vectors. Let $\xi(X)$ be the vector fields on E_3 determined by X. The above considerations inspire the following dynamic interpretation of the vector field $\xi(X)$.

a) If $X = (0, \overline{m})$ then $\xi(X)$ is the vector field the value of which in every point $Y \in \Omega$ is the moment $\overline{m}$ of some couple of forces.

b) Let $X = (\overline{f}, \overline{m})$ be a Plücker's couple, i.e. $\overline{f}.\overline{m} = 0$. The line p of X we interpret as the line of a force $\overline{f}$. Then $\xi(X)$ is the vector field the value of which in a point Y is the moment $\overline{YC} \times \overline{f}$ of the force $\overline{f}$ at Y, where $\overline{OC} = (\overline{f} \times \overline{m})/\overline{f}^2$, i. e. C is the orthogonal projection of the origin 0 into p. The value of this field in 0 is $\overline{m} = \overline{OC} \times \overline{f}$.

c) Let $X = (\overline{f}, \overline{m})$ is not Plücker's couple, i. e. $\overline{f}.\overline{m} \neq 0$. Recall that $h\overline{f}, h = (\overline{f}.\overline{m})/\overline{f}^2$, is the orthogonal projection $\overline{m}$ into $\overline{f}$. Then the vectors $\overline{f}, \overline{m} - h\overline{f}$ are orthogonal in V_3 and $(\overline{f}, \overline{m}) = (\overline{f}, \overline{m} - h\overline{f}) + (0, h\overline{f})$ where $(\overline{f}, \overline{m} - h\overline{f})$ is a Plücker's couple. So the vector field $\xi(X)$ is the sum of the vector fields $\xi((\overline{f}, \overline{m} - h\overline{f}))$ and $\xi((0, h\overline{f}))$, i. e.

$$\xi(X)(Y) = \overline{YC} \times \overline{f} + h\overline{f} = \xi((\overline{f}, \overline{m} - h\overline{f})) + \xi((0, h\overline{f})).$$

This means that the value of the field $\xi(x)$ in a point Y is the sum of the moment of the force $\overline{f}$ at Y and the moment $h\overline{f}$ of some couple of forces.

Values of the vector field $\xi(X)$ interpreted by moments of forces can be called dynamic effects of a couple X.

Recall that in literature the following notions are used. Elements of the Lie algebra A_6, i. e. couples $X = (\overline{s}, \overline{m})$, are called motors. If the vector field $\xi(X)$ of a motor X is interpreted as a vector field of velocities then X is called twist.

If $\xi(X)$ is interpreted as a vector field of moments then X is called wrench. If two wrenches $X_1, X_2 \in A_6$ belong to the same screw, i. e. if $X_2 = kX_1$ then $\xi(X_2) = k\xi(X_1)$, i. e. the dynamic effect of X_2 is a multiple of the dynamic effect of X_1. In general if wrenches X_1, X_2 have the same line of couple, i. e. if $X_2 = (k\overline{f}_1, k\overline{m}_1 + uf_1)$ then the dynamic effect of X_2 is the sum of a multiple of the dynamic effect of X_1 and of a moment of some couple of forces.

Remark 12. (about a twist-wrench interpretation of $KL(X_1, X_2)$):
Let a twist $X_1 = (\overline{w}, \overline{b}) \in A_6$ determined an equable screw motion of a body Ω around the line p_1 of X_1 with angle velocity $\overline{w}$ and with translation velocity $h\overline{w}$. Then $\overline{b}$ is the velocity of origin 0. Let $X_2 = (\overline{f}, \overline{m}) \in A_6$ is a wrench, i. e. $\overline{f}$ is a force the line of which is the line of X_2 and $\xi(X_2)$ is a such vector field that $\xi(X_2)(Y) = \overline{YC}_2 \times \overline{f} + h\overline{f}$ is the sum of the moment of $\overline{f}$ at Y and of the moment $h\overline{f}$ of some souple of forces. Recall that $\xi(X_2)(0) = \overline{m}$. The value $KL(X_1, X_2) = \overline{f}.\overline{b} + \overline{w}.\overline{m}$ can be interpreted as follows. We can say that $\overline{f}.\overline{b}$ is a translation effect of $\overline{f}$ and $\overline{w}.\overline{m}$ is a rotation effect of $\overline{f}$ at the origin 0 of the body Ω moving by a equable screw motion. Then $KL(X_1, X_2)$ can be called a power given to the solid Ω, moving under the twist X_1, by the wrench X_2 per unit of time.

Remark on a motion of the effector of a robot.. We consider the effector of a robot as a rigid solid Ω. The moving effector determines in Ω the vector field of velocities of points $Y \in \Omega$ at any time t. This vector field is the vector field of velocities of a equable screw motion around an instantaneous axis and thus it is determined by a couple $X(t) = (\overline{\omega}(t), \overline{b}(t)) \in A_6$. So a moving effector determines a curve $t \mapsto X(t)$

in A_6. Vice versa, a curve $X(t)$ in A_6 states a movement of a effector the trajectories of which are solutions of the non-autonomous differential system

$$\dot{Y} = \xi_{X(t),0} Y, \dot{\overline{y}} = \overline{\omega}(t) \times \overline{y} + \overline{b}(t).$$

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