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# THE LATTICE OF VARIETIES OF ORGRAPHS

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ABSTRACT. In [5] we investigated varieties of orgraphs (i.e. oriented graphs) as classes of orgraphs closed under isomorphic images, suborgraph identifications and induced suborgraphs, and we studied the lattice of varieties of orgraphs. We paid particular attention to varieties containing no nontrivial tournament. In this paper we pay attention to the part of the lattice of varieties of orgraphs which consists of varieties generated by sets of notrivial tournaments.

## 1. INTRODUCTION

A useful tool for investigations of some properties of graphs is a choice of suitable closure operators and examinations classes of graphs closed under these operators. For example, classes of graphs closed under induced subgraphs are called hereditary in [12] and induced hereditary in [3], and were considered in several papers. Classes of graphs closed under other operators are considered, for example, in [2] and [6]. In the paper [5] were considered classes of orgraphs closed under isomorphic images, suborgraph identification and induced suborgraphs.

By an *orgraph* we mean directed graph $\mathcal{G}(V, E)$ without loops with the following property:

for every two distinct vertices $u, v \in V$, at most one of the edges $uv$ and $vu$ is an arc from $E$.

We briefly write $uv$ instead of $[u, v]$ for vertices $u, v \in V$.

We can associate to every orgraph $\mathcal{G}(V, E)$ the graph $\mathcal{G}^\star(V^\star, E^\star)$ by omitting the orientation of all edges, i.e.

$V^\star = V$ and $\{u, v\} \in E^\star$ iff $uv \in E$ or $vu \in E$.

An orgraph $\mathcal{G}(V, E)$ is called
- weakly connected if $\mathcal{G}^\star(V^\star, E^\star)$ is connected,
- a weak cycle if $\mathcal{G}^\star(V^\star, E^\star)$ is a cycle,
- a tournament if $\mathcal{G}^\star(V^\star, E^\star)$ is a complete graph.

Let us recall that by a *suborgraph identification* of orgraphs $\mathcal{G}_1$, $\mathcal{G}_2$ we mean gluing of the orgraphs $\mathcal{G}_1$, $\mathcal{G}_2$ in their weakly connected induced suborgraphs $\mathcal{G}_1'$, $\mathcal{G}_2'$ which are isomorphic (we choose an isomorphism between $\mathcal{G}_1'$ and $\mathcal{G}_2'$ and identify the corresponding vertices of $\mathcal{G}_1'$ and $\mathcal{G}_2'$ [7]).

In this paper we follow the notation of [5]. If $\mathbb{K}$ is a set of orgraphs we denote

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- by $\Gamma(\mathbb{K})$ the smallest class of weakly connected orgraphs containing the set $\mathbb{K}$ and closed under suborgraph identifications ,

- by $S(\mathbb{K})$ the class of all weakly connected induced suborgraphs of orgraphs from $\mathbb{K}$,

- by $I(\mathbb{K})$ the class of all isomorphic images of orgraphs from $\mathbb{K}$.

**Definition 1.1.** A set $\mathbb{K}$ of orgraphs closed under isomorphic images, induced weakly connected suborgraphs and suborgraph identifications is called *a variety*; that is $\mathbb{K}$ is a variety if

$$I(\mathbb{K}) \subseteq \mathbb{K},\ S(\mathbb{K}) \subseteq \mathbb{K} \text{ and } \Gamma(\mathbb{K}) \subseteq \mathbb{K}.$$

Obviously, $I$, $S$, $\Gamma$ are closure operators on the system of all sets of weakly connected orgraphs. By [4, Theorem 5.2] we obtain the next statement.

**Theorem 1.1.** *The set of all varieties of orgraphs with set inclusion as the partial ordering is a complete lattice (denoted by* $\mathbf{L}(I, S, \Gamma)$*).*

We denote by $V(\mathbb{K})$ the smallest variety of orgraphs containing a given set $\mathbb{K}$ of orgraphs. We will say that $V(\mathbb{K})$ is generated by the set $\mathbb{K}$.

The following lemma and corollary play an important role in investigations of varieties of orgraphs.

**Lemma 1.2.** *Let $\mathcal{G}(V, E)$ be a weakly connected orgraph which is neither a tournament nor a weak cycle. Then there exist two nonadjacent vertices $u, v \in V$ such that $\mathcal{G} - \{u, v\}$ is a weakly connected orgraph.*

*Proof.* The statement immediately follows from [8] or [10, page 208]. □

**Corollary 1.3.** *If $\mathcal{G}(V, E)$ is a weakly connected orgraph which is neither a weak cycle nor a tournament, then $\mathcal{G}$ is isomorphic to a suborgraph identification of two proper weakly connected suborgraphs of $\mathcal{G}$.*

*Proof.* By Lemma 1.2 there are two nonadjacent vertices $u, v \in V$ such that $\mathcal{G} - \{u, v\}$ is weakly connected. Let $f$ be the identity on the suborgraph $\mathcal{G} - \{u, v\}$. The orgraphs $\mathcal{G}_1 = \mathcal{G} - \{u\}$ and $\mathcal{G}_2 = \mathcal{G} - \{v\}$ are proper weakly connected induced suborgraphs of $\mathcal{G}$, and obviously $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2$. □

Whenever $uv$ is an arc of an orgraph $\mathcal{G}(V, E)$ , the vertex $u$ is called an *adjacent vertex to* $v$ and $v$ is called an *adjacent vertex from* $u$. An *outdegree* (an *indegree*) of a vertex $v \in V$ in the orgraph $\mathcal{G}(V, E)$ is the number of vertices adjacent from $v$ (to $v$). When outdegree of a vertex $v$ is $i$ and indegree of $v$ is $j$, we will say that $v$ is of type $v^{(i)}_{(j)}$ and write simply $v^i_j$, when no confusion can arise.

A tournament will be denoted by $\mathcal{T}_n(V, E)$ or briefly by $\mathcal{T}_n$. We say that a tournament $\mathcal{T}_n(V, E)$ is of type $\mathcal{T}^{(o_1, o_2, \dots, o_k)}$, $o_i \leq o_{i+1}$ for each $i = 1, \dots, k-1$, if $V = \{v_1, \dots, v_k\}$ and $o_1$, $o_2$, $\dots$ , $o_k$ are the outdegres of the vertices $v_1$, $v_2$, $\dots$ , $v_k$, respectively. When the tournament $\mathcal{T}_n$ is of the type $\mathcal{T}^{(o_1, o_2, \dots, o_k)}$, we more precisely write $\mathcal{T}_n = \langle v_1^{(o_1)}, v_2^{(o_2)}, \dots, v_k^{(o_k)} \rangle$. Let us note that the notation $\mathcal{T}^{(o_1, o_2, \dots, o_k)}$of k-vertex tournament is ambiguous for $k \geq 5$. We identify a tournament with its type if $k \leq 4$. The tournament $\mathcal{T}^{(1,1,1)}$ was denoted (as the weak cycle) by $\mathcal{C}_{(3,0)}$ and the tournament $\mathcal{T}^{(0,1,2)}$ was denoted by $\mathcal{C}_{(2,1)}$ in [5]. We say that a tournament $\mathcal{T}_n(V, E)$ is *nontrivial* if $|V| \geq 3$.

According to [5] we denote

- by $\mathcal{C}_{(4,1)}$ the weak cycle with two adjacent vertices of the types $v_0^2$, $v_2^0$ and three vertices of the type $v_1^1$ (see Figure 1a),

- by $\mathcal{C}_{(3,2)}$ the weak cycle with two nonadjacent vertices of the types $v_0^2$, $v_2^0$ and three vertices of the type $v_1^1$ (see Figure 1b).

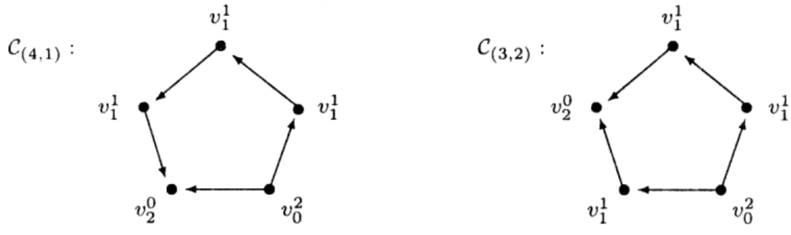


Figure 1a-b

We denote by $\mathbf{0}$ the smallest element of the lattice $\mathbf{L}(I,S,\Gamma)$ and by $\mathbf{1}$ the greatest element of the lattice $\mathbf{L}(I,S,\Gamma)$.

In the paper [5] we showed that the interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ of the lattice $\mathbf{L}(I,S,\Gamma)$ is isomorphic to the lattice $\mathbf{3} \oplus \mathbf{D}^d$ where $\oplus$ is the linear (ordinal) sum of the 3-element chain and the lattice $\mathbf{D}^d$, where $\mathbf{D}^d$ is the dual lattice of the lattice $\mathbf{D}$ of all nonnegative integers with the divisibility relation as the partial ordering. A variety of orgraphs belongs to the interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ iff it contains no nontrivial tournament.

In the next section we pay attention to the interval $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ of the lattice $\mathbf{L}(I,S,\Gamma)$.

In [5] we used a characteristic of a weak cycle. Let $\mathcal{C}(V,E)$ be a weak cycle of the length $n$. If all arcs of $\mathcal{C}$ have the same orientation, we say that the characteristic of $\mathcal{C}$ is $n$. On the other hand, if arcs of $\mathcal{C}$ have not the same orientation, we choose an arc $vw \in E$, and we call all arcs of $\mathcal{C}$ having the same orientation as $vw$ *positive*; the other arcs are *negative*. The *characteristic* $ch(\mathcal{C})$ of the weak cycle $\mathcal{C}$ is $|p-n|$, where $p$ is the number of all positive arcs of $\mathcal{C}$ and $n$ is the number of all negative arcs of $\mathcal{C}$.

The next lemmas were proved in [5] and will be used in this paper. First, we denote analogously as in [5]

- by $\mathcal{C}_{(1,1,\dots,1)}$ a weak cycle containing no vertex of the type $v_1^1$,
- by $\mathcal{C}_{(n,0)}$, $n \geq 3$, an n-vertex weak cycle containing only vertices of the type $v_1^1$,
- by $\mathcal{C}_{(3,1)}$ the weak cycle with two adjacent vertices of the type $v_1^1$, one vertex of the type $v_0^2$ and one vertex of the type $v_2^0$.

**Lemma 1.4.** *Let $\mathbf{V}$ be a variety of orgraphs. Let $\mathcal{C}$ be a weak cycle different from weak cycles of the type $\mathcal{C}_{(1,1,\dots,1)}$. If $\mathcal{C} \in \mathbf{V}$ and $\mathcal{C}'$ is a weak cycle for which*

$$ch(\mathcal{C}') = ch(\mathcal{C}), \quad \mathcal{C}' \neq \mathcal{C}_{(3,0)} \text{ and } \mathcal{C}' \neq \mathcal{C}_{(2,1)}$$

*(i.e. the characteristics of the weak cycles $\mathcal{C}$, $\mathcal{C}'$ are the same and $\mathcal{C}'$ is not a tournament) then $\mathcal{C}' \in \mathbf{V}$, too.*

**Lemma 1.5.** *Let $\mathbf{V}$ be a variety generated by a weak cycle $\mathcal{C}_{(n,0)}$, $n \geq 3$, or by $\mathcal{C}_{(3,1)}$, or by $\mathcal{C}_{(3,2)}$. Let $\mathcal{G}$ be an orgraph containing no nontrivial tournament as an induced suborgraph . Then $\mathcal{G} \in \mathbf{V}$ iff the characteristic of each weak cycle of the orgraph $\mathcal{G}$ is a multiple of the characteristic of the generating weak cycle.*

## 2. Varieties containing some nontrivial tournaments

By Corollary 1.3 every variety of orgraphs is generated by a set of weak cycles and tournaments. Therefore the next lemmas related to minimal nontrivial tournaments (minimal with respect the relation of being a subtournament) will prove useful.

**Lemma 2.1.** *A tournament $\mathcal{T}_n$ does not contain the subtournament $\mathcal{T}^{(1,1,1)}$ if and only if the tournament $\mathcal{T}_n$ is of the type $\mathcal{T}^{(0,1,\dots,k)}$.*

*Proof.* We prove the statement by induction on the number of vertices of tournaments.

The statement is evidently true for 3-vertex tournaments.

Let the statement be true for any k-vertex tournaments.

1. Let $\mathcal{T}_n = \langle v_0^{(0)}, v_1^{(1)}, \dots, v_k^{(k)} \rangle$ be a tournament of the type $\mathcal{T}^{(0,1,\dots,k)}$. We prove that $\mathcal{T}_n$ does not contain the tournament $\mathcal{T}^{(1,1,1)}$. Omitting of the vertex $v_k^{(k)}$ of $\mathcal{T}_n$ (the outdegree of the vertex $v_k$ is $k$) we obtain k-vertex tournament $\mathcal{T}_n'$ of the type $\mathcal{T}^{(0,1,\dots,k-1)}$. Evidently any 3-vertex subtournament of the tournament $\mathcal{T}_n$ is either subtournament of the tournament $\mathcal{T}_n'$ or a subtournament containing the vertex $v_k$. The tournament $\mathcal{T}_n'$ contains no subtournament of the type $\mathcal{T}^{(1,1,1)}$ (by induction hypothesis) and the indegree of the vertex $v_k$ is zero, therefore the statement follows.

2. Let $\mathcal{T}_n$ be a k+1-vertex tournament of type different from the type $\mathcal{T}^{(0,1,\dots,k)}$. We prove that $\mathcal{T}^{(1,1,1)}$ is its subtournament. Omitting a vertex $v$ of $\mathcal{T}_n$ we obtain k-vertex tournament $\mathcal{T}_n'$.

a) If $\mathcal{T}_n'$ contains the subtournament $\mathcal{T}^{(1,1,1)}$ then $\mathcal{T}^{(1,1,1)}$ is the subtournament of the tournament $\mathcal{T}_n$, too.

b) If $\mathcal{T}_n'$ contains no subtournament of the type $\mathcal{T}^{(1,1,1)}$ then $\mathcal{T}_n'$ is a tournament of the type $\mathcal{T}^{(0,1,\dots,k-1)}$ by induction hypothesis. Let $\mathcal{T}_n' = \langle u_0^{(0)}, u_1^{(1)}, \dots, u_{k-1}{}^{(k-1)} \rangle$.

If there exist two vertices $u_i, u_j \in V(\mathcal{T}_n')$, $i < j$, such that $u_i v \in E(\mathcal{T}_n)$ and $v u_j \in E(\mathcal{T}_n)$ then the tournament $\langle v, u_j, u_i \rangle = \mathcal{T}^{(1,1,1)}$ is the subtournament of $\mathcal{T}_n$.

Otherwise, the tournament $\langle v, u_0, u_1, \dots, u_{k-1} \rangle$ or $\langle u_0, u_1, \dots, u_{k-1}, v \rangle$ or $\langle u_0, u_1, \dots, u_s, v, u_{(s+1)}, \dots, u_{k-1} \rangle$, $0 \le s \le k-1$, is of the type $\mathcal{T}^{(0,1,\dots,k)}$, a contradiction. □

It is easy to verify that the next statement is true.

**Lemma 2.2.** *The tournament $\mathcal{T}^{(0,1,2)}$ is a subtournament of every nontrivial tournament $\mathcal{T}_n \neq \mathcal{T}^{(1,1,1)}$.*

Now, we focus our attention to varieties containing at least one nontrivial tournament.

**Lemma 2.3.** *Let $\mathbb{M}$ be a set of orgraphs . If a nontrivial tournament $\mathcal{T}_n$ is not a suborgraph of any orgraph from $\mathbb{M}$ then $\mathcal{T}_n \notin \mathbf{V}(\mathbb{M})$ (i.e. $\mathcal{T}_n$ does not belong to the variety generated by the set $\mathbb{M}$).*

*Proof.* It immediately follows from the following fact. If $\mathcal{T}_n$ is neither a suborgraph of an orgraph $\mathcal{G}_1$ nor a suborgraph of an orgraph $\mathcal{G}_2$, then obviously $\mathcal{T}_n$ is not a suborgraph of any suborgraph identification of the orgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$. □

**Lemma 2.4.** *The variety $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ contains every weak cycle $\mathcal{C} \neq \mathcal{C}_{(3,0)}$, and it covers the variety $\mathbf{V}(\mathcal{C}_{(3,2)})$.*

*Proof.* First we recall that $\mathcal{T}^{(0,1,2)} = \mathcal{C}_{(2,1)}$ and $\mathcal{C}_{(3,0)} = \mathcal{T}^{(1,1,1)}$. The weak cycle $\mathcal{C}_{(3,2)}$ belongs to the variety $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ by Lemma 1.4, and therefore the variety $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ contains every weak cycle $\mathcal{C} \neq \mathcal{C}_{(3,0)}$ by Lemma 1.5. The variety $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ contains only one tournament (by Lemma 2.3) and the statemant follows. □

**Corollary 2.5.** *Every variety* $\mathbf{V} \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$ *is generated by a suitable set of tournaments.*

*Proof.* The variety $\mathbf{V} \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$ is generated by a set $\mathbb{M} = \mathbb{M}_1 \cup \mathbb{M}_2$, where $\mathbb{M}_1$ is a set of weak cycles and $\mathbb{M}_2$ is a set of tournaments (and we suppose $\mathcal{T}^{(1,1,1)} \in \mathbb{M}_2$ if $\mathcal{C}_{(3,0)} \in \mathbb{M}_1$). We can assume that the set $\mathbb{M}_2$ of tournaments is closed under subtournaments (and so $\mathcal{T}^{(0,1,2)} \in \mathbb{M}_2$). It follows $\mathbf{V}(\mathbb{M}_1 \cup \mathbb{M}_2) = \mathbf{V}(\mathbb{M}_2)$ by Lemma 2.4. □

**Corollary 2.6.** *Let* $\mathbb{M}_1$, $\mathbb{M}_2$ *be sets of nontrivial tournaments closed under nontrivial subtournaments and let* $\mathbf{V}(\mathbb{M}_1) \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$ *and* $\mathbf{V}(\mathbb{M}_2) \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$. *The variety* $\mathbf{V}(\mathbb{M}_1)$ *is covered by the variety* $\mathbf{V}(\mathbb{M}_2)$ *if and only if there exists a tournament* $\mathcal{T}_n^{\blacktriangledown}$ *such that* $\mathbb{M}_2 = \mathbb{M}_1 \cup \{\mathcal{T}_n^{\blacktriangledown}\}$ *and* $\mathcal{T}_n^{\blacktriangledown} \notin \mathbb{M}_1$.

Now we investigate relations between varieties which contain the tournament $\mathcal{T}^{(1,1,1)}$.

**Lemma 2.7.** *a) The variety* $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ *covers only one variety* $\mathbf{V}(\mathcal{C}_{(4,1)})$.
*b) The variety* $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)})$ *covers only two varieties* $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ *and* $\mathbf{V}(\mathcal{C}_{(3,2)})$.
*c) The variety* $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)})$ *is covered by the variety* $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{T}^{(0,1,2)})$.

*Proof.* a) The variety $\mathbf{V}(\mathcal{T}^{(1,1,1)}) = \mathbf{V}(\mathcal{C}_{(3,0)})$ does not contain any nontrivial tournament $\mathcal{T}_n \neq \mathcal{T}^{(1,1,1)}$ and the weak cycle $\mathcal{C}_{(4,1)}$ belongs to $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ by Lemma 1.4. On the other hand the tournament $\mathcal{T}^{(1,1,1)}$ does not belong to the variety $\mathcal{C}_{(4,1)}$ by Lemma 2.3. A weak cycle $\mathcal{C}$ belongs to the variety $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ if and only if the characteristic of $\mathcal{C}$ is a multiple of the number 3 (by Lemma 1.5) and the statement follows.
b) The variety $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)})$ does not contain nontrivial tournament $\mathcal{T}_n \neq \mathcal{T}^{(1,1,1)}$ and contains every weak cycle $\mathcal{C} \neq \mathcal{C}_{(2,1)}$ (the weak cycle $C_{(2,1)} = \mathcal{T}^{(0,1,2)}$ is the tournament). Let us recall again that the variety $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ contains a weak cycle only if and only if its characteristic is a multiple of the number 3 and the variety $\mathbf{V}(\mathcal{C}_{(3,2)})$ contains every weak cycle $\mathcal{C} \neq \mathcal{C}_{(3,0)}$, by Lemma 1.5.
c) The variety $\mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{T}^{(0,1,2)})$ contains only two nontrivial tournaments $\mathcal{T}^{(1,1,1)}$ and $\mathcal{T}^{(0,1,2)}$ and all weak cycles, and the above considerations yield the statement. □

Thus, we have proved the next statement.

**Theorem 2.8.** *The lattice* $\mathbf{L}(I, S, \Gamma)$ *consists of the interval* $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ *(containing all varieties without nontrivial tournaments) and order filter with two minimal elements* $\mathbf{V}(\mathcal{T}^{(1,1,1)})$ *and* $\mathbf{V}(\mathcal{T}^{(0,1,2)})$ *(see Figure 2). The pair of varieties* $\langle \mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)}), \mathbf{V}(\mathcal{T}^{(0,1,2)}) \rangle$ *is the splitting pair of the lattice* $\mathbf{L}(I, S, \Gamma)$ *(i.e. for every variety* $\mathbf{V}$ *either* $\mathbf{V} \leq \mathbf{V}(\mathcal{T}^{(1,1,1)}, \mathcal{C}_{(3,2)})$ *or* $\mathbf{V} \geq \mathbf{V}(\mathcal{T}^{(0,1,2)})$*).*

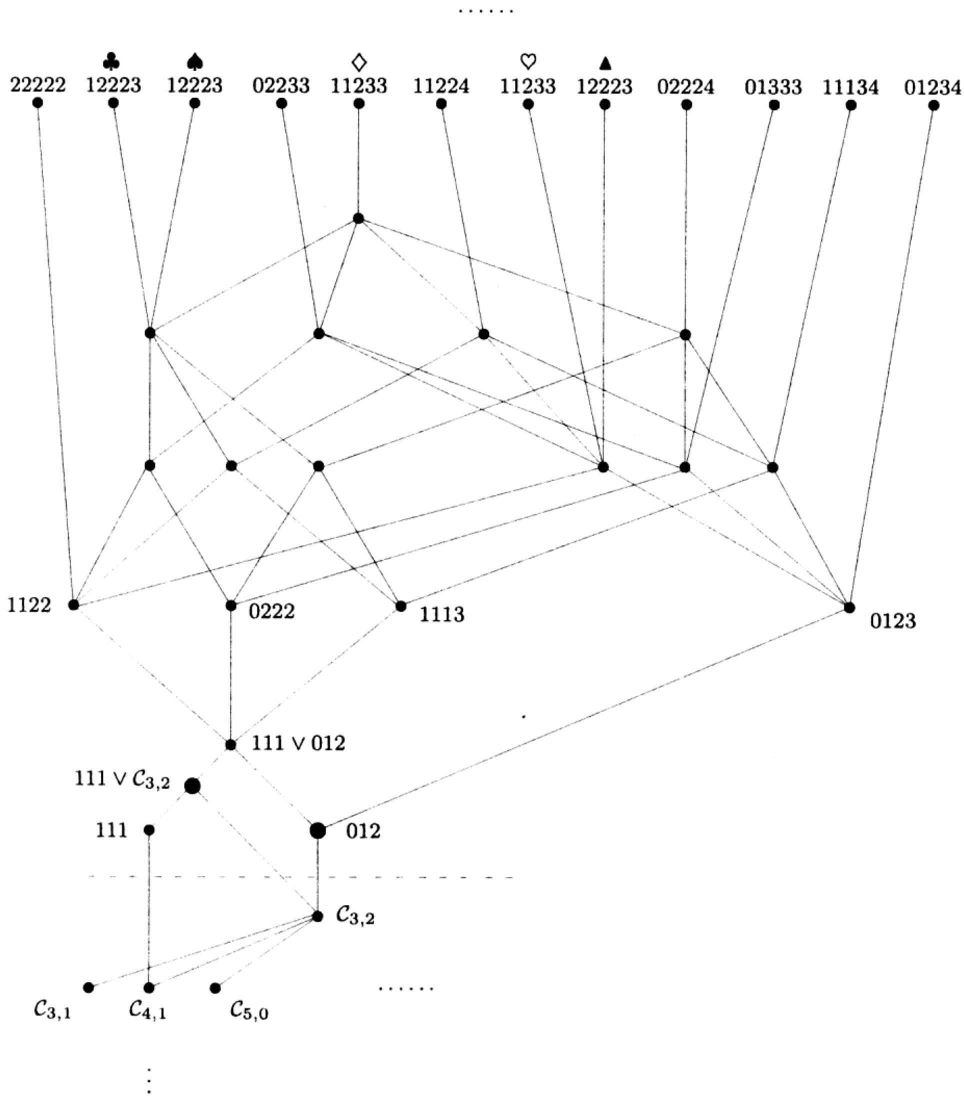


Figure 2

In Figure 2, the generators are used to denote the corresponding varieties, where tournaments are denoted by their types. Since some different tournaments with at least 5 vertices have the same type we depicted all 5-vertex tournaments in Figure 3 (by [11]).

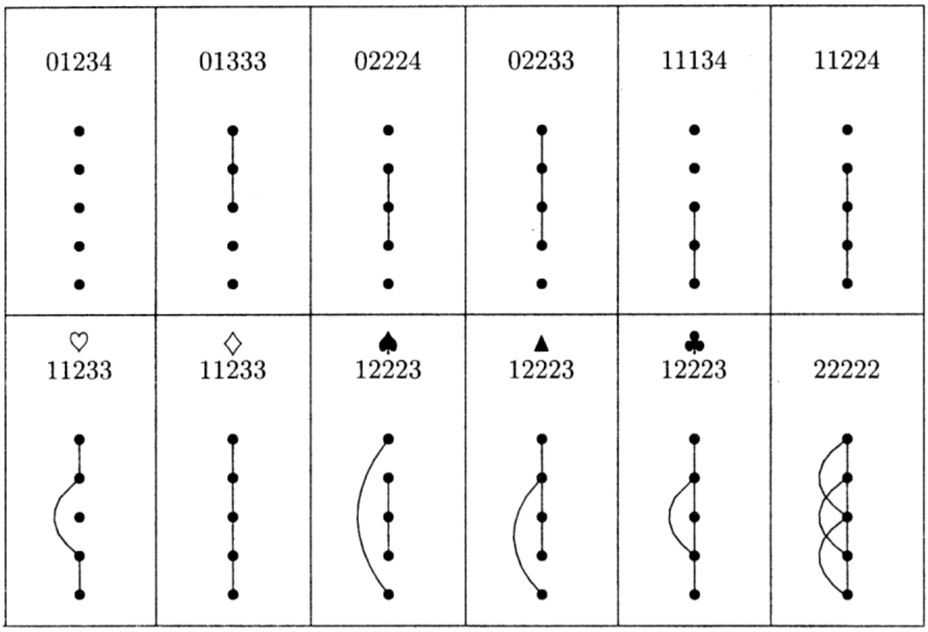


Figure 3
*(Upward arcs are shown; downward arcs are implied)*

In [5] we showed that the sublattice (the interval) $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ of the lattice $\mathbf{L}(I, S, \Gamma)$ is distributive. Now we will strengthen the statement.

**Theorem 2.9.** *The lattice* $\mathbf{L}(I, S, \Gamma)$ *is distributive.*

*Proof.*
a) First, we show that the sublattice (the interval) $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ is a distributive lattice. Let $\mathbb{M}_1$, $\mathbb{M}_2$ be sets of nontrivial tournaments closed under subtournaments and let $\mathbf{V}_1 = \mathbf{V}(\mathbb{M}_1)$, $\mathbf{V}_2 = \mathbf{V}(\mathbb{M}_2)$ be varieties generated by the sets $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively. By the above lemmas we have
$\mathbf{V}_1 \vee \mathbf{V}_2 = \mathbf{V}(\mathbb{M}_1 \cup \mathbb{M}_2)$ and $\mathbf{V}_1 \wedge \mathbf{V}_2 = \mathbf{V}(\mathbb{M}_1 \cap \mathbb{M}_2)$ if $\mathbb{M}_1 \cap \mathbb{M}_2 \neq \emptyset$.
Notice that $\mathbb{M}_1 \cap \mathbb{M}_2 = \emptyset$ if one of these sets is $\{\mathcal{T}^{(1,1,1)}\}$ and the other contains only tournaments of the type $\mathcal{T}^{(0,\dots,k)}$. It implies that the sublattice $[\mathbf{V}(\mathcal{T}^{(0,1,2)}), \mathbf{1}]$ of the lattice $\mathbf{L}(I, S, \Gamma)$ is distributive. Therefore the sublattice $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ is also distributive as is easy to check.
b) We show that the lattice $\mathbf{L}(I, S, \Gamma)$ contains neither the pentagon $\mathbf{N}_5$ nor the diamant $\mathbf{M}_3$.

Suppose, on the contrary, that the diamant $\mathbf{M}_3$ is a sublattice of the lattice $\mathbf{L}(I, S, \Gamma)$. At least two noncomparable elements of $\mathbf{M}_3$ belong to the interval $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ or to the interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)}]$. It follows that the sublattice $\mathbf{M}_3$ is a sublattice of the interval $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ or of the interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)}]$, and both mentioned intervals are distributive lattices, a contradiction.

Suppose, on the contrary, that the pentagon $\mathbf{N}_5$ is a sublattice of the lattice $\mathbf{L}(I, S, \Gamma)$. Let $a, b, c$ be elements of $\mathbf{N}_5$, $a < c$, $a \parallel b$, $c \parallel b$, $c$ covers $a$. Then both elements $a$, $c$ belong either to interval $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ or to interval $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$. Since the intervals $[\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ and $[\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ are distributive the element $b$ belongs to the other interval. There are only two possibilities: $a = \mathbf{V}(\mathcal{C}_{(4,1)})$ and

$c = \mathbf{V}(\mathcal{T}^{(1,1,1)})$ and $b \in [\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ (for example $b = \mathbf{V}(\mathcal{C}_{(5,0)})$) or $b = \mathbf{V}(\mathcal{C}_{(3,0)}) \in [\mathbf{V}(\mathcal{C}_{(4,1)}), \mathbf{1}]$ and $a, c \in [\mathbf{0}, \mathbf{V}(\mathcal{C}_{(3,2)})]$ (for example $c = \mathbf{V}(\mathcal{C}_{(5,0)})$, $a = \mathbf{V}(\mathcal{C}_{(10,0)})$). In this case we have $b \vee a < b \vee c$ or $b \wedge a < b \wedge c$ (see Lemma 1.5), a contradiction. □

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