Acta Univ. M. Belii
Math. no. 9 (2001), pp. 51–60

ON α-PSEUDODIMENSION OF MONOUNARY ALGEBRAS

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Abstract. In this paper the notion of α-realizer is defined. There are found necessary and sufficient conditions under which an α-realizer of a connected monounary algebra exists. Next we deal with α-pseudodimension of a product of some special types of monounary algebras.

1 Introduction

Let $\mathcal{U}$ be the class of all monounary algebras and let $\alpha = (L, f)$ be a fixed element of $\mathcal{U}$. To each $(A, f) \in \mathcal{U}$ we assign a cardinal which will be denoted by α-pdim(A, f); we say that this cardinal is the α-pseudodimension of (A, f).

Our definition is in accordance with that used by V. Novák and M. Novotný [6] (cf. especially Example 6.4 of [6]).

The most of results concern the case when both (A, f) and (L, f) are finite connected monounary algebras.

First we study α-realizers of $(A, f) \in \mathcal{U}$. There are found necessary and sufficient conditions under which an α-realizer of a connected monounary algebra exists. Next some special types α are dealt with and we determine α-pdim(A, f) in the case when (A, f) is a direct product of sticks.

After the World War II, O.Borůvka formulated a problem concerning matrices commuting with a given matrix, that led to study homomorphisms of monounary algebras. His problem stimulated the investigation of these algebras; monounary algebras were investigated e.g. by M.Novotný [7],[8], O.Kopeček [3], E.Nelson [4], D.Jakubíková–Studenovská [1],[2]. The concept of pseudodimension was introduced in [5] for ordered sets. Later it was extended by Novák and Novotný [6] to the concept of α-pseudodimension of arbitrary relational structures.

For the terminology and definitions cf. Section 2.

2 α-realizer

In this section we start with defining of the notions we will use below. Then we investigate α-realizers of (A, f).

2000 *Mathematics Subject Classification.* 08A60.
Key words and phrases. monounary algebra, realizer, pseudodimension
Supported by grant VEGA 1/7468/20.

Definition 2.1. Let $n \in N, k \in N \cup \{0\}$. The algebra of the type (n, k) is the monounary algebra (B, f), where $B = Z_n \cup \{m \in N : m \leq k\}$ (where $Z_n = \{0_n, \ldots, (n-1)_n\}$ is the set of all integers mod n),

$$f(i_n) = (i+1)_n \text{ for each } i \in Z, f(1) = 0_n,$$

$$f(m) = m - 1 \text{ for each } m \in N, 1 < m \leq k.$$

In the case when $n = 1$, the algebra (B, f) is called a stick or a stick of type k.

Notation 2.2. Let (B, f) be a connected monounary algebra. We denote by $C(B)$ the set of all cyclic elements of (B, f) and $R(B) = |C(B)|$.

The degree $s(x)$ of an element $x \in B$ was defined in [7] (cf. also [1]) as follows:

Let us denote by $B^{(\infty)}$ the set of all elements $x \in B$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of elements belonging to B with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $B^{(0)} = \{x \in B : f^{-1}(x) = \emptyset\}$. Now we define a set $B^{(\lambda)} \subseteq B$ for each ordinal λ by induction. Let $\lambda > 0$ be an ordinal. Assume that we have defined $B^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$B^{(\lambda)} = \{x \in B - \bigcup_{\alpha<\lambda} B^{(\alpha)} : f^{-1}(x) \subseteq \bigcup_{\alpha<\lambda} B^{(\alpha)}\}.$$

The sets $B^{(\lambda)}$ (where λ is an ordinal or $\lambda \doteq \infty$) are pairwise disjoint. For each $x \in B$, either $x \in B^{(\infty)}$ or there is an ordinal λ with $x \in B^{(\lambda)}$. In the former case we put $s(x) = \infty$, in the latter we set $s(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

Suppose that $R(B) \neq 0$. If $B = C(B)$, then we put $h(B) = 0$. If $B \neq C(B)$, then we define $h(B) = 1 +$ sup $\{s(x) : x \in B - C(B)\}$.

Notice that the definition of $s(x)$ implies that if $B \neq C(B)$, then

$$h(B) = 1 + \sup\{s(x) : x \in B - C(B), f(x) \in C(B)\}.$$

Remark. Let us remark that we considerably apply results of M.Novotný [7],[8] concerning homomorphisms of monounary algebras. E.g., without further reference we will use that if (A, f) and (B, f) are monounary algebras, then

(1) if φ is a homomorphism of (A, f) into (B, f), then $s(\varphi(x)) \geq s(x)$ for each $x \in A$,
(2) if φ is a homomorphism of (A, f) into (B, f) and $x \in A$ belongs to a cycle C, then $\varphi(x)$ belongs to a cycle $D \subseteq B$ such that $|D|$ divides $|C|$.

Notation 2.3. We will denote by (Z, f) and (N, f) the monounary algebra such that $f(i) = i + 1$ for each $i \in Z$ or $i \in N$, respectively.

Further, for a cardinal k let (N_k, f) be a fixed monounary algebra such that

$$N_k = N \cup D,\ N \cap D = \emptyset,\ |D| = k,$$

$$f(a) = \begin{cases} a+1 & \text{if } a \in N, \\ 1 & \text{if } a \in D. \end{cases}$$

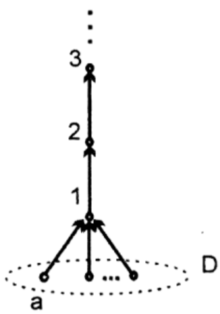

FIG. 1

Definition 2.4. Let $(A, f) \in \mathcal{U}$ and let $\{\varphi_j : j \in J\}$ be a nonempty system of mappings of A into L such that for any $x, y \in A$ we have

$$y = f(x) \iff (\forall j \in J)(\varphi_j(y) = f(\varphi_j(x)).$$

Then $\{\varphi_j : j \in J\}$ is said to be an α-realizer of (A, f).

If no α-realizer of (A, f) exists, then we set

$$\alpha\text{-pdim } (A, f) = 0.$$

Further, suppose that there exists some α-realizer of (A, f); then we put

$$\alpha \text{ -pdim } (A, f) = \min\{|J| : \{\varphi_j : j \in J\} \text{ is an } \alpha \text{ -realizer of } (A, f)\}.$$

This cardinal is called α-pseudodimension of (A, f).

This definition immediately yields the following two assertions:

Lemma 2.5. Let $(A, f) \in \mathcal{U}$ and suppose that $\{\varphi_j : j \in J\}$ is an α-realizer of (A, f). For $j \in J$, the mapping φ_j is a homomorphism of (A, f) into (L, f).

Lemma 2.6. Let $(A, f) \in \mathcal{U}$ and let Υ be a nonempty system of homomorphisms of (A, f) into (L, f). Then Υ is an α-realizer of (A, f) if and only if the following implication is valid for each $x, y \in A$

$$(*) \qquad ((\forall \varphi \in \Upsilon)(\varphi(y) = \varphi(f(x)))) \Rightarrow y = f(x).$$

Corollary 2.7. If $(A, f) \in \mathcal{U}$ and there exists an injective homomorphism of (A, f) into (L, f), then α-pdim$(A, f) = 1$.

Lemma 2.8. Let (A, f) and (L, f) be connected monounary algebras. If there exists an α-realizer of (A, f), then $R(A) = R(L)$.

Proof. Suppose that Υ is an α-realizer of (A, f).

a) First assume that $R(L) = m \in N$. Then there exists $x \in A$ such that $\varphi(f(x)) \in C(L)$. Put $y = f^{m+1}(x)$. For $\varphi \in \Upsilon$ we get

$$\varphi(y) = \varphi(f^{m+1}(x)) = f^{m+1}(\varphi(x)) = f^m(\varphi(f(x))) = \varphi(f(x)).$$

Since Υ is an α-realizer, $(*)$ of 2.6 yields that $y = f(x)$, i.e., $f^m(f(x)) = f(x)$. Therefore $R(A)$ divides m. From 2.5 it follows that $R(L)$ divides $R(A)$ (because each φ is a homomorphism of (A, f) into (L, f)), thus we obtain $R(A) = R(L)$.

b) Now suppose that $R(L) = 0$. According to 2.5, $R(A) = 0$, too. □

Theorem 2.9. Let (A, f) and (L, f) be connected monounary algebras such that $R(A) = R(L) \neq 0$. An α-realizer of (A, f) exists if and only if $h(A) \leq h(L)$.

Proof. Let Υ be an α-realizer of (A, f). Let $a \in A - C(A)$ be such that $f(a) \in C(A)$. There exists $x \in C(A)$ with $f^2(x) = f(a)$. Put $c = f(x)$. Then $f(c) = f(a) \in C(A)$. First suppose that $\varphi(a) \in C(L)$ for each $\varphi \in \Upsilon$. This implies that for $\varphi \in \Upsilon$ we have

$$\varphi(a) = \varphi(c) = \varphi(f(x)),$$

thus by $(*)$, $a = f(x)$, a contradiction. Hence there exists $\psi \in \Upsilon$ such that $b = \psi(a) \notin C(L)$. Since ψ is a homomorphism, the element $f(b)$ is cyclic and

$$s(a) \leq s(\psi(a)) = s(b).$$

Therefore $h(A) \leq h(L)$.

Conversely, assume that $h(A) \leq h(L)$. If $h(A) = 0$, then let φ_0 be an arbitrary isomorphism of A onto $C(L)$. It is obvious that $\Upsilon = \{\varphi_0\}$ is an α-realizer of A. Now let $h(A) \neq 0$. Let $u \in A - C(A)$. Then there is $a \in A - C(A)$ with $f(a) \in C(A)$ and $u \in f^{-n}(a)$ for some $n \in N \cup \{0\}$. The relation $h(A) \leq h(L)$ implies that there is $b \in L - C(L)$ such that $f(b) \in C(L)$ and that $s(a) \leq s(b)$. Let b be a fixed element with this property. Obviously, $s(u) \leq s(b)$. By [8], Thm., p.157 there exists a homomorphism ψ_u of (A, f) into (L, f) having the following properties:

(1) $\psi_u(u) = b$,

(2) if $v \notin \bigcup_{m \in N \cup \{0\}} f^{-m}(u) \cup \{f^k(u) : k \in N\}$, then $\psi_u(v) \in C(L)$.

Denote $\Upsilon = \{\psi_u : u \in A - C(A)\}$. Let us verify that Υ is an α-realizer of (A, f) according to $(*)$. Assume that $x, y \in A$ and that $\psi_u(y) = \psi_u(f(x))$ for each $\psi_u \in \Upsilon$.

a) If $f(x) \notin C(A)$, then take $u = f(x)$. We get $\psi_u(y) = \psi_u(u)$. Since $\psi_u^{-1}(\psi_u(u))$ is a one-element set $\{u\}$ by (2), this implies that $y = u$, i.e., $y = f(x)$.

b) Let $f(x) \in C(A)$. If $y \notin C(A)$ then $\psi_y(y) \notin C(L)$, hence $\psi_y(y) \neq \psi_y(f(x))$, a contradiction. Thus $y \in C(A)$. Take an arbitrary $\varphi \in \Upsilon$. Then φ is an isomorphism of $C(A)$ onto $C(L)$, thus the relation $\varphi(f(x)) = \varphi(y)$ yields that $f(x) = y$.

Therefore Υ is an α-realizer of (A, f). □

Theorem 2.10. Let (A, f) and (L, f) be connected monounary algebras such that $R(A) = R(L) = 0$. Let $P = \{u \in A : |f^{-1}(u)| > 1,\ f^{-2}(u) \neq \emptyset\}$. An α-realizer of (A, f) exists if and only if one of the following conditions is satisfied:

(a) $(A, f) \cong (N, f)$ or $(A, f) \cong (N_k, f)$ for some $k \in$ Card;

(b) $(A, f) \cong (Z, f)$ and there is a subalgebra of (L, f) isomorphic to (Z, f);

(c) $P \neq \emptyset$ and for each $u \in A$, $q_1, q_2 \in f^{-1}(u)$, $q_1 \neq q_2$ such that $f^{-1}(q_1) \neq \emptyset$ there are $v \in L$ and distinct elements $t_1, t_2 \in f^{-1}(v)$ such that $s(f^k(u)) \leq s(f^k(v))$, $s(q_i) \leq s(t_i)$ for each $k \in N \cup \{0\}$, $i \in \{1, 2\}$.

Proof. Let Υ be an α-realizer of (A, f). First suppose that $P \neq \emptyset$. Take $u \in P$, $x \in f^{-2}(u)$, $q_2 \in f^{-1}(u) - \{f(x)\}$. Let $q_1 = f(x)$. Since Υ is an α-realizer, we obtain that there is $\varphi \in \Upsilon$ such that $\varphi(q_2) \neq \varphi(q_1)$. Put $v = \varphi(u)$. Then $s(f^k(u)) \leq s(\varphi(f^k(u))) = s(f^k(v))$, $s(q_i) \leq s(\varphi(q_i))$ for each $k \in N \cup \{0\}$, $i \in \{1, 2\}$, hence (c) is valid. Now let $P = \emptyset$. Then (A, f) is isomorphic to one of the algebras $(Z, f), (N, f), (N_k, f)$ for some $k \in$ Card, i.e., either (a) is valid or $(A, f) \cong (Z, f)$.

Each $\varphi \in \Upsilon$ is a homomorphism, thus if $(A, f) \cong (Z, f)$ then (Z, f) is isomorphic to some subalgebra of (L, f). Therefore one of the conditions $(a) - (c)$ is satisfied.

Conversely, let one of the conditions $(a) - (c)$ be valid. If (a) or (b) is valid, then there exists a homomorphism φ_0 of (A, f) into (L, f); put $\Upsilon = \{\varphi_0\}$. Let $x, y \in A$, $\varphi_0(y) = \varphi_0(f(x))$, $y \neq f(x)$. If (A, f) is isomorphic to (N, f) or to (Z, f), then each homomorphism of (A, f) into (L, f) is injective. Let (A, f) be (up to isomorphism) (N_k, f) for some $k \in$ Card. Further, the relation $\varphi_0(y) = \varphi_0(f(x))$ implies that $\{y, f(x)\} \subseteq D$, which is a contradiction, since $f(x) \in N - D$.

Let (c) hold. If $u \in P$, $p \in f^{-2}(u)$, $q \in f^{-1}(u) - \{f(p)\}$ then take $q_1 = f(p)$, $q_2 = q$; by (c) (according to [8], as in the proof of Thm.2.9) there exists a homomorphism ψ_{upq} of (A, f) into (L, f) such that

(1) $\psi_{upq}(u) = v$,

(2) $\psi_{upq}(q) \neq \psi_{upq}(f(p))$.

Let Υ the set of all homomorphisms of the form ψ_{upq}. We will show that Υ is an α-realizer of (A, f). Let $x, y \in A$ and suppose that $\varphi(y) = \varphi(f(x))$ for each $\varphi \in \Upsilon$. Put $z = f(x)$. From the connectedness we infer that $f^m(y) = f^n(z)$ for some $m, n \in N \cup \{0\}$; we can assume that m, n are the smallest nonnegative integers with this property. Since $\Upsilon \neq \emptyset$ and $\varphi(y) = \varphi(z)$ for $\varphi \in \Upsilon$, we get that $m \neq 0$ and $n \neq 0$. Denote $u = f^m(y)$, $p = f^{n-1}(x)$, $q = f^{m-1}(y)$. In view of the relation $\varphi_{upq} \in \Upsilon$ we have

(3) $\psi_{upq}(y) = \psi_{upq}(z)$.

This implies
$f^m(\psi_{upq}(z)) = f^m(\psi_{upq}(y)) = \psi_{upq}(f^m(y)) = \psi_{upq}(u) = \psi_{upq}(f^n(z)) = f^n(\psi_{upq}(z))$,
hence $m = n$. Assume that $y \neq z$. Next,
$\psi_{upq}(q) = \psi_{upq}(f^{n-1}(y)) = f^{n-1}(\psi_{upq}(y))$, i.e., $\psi_{upq}(y) \in f^{-(n-1)}(\psi_{upq}(q))$.
Similarly we obtain $\psi_{upq}(f(p)) = \psi_{upq}(f(f^{n-1}(x)) = f^{n-1}(\psi_{upq}(f(x)))$,
i.e., $\psi_{upq}(f(x)) \in f^{-(n-1)}(\psi_{upq}(f(p))$. In view of (3) we get

$$f^{-(n-1)}(\psi_{upq}(q)) \cap f^{-(n-1)}(\psi_{upq}(f(p))) \neq \emptyset,$$

which is a contradiction to (2). This concludes the proof. □

3 α-PDIMENSION AND A PRODUCT OF STICKS

In this section we deal with realizers of type (n, k), $n \in N, k \in N \cup \{0\}$. Further, we find the value of $(1, k)$-pseudodimension of a direct product of sticks.

Lemma 3.1. Let $n \in N, k \in N \cup \{0\}$. An (n, k)-realizer of a connected monounary algebra (A, f) exists if and only if $R(A) = n$ and $f^k(a) \in C(A)$ for each $a \in A$.

Proof. The assertion is a corollary of 2.9. □

Theorem 3.2. Let $n \in N, k \in N \cup \{0\}$.

a) If $k = 0$ or $n = 1, k = 1$, then (n, k)-pdim $(A, f) = 1$ for each monounary algebra such that an (n, k)-realizer exists.

b) Let $k = 1, n \geq 2$. If $m \in \{1, 2, \ldots, n\}$, then there exists $(A, f) \in \mathcal{U}$ such that (n, k)-pdim $(A, f) = m$. If $m \in N, m > n$, then (n, k)-pdim $(A, f) \neq m$ for each $(A, f) \in \mathcal{U}$.

c) If $k \geq 2, m \in N$, then there exists $(A, f) \in \mathcal{U}$ such that (n, k)-pdim $(A, f) = m$.

Proof.

a) Assume that an (n,k)-realizer of (A,f) exists. If $k=0$, then $|A| = |C(A)| = n$ by 3.1 and there is an isomorphism φ_0 of A onto Z_n. Then 2.7 implies that $(n,0)$-pdim $(A,f) = 1$. If $n=1$, $k=1$, then 3.1 implies that there is $c \in A$ such that $f(a) = c$ for each $a \in A$. Put $\varphi(c) = 0_1$, $\varphi(a) = 1$ for each $a \in A - \{c\}$. Then $\{\varphi\}$ is a $(1,1)$-realizer of (A,f) and $(1,1)$-pdim $(A,f) = 1$.

b) From the assumption it follows that $L = Z_n \cup \{1\}$.
Let $m \in \{1,\dots,n\}$. We put $A = Z_n \cup \{1,\dots,m\}$, $f(i_n) = (i+1)_n$ for each $i \in Z$, $f(l) = l_n$ for each $l \in \{1,\dots,m\}$. For $j \in \{1,\dots,m\}$ we define a mapping $\varphi_j : A \to L$ as follows: $\varphi_j(i_n) = (i-j)_n$ for each $i \in Z$, $\varphi_j(j) = 1$, $\varphi_j(l) = (l-1-j)_n$ for each $l \in \{1,\dots,m\} - \{j\}$. (cf. Fig.2.)

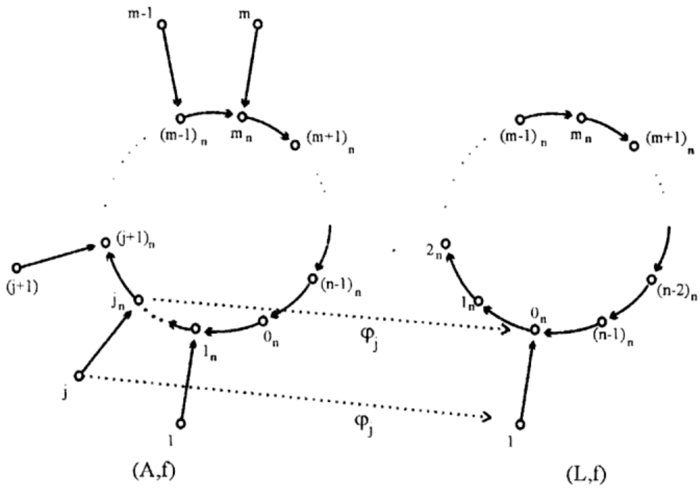

FIG. 2

It is easy to verify that φ_j is a homomorphism for each $j \in \{1,\dots,m\}$. Denote $\Upsilon = \{\varphi_j : j \in \{1,\dots,m\}\}$. Let $x, y \in A$, $\varphi_j(y) = \varphi_j(f(x))$ for each $j \in \{1,\dots,m\}$. We have $\varphi_j^{-1}(1) = \{j\}$ for each j, thus $y \neq j$ for each j. Thus $y \in Z_n$. Next, $f(x) \in Z_n$, hence in view of the fact that any φ_j is a bijection of $C(A)$ onto $C(L)$, the relation $\varphi_j(y) = \varphi_j(f(x))$ implies that $y = f(x)$. Therefore Υ is an $(n,1)$-realizer of (A,f) and $(n,1)$-pdim $(A,f) \leq m$.

Suppose that Υ' is an $(n,1)$-realizer of (A,f). Let $j \in \{1,\dots,m\}$. If $\psi(j) \in Z_n$ for each $\psi \in \Upsilon'$, then

$$\psi(j) = \psi((j-1)_n) = \psi(f((j-2)_n)),$$

$$j = f((j-2)_n),$$

which is a contradiction. Thus there exists $\psi_j \in \Upsilon'$ such that $\psi_j(j) = 1$. If $\psi_j = \psi_l$ for $j, l \in \{1,\dots,m\}$, then $\psi_j(l) = 1 = \psi_j(j)$, which implies

$$\psi_j(j_n) = \psi_j(f(j)) = f(\psi_j(j)) = f(1) = 0_n.$$

Similarly, $\psi_j(l_n) = 0_n$. Further, we have

$$0_n = \psi_j(l_n) = \psi_j(f^{l-j}(j_n)) = f^{l-j}(\psi_j(j_n)) = f^{l-j}(0_n) = (l-j)_n,$$

thus $l = j$. Hence $|\Upsilon'| \geq m$, therefore $(n,1)$-pdim $(A,f) = m$.

Let $m > n$ and suppose that there is $(A,f) \in \mathcal{U}$ with $(n,1)$-pdim $(A,f) = m$. Thus there is an $(n,1)$-realizer Υ of (A,f) with $|\Upsilon| = m$. According to 3.1 we have $R(A) = n$ and $f(a) \in C(A)$ for each $a \in A$. Up to isomorphism,

$$A = Z_n \cup D_1 \cup D_2 \cup \cdots \cup D_n,$$

$$f(i_n) = (i+1)_n \text{ for each } i \in Z,$$

$$f(d) = l_n \text{ for each } d \in D_l,\ l \in \{1, \ldots, n\}.$$

Let $j \in \{1, \ldots, n\}$. Define a mapping $\varphi_j : A \to L$ as follows:

$$\varphi_j(i_n) = (i-j)_n \text{ each } i \in Z,$$

$$\varphi_j(d) = \begin{cases} 1 & \text{if } d \in D_j, \\ (l-1-j)_n & \text{if } d \in D_l,\ l \neq j. \end{cases}$$

It is easy to verify that $\{\varphi_j : j \in \{1, \ldots, n\}\}$ is an $(n,1)$-realizer of (A,f), hence $(n,1)$-pdim $(A,f) \leq n$, which is a contradiction.

c) Let $k \geq 2, m \in N$. There exists $t \in N$ such that $2^{m-1} < t+1 \leq 2^m$. We denote by (A,f) a monounary algebra such that $A = Z_n \cup \{a_1, \ldots, a_t\} \cup \{b_1, \ldots, b_t\}$ (suppose that all these elements are distinct and they do not belong to Z_n), where $f(i_n) = (i+1)_n$ for each $i \in Z$, $f(a_l) = 0_n$, $f(b_l) = a_l$ for each $l = 1, \ldots, t$.

There exist 2^m distinct m-tuples of the elements $(n-1)_n, 1$. Thus there exists a set $Q = \{q_1, \ldots, q_t\}$ of m-tuples of the elements $(n-1)_n$, 1 with $|Q| = t$, $q \neq ((n-1)_n, \ldots, (n-1)_n)$ for each $q \in Q$. For $j \in \{1, \ldots, m\}$, $l \in \{1, \ldots, t\}$ let $q_l(j)$ be the projection of q_l into the j-th coordinate. Let $j \in \{1, \ldots, m\}$; we will define a mapping φ_j as follows. For $l \in \{1, \ldots, t\}$ we put

$$\varphi_j(i_n) = i_n \text{ for each } i \in Z,$$

$$\varphi_j(a_l) = q_l(j),$$

$$\varphi_j(b_l) = \begin{cases} 2 & \text{if } q_l(j) = 1, \\ (n-2)_n & \text{otherwise.} \end{cases}$$

It is easy to verify that $\{\varphi_j : j \in \{1, \ldots, m\}\}$ is a set of homomorphisms and that it is an (n,k)-realizer of (A,f) (cf. Example 1).

Next suppose that Υ is an (n,k)-realizer of (A,f), $\Upsilon = \{\psi_1, \ldots, \psi_r\}$, $r = |\Upsilon| < m$. For $l \in \{1, \ldots, t\}$ consider an r-tuple $p^{(l)}$ such that for $j \in \{1, 2, \ldots, r\}$

$$p^{(l)}(j) = \begin{cases} 0 & \text{if } \psi_j(a_l) \in Z_n, \\ 1 & \text{otherwise.} \end{cases}$$

Let $l \in \{1, \ldots, t\}$. If $p^{(l)}(j) = 0$ for each $j \in \{1, \ldots, r\}$, then $\psi_j(a_l) \in Z_n$ for each $j \in \{1, \ldots, r\}$; then $\psi_j(a_l) = \psi_j((n-1)_n) = \psi_j(f((n-2)_n))$ and the definition of an (n,k)-realizer implies that $a_l = f((n-2)_n)$, a contradiction. Therefore each r-tuple $p^{(l)}$ does not consist of zeros only.

Since ψ_j is a homomorphism, $\psi_j(a_l) = 1$ or $\psi_j(a_l) \in Z_n$. If $l, l' \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, r\}$, then either

(1) $$\psi_j(a_l) = \psi_j(a_{l'}) \text{ or}$$

(2) $$\psi_j(a_l) = 1, \psi_j(a_{l'}) = (n-1)_n \text{ or}$$

(3) $$\psi_j(a_l) = (n-1)_n, \psi_j(a_{l'}) = 1.$$

By the assumption, $r \le m-1$, $2^r - 1 \le 2^{m-1} - 1 < t$, thus there exist $l, l' \in \{1, \ldots, t\}$, $l \ne l'$ such that $p^{(l)} = p^{(l')}$. Then $p^{(l)}(j) = p^{(l')}(j)$ for each $j \in \{1, \ldots, r\}$. Then we obtain that the cases (2) and (3) yield a contradiction, thus $\psi_j(a_l) = \psi_j(a_{l'})$ for each $j \in \{1, \ldots, r\}$. This implies that for each $j \in \{1, \ldots, r\}$, $\psi_j(f(b_l)) = \psi_j(a_l) = \psi_j(a_{l'})$. According the fact that Υ is an (n,k)-realizer we get $f(b_l) = a_{l'}$, which is a contradiction.

Thus we have shown that (n,k)-pdim$(A, f) = m$. □

Example 1. Let $n = 2$ and $k \ge 2$. For $m = 3$ we will define (A, f) such that $(2,k)$-pdim(A, f) is equal to m. Let us follow the proof of theorem 3.2c). The relation $2^2 < t+1 \le 2^3$ implies $t \in \{4, 5, 6, 7\}$. Let $t = 4$.

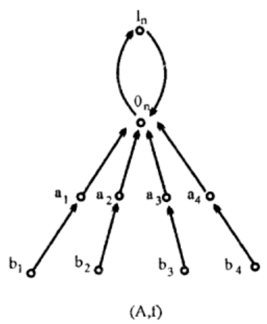

FIG. 3

There exists 2^3 of 3-tuples of elements $1, 1_2$: $(1,1,1)$, $(1,1,1_2)$, $(1,1_2,1)$, $(1_2,1,1)$, $(1,1_2,1_2)$, $(1_2,1,1_2)$, $(1_2,1_2,1)$, $(1_2,1_2,1_2)$. Next we choose four elements of them $(t=4)$, e.g., let $q_1 = (1,1,1)$, $q_2 = (1,1,1_2)$, $q_3 = (1,1_2,1)$ and $q_4 = (1_2,1,1)$. Put $Q = \{q_1, q_2, q_3, q_4\}$. We can define three $(m=3)$ mappings $\varphi_1, \varphi_2, \varphi_3$.

	0_2	1_2	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
φ_1	0_2	1_2	1	1	1	1_2	2	2	2	0_2
φ_2	0_2	1_2	1	1	1_2	1	2	2	0_2	2
φ_3	0_2	1_2	1	1_2	1	1	2	0_2	2	2

It can be verified that $\{\varphi_1, \varphi_2, \varphi_3\}$ is a $(2,k)$-realizer of algebra (A, f) and $(2,k)$-pdim$(A, f) = 3$.

Theorem 3.3. Let (A, f) be a direct product of sticks $(A_1, f), (A_2, f), \ldots, (A_m, f)$ of types $k_1, k_2, \ldots, k_m$ such that $|\{i \in \{1, \ldots, m\} : k_i = 1\}| \leq 1$. If $k \geq k_i$ for each $i \in \{1, \ldots, m\}$, then $(1, k)$-pdim $(A, f) = m$.

Proof. Let (L, f) be a monounary algebra of type k, $k \geq k_i$ for each $i \in \{1, \ldots, m\}$. We can suppose that $L = \{0, 1, \ldots, k\}$, $A_i = \{0, 1, \ldots, k_i\}$,

$$f(j) = \begin{cases} j-1 & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

in L and in A_i for $i \in \{1, \ldots, m\}$. For $j \in \{1, \ldots, m\}$ we define a mapping $\varphi_j : A_1 \times \cdots \times A_m \to L$ such that $\varphi_j((a_1, \ldots, a_m)) = a_j$. Put $\Upsilon = \{\varphi_j : j \in \{1, \ldots, m\}\}$. If $x, y \in A$, $\varphi_j(y) = \varphi_j(f(x))$ for each $j \in \{1, \ldots, m\}$, then $y = f(x)$. Thus Υ is a $(1, k)$-realizer of (A, f) and $(1, k)$-pdim $(A, f) \leq m$.
Suppose that Υ' is a $(1, k)$-realizer of (A, f), $|\Upsilon'| < m$. Denote $\bar{0} = (0, 0, \ldots, 0) \in A$. Obviously, $\psi(\bar{0}) = 0$ for each $\psi \in \Upsilon'$. We have $|f^{-1}(\bar{0})| = |\{a = (a_1, \ldots, a_m) : a_i \in \{0, 1\}$ for each $i \in \{1, \ldots, m\}\}| = 2^m$. Next, if $a \in f^{-1}(\bar{0})$, then $\psi(a) \in \{0, 1\}$ for each $\psi \in \Upsilon'$. Since $2^{|\Upsilon'|} < 2^m$, there are $a, b \in f^{-1}(\bar{0})$, $a \neq b$ such that $\psi(a) = \psi(b)$ for each $\psi \in \Upsilon'$. Without loss of generality, in view of the assumption that $|\{i \in \{1, \ldots, m\} : k_i = 1\}| \leq 1$ we get that $f^{-1}(b) \neq \emptyset$; let $x \in f^{-1}(b)$. Then $\psi(a) = \psi(f(x))$ for each $\psi \in \Upsilon'$. The set Υ' is a $(1, k)$-realizer, of (A, f), thus $(*)$ implies $a = f(x) = b$, which is a contradiction. Therefore $(1, k)$-pdim $(A, f) = m$. □

Lemma 3.4. Let (B, f) be a monounary algebra fulfilling the condition

(c) if $b \in B$, then there is $b' \in B$ with $f(b) = f(b')$, $f^{-1}(b') = \emptyset$.

Let (E, f) be a 1-stick. Then $(B, f) \times (E, f)$ fulfils (c) and if $(1, k)$-pdim$(B, f) = p$, then $(1, k)$-pdim$((B, f) \times (E, f)) = p$.

Proof. Without loss of generality, $E = \{0, 1\}$. First we show (c) for the algebra $(B, f) \times (E, f)$. Let $(b, e) \in B \times E$. By (c), there is $b' \in B$ with $f(b) = f(b')$, $f^{-1}(b') = \emptyset$. Take $(b', e) \in B \times E$. Then

$$f((b', e)) = (f(b'), f(e)) = (f(b), f(e)) = f((b, e)),$$

$$f^{-1}((b', e)) = \{(x_1, x_2) : x_1 \in f^{-1}(b'),\ x_2 \in f^{-1}(e)\} = \emptyset.$$

Further suppose that $(1, k)$-pdim$(B, f) = p$ and that Υ is a $(1, k)$-realizer of (B, f). For $\varphi \in \Upsilon$ we define a mapping $\bar{\varphi} : B \times E \to L$ as follows. Let $(b, e) \in B \times E$, b' be the element corresponding to b in view of the condition (c). We put

$$\bar{\varphi}((b, e)) = \begin{cases} \varphi(b) \text{ if } e = 0, \\ \varphi(b') \text{ if } e = 1; \end{cases}$$

$\bar{\Upsilon} = \{\bar{\varphi} : \varphi \in \Upsilon\}$. To prove that $\bar{\Upsilon}$ is a $(1, k)$-realizer of $(B, f) \times (E, f)$ assume that (x, e), $(y, j) \in B \times E$ and that $\bar{\varphi}(f((x, e))) = \bar{\varphi}((y, j))$ for each $\bar{\varphi} \in \bar{\Upsilon}$. For any $\varphi \in \Upsilon$ we have

$$\bar{\varphi}(f((x, e))) = \bar{\varphi}((f(x), 0)) = \varphi(f(x)).$$

If $j = 0$, then

$$\bar{\varphi}((y, j)) = \bar{\varphi}((y, 0)) = \varphi(y),$$

thus the fact that $x, y \in B$ and that Υ is a $(1,k)$-realizer of (B, f) implies that $y = f(x)$, hence

$$(y, j) = (f(x), 0) = f((x, 0)) = f((x, e)).$$

Let $j = 1$. To $y \in B$ there is $y' \in B$ with $f(y) = f(y'),\ f^{-1}(y') = \emptyset$. For any $\varphi \in \Upsilon$,

$$\bar{\varphi}((y, j)) = \bar{\varphi}((y, 1)) = \varphi(y'),$$

i.e., $\varphi(y') = \varphi(f(x))$ for each $\varphi \in \Upsilon$. Since Υ is a $(1,k)$-realizer of (B, f), this implies that $y' = f(x)$, which is a contradiction, because $f^{-1}(y') = \emptyset$. Thus j cannot be 1. Therefore

$$(1,k)\text{-pdim}((B, f) \times (E, f)) \le (1,k)\text{-pdim}(B, f).$$

The converse relation is obvious, thus
$(1,k)\text{-pdim}((B, f) \times (E, f)) = (1,k)\text{-pdim}(B, f)$. □

Corollary 3.5. Let (A, f) be a direct product of sticks $(A_1, f), \dots, (A_n, f)$ of types $k_1, k_2, \dots, k_m$ and assume that $|\{i \in \{1, \dots, m\} : k_i = 1\}| = t > 1$. If $k \ge k_i$ for each $i \in \{1, \dots, m\}$, then $(1,k)$-pdim $(A, f) = m - t + 1$.

Proof. The assertion is a consequence of 3.3 and 3.4; we can proceed by induction.

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(Received October 10, 2001)

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