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SOME ERROR ESTIMATES IN THE NEWTON METHOD

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Abstract. For the numerical solution of the equation $f(x) = 0$ by the Newton method the inequality $|x_{n-1} - x_n| \le \delta$ is often used as a stopping rule, where $\delta > 0$ is prescribed. We show that this inequality yields no information about $|x_n - x_0|$, where x_0 is a root, because the inequality $|x_n - x_0| \le |x_{n-1} - x_n|$ is not true in a general case. We give several simple estimates for $|x_n - x_0|$. Particularly, we give a sufficient condition under which $|x_n - x_0| \le |x_{n-1} - x_n|$.

We consider the equation $f(x) = 0$, where f is a convex or concave strictly monotone function of the class C^1 on the interval $\langle a, b\rangle$ such that $f(a)f(b) < 0$ and $\min(|f'(a)|, |f'(b)|) > 0$. To find a numerical solution x_0 of this equation the Newton method is often used. Namely, put

$$x_1 = \begin{cases} b \text{ if } f \text{ is convex and increasing or concave and decreasing} \\ a \text{ if } f \text{ is convex and decreasing or concave and increasing} \end{cases}$$

and

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \qquad \text{for } n > 1 \ .$$

Then we obtain a monotone sequence $(x_n)_{n=1}^{\infty}$ which converges to the root x_0. The inequality $|x_{n-1} - x_n| \le \delta$ is often used as a stopping rule, see [2, p. 87]. This rule is only formal, i.e. it does not imply $|x_n - x_0| \le \delta$, because the estimate $|x_n - x_0| \le |x_{n-1} - x_n|$ is false in a general case. This fact is illustrated on Figure 1.

To disprove this estimate we also consider the equation $\tan x = x$, or equivalently $\tan x - x = 0$, on the interval $\langle \pi, \frac{3\pi}{2})$. Put

$$f(x) = \tan x \ - \ x \ .$$

Since

$$f(\pi) < 0 \text{ and } \lim_{x \nearrow \frac{3\pi}{2}} f(x) = +\infty \ ,$$

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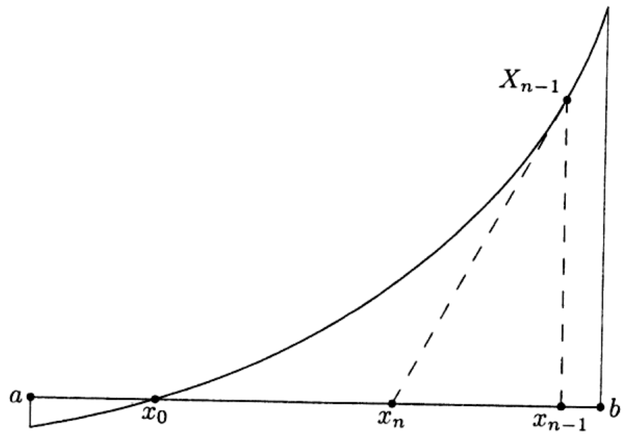

FIGURE 1 *Newton method*

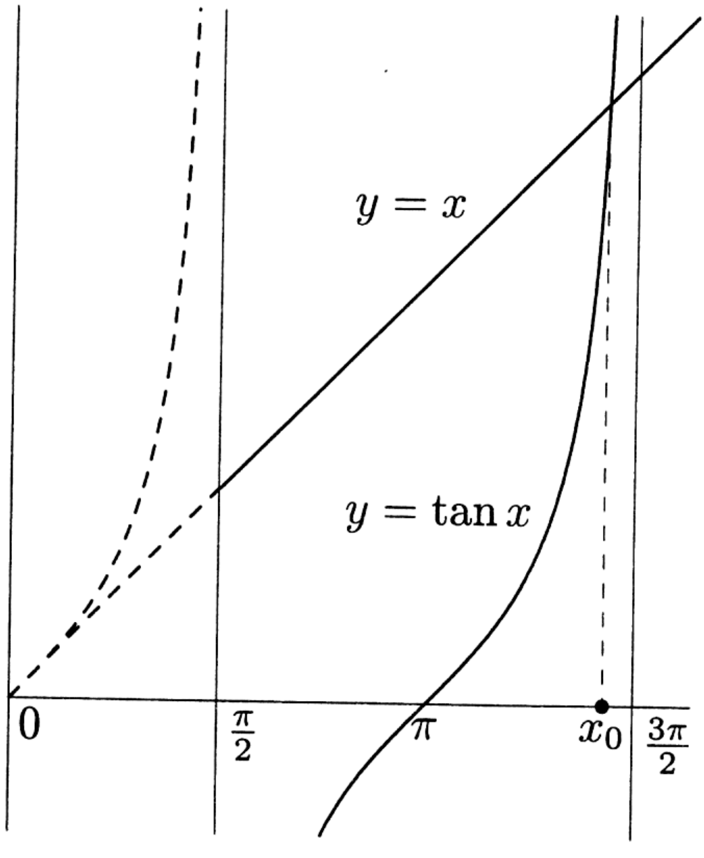

FIGURE 2 *Equation* $\tan x = x$

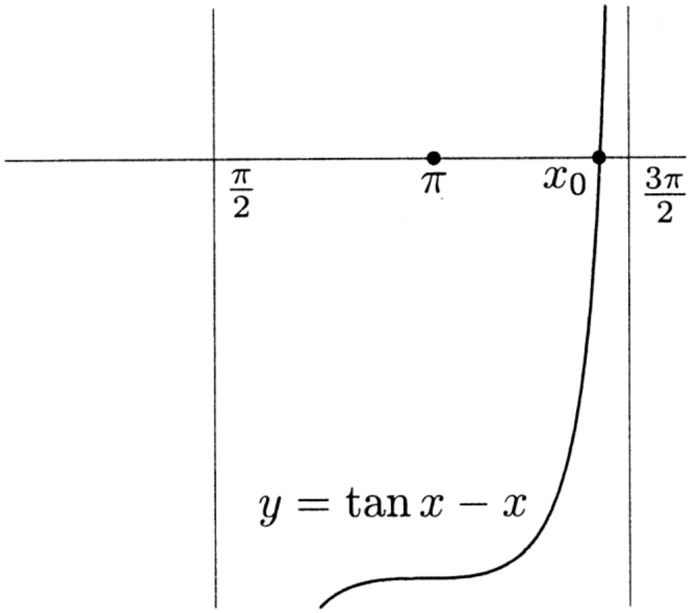

FIGURE 3 *Equation* $\tan x - x = 0$

our equation has a root in the interval $\langle \pi, \frac{3\pi}{2})$.

The derivative

$$f'(x) = \frac{1}{\cos^2 x} - 1 = \tan^2 x$$

is positive and increasing on $(\pi, \frac{3\pi}{2})$. Therefore, f is an increasing and convex function on this interval. We solve this equation by the Newton method starting with $x_1 = \frac{3\pi}{2} - 10^{-4}$. The results are presented in Table 1.

We see that the estimate $|x_n - x_0| \leqslant |x_n - x_{n-1}|$ is false. For the behaviour of the sequence $(x_n)_{n=1}^{\infty}$ note that

$$x_n = g(x_{n-1}), \text{ where } g(x) = \frac{x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} .$$

The point $\frac{3\pi}{2}$ is not a root of our equation, but it is a repulsive fixed point of the function g, because

$$g'\left(\frac{3\pi}{2}\right) = 2 > 1 .$$

For any $x_1 \in \left(x_0, \frac{3\pi}{2}\right)$ we obtain a sequence converging to x_0, but if the initial point x_1 is closed to $\frac{3\pi}{2}$ then also x_n is closed to $\frac{3\pi}{2}$ for many n. Particularly, if we take $x_1 = \frac{3\pi}{2} - 10^{-9}$, then we obtain an example which shows that the convergence of the Newton method may be slower than the convergence of the bisection method.

Remark. All values in this paper were evaluated onto 18-20 significant digits and then all outputs were rounded onto 10 significant digits.

For the evaluation of x_0 with a given accuracy δ there are several possibilities. The simplest way is the evaluation of x_n until the inequality $f(x_n - \delta) f(x_n + \delta) < 0$

n	x_n	$x_{n-1} - x_n$	$x_n - x_0$
1	4.712288980		0.218879522
2	4.712189028	0.000099953	0.218779570
3	4.711989263	0.000199764	0.218579805
4	4.711590298	0.000398965	0.218180841
5	4.710794622	0.000795677	0.217385164
6	4.709212237	0.001582385	0.215802779
7	4.706083007	0.003129230	0.212673549
8	4.699964094	0.006118913	0.206554636
9	4.688264213	0.011699881	0.194854755
10	4.666864413	0.021399800	0.173454955
11	4.630993761	0.035870652	0.137584303
12	4.580235510	0.050758252	0.086826052
13	4.528239646	0.051995864	0.034830188
14	4.499076575	0.029163071	0.005667117
15	4.493560666	0.005515909	0.000151208
16	4.493409566	0.000151100	0.000000108
17	4.493409458	0.000000108	0.000000000
18	4.493409458	0.000000000	0.000000000

TABLE 1

is satisfied. The second possibility is a combination of the Newton method with a modified method of false position, [1, p.183], i.e. together with x_n we evaluate also ξ_n by

$$\xi_1 = a \quad \text{and}$$

$$\xi_n = x_n - \frac{f(x_n)(x_n - \xi_{n-1})}{f(x_n) - f(\xi_{n-1})} \qquad \text{for } n > 1\,.$$

Then we have $\xi_n < x_0 < x_n$, (when the function f is increasing and convex). It means that the inequality $x_n - \xi_n < \delta$ guarantees $x_n - x_0 < \delta$. Table 2 contains the results of evaluation of ξ_n and x_n for the equation $\tan x = x$ starting with $\xi_1 = 4.3$ and $x_1 = 4.7$.

n	ξ_n	x_n	$x_n - \xi_n$
1	4.300000000	4.700000000	0.400000000
2	4.320114416	4.688331848	0.368217432
3	4.354413674	4.666984472	0.312570798
4	4.404248369	4.631183287	0.226934918
5	4.456982727	4.580473096	0.123490370
6	4.487397534	4.528429052	0.041031518
7	4.493247036	4.499138109	0.005891073
8	4.493409340	4.493563964	0.000154625
9	4.493409458	4.493409570	0.000000113
10	4.493409458	4.493409458	0.000000000

TABLE 2 *Left and right estimates of the root*

For the comparison we evaluate ξ_n by the (simple) method of false position, i.e.

$$\xi_n = b - \frac{f(b)(b - \xi_{n-1})}{f(b) - f(\xi_{n-1})}, \text{ where}$$
$$b = 4.7 \quad \text{and} \quad \xi_1 = 4.3 .$$

n	ξ_n
1	4.300000000
2	4.310325422
3	4.320114062
4	4.329392330
5	4.338185494
6	4.346517706
7	4.354412045
8	4.361890542
9	4.368974227
10	4.375683153

TABLE 3 *Simple method of false position*

Tables 2 and 3 show that the convergence of the modified method of false position convergence is faster then the convergence of the simple method. Roughly speaking, the Newton method accelerates the convergence of the modified method of false position.

Another left estimates ξ_n of the root may be evaluated by the following scheme, see [4, p.180]. Put

$$\xi_1 = a \quad \text{and}$$
$$\xi_n = \xi_{n-1} - \frac{f(\xi_{n-1})}{f'(x_{n-1})} \quad \text{for } n > 1 .$$

The results are contained in Table 4.

n	ξ_n	x_n	$x_n - \xi_n$
1	4.300000000	4.700000000	0.400000000
2	4.301166132	4.688331848	0.387165716
3	4.305311541	4.666984472	0.361672931
4	4.318465687	4.631183287	0.312717600
5	4.352138102	4.580473096	0.228334994
6	4.410902541	4.528429052	0.117526511
7	4.466942647	4.499138109	0.032195462
8	4.490428002	4.493563964	0.003135962
9	4.493368097	4.493409570	0.000041473
10	4.493409450	4.493409458	0.000000008
11	4.493409458	4.493409458	0.000000000

TABLE 4 *Another left estimates of the root*

The another possibility of the estimate of $|x_n - x_0|$ is given by the following theorem, cf. [1, p.163] and [4, p.183].

Theorem 1. *Let f be a convex or concave strictly monotone function of the class C^1 on the interval $\langle a, b\rangle$ such that $f(a)f(b) < 0$, $|f'(a)| > 0$ and $|f'(b)| > 0$. Let $x_0 \in \langle a, b\rangle$ be the root of f and the sequence $(x_n)_{n=1}^{\infty}$ be of the Newton method. Then*

$$|x_n - x_0| \le \frac{|f(x_n)|}{A}\,, \quad \text{where} \quad A = \min(|f'(a)|, |f'(b)|)\,.$$

Proof. Since f is strictly monotone, $f'(a)$ and $f'(b)$ have the same sign. Moreover,

$$\min_{x\in\langle a,b\rangle} f'(x) = \min(|f'(a)|, |f'(b)|)\,,$$

because f is convex or concave. Therefore,

$$|f(x_n)| = |f(x_n) - f(x_0)| = |f'(\xi)||x_n - x_0| \geqslant A(|x_n - x_0|\,,$$

where ξ is between x_n and x_0.

Example. Take values from Table 1 and $a = 4.45$. Then $a < x_0$, because $f(a) < 0$. Since $A = f'(a)$, we obtain

$$0 \le x_{18} - x_0 \le \frac{f(x_{18})}{f'(4.45)} \doteq 1.3 \cdot 10^{-10}$$

Finally, we prove the following result.

Theorem 2. *Let f be a convex or concave strictly monotone function of the class C^1 on the interval $\langle a, b\rangle$ such that $f(a)f(b) < 0$, $|f'(a)| > 0$ and $|f'(b)| > 0$. Let $x_0 \in \langle a, b\rangle$ be the root of f and the sequence $(x_n)_{n=1}^{\infty}$ be of the Newton method. Then*

$$0 \le |x_n - x_0| \leqslant \left(\frac{|f'(x_{n-1})|}{A} - 1\right)|x_{n-1} - x_n| \leqslant \left(\frac{B}{A} - 1\right)|x_{n-1} - x_n|\,,$$

where

$$A = \min(|f'(a)|, |f'(b)|) \text{ and } B = \max(|f'(a)|, |f'(b)|)\,.$$

Particularly,

$$|x_n - x_0| \leqslant |x_{n-1} - x_n| \text{ whenever } B \leqslant 2A \text{ or } f'(x_{n-1}) \leqslant 2A\,.$$

Proof. Without loss of generality we may assume that the function f is increasing and convex. We have

$f(x_{n-1}) = f'(x_{n-1})(x_{n-1} - x_n)$
$f(x_{n-1}) = f(x_{n-1}) - f(x_0) = f'(\xi)(x_{n-1} - x_0) = f'(\xi)(x_{n-1} - x_n) + f'(\xi)(x_n - x_0)$,
where $x_0 < \xi < x_{n-1}$.

Therefore,

$$f'(\xi)(x_n - x_0) = f'(x_{n-1})(x_{n-1} - x_n) - f'(\xi)(x_{n-1} - x_n)$$

and

$$x_n - x_0 = \frac{f'(x_{n-1})}{f'(\xi)}(x_{n-1} - x_n) - (x_{n-1} - x_n) = \left(\frac{f'(x_{n-1})}{f'(\xi)} - 1\right)(x_{n-1} - x_n) .$$

Since f is convex and $a < x_0 < \xi < x_{n-1} < b$, we have $f'(a) \leqslant f'(\xi)$ and $f'(x_{n-1}) \leqslant f'(b)$. So, we obtain the desired inequality.

Example. Take values from Table 1 and $a = 4.45$. We obtain

$$0 \leq x_{18} - x_0 \leq \left(\frac{f'(x_{17})}{f'(4.45)} - 1\right)(x_{17} - x_{18}) \leq 4.5 \cdot 10^{-10} .$$

(By Table 1 we have $x_{17} - x_{18} = 0$. However, both values x_{17} and x_{18} are rounded. Therefore, we have used $x_{17} - x_{18} \leq 10^{-9}$.)

Remark. If the inequality $B \leqslant 2A$ is not satisfied, then the interval $\langle a, b\rangle$ is too wide. We may use the bisection method until this inequality is satisfied and then the Newton method may be used with a true estimate $x_n - x_0 \leq x_{n-1} - x_n$.

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