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ECCENTRIC SEQUENCES AND CYCLES IN GRAPHS

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Abstract. An eccentric sequence is called minimal if it has no proper eccentric subsequence with the same number of distinct eccentricities. A graph is said to be a minimal graph if it realizes a minimal eccentric sequence. Some minimal graphs and some minimal eccentric sequences are described. It is shown that a graph with radius r, diameter $d \leq 2r-2$ and with at most $3r-2$ vertices contains a cycle of length $2r$ or $2r+1$.

1. Introduction

The eccentricity of a vertex v of a connected graph is the distance between v and a vertex furthest from v. To any finite connected graph G one can assign the sequence of the eccentricities of its vertices, called the eccentric sequence of G. To decide whether a sequence of positive integers is the eccentric sequence of some graph is a difficult task. Only very few results are known in this direction (see [1] and the recent survey [2]). L. Lesniak showed that a sequence S of positive integers is the eccentric sequence of some graph if and only if some subsequence S' of S with the same number of distinct values is eccentric. This result naturally leads to the concept of minimal eccentric sequences. Throughout the paper, any graph which realizes a minimal eccentric sequence is said to be a minimal graph.

In the paper we describe large classes of minimal unicyclic graphs with an even cycle (see Theorem 3.1) and with an odd cycle (see Theorem 3.2). From this we obtain an explicit description of minimal eccentric sequences with two distinct values and with length at most $\left\lfloor \frac{8r+5}{3} \right\rfloor$ (see Theorem 3.4). The key result of the present paper is our Theorem 2.6, which asserts that a graph with radius r, diameter $d \leq 2r-2$ and with at most $3r-2$ vertices contains a cycle of length $2r$ or $2r+1$. In fact, it significantly reduces the number of possibilities which one needs to consider to prove Theorems 3.1 and 3.2.

We are going to recall terminology and fix notations. We consider undirected connected finite graphs without loops and multiple edges. We will use standard notations of the graph theory (see for example [4]). We recall some of them. We denote by $V(G)$ the vertex set and by $E(G)$ the set of edges of a graph G. The

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symbol $|V(G)|$ is used for cardinality of $V(G)$. Let $u, v \in V(G)$, by a $u - v$ *path* we mean the finite alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ of vertices and edges beginning with vertex u and ending with vertex v such that $e_i = u_{i-1}u_i$ for $i = 1, 2, \ldots, k$, in which no vertex is repeated. It is also denoted by $P_k = (u_0, u_1, u_2, \ldots, u_k)$; the number k is its *length*. If $P_{n-1} = (u_1, u_2, \ldots, u_n)$ is a path and $u_n u_1 \in E(G)$ then $C_n = (u_1, u_2, \ldots, u_n, u_1)$ is a cycle of length n (throughout the paper, C_n will always denote a cycle of length n). The subgraph of a graph G induced by the edges of a path or a cycle is also referred to as a path or a cycle of G. The *distance* $d(u, v)$ between two vertices u and v is the minimum of the lengths of the $u - v$ paths of G. A shortest $u - v$ path is called a $u - v$ *geodesic path*. We denote by $d_{G'}(u, v)$ the distance between vertices $u, v \in V(G')$ in the subgraph G' of the graph G. The distance between a vertex $u \in V(G)$ and a subgraph G' of a graph G will be denoted by $d_G(v, G')$, i.e. $d_G(v, G') = \min\{d_G(v, x); x \in V(G')\}$.

Denote the degree of a vertex $u \in V(G)$ by $\deg_G(u)$ and the eccentricity of a vertex $u \in V(G)$ by $e_G(u)$. Recall that

$$e_G(u) = \max\{d_G(u, v);\ v \in V(G)\}.$$

We denote it briefly by $e(u)$ when no confusion can arise. We will use the symbol rad G to denote the radius of the graph G (i.e. the minimum of eccentricities of vertices of the graph G). The symbol diam G is used for the diameter of the graph G (i.e. the maximum of eccentricities of its vertices). We write simply r and d when there is no confusion.

The eccentric sequence of a graph G is a list of the eccentricities of its vertices in nondecreasing order. Since often there are some vertices having the same eccentricity we will denote it simply

$$e(G) = (e_1^{m_1}, e_2^{m_2}, \ldots, e_k^{m_k}) = (e_i^{m_i})_{i=1}^{k}$$

where e_i are eccentricities for which $e_i < e_{i+1}$ and m_i is the multiplicity of e_i. A sequence of positive integers is called eccentric if there is a graph which realizes the considered sequence. By Lesniak [5] the following statement holds.

Theorem 1.1. *A nondecreasing sequence* $(e_1^{m_1}, e_2^{m_2}, \ldots, e_k^{m_k})$ *is eccentric if and only if some of its subsequences with* k *distinct values is eccentric.*

An eccentric sequence is called minimal (by R. Nandakumar, see [1]) if it has no proper eccentric subsequence with the same number of distinct eccentricities. A graph is said to be a minimal graph if it realizes a minimal eccentric sequence.

Let $e(G_1) = (e_1^{m_1}, e_2^{m_2}, \ldots, e_k^{m_k})$ and $e(G_2) = (e_1^{n_1}, e_2^{n_2}, \ldots, e_k^{n_k})$ be eccentric sequences of graphs G_1 and G_2. We write $e(G_1) \leq e(G_2)$ if $1 \leq m_i \leq n_i$ for each $i \in \{1, \ldots, k\}$. We write $e(G_1) < e(G_2)$ if $e(G_1) \leq e(G_2)$ and moreover there is $i \in \{1, \ldots, k\}$ for which $m_i < n_i$. If $e(G_1) < e(G_2)$ then it is obvious that $|V(G_1)| < |V(G_2)|$.

Definition 1.2. A graph G is said to be a *sun-graph* if it is unicyclic (i.e. it has exactly one cycle C), $\deg_G(u) \leq 3$ for $u \in V(C)$ and $\deg_G(u) \leq 2$ for $u \in V(G) - V(C)$. The mentioned cycle C is called the *kernel* of the sun-graph G.

Definition 1.3. Let G be a sun-graph with the kernel C. A $u - v$ path in the graph G is said to be a *ray* of the graph G if $\deg_G(u) = 1$ and v is the only vertex of the path belonging to $V(C)$. If G is a sun-graph with n rays we briefly say that G is an *n-rays sun-graph*.

In Figure 1.1 it is depicted a 4-rays sun-graph.

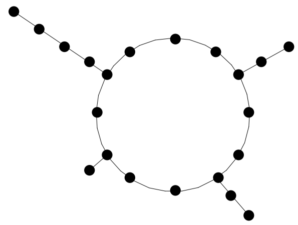


Fig. 1.1

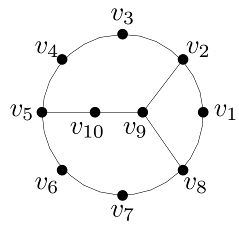


Fig. 1.2

Definition 1.4. A cycle C in a graph G is called a *geodesic cycle* if for each two vertices x, y of the cycle C it holds $d_C(x,y) = d_G(x,y)$.

In Figure 1.2 the cycle $(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1)$ is a geodesic cycle and $(v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_2)$ is not a geodesic cycle.

Definition 1.5. Let C be a cycle of a graph G and let its length be $2k$ or $2k+1$. A vertex of the cycle C is said to be C-*excited* (in the graph G) if its eccentricity is larger than k. The number of the C-excited vertices of G will be denoted by $\text{exc}_G(C)$.

Lemma 1.6. *Let C be a cycle of a graph G and $|V(G)| - |V(C)| = m$. Then*
a) $exc_G(C) \leq 2m - 1$ if length of the cycle C is even and $m \geq 1$,
b) $exc_G(C) \leq 2m$ if length of the cycle C is odd,
c) $exc_G(C) \leq 2m - n$ if C is an even cycle and there are at least n vertices from $V(G) - V(C)$ such that each of them is adjacent to at least one vertex of the cycle C.

Proof. a) Consider a sequence $G_0 = C$, G_1, ... , G_m of subgraphs of the graph G such that the subgraph G_{i+1} is obtained from the subgraph G_i (for $i < m$) by adding some vertex $u \in V(G) - V(G_i)$ and some edge $uv \in E(G)$ where $v \in V(G_i)$ (see Figure 1.3). Obviously, $\text{exc}_{G_1}(C) = 1$ and $\text{exc}_{G_{i+1}}(C) \leq \text{exc}_{G_i}(C) + 2$. Therefore the graph G_m contains at most $2m - 1$ C-excited vertices. Since $V(G) = V(G_m)$ and $E(G_m) \subset E(G)$ we get $\text{exc}_G(C) \leq \text{exc}_{G_m}(C)$.

b) In this case $\text{exc}_{G_1}(C) = 2$ and we can prove the statement analogously as in the case a).

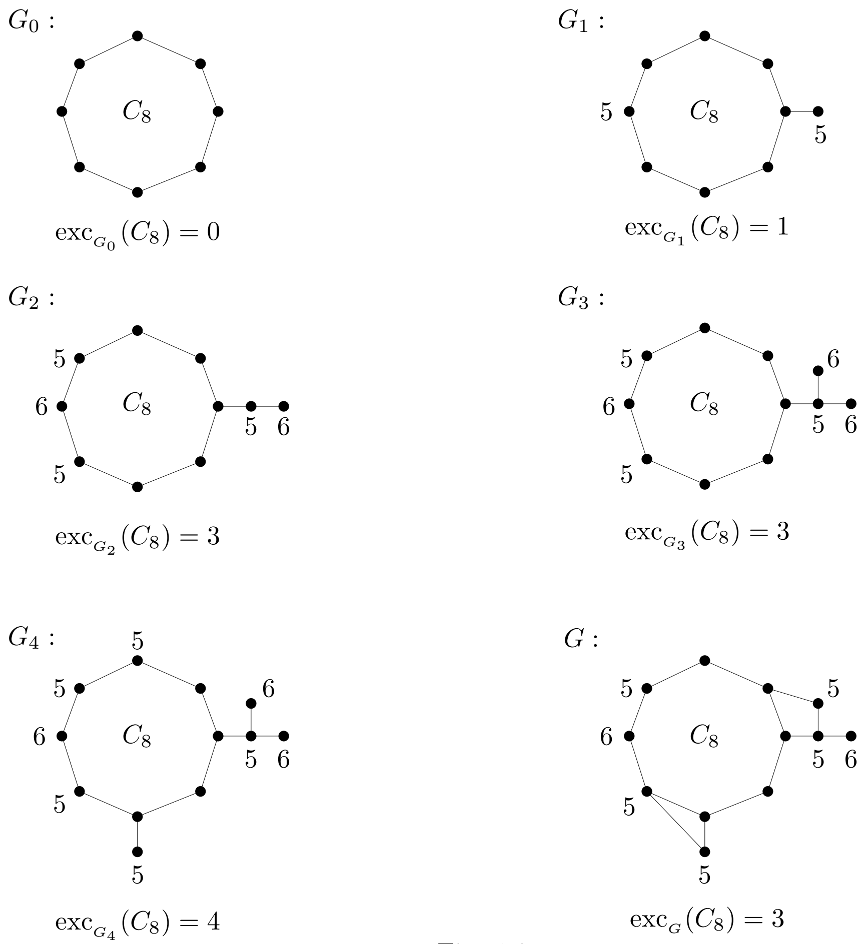


Fig. 1.3

c) The required sequence can be chosen in such a way that each vertex from $V(G_n) - V(C)$ is adjacent to a vertex of the cycle C. In this case $\mathrm{exc}_{G_n}(C) \leq n$ and then obviously $\mathrm{exc}_G(C) \leq 2m - n$. $\square$

The following statement can be proved analogously to Lemma 1.6.

Lemma 1.7. *Let G_1 be a unicyclic subgraph of G with the cycle C. If $A = \{v \in V(C); e_{G_1}(v) < rad\ G\}$ then $|V(G)| - |V(G_1)| \geq \left\lceil \frac{|A|}{2} \right\rceil$ ($\lceil x \rceil$ is the least integer $i \geq x$).*

2. On cycles in graphs.

In this section we show that a graph G with radius r and diameter d, for which $d \leq 2r - 2$ and $|V(G)| \leq 3r - 2$, contains a cycle C_{2r} or C_{2r+1}. The statement is a consequence of the following three lemmas.

Lemma 2.1. *Let G be a graph with radius r and let G contain the subgraph G_1 depicted in Fig. 2.1. If $m + n + 2 \geq 2r$, $m + k + 2 < 2r$ and $n + k + 2 < 2r$ ($k = 0$ is possible and $uv \in E(G_1)$ in this case) then*

$$|V(G)| > 2r - 1 + \frac{m+n}{2}.$$

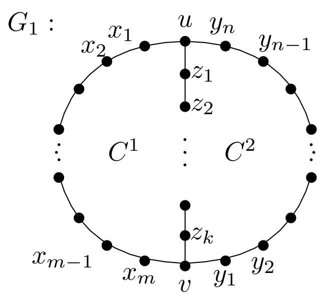


Fig. 2.1

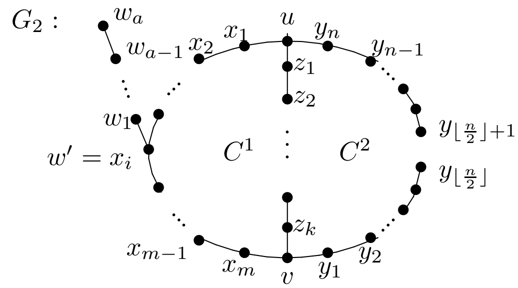


Fig. 2.2

Proof. Since $e_G(u) \geq r$ there is a vertex $w \in V(G)$ such that $d_G(u, w) \geq r$ and it is clear that $w \notin V(G_1)$. Let P be a $u - w$ geodesic path. Let w' be the last vertex of P such that $w' \in V(G_1)$. Without loss of generality we can assume that $w' \in \{u, v, x_1, \dots, x_m, z_1, \dots, z_k\}$. Let $P = (u, \dots, w', w_1, w_2, \dots, w_j)$ where $w_j = w$.

Let $r_1 = \lfloor \frac{m+k+2}{2} \rfloor$. By the assumption $m+k+2 < 2r$ and therefore the number $a = r - 1 - r_1$ is nonnegative and obviously $a \leq j$.

Consider a subgraph G_2 of the graph G (see Figure 2.2) for which

$V(G_2) = V(G_1) \cup \{w_1, w_2, \dots, w_a\}$

$E(G_2) = E(G_1) \cup \{w'w_1, w_1w_2, \dots, w_{a-1}w_a\} - \{y_{\lfloor \frac{n}{2} \rfloor} y_{\lfloor \frac{n}{2} \rfloor+1}\}$ if $a > 0$.

If $a = 0$ then $V(G_2) = V(G_1)$ and $E(G_2) = E(G_1) - \{y_{\lfloor \frac{n}{2} \rfloor} y_{\lfloor \frac{n}{2} \rfloor+1}\}$ (we put $y_0 = v$ if $n = 1$). Now consider the cycle $C^1 = (u, x_1, \dots, x_m, v, z_k, \dots, z_1, u)$. If $x \in V(C^1)$ then $d_{G_2}(w_i, x) \leq a + r_1 = r - 1$ for $i = 1, 2, \dots, a$. Therefore $e_{G_2}(x) \geq r$ for $x \in V(C^1)$ if and only if $d_{G_2}(x, y_{\lfloor \frac{n}{2} \rfloor}) \geq r$ or $d_{G_2}(x, y_{\lfloor \frac{n}{2} \rfloor+1}) \geq r$.

Let $A = \{x \in V(C^1); e_{G_2}(x) < r\}$. We show that $|A| = 2r - n - k - 2$. Consider two cases.

a) $\lfloor \frac{n}{2} \rfloor + k + 1 \geq r$

In this case $k > 0$ and since $m + n + 1 \geq 2r - 1$ we get $A \subseteq \{z_1, \dots, z_k\}$. Let $A_1 = \{x \in A; d_{G_2}(x, y_{\lfloor \frac{n}{2} \rfloor+1}) < r\}$ and $A_2 = \{x \in A; d_{G_2}(x, y_{\lfloor \frac{n}{2} \rfloor}) < r\}$. Then $|A| = |A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2| = (r - (n + 1 - \lfloor \frac{n}{2} \rfloor)) + (r - (\lfloor \frac{n}{2} \rfloor + 1)) - k$ and so we have $|A| = 2r - n - k - 2$.

b) $\lfloor \frac{n}{2} \rfloor + k + 1 < r$

In this case $\{z_1, z_2, \dots, z_k\} \subseteq A$ and we get

$|A| = k + (r - (k + n + 1 - \lfloor \frac{n}{2} \rfloor)) + (r - (k + 1 + \lfloor \frac{n}{2} \rfloor)) = 2r - n - k - 2.$

Since the eccentricity of each vertex of G is at least r, by Lemma 1.7 we have $|V(G)| - |V(G_2)| \geq \left\lceil \frac{|A|}{2} \right\rceil$. It follows that

$$
\begin{aligned}
|V(G)| &\geq m + n + k + 2 + a + \left\lceil \frac{|A|}{2} \right\rceil \\
&= m + n + k + 2 + r - 1 - \left\lfloor \frac{m+k+2}{2} \right\rfloor + \left\lceil \frac{2r - n - k - 2}{2} \right\rceil \\
&\geq m + n + k + 2 + r - 1 - \frac{m+k+2}{2} + \frac{2r - n - k - 2}{2} \\
&= 2r - 1 + \frac{m+n}{2}.
\end{aligned}
$$

It is clear that the equality does not hold if at least one from the integers $m+k$ and $n+k$ is odd. In the opposite case the lengths of the cycles C^1 and C^2 (see Figure 2.1) are even and $|A|$ is also an even integer. It is easy to see that in this case $|V(G)| - |V(G_2)| > \frac{|A|}{2} = \left\lceil \frac{|A|}{2} \right\rceil$. It implies $|V(G)| > 2r - 1 + \frac{m+n}{2}$ and the proof is complete. $\square$

Corollary 2.2. *Let a graph G contain a subgraph isomorphic to the graph in Fig. 2.1 and rad $G = r$. If $m+n+2 \geq 2r$, $m+k+2 < 2r$ and $n+k+2 < 2r$ then $|V(G)| \geq 3r - 1$.*

Proof. Since $m+n \geq 2r-2$, by Lemma 2.1 we get
$|V(G)| > 2r - 1 + \frac{m+n}{2} \geq 2r - 1 + \frac{2r-2}{2} = 3r - 2$. $\square$

Remark. The graph G in Figure 2.3 satisfies the assumption of Corollary 2.2 ($m = n = 2$, $k = 0$, $r = 3$) and $|V(G)| = 3r - 1$.

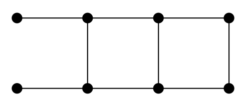


Fig. 2.3

Lemma 2.3. *Let G be a graph with radius r which contains a geodesic cycle C_m. If $m \geq 2r+2$ then $|V(G)| \geq \frac{3m-4}{2}$.*

Proof. Let $u \in V(G)$ be a vertex with eccentricity $e(u) = r$. Since the eccentricity of each vertex of C_m is at least $r+1$ it follows that $u \notin V(C_m)$. Denote by G' the component of the graph $G - C_m$ containing the vertex u.

We distinguish two cases.

a) Each path of the cycle C_m having the length $\left\lceil \frac{m}{2} \right\rceil - 2$ contains a vertex adjacent to a vertex of G'.
In this case there exist vertices $w_1, w_2, w_3 \in V(C_m)$ and (not necessarily distinct) vertices $w_1', w_2', w_3' \in V(G')$ satisfying
(i) $w_1w_1', w_2w_2', w_3w_3' \in E(G)$,
(ii) $d_{C_m}(w_1, w_2) + d_{C_m}(w_2, w_3) + d_{C_m}(w_3, w_1) = m$.
Let P^1 be a $w_1' - w_2'$ geodesic path in G'. Let v be a vertex of the path P^1 such that $d_{G'}(w_3', v) = d_{G'}(w_3', P^1)$.

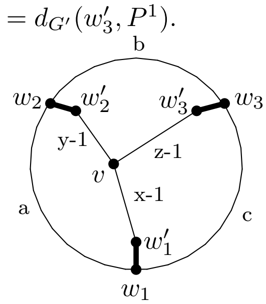


Fig. 2.4

Denote $a = d_G(w_1, w_2) = d_{C_m}(w_1, w_2)$, $b = d_G(w_2, w_3) = d_{C_m}(w_2, w_3)$, $c = d_G(w_1, w_3) = d_{C_m}(w_1, w_3)$, $x = d_{G'}(v, w_1') + 1$, $y = d_{G'}(v, w_2') + 1$, $z = d_{G'}(v, w_3') + 1$ (see schematic Figure 2.4).

Since C_m is a geodesic cycle we get

$a \le x + y, \qquad b \le y + z, \qquad c \le x + z.$

Hence,

$x + y + z \ge \frac{a+b+c}{2} = \frac{m}{2}$. It implies

$|V(G)| \ge a + b + c + x + y + z - 2 \ge m + \frac{m}{2} - 2 = \frac{3m-4}{2}$.

b) There exists a path A of C_m satisfying the following two conditions

(j) the length of A is at least $\lceil \frac{m}{2} \rceil - 2$,

(jj) no vertex of A is adjacent to a vertex of G'.

We are going to show that this assumption yields a contradiction. In this case there exists a path of C_m of length at most $m - (\lceil \frac{m}{2} \rceil - 2) - 2 = \lfloor \frac{m}{2} \rfloor$ containing every vertex $v \in V(C_m)$ adjacent to at least one vertex of G'. Therefore there is a path $(v_1, v_2, \ldots, v_k)$ of C_m satisfying the following three conditions

(k) $0 \le k - 1 \le \lfloor \frac{m}{2} \rfloor$,

(kk) the vertex v_1 is adjacent to a vertex from $V(G')$ and the vertex v_k is also adjacent to a vertex from $V(G')$,

(kkk) no vertex $v \in V(C_m) - \{v_1, v_2, \ldots, v_k\}$ is adjacent to a vertex from $V(G')$.

Denote by P^1 a $u - v_1$ geodesic path and by P^2 a $u - v_k$ geodesic path in the subgraph of G induced by the set $V(C_m) \cup V(G')$. Note that P^1 and P^2 are also geodesic paths in G. Let v_i be the first vertex of P^1 belonging to C_m and v_j be the first vertex of P^2 belonging to C_m. Since C_m is a geodesic cycle it is easy to see that $i \le j$ and we can assume that $P^1 = (u, \ldots, v_i, v_{i-1}, v_{i-2}, \ldots, v_1)$ and $P^2 = (u, \ldots, v_j, v_{j+1}, v_{j+2}, \ldots, v_k)$. Let u_1 be the last vertex of P^1 belonging to P^2. Since C_m is a geodesic cycle there exists (if $d_G(v_i, v_j) < \lfloor \frac{m}{2} \rfloor$; the case $d_G(v_i, v_j) = \lfloor \frac{m}{2} \rfloor = d_G(v_1, v_k)$ is obvious) a vertex $v_p \in V(C_m)$ such that the following three conditions are satisfied (see schematic Figure 2.5)

$i \le p \le j$, $d_G(v_i, v_p) \le d_G(v_i, u_1)$, $d_G(v_j, v_p) \le d_G(v_j, u_1)$.

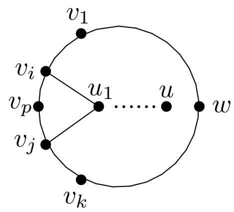


Fig. 2.5

Further, there exists a vertex $w \in C_m$, $w \notin \{v_1, v_2, \ldots, v_k\}$ such that $d_G(w, v_p) > r$. Obviously, every $u - w$ path contains at least one vertex from the set $\{v_1, v_2, \ldots, v_k\}$. Since C_m is a geodesic cycle, due to symmetry we may without loss of generality assume that some $u - w$ geodesic path in G contains the vertex v_1. Thus we get

$$d(u, w) = d(u, u_1) + d(u_1, v_i) + d(v_i, w) \geq d(u, u_1) + d(v_p, v_i) + d(v_i, w)$$
$$\geq d(u, u_1) + d(v_p, w) \geq d(v_p, w) > r.$$

A contradiction with the assumption $e(u) = r$.

□

Corollary 2.4. *Let G be a graph with radius r which contains a geodesic cycle C_m. If $m \geq 2r + 2$ then $|V(G)| \geq 3r + 1$.*

Proof. Since $m \geq 2r + 2$, by Lemma 2.3 we get $|V(G)| \geq \frac{3m-4}{4} \geq \frac{3(2r+2)-4}{2} = 3r + 1$. □

Remark. The graph in Fig. 1.2 contains a geodesic cycle of length $2r + 2$ and $|V(G)| = 3r + 1$ $(r = 3)$.

Lemma 2.5. *If a graph G satisfies the following conditions*

(i) *rad $G = r$,*
(ii) *diam $G \leq 2r - 2$,*
(iii) $|V(G)| \leq 3r - 2$

then the circumference of G (i.e. the length of any longest cycle of G) is at least $2r$.

Proof. Suppose on the contrary that the length of a longest cycle C_m of the graph G is $m < 2r$.

Let G^+ be the block (i.e. maximal 2-connected subgraph) of G containing the cycle C_m. Consider a vertex v_1 such that $d(v_1, G^+) = \max\{d(v, G^+); v \in V(G)\}$. Since rad $G = r$, $m < 2r$ and G^+ is a 2-connected graph we get $d_G(v_1, G^+) > 0$, i.e. $v_1 \notin V(G^+)$. Let $v_1' \in V(G^+)$ be a vertex for which $d(v_1, v_1') = d(v_1, G^+)$. Since G^+ is the block of G, the vertex v_1' is a cut-vertex of the graph G. Denote by G_1' the component of the graph $G - v_1'$ containing the vertex v_1 and let $G_1 = \langle V(G_1') \cup \{v_1'\} \rangle$, i.e. G_1 is the subgraph of G induced by the set $V(G_1') \cup \{v_1'\}$. Consider a vertex v_2 such that $d(v_2, G^+) = \max\{d(v, G^+); v \in V(G) - V(G_1)\}$. If $d(v_1, v_1') \leq r - 1$ then $d(v_2, G^+) > 0$ (otherwise $e_G(v_1') < r$). Denote by v_2' a vertex of G^+ for which $d(v_2, v_2') = d(v_2, G^+)$. The vertex v_2' is a cut-vertex of G. Let P^1 be a $v_1 - v_1'$ geodesic path and P^2 be a $v_2 - v_2'$ geodesic path. Now we distinguish two cases.

a) $d(v_1, v_1') \leq r - 1$

Since $d(v_1, v_2) \leq 2r - 2$ and $d(v_1, G^+) \geq d(v_2, G^+)$ there is a vertex

$u \in V(G^+)$ such that $d(v_1, u) \leq r-1$ and $d(v_2, u) \leq r-1$. We are going to show that $e_G(u) \leq r-1$ and this contradicts our assumption.
We show that $d(u,w) \leq r-1$ for every vertex $w \in V(G)$. Clearly, if $w \in V(G^+)$ then $d(w,u) < r$. So, let $w \in V(G) - V(G^+)$ and P be a $w-u$ geodesic path. If $v'_1 \in V(P)$ or $v'_2 \in V(P)$ then $d(w,u) \leq \max\{d(u,v_1), d(u,v_2)\} \leq r-1$. Assume that P contains neither the vertex v'_1 nor the vertex v'_2. Denote by w' the first vertex of the path P belonging to the graph G^+. Denote $a = d(w,w')$. Since $m < 2r$ there exists a positive integer k such that $m = 2r-2k$ (if m is even) or $m = 2r-2k+1$ (if m is odd). The vertices u and w' belong to the block G^+ which contains the longest cycle C_m, hence $d(u,w') \leq r-k$. If $a \leq k-1$ then $d(u,w) \leq (r-k)+(k-1) = r-1$. We will show that $a \geq k$ is impossible since it contradicts our assumption (iii). If $a \geq k$ then obviously there is a unicycle subgraph H of G containing the cycle C_m and k vertices from each of the paths P^1, P^2, P such that either at most 3 vertices of C_m have eccentricities at least r (if $m = 2r-2k$) or at most 6 vertices of C_m have eccentricities at least r (if $m = 2r-2k+1$). Then by Lemma 1.7 we get $|V(G)| \geq 3r-1$. Really,
if $m = 2r-2k$ then
$|V(G)| \geq (2r-2k)+3k+\lceil\frac{2r-2k-3}{2}\rceil = 2r+k+r-k-1 = 3r-1$, and
if $m = 2r-2k+1$ then
$|V(G)| \geq (2r-2k+1)+3k+\lceil\frac{(2r-2k+1)-6}{2}\rceil = 2r+k+1+r-k-2 = 3r-1$.

b) $d(v_1, v'_1) \geq r$
Let u be the vertex of the path P^1 such that $d(u,v_1) = r-1$. If $w \in V(G) - V(G_1)$ then $d(u,w) \leq r-1$ (diam $G \leq 2r-2$ and v'_1 is a cut-vertex of the graph G). If for each vertex $w \in V(G_1)$ it holds $d(u,w) \leq r-1$ then $e_G(u) \leq r-1$, a contradiction. Let $w \in V(G_1)$ be such that $d(u,w) \geq r$. Since $d(w,v_1) \leq 2r-2$ and $d(w,v'_1) \leq d(v_1,v'_1)$ then there is a cycle C' of G such that $u \in V(C')$. Let G' be the block of the graph G containing the cycle C'. Let v_0 be a vertex of the graph G such that $d(v_0, G') = \max\{d(x,G'); x \in V(G)\}$. Let $v'_0 \in V(G')$ be a vertex such that $d(v_0,v'_0) = d(v_0,G')$. Then v'_0 is a cut-vertex of the graph G. If $d(v_0,v'_0) \leq r-1$ then similarly as in the case a) (take the block G' instead of the block G^+) one can find a vertex with eccentricity less than r in the graph G, which is impossible. Consider now the case $d(v_0,v'_0) \geq r$. Let P' be a $v_0 - v'_0$ geodesic path and u' be the vertex of the path P' such that $d(v_0,u') = r-1$. For every vertex z from any component of the graph $G - v'_0$ such that the vertex v_0 does not belong to this component we have $d(z,u') \leq r-1$ (otherwise $d(z,v_0) >$ diam G, which is impossible). Let K be the component of $G - v'_0$ which contains the vertex v_0. Since $e_G(u') \geq r$, K contains a vertex s with $d(s,u') \geq r$. We show that this contradicts the assumption (iii). First realize that K has at least $2r$ vertices. Further, $d(v_1,G') < r$ and so v_0 and v_1 belong to different components of $G - v'_0$ (otherwise we would have $d(v_0,G^+) > d(v_1,G^+)$). It follows that the path P^1 contains at least r vertices different from the above mentioned $2r$ vertices. Therefore $|V(G)| \geq 2r+r = 3r$, a contradiction.

□

Remark. The graph in Figure 2.6 has $3r-1$ vertices and the circumference of the graph is less than $2r$. So, the inequality (iii) in Lemma 2.5 is tight, it cannot be improved.

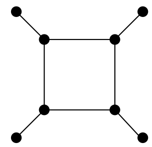


Fig. 2.6

The following important statement is an immediate consequence of Corollaries 2.2, 2.4 and Lemma 2.5.

Theorem 2.6. *If a graph G satisfies the conditions*

(i) *$rad\ G = r$,*
(ii) *$diam\ G \leq 2r - 2$,*
(iii) *$|V(G)| \leq 3r - 2$,*

then G contains a geodesic cycle of length $2r$ or $2r+1$.

Corollary 2.7. *(P.A. Ostrand, see [6])*
For all positive integers r and d satisfying $r \leq d \leq 2r-2$ there exist graphs of radius r and diameter d. The minimum order of such a graph is $d+r$. There are exactly $\lfloor \frac{d-r}{2} \rfloor + 1$ non-isomorphic graphs of order $d+r$, radius r and diameter d. They are characterized as being the sun-graphs with the kernel C_{2r} and with one or two rays (see Fig. 2.7). All isomorphic classes are obtained as s ranges from 0 to $\lfloor \frac{d-r}{2} \rfloor$.

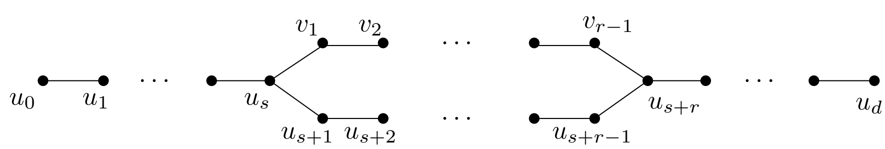


Fig. 2.7

Proof. By Theorem 2.6 a graph G such that rad $G = r$, diam $G = d \leq 2r - 2$ and $|V(G)| \leq 3r - 2$ contains a geodesic cycle C_{2r} or C_{2r+1}. Since diam $G = d$ the graph G has to contain at least $d - r$ vertices except vertices of the cycle. The corollary follows. □

3. Minimal graphs and minimal eccentric sequences.

Theorem 3.1. *Let G be a sun-graph with at most 2 rays and with the kernel C_{2r}. If $|V(G)| = 2r + k$, $1 \leq k \leq r - 1$, diam $G \leq 2r - 2$ and $exc_G(C_{2r}) \geq 2k - 2$ then the graph G is minimal.*

Proof. We will show that there is no graph H such that $e(H) < e(G)$ and $|V(H)| = 2r + k - 1$. Suppose, contrary to our claim, that such a graph H exists. Since rad $H = r$ and $|V(H)| \leq 3r - 2$, by Theorem 2.6 the graph H contains a cycle C of

length $2r$ or $2r+1$. By the assumption there are at most $2r-(2k-2)=2r-2k+2$ vertices with eccentricity r in G. We distinguish two cases.
a) $|V(C)|=2r+1$
Since $|V(H)|-|V(C)|=k-2$, by Lemma 1.6b we get $\mathrm{exc}_H(C)\leq 2(k-2)$. Therefore there are at least $2r+1-2(k-2)=2r-2k+5$ vertices with eccentricity r in H, contrary to $e(H)<e(G)$.
b) $|V(C)|=2r$
In this case $|V(H)|-|V(C)|=k-1$, whence by Lemma 1.6a we get $\mathrm{exc}_H(C)\leq 2(k-1)-1=2k-3$ for $k\geq 2$ (the case $k=1$ is trivial). Therefore there are at least $2r-(2k-3)=2r-2k+3$ vertices with eccentricity r in the graph H, contrary to $e(H)<e(G)$. □

Remark. Let G be a sun-graph with at least 3 rays and with the kernel C_{2r}. If $|V(G)|-|V(C)|=k$ then $\mathrm{exc}_G(C)\leq 2k-3$ by Lemma 1.6c. The eccentricity sequence of G need not be minimal even in the case of the equality $\mathrm{exc}_G(C)=2k-3$ (see Figure 3.1).

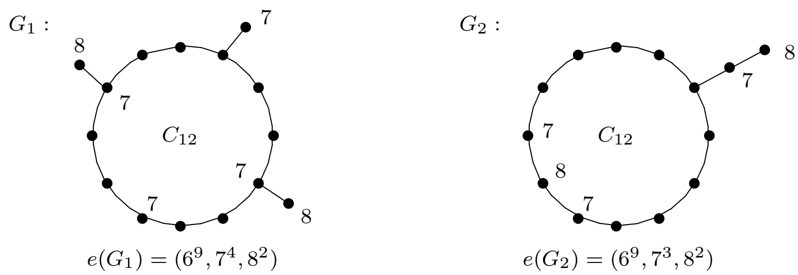


Fig. 3.1

Theorem 3.2. *Let G be a sun-graph with at least 5 rays and with the kernel C_{2r+1}. If $|V(G)|=2r+1+k$, $k\leq r-2$ and $exc_G(C_{2r+1})\geq 2k-1$ then the graph G is minimal.*

Proof. Since $\mathrm{exc}_G(C_{2r+1})\geq 2k-1$ we get that $\mathrm{exc}_G(C_{2r+1})$ is equal to either $2k-1$ or $2k$ (by Lemma 1.6b). It is sufficient to show that there is no graph H such that $e(H)<e(G)$ and $|V(H)|=2r+k$. Suppose, contrary to our claim, that such a graph H exists. By Theorem 2.6 H contains a cycle C of length $2r$ or $2r+1$. By the assumption there are at most $2r+1-(2k-1)=2r-2k+2$ vertices with eccentricity r in G. We distinguish several cases.
a) $|V(C)|=2r+1$
In this case $\mathrm{exc}_H(C_{2r+1})\leq 2(k-1)=2k-2$ (by Lemma 1.6b), hence there are at least $2r+1-(2k-2)=2r-2k+3$ vertices of the graph H with eccentricity r, a contradiction.
b) $|V(C)|=2r$
Denote by m the number of vertices from $V(H)-V(C)$ which are adjacent to at least one vertex of the cycle C.

- b_1) Let $m\geq 3$. In this case, by Lemma 1.6.c we get $\mathrm{exc}_H(C)\leq 2k-3$ whence there are at least $2r-(2k-3)=2r-2k+3$ vertices with eccentricity r in H, a contradiction.
- b_2) Let $m=2$. Since $e(H)<e(G)$ we have $\mathrm{exc}_H(C)\geq 2k-2$. By Lemma 1.6.c $\mathrm{exc}_H(C)\leq 2k-2$ and so we get $\mathrm{exc}_H(C)=2k-2$. It is easy to verify

that H is a 2-rays sun-graph with the kernel C_{2r}. Hence there are at most 6 vertices of the graph H with eccentricity $r+1$. Since $e(H) < e(G)$ and $|V(G)| = |V(H)| + 1$, there are at most 7 vertices with eccentricity $r+1$ in the graph G. Since $\mathrm{exc}_G(C_{2r+1}) \geq 2k-1$ and $\mathrm{exc}_G(C_{2r+1}) \leq 2k$ (by Lemma 1.6b)) we have two possibilities $\mathrm{exc}_G(C_{2r+1}) = 2k$ or $\mathrm{exc}_G(C_{2r+1}) = 2k-1$. G is an n-rays sun-graph for $n \geq 5$ and we get that the number of vertices of C_{2r+1} with eccentricity $r+1$ is $2n$ or $2n-1$. So, G has at least 9 vertices with eccentricity $r+1$, a contradiction.

b_3) Let $m = 1$. The vertex from $V(H) - V(C)$ adjacent with at least one vertex of the cycle C is a cut-vertex of the graph H. Since $e(H) < e(G)$ we get $\mathrm{exc}_H(C) \geq 2k-2$. Therefore it is easy to verify that there are at most 3 vertices of the graph H with eccentricity $r+1$. But we know that there are at least 9 vertices with eccentricity $r+1$ in the graph G, a contradiction.

□

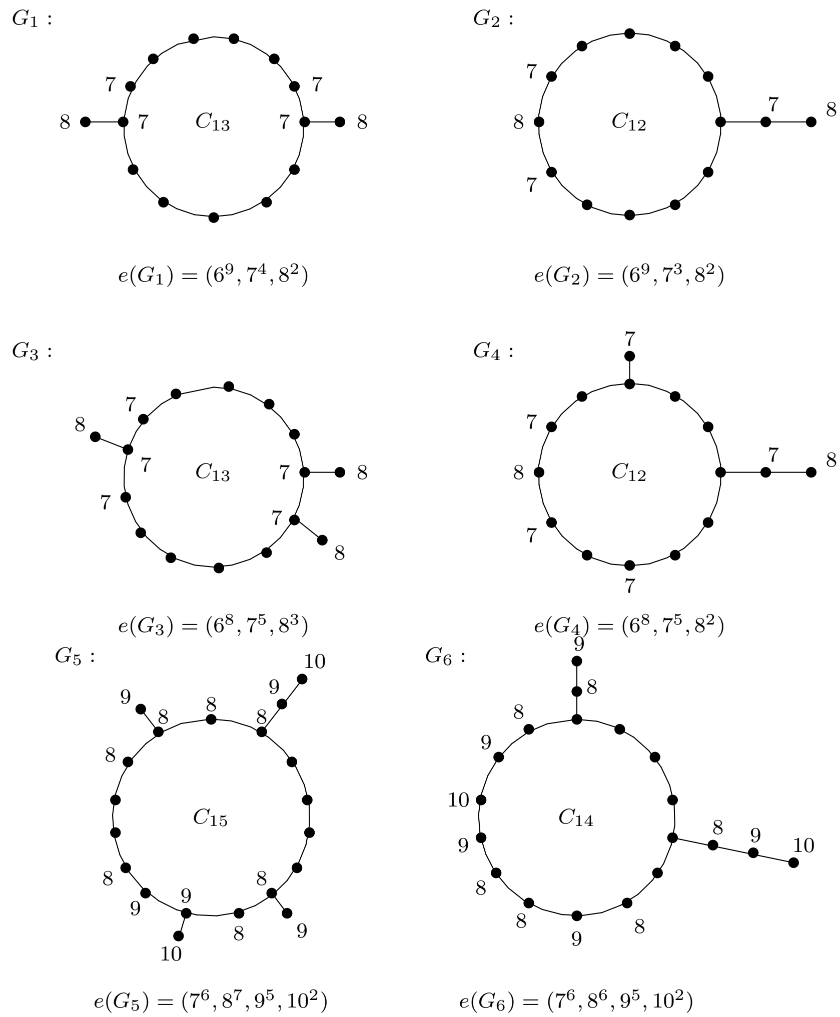


Fig. 3.2

Remarks.

1. If G is a sun-graph with one ray and with the kernel C_{2r+1} then obviously the graph G is not minimal.
2. If G is an n-ray sun-graph, $n \in \{2, 3, 4\}$ and G satisfies the remaining assumptions of Theorem 3.2 then G may or may not be minimal (see Lemma 3.3 and Fig. 3.2).

Lemma 3.3. *The following sequences*

$$\begin{array}{lll} a) & (r^{2r-3}, (r+1)^6), & r \geq 4, \\ b) & (r^{2r-4}, (r+1)^8), & r \geq 5, \\ & (r^{2r-5}, (r+1)^9), & r \geq 5, \\ c) & (r^{2r-6}, (r+1)^{11}), & r \geq 6, \\ & (r^{2r-7}, (r+1)^{12}), & r \geq 6, \end{array}$$

are minimal eccentric sequences.

Proof. The sequences from the lemma are the eccentric sequences of the graphs in Figure 3.3.

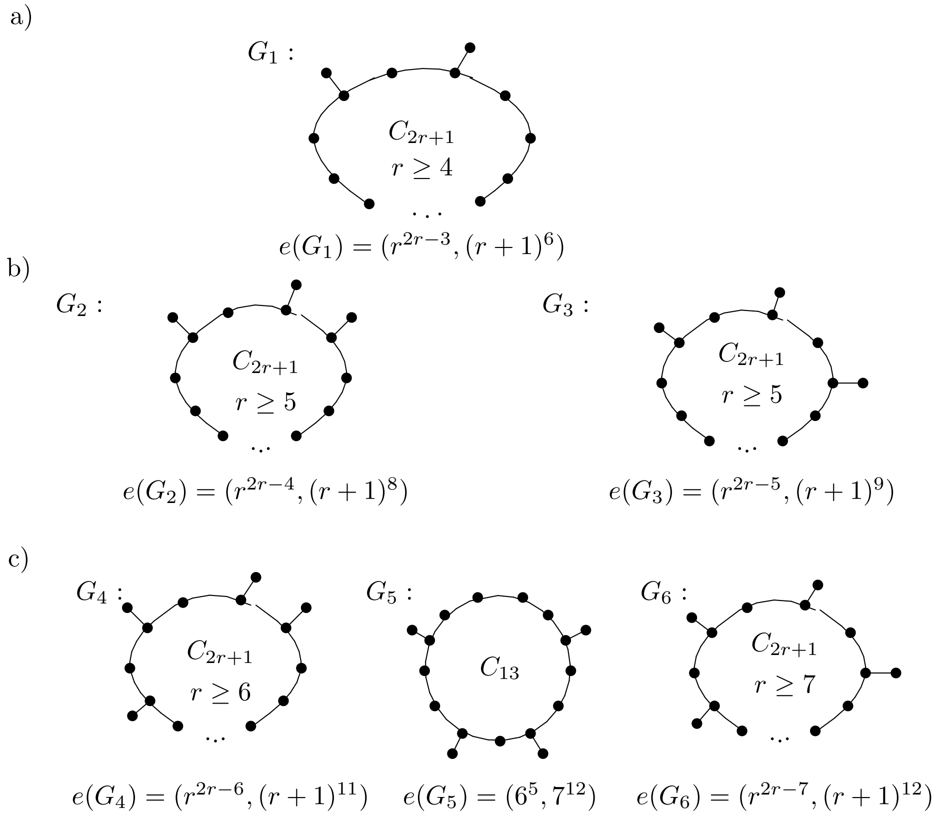


Fig. 3.3

We will show that these sequences are minimal. It is sufficient to show that if a considered sequence is the eccentric sequence of a graph G then there is no graph H such that $|V(H)| = |V(G)| - 1$ and $e(H) < e(G)$. Suppose, contrary to our claim, that such a graph H exists. The eccentricity of each vertex of H is either r or $r+1$. By Theorem 2.6 the graph H contains a cycle C_{2r} or C_{2r+1}. We distinguish three cases (see Figure 3.3).

a) In this case $|V(G_1)| = 2r+3$ and $|V(H)| = 2r+2$.

- a_1) If H contains C_{2r+1} then $\mathrm{exc}(C_{2r+1}) \leq 2$ by Lemma 1.6b. Hence there are at least $2r-1$ vertices with eccentricity r in H, contrary to $e(H) < e(G_1)$.
- a_2) Let H contain a cycle C_{2r}. Since $\mathrm{exc}_H(C_{2r}) \leq 3$ (by Lemma 1.6a) and $e(H) < e(G)$, the eccentricity of each of two vertices in $V(H) - V(C_{2r})$ is $r+1$. It yields that each of these vertices has to be adjacent with some vertex of the cycle C_{2r}. Hence $\mathrm{exc}_H(C_{2r}) \leq 2$ (Lemma 1.6c), contrary to $e(H) < e(G_1)$.

b) In this case $|V(G_2)| = |V(G_3)| = 2r+4$ and therefore $|V(H)| = 2r+3$.

- b_1) If H contains a cycle C_{2r+1} then $\mathrm{exc}_H(C_{2r+1}) \leq 4$ (by Lemma 1.6b), a contradiction.
- b_2) Let H contain a cycle C_{2r}.

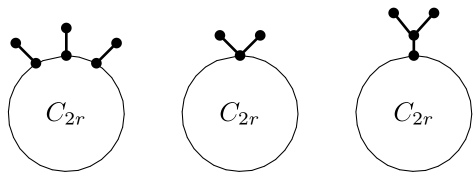


Fig. 3.4

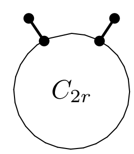


Fig. 3.5

If some of the graphs (for simplicity, the vertices of C_{2r} with degree two are not marked; their position on C_{2r} will not be important) represented by Figure 3.4 is a subgraph of the graph H then it is easy to see that there are at least $2r-3$ vertices with eccentricity r in H, a contradiction. So we can assume that no graph represented by Figure 3.4 is a subgraph of H. If a graph represented by Figure 3.5 is a subgraph of the graph H then $\mathrm{exc}_H(C_{2r}) \leq 4$. Therefore we have a contradiction with $e(H) < e(G_3)$ and the inequality $e(H) < e(G_2)$ is possible only under the assumption that the eccentricity of each vertex from $V(H) - V(C_{2r})$ is $r+1$. Hence none of these three vertices is a cut-vertex of H (otherwise we would have a vertex with eccentricity r+2). Now it is easy to check that some of the graphs represented by Figure 3.6 has to be a subgraph of H, contrary to $e(H) < e(G_2)$.

The last possibility is that a graph represented by Figure 3.7 is a subgraph of H and the vertices v_1, v_2 are cut-vertices of H. Hence $e_H(v_3) \geq r+2$, a contradiction.

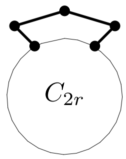


Fig. 3.6

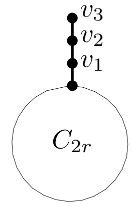


Fig. 3.7

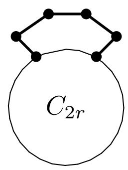


Fig. 3.8

c) In this case $|V(G_4)| = |V(G_6)| = 2r+5$, $|V(G_5)| = 2.\text{rad}\ G_5 + 5$ and therefore $|V(H)| = 2r+4$.

c_1) If H contains C_{2r+1} then $\text{exc}_H(C_{2r+1}) \leq 6$ (Lemma 1.6b). Therefore there are at least $2r-5$ vertices with eccentricity r in H, a contradiction.

c_2) Let H contain a cycle C_{2r}.
If some of the graphs represented by Figure 3.4 is a subgraph of the graph H then $\text{exc}_H(C_{2r}) \leq 5$, a contradiction. So we can suppose that none of the graphs represented by Figure 3.4 is a subgraph of H. If some of the graphs represented by Figure 3.5 is a subgraph of H then $\text{exc}_H(C_{2r}) \leq 6$. Therefore the eccentricity of each vertex from $V(H) - V(C_{2r})$ is $r+1$. It follows that none of these four vertices can be a cut-vertex of H. Hence it is obvious that some of the graphs represented by Figure 3.8 is a subgraph of H and it is easy to check that the condition $e(H) < e(G)$ does not hold. The last possibility is that a graph represented by Figure 3.7 is a subgraph of H and the vertices v_1, v_2 are cut-vertices of H. Hence $e_H(v_3) \geq r+2$, a contradiction.

□

Theorem 3.4.
a) The sequences

$$\begin{array}{lllll}(3^5, 4^2), & (3^4, 4^4), & & & \\ (4^7, 5^2), & (4^6, 5^4), & (4^5, 5^6), & & \\ (5^9, 6^2), & (5^8, 6^4), & (5^7, 6^6), & (5^6, 6^8), & (5^5, 6^9)\end{array}$$

are minimal eccentric sequences.

b) The sequences

$$(r^{2r-1}, (r+1)^2),$$
$$(r^{2r-2}, (r+1)^4),$$
$$(r^{2r-2i+1}, (r+1)^{3i}), i = 2, 3, \ldots, \left\lfloor \frac{2r+1}{3} \right\rfloor,$$
$$(r^{2r-2i}, (r+1)^{3i+2}), i = 2, 3, \ldots, \left\lfloor \frac{2r-1}{3} \right\rfloor$$

are minimal eccentric sequences of type $(r^{\alpha}, (r+1)^{\beta})$ *for* $r \geq 6$ *and* $\alpha + \beta \leq \frac{8r+5}{3}$*.*

Proof. The sequences $(r^{2r-1}, (r+1)^2)$ and $(r^{2r-2}, (r+1)^4)$ are the eccentric sequences of the graphs in Figures 3.9 and 3.10, respectively, and by Theorem 3.1 they are minimal (for $r \geq 3$).

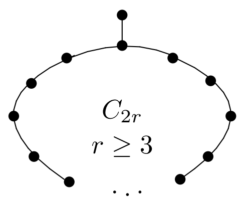


Fig. 3.9

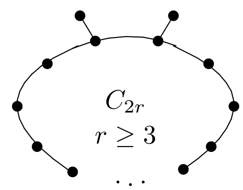
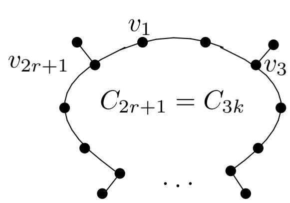


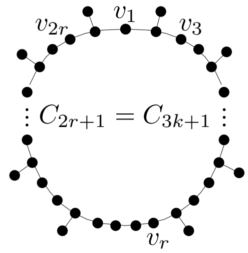
Fig. 3.10

The sequences $(4^5, 5^6)$, $(5^7, 6^6)$, $(5^6, 6^8)$ and $(5^5, 6^9)$ are minimal by Lemma 3.3. We are going to show that also the sequences of the type $(r^{2r-2i+1}, (r+1)^{3i})$, $r \geq 6$ are minimal. For $i \in \{2, 3, 4\}$ it holds by Lemma 3.3. For $i \geq 5$ according to Theorem 3.2 it is sufficient to consider a subgraph of one of the graphs in Figures 3.11, 3.12 or 3.13 (depending on whether $2r+1 = 3k$, $2r+1 = 3k+1$ or $2r+1 = 3k+2$, respectively). The subgraph has to be a sun-graph with the required number of vertices. For instance the eccentricity sequence $(12^{15}, 13^{15})$ is realized by a graph with 30 vertices and with radius $r = 12$. So we have to consider the graph in Figure 3.12 $(2.12 + 1 = 3.8 + 1)$ which has 33 vertices for $r = 12$. Then the graph which realizes the sequence $(12^{15}, 13^{15})$ can be obtained from the previous graph with 33 vertices by deleting any three end-vertices. Two of the possibilities are depicted in Figure 3.14.



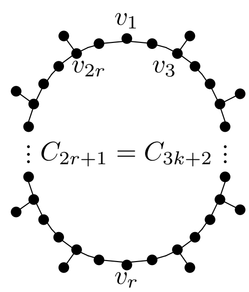
$\deg(v_{3i}) = 3$
$i = 1, 2, \ldots, \frac{2r+1}{3}$

Fig. 3.11



$$\deg(v_{3i-1}) = \deg(v_{2r+4-3i}) = 3,$$
$$i = 1, 2, \ldots, \tfrac{r}{3}$$

Fig. 3.12



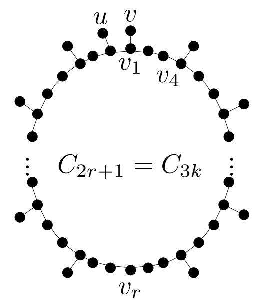
$$\deg(v_{3i}) = \deg(v_{2r-3i}) = \deg(v_{2r}) = 3,$$
$$i = 1, 2, \ldots, \tfrac{r-2}{3}$$

Fig. 3.13

$e(H_1) = (12^{15}, 13^{15})$ $\qquad$ $e(H_2) = (12^{15}, 13^{15})$

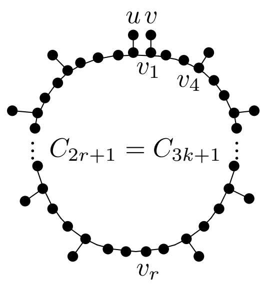
Fig. 3.14

It remains to show that the sequences of type $(r^{2r-2i}, (r+1)^{3i+2})$, $r \geq 6$ are minimal. For $i \in \{2, 3\}$ it holds by Lemma 3.3. For $i \geq 4$ (according to Theorem 3.2) it is sufficient to consider a subgraph of one of the graphs in Figures 3.15, 3.16 or 3.17 (depending on whether $2r+1 = 3k$, $2r+1 = 3k+1$ or $2r+1 = 3k+2$, respectively). The subgraph has to be a sun-graph with the required number of vertices and containing the vertices u, v. $\square$



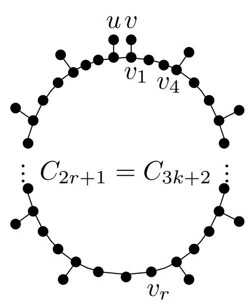
$$\deg(v_{2r+1}) = \deg(v_{3i-2}) = \deg(v_{2r+2-3i}) = 3,$$
$$i = 1, 2, \ldots, \tfrac{r-1}{3}$$

Fig. 3.15



$$\deg(v_{2r+1}) = \deg(v_{r-2}) = \deg(v_{3i-2}) = \deg(v_{2r-3i}) = 3,$$
$$i = 1, 2, \ldots, \tfrac{r}{3} - 1$$

Fig. 3.16



$$\deg(v_{3i+1}) = \deg(v_{2r+1-3i}) = 3,$$
$$i = 0, 1, 2, \ldots, \tfrac{r-2}{3}$$

Fig. 3.17

Remark. All minimal eccentric sequences for $r = 3$ and with only two values are known (see [3]):

$$(3^5, 4^2),\ (3^4, 4^4),\ (3^3, 4^6),\ (3^2, 4^8),\ (3, 4^{10}).$$

The sequence $(3^3, 4^6)$ is realizable by a sun-graph but the last two sequences are not. In Figure 3.18 two realizations of the sequence $(3^3, 4^6)$ are depicted. Note that the minimality of this sequence does not follow from previous considerations $(3 + 6 = 3r)$.

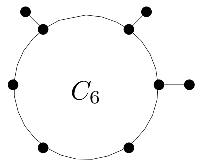


Fig. 3.18

In the Figure 3.19 a few realizations of the sequence $(3^4, 4^4)$ are depicted.

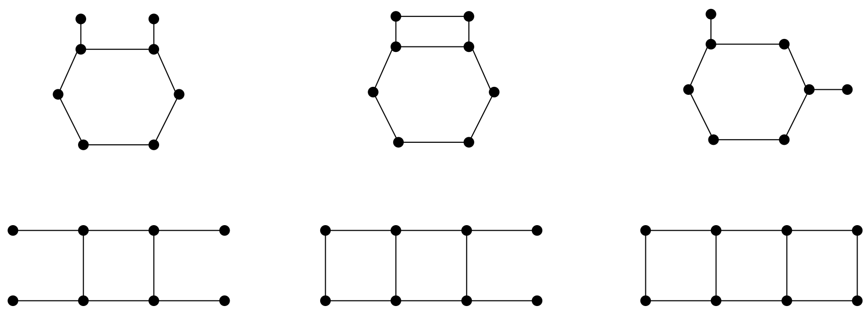


Fig. 3.19

References

[1] Buckley, F. and Harary F., *Distance in Graphs*, Addison-Wesley, Redwood City, California, 1990.

[2] Buckley, F., *Eccentric sequences, eccentric sets, and graph centrality. Dedicated to the memory of Robert E. Sacks (Purchase, NY, 2000)*, Graph Theory Notes N.Y. 40, 2001, pp. 18-22.

[3] Haviar, A., Hrnčiar, P. and Monoszová, G., *Minimal eccentric sequences with least eccentricity three*, Acta Univ. M. Belii, Math. no 5, 1997, pp. 27-50.

[4] Chartrand, G. and Lesniak, L., *Graphs and Digraphs*, Chapman & Hall, London, 1996.

[5] Lesniak, L., *Eccentric sequences in graphs*, Period. Math. Hungar. 6, 1975, pp. 287-293.

[6] Ostrand, P.A., *Graphs with specified radius and diameter*, Discrete Math. 4, 1973, pp. 71-75.

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