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NATURAL DUALITIES FOR STRUCTURES

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Dedicated to George Grätzer and E. Tamás Schmidt on their 70th birthdays.

Abstract. Following on from results of Hofmann [27], we investigate the extension of the theory of natural dualities to quasivarieties generated by finite structures that can have operations, partial operations and relations in their type. It turns out that the usual proofs of the Second Duality Theorem, the Duality Compactness Theorem and the NU Duality Theorem extend to this setting with only minor changes. We present simple proofs of Two-for-One Full Duality and Strong Duality Theorems, and show how our techniques can be applied to yield new dualities from known strong dualities by simply swapping the topology from one side to the other.

While developing the theory of natural dualities, the author and his various coauthors made a conscious decision to aim the theory at those who use universal algebraic ideas, for whom a duality may be a useful tool; see, for example, Davey and Werner [20, 21, 22], Davey and Priestley [18], Clark and Davey [5] and Pitkethly and Davey [30]. This required a careful choice of an appropriate level of generality for both the starting algebraic category $\boldsymbol{\mathcal{A}}$ and the topological dual category $\boldsymbol{\mathfrak{X}}$. To maximise the algebraic content we decided to concentrate on the case where $\boldsymbol{\mathcal{A}}$ was an $\mathbb{ISP}$-closed class of algebras generated by a single finite (total) algebra. Later, we extended the theory to include multi-sorted dualities, where $\boldsymbol{\mathcal{A}}$ is an $\mathbb{ISP}$-closed class generated by a finite set of finite algebras; see [18] and [5, Chapter 7]. We were aware that the restriction to *finite* generating algebras was not necessary; see, for example, Davey [12], the appendix of Davey and Werner [20], Davey and Werner [21, 22], Davey and Priestley [18], and Clark and Davey [5, Exercise 2.9]. Nevertheless, we felt that the theory for finitely generated classes was sufficiently rich to warrant special attention. Based on our experience with many examples, we chose the dual category $\boldsymbol{\mathfrak{X}}$ to be the $\mathbb{IS}_c\mathbb{P}^+$-closed class generated by a finite topological structure whose type allowed finitary total operations, partial operations and relations.

The lack of symmetry between the allowable types of structures on the original algebraic side and on the dual topological side was intentional but to a large extent unnecessary. Much of the theory goes through if we allow total operations, partial operations and relations on both sides. The first development in this direction was by Hofmann [27], who presented a generalisation of the Duality Compactness Theorem and a Two-for-One Full Duality Theorem. Hofmann's results make an important contribution to the theory of natural dualities. He generalises the

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basic setting of natural dualities by allowing both the "algebraic" and "topological" categories to be determined in an appropriate way by finitary limit sketches. (See [27] for the details and Adámek and Rosický [1] for the required background on sketches.) In this way, Hofmann eliminates the asymmetry inherent in the original theory of natural dualities. In particular, his approach allows structures with operations, partial operations and relations on both sides of the duality.

Hofmann's results suggest many avenues for further investigation within the theory of natural dualities. For example,

(1) extend results in the original theory to the sketch-based setting,
(2) extend results in the original theory to the case where the algebraic category $\boldsymbol{\mathcal{A}}$ and the topological category $\boldsymbol{\mathfrak{X}}$ consist of structures, with operations, partial operations and relations,
(3) give examples of dualities for categories that are truly sketch-based and not covered by the extension of the theory to classes of structures,
(4) give new examples of dualities for classes of structures.

This paper addresses (2) and (4). These are important as they make the theory, expanded to structures, available to users of natural dualities without requiring them first to absorb the theory of sketches. Moreover, work on (2) and (4) will set the boundaries of the theory of natural dualities for structures and thereby indicate what might be possible in (1) and (3).

As a contribution to (2), we present the basics of the theory of natural dualities for structures. We show that a generalisation of the Second Duality Theorem (Davey and Werner [20, 1.16]), the version of Hofmann's Duality Compactness Theorem that applies to structures, and a generalisation of the NU Duality Theorem (Davey and Werner [20, 1.19]) can all be obtained by easy modifications of the proofs given in Clark and Davey [5]. We also present proofs of Hofmann's Two-for-One Full Duality Theorem for structures and of a generalisation of Clark and Davey's Two-for-One Strong Duality Theorem [4]. As a contribution to (4), we close the paper with a number of applications to quasivarieties of structures of the two-for-one duality theorems and an application of the generalised NU Duality Theorem.

1. Structures of type $\langle G, H, R\rangle$

This section is a brief refresher on quasivarieties and universal Horn classes generated by finite structures. We present just enough to meet our needs. For detailed treatments in the setting of partial algebras and more general categories we refer to Burmeister [3] and Adámek and Rosický [1], respectively.

We begin with sets G of finitary total operation symbols, H of finitary partial operation symbols, and R of finitary relation symbols. The total operation symbols may be nullary, but, to remove unnecessary complications, we shall assume that the arity of each partial operation and relation symbol is positive. A *structure*,

$$\mathbf{A} = \langle A; G^{\mathbf{A}}, H^{\mathbf{A}}, R^{\mathbf{A}}\rangle,$$

of type $\langle G, H, R\rangle$ is defined in the usual way; see Clark and Davey [5]. If H is empty, then we refer to $\mathbf{A}$ as a *total structure*; if both H and R are empty, then we simply refer to $\mathbf{A}$ as an *algebra*. We allow the underlying set A of $\mathbf{A}$ to be empty only if there are no nullary symbols in G. An *atomic formula* of type $\langle G, H, R\rangle$ is an expression of the form

$$t_1 \approx t_2 \quad \text{or} \quad r(t_1, \dots, t_n),$$

where $t_1, t_2, \ldots, t_n$ are terms of type $G \cup H$ and $r \in R$ is n-ary. For a detailed discussion of the validity of first-order formulæ in a structure, see Burmeister's survey article [3]. Note that in [3], no distinction is made between total and partial operation symbols in the type. A brief introduction suitable to our needs can be found in Clark and Davey [5, page 25]. Two important features to note are that, given a term t of type $G \cup H$, the domain of the corresponding term function $t^{\mathbf{A}}$ on A is its maximum domain, and that, given n-ary terms t_1 and t_2 and $a_1, \ldots, a_n \in A$, the structure $\mathbf{A}$ satisfies $t_1^{\mathbf{A}}(a_1, ..., a_n) = t_2^{\mathbf{A}}(a_1, ..., a_n)$ if and only if both sides are defined and equal. In particular, $\mathbf{A}$ satisfies $t^{\mathbf{A}}(a_1, ..., a_n) = t^{\mathbf{A}}(a_1, ..., a_n)$ if and only if $(a_1, \ldots, a_n) \in \mathrm{dom}(t^{\mathbf{A}})$. A *universal Horn sentence* of type $\langle G, H, R\rangle$ is a universally quantified formula of the form

$$\varphi, \qquad \Big(\bigwedge_{i=1}^{n} \varphi_i\Big) \to \varphi, \quad \text{or} \quad \bigvee_{i=1}^{n} \neg\varphi_i,$$

where φ and all the φ_i's are atomic formulæ of type $\langle G, H, R\rangle$, and $n \in \mathbb{N}$. We shall refer to universal Horn sentences of the first and second kinds as *atomic sentences* and *quasi-atomic sentences*, respectively.

Let Σ be a set of universal Horn sentences of type $\langle G, H, R\rangle$. Then $\mathrm{Mod}(\Sigma)$ denotes the class consisting of all *non-empty* models of Σ, while $\mathrm{Mod}^0(\Sigma)$ includes the empty structure $\boldsymbol{\varnothing}$ of type $\langle G, H, R\rangle$ in the case that G contains no nullary symbols. A class $\boldsymbol{\mathcal{A}}$ of structures is called a *universal Horn class* if $\boldsymbol{\mathcal{A}} = \mathrm{Mod}(\Sigma)$ or $\boldsymbol{\mathcal{A}} = \mathrm{Mod}^0(\Sigma)$, for some set Σ of universal Horn sentences. A class defined by atomic and quasi-atomic sentences is called a *quasivariety*, and a class defined by atomic sentences is called either an *atomic class* or a *variety*.

Let $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$ be the usual class operators. We adopt the normal algebraic convention that $\mathbb{S}(\boldsymbol{\mathcal{K}})$ denotes the class consisting of all *non-empty* substructures of structures from $\boldsymbol{\mathcal{K}}$. Note that $\mathbb{P}(\boldsymbol{\mathcal{K}})$ includes products over an empty index set, whence $\mathbb{P}(\boldsymbol{\mathcal{K}})$ includes the *complete one-element structure* $\mathbf{1}$ of type $\langle G, H, R\rangle$ with underlying set $\{\varnothing\}$ and every relation and the domain of every partial operation non-empty. We also require two further operators, namely $\mathbb{S}^0$, which includes empty substructures in case G contains no nullary operation symbols, and $\mathbb{P}^+$, which excludes products over an empty index set.

Theorem 1.1. *Let $\boldsymbol{\mathcal{K}}$ be a finite non-empty set of finite structures of type $\langle G, H, R\rangle$.*

- (i) *The smallest universal Horn class containing $\boldsymbol{\mathcal{K}}$ is the class $\mathbb{ISP}^+(\boldsymbol{\mathcal{K}})$, if the empty structure is not allowed, and is $\mathbb{IS}^0\mathbb{P}^+(\boldsymbol{\mathcal{K}})$, if the empty structure is allowed.*
- (ii) *The smallest quasivariety containing $\boldsymbol{\mathcal{K}}$ is the class $\mathbb{ISP}(\boldsymbol{\mathcal{K}})$, if the empty structure is not allowed, and is $\mathbb{IS}^0\mathbb{P}(\boldsymbol{\mathcal{K}})$, if the empty structure is allowed.*
- (iii) *The smallest atomic class containing $\boldsymbol{\mathcal{K}}$ is the class $\mathbb{HSP}(\boldsymbol{\mathcal{K}})$, if the empty structure is not allowed, and is $\mathbb{HS}^0\mathbb{P}(\boldsymbol{\mathcal{K}})$, if the empty structure is allowed.*

Proof. This is the finitely generated version of Theorems 3.4(i)(iii) and 4.5 of Burmeister [3]. It is a good exercise to write out a direct proof based on the version for algebras in Clark and Davey [5, 1.3.4 and Appendix A]. □

If and when to include the empty structure in a class of structures is a matter of some debate. It still leads to heated discussions between category theorists, who want their categories to be complete and cocomplete, and algebraists, who are usually happy to live without the free structure generated by the empty set when it happens to be empty, and even without the complete one-element structure when considering universal Horn classes. Our usual convention will be to exclude the empty structure when considering algebras ($H = R = \varnothing$), and to include the empty structure when considering purely relational structures ($G = H = \varnothing$), and to make a decision on a case-by-case basis otherwise. Moreover, we almost always allow the empty structure when working with topological structures; see the discussion at the start of Section 6 and particularly Lemma 6.2.

The following result is a completely standard but often-required characterisation of the structures in the class $\mathbb{ISP}^+(\mathcal{K})$. It is the non-topological version of the Separation Theorem [5, 1.4.4], see Lemma 3.1 below.

Lemma 1.2. *Let $\mathcal{K}$ be a non-empty set of structures of type $\langle G, H, R\rangle$ and let $\mathbf{A}$ be a non-empty structure of the same type.*

(i) *The complete one-element structure $\mathbf{1}$ belongs to $\mathbb{ISP}^+(\mathcal{K})$ if and only if some $\mathbf{M} \in \mathcal{K}$ has a substructure isomorphic to $\mathbf{1}$.*

(ii) *Assume that $\mathbf{A}$ is not isomorphic to the complete one-element structure $\mathbf{1}$. Then $\mathbf{A} \in \mathbb{ISP}^+(\mathcal{K})$ if and only if, for all $r \in \{=\} \cup \{\operatorname{dom}(h) \mid h \in H\} \cup R$ of arity n, and all $a_1, \dots, a_n \in A$ with $(a_1, \dots, a_n) \notin r^{\mathbf{A}}$, there exists $\mathbf{M} \in \mathcal{K}$ and a homomorphism $\varphi : \mathbf{A} \to \mathbf{M}$ such that $(\varphi(a_1), \dots, \varphi(a_n)) \notin r^{\mathbf{M}}$.*

Later we shall need the fact that every universal Horn class of structures is closed under direct limits. Indeed, the usual construction for algebras still applies. Let $\mathcal{A}$ be a class of structures of type $\langle G, H, R\rangle$, let $\mathbf{S} = \langle S; \leqslant\rangle$ be a non-empty directed ordered set and let $\{\mathbf{A}_s \mid s \in S\}$ be a direct system in $\mathcal{A}$ with connecting homomorphisms $\varphi_{st} : \mathbf{A}_s \to \mathbf{A}_t$, for all $s \leqslant t$ in $\mathbf{S}$. Define an equivalence relation $\equiv$ on the disjoint union $\dot{\bigcup}\{A_s \mid s \in S\}$ as follows: for $a \in A_s$ and $b \in A_t$, define $a \equiv b$ if there exists an upper bound u of $\{s, t\}$ in $\mathbf{S}$ such that $\varphi_{su}(a) = \varphi_{tu}(b)$, and denote the equivalence class of a by $[a]$. We convert $B := \dot{\bigcup}\{A_s \mid s \in S\}/\equiv$ into a structure $\mathbf{B}$ of type $\langle G, H, R\rangle$ as follows. For each $h \in H$ of arity n, and all $s_1, \dots, s_n \in S$ and $a_1, \dots, a_n$, with $a_i \in A_{s_i}$, define

$$([a_1], \dots, [a_n]) \in \operatorname{dom}(h^{\mathbf{B}}) \iff \begin{cases} (\varphi_{s_1 t}(a_1), \dots, \varphi_{s_n t}(a_n)) \in \operatorname{dom}(h^{\mathbf{A}_t}), \\ \text{for some upper bound } t \text{ of } \{s_1, \dots, s_n\} \text{ in } \mathbf{S}, \end{cases}$$

in which case

$$h^{\mathbf{B}}([a_1], \dots, [a_n]) := \big[h^{\mathbf{A}_t}(\varphi_{s_1 t}(a_1), \dots, \varphi_{s_n t}(a_n))\big].$$

For each $g \in G$ and each $r \in R$, the total operation $g^{\mathbf{B}}$ and the relation $r^{\mathbf{B}}$ are defined analogously.

Theorem 1.3. *Let $\mathcal{A}$ be a universal Horn class of structures of type $\langle G, H, R\rangle$. Let $\mathbf{S} = \langle S; \leqslant\rangle$ be a non-empty directed ordered set and let $\{\mathbf{A}_s \mid s \in S\}$ be a direct system in $\mathcal{A}$ with connecting homomorphisms $\varphi_{st} : \mathbf{A}_s \to \mathbf{A}_t$, for all $s \leqslant t$ in $\mathbf{S}$. Then the structure $\mathbf{B}$ defined above belongs to $\mathcal{A}$ and is a direct limit in $\mathcal{A}$ of the direct system $\{\mathbf{A}_s \mid s \in S\}$.*

Proof. It is an easy exercise to show that a universal Horn sentence satisfied by all of the structures $\mathbf{A}_s$, for $s \in S$, is also satisfied by $\mathbf{B}$, whence $\mathbf{B}$ belongs to $\mathcal{A}$.

A simple argument now shows that $\mathbf{B}$ satisfies the universal mapping definition of the direct limit in $\boldsymbol{\mathcal{A}}$ of the direct system. $\square$

Henceforth, we denote the structure $\mathbf{B}$ constructed above by $\varinjlim_{s\in S} \mathbf{A}_s$. The following useful fact is well known for algebras (see Grätzer [24]). Its proof is no more difficult in this more general setting, and we leave it for the reader.

Lemma 1.4. *Let* $\mathbf{A}$ *be a structure of type* $\langle G, H, R\rangle$. *Then* $\mathbf{A}$ *is isomorphic to* $\varinjlim_{s\in S} \mathbf{A}_s$, *where* $\{\, \mathbf{A}_s \mid s \in S \,\}$ *is the direct system consisting of the finitely generated substructures of* $\mathbf{A}$ *ordered by inclusion.*

A structure $\mathbf{A}$ is said to be *locally finite* if every finitely generated substructure of $\mathbf{A}$ is finite, and a class $\boldsymbol{\mathcal{A}}$ is locally finite if every structure in $\boldsymbol{\mathcal{A}}$ is locally finite. As in the case of algebras, finitely generated quasivarieties are locally finite.

Lemma 1.5. *Let* $\boldsymbol{\mathcal{K}}$ *be a finite non-empty set of finite structures of type* $\langle G, H, R\rangle$. *Then the quasivariety* $\mathbb{ISP}(\boldsymbol{\mathcal{K}})$ *generated by* $\boldsymbol{\mathcal{K}}$ *is locally finite.*

Proof. Let $\boldsymbol{\mathcal{K}} = \{\mathbf{M}_1, \dots, \mathbf{M}_k\}$. Every n-ary term t of type $G \cup H$ induces a k-tuple $(t^{\mathbf{M}_1}, \dots, t^{\mathbf{M}_k})$, where $t^{\mathbf{M}_i}$ is an n-ary partial operation on M_i. The number of such k-tuples of n-ary term functions is bounded above by $\ell := m^{2^{m^n} k}$, where m is the maximum size of a structure in $\boldsymbol{\mathcal{K}}$. Hence, there exist n-ary terms $t_1, t_2, \dots, t_s$ of type $G \cup H$, with $s \leqslant \ell$, such that, for every n-ary term t of type $G \cup H$, there exists $i \in \{1, \dots, s\}$ such that, for every $\mathbf{M} \in \boldsymbol{\mathcal{K}}$, we have $t^{\mathbf{M}} = t_i^{\mathbf{M}}$. So the class $\boldsymbol{\mathcal{K}}$ satisfies the quasi-equations

$$
\begin{aligned}
t_i(x_1, \dots, x_n) \approx t_i(x_1, \dots, x_n) &\implies t(x_1, \dots, x_n) \approx t(x_1, \dots, x_n),\\
t(x_1, \dots, x_n) \approx t(x_1, \dots, x_n) &\implies t(x_1, \dots, x_n) \approx t_i(x_1, \dots, x_n),
\end{aligned}
$$

which express the fact that t and t_i induce identical term functions on every structure in $\boldsymbol{\mathcal{K}}$. Since every structure in $\mathbb{ISP}(\boldsymbol{\mathcal{K}})$ satisfies every quasi-equation true in $\boldsymbol{\mathcal{K}}$, we conclude that t and t_i induce identical term functions on every structure in $\mathbb{ISP}(\boldsymbol{\mathcal{K}})$. It now follows easily that, if $\mathbf{A} \in \mathbb{ISP}(\boldsymbol{\mathcal{K}})$ is n-generated, then $|A| \leqslant s$. $\square$

The following standard result is a useful consequence of the previous three results. As usual, we denote the finite members of a class $\boldsymbol{\mathcal{C}}$ by $\boldsymbol{\mathcal{C}}_{\mathrm{fin}}$.

Lemma 1.6. *Let* $\mathbf{M} = \langle M; G, H, R\rangle$ *be a finite structure and let* Σ *be a set of universal Horn sentences of type* $\langle G, H, R\rangle$. *If* $[\mathbb{ISP}^+(\mathbf{M})]_{\mathrm{fin}} = [\mathrm{Mod}(\Sigma)]_{\mathrm{fin}}$ *and every finitely generated model of* Σ *is finite, then* $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$.

Proof. By Theorem 1.3 and Lemma 1.4, every universal Horn class is uniquely determined by its finitely generated structures. By Lemma 1.5, the universal Horn class $\mathbb{ISP}^+(\mathbf{M})$ is locally finite and, by assumption, the universal Horn class $\mathrm{Mod}(\Sigma)$ is locally finite. It follows at once that $[\mathbb{ISP}^+(\mathbf{M})]_{\mathrm{fin}} = [\mathrm{Mod}(\Sigma)]_{\mathrm{fin}}$ implies $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$. $\square$

In order to extend the First and Second Duality Theorems (see Clark and Davey [5, 2.2.2 and 2.2.7]) to this more general setting, we need to know that the usual description of free algebras in the class $\mathbb{ISP}(\mathbf{M})$ generated by an algebra $\mathbf{M}$ extends to this setting. The proof of the following result is an easy modification of the proof of the corresponding result for algebras (see, for example, Appendix A of [5]). Let $\mathbf{M} = \langle M; G, H, R\rangle$ be a structure. The term function induced by an n-ary term of type $G \cup H$ is an n-ary partial operation $t^{\mathbf{M}} : D \to M$, where $D \subseteq M^n$. If $D = M^n$, then we refer to $t^{\mathbf{M}} : D \to M$ as a *total n-ary term function of* $\mathbf{M}$. In this case, even though $t^{\mathbf{M}}$ is a total operation, the term t may include partial operation symbols in H for which the corresponding partial operation $h^{\mathbf{M}}$ is not total. Let S be a non-empty set. A function $f : M^S \to M$ is a *total S-ary term function of* $\mathbf{M}$ if, for some $n \geqslant 0$, there exist $s_1, \dots, s_n \in S$ and a total n-ary term function $t^{\mathbf{M}} : M^n \to M$ of $\mathbf{M}$ such that $f(a) = t^{\mathbf{M}}(a(s_1), \dots, a(s_n))$, for all $a \in M^S$. Let $\mathbf{F}_{\mathbf{M}}(S)$ denote the substructure of $\mathbf{M}^{M^S}$ consisting of the total S-ary term functions of $\mathbf{M}$. Clearly, $\mathbf{F}_{\mathbf{M}}(S)$ is the substructure of $\mathbf{M}^{M^S}$ generated by the projections.

Lemma 1.7. *Let* $\mathbf{M}$ *be a non-empty structure and let* $\mathcal{V} := \mathbb{HSP}(\mathbf{M})$ *be the atomic class generated by* $\mathbf{M}$. *For every non-empty set S, the structure* $\mathbf{F}_{\mathbf{M}}(S)$ *is freely generated in* $\mathcal{V}$ *by the set* $\{\, \pi_s : M^S \to M \mid s \in S \,\}$ *of projections.*

2. Alter egos

In this section we indicate how the definition of an alter ego of a finite algebra can be extended to finite structures. Let

$$\mathbf{M}_1 = \langle M; G_1, H_1, R_1\rangle \quad \text{and} \quad \mathbf{M}_2 = \langle M; G_2, H_2, R_2\rangle$$

be two structures defined on the same set M. To avoid technicalities, we assume that M is non-empty and that the relations in R_1 and R_2 and the domains of the partial operations in H_1 and H_2 are non-empty. Then $\mathbf{M}_2$ is said to be *compatible with* $\mathbf{M}_1$ if

(a) for all $n \geqslant 0$, each n-ary operation $g \in G_2$ is a homomorphism from $\mathbf{M}_1^n$ to $\mathbf{M}_1$,
(b) for all $n \geqslant 1$ and each n-ary partial operation $h \in H_2$, the domain of h forms a substructure $\mathbf{dom}(h)$ of $\mathbf{M}_1^n$ and h is a homomorphism from $\mathbf{dom}(h)$ to $\mathbf{M}_1$, and
(c) for all $n \geqslant 1$, each n-ary relation $r \in R_2$ forms a substructure of $\mathbf{M}_1^n$.

Note that it follows from (a) that, if $\mathbf{M}_2$ is compatible with $\mathbf{M}_1$ and c is (the value of) a nullary operation of $\mathbf{M}_2$, then $\{c\}$ forms a substructure of $\mathbf{M}_1$ isomorphic to the complete one-element structure $\mathbf{1}_1$ of type $\langle G_1, H_1, R_1\rangle$.

Lemma 2.1. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be structures defined on the same underlying set. Then* $\mathbf{M}_2$ *is compatible with* $\mathbf{M}_1$ *if and only if* $\mathbf{M}_1$ *is compatible with* $\mathbf{M}_2$.

Proof. This is a symbol-pushing exercise. The crux of the proof is the fact that, given partial operations h_1 and h_2 of arities m and n on a set M, the domain of h_1 is closed under h_2 and h_1 preserves h_2 if and only if the same thing holds with h_1 and h_2 interchanged. □

Since compatibility is a symmetric relation, we shall simply say that $\mathbf{M}_1$ and $\mathbf{M}_2$ are *compatible.* If $\mathbf{M} = \langle M; G, H, R\rangle$ is a finite structure, then $\utilde{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ will denote the topological structure obtained by adding the discrete topology $\mathcal{T}$ to $\mathbf{M}$. If M is a finite non-empty set and $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible structures with underlying set M, then we shall refer to the discrete topological structure $\utilde{\mathbf{M}}_2$ as an *alter ego* of the structure $\mathbf{M}_1$. (In the case that $\mathbf{M}_1$ is an algebra, instead of saying that $\underset{\sim}{\mathbf{M}}_2$ is an alter ego of $\mathbf{M}_1$, many authors say that $\underset{\sim}{\mathbf{M}}_2$ is *algebraic over* $\mathbf{M}_1$.)

It is now completely straightforward to check that the basics of the theory of natural dualities extend to structures. We will sketch the details.

Let $\mathbf{M}_1 = \langle M; G_1, H_1, R_1\rangle$ be a finite structure and let $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, H_2, R_2, \mathcal{T}\rangle$ be an alter ego of $\mathbf{M}_1$. Define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ to be the quasivariety generated by $\mathbf{M}_1$, and let $\boldsymbol{\mathfrak{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ be the class consisting of all topological structures of the same type as $\underset{\sim}{\mathbf{M}}_2$ that are isomorphic to a possibly empty closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}_2$. With a slight abuse of terminology, the class $\boldsymbol{\mathfrak{X}}$ is usually referred to as the *topological quasivariety* generated by $\underset{\sim}{\mathbf{M}}_2$. We also denote by $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$ the corresponding categories obtained by adding as morphisms all homomorphisms and all continuous homomorphisms, respectively. (As an aid to the reader, we shall refer to morphisms in $\boldsymbol{\mathcal{A}}$ as homomorphisms and reserve the name morphism for the category $\boldsymbol{\mathfrak{X}}$.)

The fact that the structures $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible guarantees that we can set up a dual adjunction $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$. The verification of the many claims below, both implicit and explicit, is straightforward. (See Section 1.5 of [5] for the details in the case that $\mathbf{M}_1$ is an algebra.) Define contravariant hom-functors $\mathrm{D} : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathfrak{X}}$ and $\mathrm{E} : \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathcal{A}}$ as follows:

- for each structure $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, the *dual of* $\mathbf{A}$ is the topologically closed substructure $\mathrm{D}(\mathbf{A})$ of $\underset{\sim}{\mathbf{M}}_2^A$ formed by the set $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_1)$ of all homomorphisms from $\mathbf{A}$ to $\mathbf{M}_1$,
- for each structure $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$, the *dual of* $\mathbf{X}$ is the substructure $\mathrm{E}(\mathbf{X})$ of $\mathbf{M}_1^X$ formed by the set $\boldsymbol{\mathfrak{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}}_2)$ of all morphisms from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}_2$,
- given a homomorphism $u : \mathbf{A} \to \mathbf{B}$ in $\boldsymbol{\mathcal{A}}$, the morphism $\mathrm{D}(u) : \mathrm{D}(\mathbf{B}) \to \mathrm{D}(\mathbf{A})$ is defined by $\mathrm{D}(u)(x) := x \circ u$, for all $x \in \boldsymbol{\mathcal{A}}(\mathbf{B}, \mathbf{M}_1)$,
- given a morphism $\psi : \mathbf{X} \to \mathbf{Y}$ in $\boldsymbol{\mathfrak{X}}$, the homomorphism $\mathrm{E}(\psi) : \mathrm{E}(\mathbf{Y}) \to \mathrm{E}(\mathbf{X})$ is defined by $\mathrm{E}(\psi)(\alpha) := \alpha \circ \psi$, for all $\alpha \in \boldsymbol{\mathfrak{X}}(\mathbf{Y}, \underset{\sim}{\mathbf{M}}_2)$.

For each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and each $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$, define the *evaluation maps*

$$e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A}) \quad \text{and} \quad \varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$$

by $e_{\mathbf{A}}(a)(x) := x(a)$, for all $a \in A$ and all $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_1)$, and $\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$, for all $x \in X$ and all $\alpha \in \boldsymbol{\mathfrak{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}}_2)$. This defines a pair of natural transformations $e : \mathrm{id}_{\boldsymbol{\mathcal{A}}} \to \mathrm{ED}$ and $\varepsilon : \mathrm{id}_{\boldsymbol{\mathfrak{X}}} \to \mathrm{DE}$. Moreover, the construction of $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$ via $\mathbb{ISP}$ and $\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+$, respectively, ensures that the maps $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ and $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ are embeddings, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and all $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$. (For us, *embedding* in $\boldsymbol{\mathcal{A}}$ means 'isomorphism onto a substructure' and in $\boldsymbol{\mathfrak{X}}$ means 'isomorphism onto a topologically closed substructure'.)

The following theorem summarises the basic properties of this construction that we shall need later.

Theorem 2.2. *Let* $\mathbf{M}_1$ *be a finite structure, let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$ *and define* $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathfrak{X}} = \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. *Then*

(i) *products in both* $\boldsymbol{\mathcal{A}}$ *and* $\boldsymbol{\mathfrak{X}}$ *are constructed pointwise,*
(ii) $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$*, as defined above, is a dual adjunction between* $\boldsymbol{\mathcal{A}}$ *and* $\boldsymbol{\mathfrak{X}}$*,*
(iii) *for all* $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ *and* $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$*, the maps* $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ *and* $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ *are embeddings,*

(iv) *for every non-empty set S, the natural bijection between $\boldsymbol{\mathcal{A}}(\mathbf{F}_{\mathbf{M}_1}(S), \mathbf{M}_1)$ and M^S is an isomorphism between $\mathrm{D}(\mathbf{F}_{\mathbf{M}_1}(S))$ and $\underset{\sim}{\mathbf{M}}{}_2^S$,*

(v) *if $u : \mathbf{A} \to \mathbf{B}$ is a surjection in $\boldsymbol{\mathcal{A}}$, then $\mathrm{D}(u) : \mathrm{D}(\mathbf{B}) \to \mathrm{D}(\mathbf{A})$ is an embedding in $\boldsymbol{\mathfrak{X}}$,*

(vi) *if $\psi : \mathbf{X} \to \mathbf{Y}$ is a surjection in $\boldsymbol{\mathfrak{X}}$, then $\mathrm{E}(\psi) : \mathrm{E}(\mathbf{Y}) \to \mathrm{E}(\mathbf{X})$ is an embedding in $\boldsymbol{\mathcal{A}}$.*

Proof. Part (iv) follows from the fact that a dual adjunction maps coproducts to products, along with the additional fact that $\mathbf{F}_{\mathbf{M}_1}(S)$ is an S-fold coproduct of copies of $\mathbf{F}_{\mathbf{M}_1}(1)$ and $\mathrm{D}(\mathbf{F}_{\mathbf{M}_1}(1))$ is isomorphic to $\underset{\sim}{\mathbf{M}}_2$. The remaining parts of the theorem are straightforward calculations. □

If the map $e_{\mathbf{A}}$ is surjective and therefore an isomorphism, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, then we say that the alter ego $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}$ or simply that $\underset{\sim}{\mathbf{M}}_2$ *dualises* $\mathbf{M}_1$. If $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$, then $\boldsymbol{\mathcal{A}}$ is dually equivalent to a full subcategory of the category $\boldsymbol{\mathfrak{X}}$. In this case, we have a representation for $\boldsymbol{\mathcal{A}}$: each structure $\mathbf{A}$ in $\boldsymbol{\mathcal{A}}$ is isomorphic to the structure $\mathrm{ED}(\mathbf{A})$ consisting of all continuous homomorphisms from $\mathrm{D}(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}_2$. If $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$ and, in addition, $\varepsilon_{\mathbf{X}}$ is surjective and therefore an isomorphism, for all $\mathbf{X}$ in $\boldsymbol{\mathfrak{X}}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality on* $\boldsymbol{\mathcal{A}}$ or, more fully, that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\boldsymbol{\mathcal{A}}$ *and* $\boldsymbol{\mathfrak{X}}$ or, more simply, that $\underset{\sim}{\mathbf{M}}_2$ *fully dualises* $\mathbf{M}_1$. In this case, the functors D and E give a dual equivalence between the categories $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$. If $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$ and, moreover, $\underset{\sim}{\mathbf{M}}_2$ is injective in the category $\boldsymbol{\mathfrak{X}}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality on* $\boldsymbol{\mathcal{A}}$ or that $\underset{\sim}{\mathbf{M}}_2$ *strongly dualises* $\mathbf{M}_1$. (Recall that $\underset{\sim}{\mathbf{M}}_2$ is *injective* in a subclass $\boldsymbol{\mathfrak{C}}$ of $\boldsymbol{\mathfrak{X}}$ if $\underset{\sim}{\mathbf{M}}_2 \in \boldsymbol{\mathfrak{C}}$ and, for every embedding $\varphi : \mathbf{X} \to \mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \boldsymbol{\mathfrak{C}}$, every morphism $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}_2$ extends to a morphism $\beta : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}_2$, that is, $\beta \circ \varphi = \alpha$.)

Remark 2.3. In the case that G_1 contains no nullary symbols, we may wish to include the empty structure $\boldsymbol{\varnothing}_1$ of type $\langle G_1, H_1, R_1 \rangle$ in $\boldsymbol{\mathcal{A}}$. In that case, we must also add (all isomorphic copies of) the complete structure $\mathbf{1}_2$ of type $\langle G_2, H_2, R_2 \rangle$ to $\boldsymbol{\mathfrak{X}}$. Thus, we would redefine $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$ to be $\boldsymbol{\mathcal{A}} := \mathbb{IS}^0\mathbb{P}(\mathbf{M}_1)$ and $\boldsymbol{\mathfrak{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{M}}_2)$. Some care must be taken, as this simple change can destroy a duality. Indeed, assume that $\mathbf{M}_1$ has no nullary operations but does have constant total unary term functions. Then the empty structure $\boldsymbol{\varnothing}_1$ and the substructure $\mathbf{C}^1_1$ of $\mathbf{M}_1$, consisting of the values of the constant total unary term functions of $\mathbf{M}_1$, satisfy $\mathrm{D}(\boldsymbol{\varnothing}_1) \cong \mathrm{D}(\mathbf{C}^1_1) \cong \mathbf{1}_2$, with $\boldsymbol{\varnothing}_1 \ncong \mathbf{C}^1_1$. See Lemma 6.2 below for further details.

3. Axiomatizing topological quasivarieties

Let $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}} \rangle$ be a structure of type $\langle G, H, R \rangle$ with a topology added. We say that $\mathbf{X}$ is a *Boolean topological structure* of type $\langle G, H, R \rangle$ if

- $\mathcal{T}^{\mathbf{X}}$ is a Boolean topology on X (that is, $\mathcal{T}^{\mathbf{X}}$ is compact, Hausdorff and has a basis of clopen sets),
- every relation in $R^{\mathbf{X}}$ and the domain of each partial operation in $H^{\mathbf{X}}$ is a closed subspace of the appropriate power of $\langle X; \mathcal{T}^{\mathbf{X}} \rangle$, and
- every map in $G^{\mathbf{X}} \cup H^{\mathbf{X}}$ is continuous with respect to $\mathcal{T}^{\mathbf{X}}$.

The following result, known as the Separation Theorem [5, 1.4.4], is the topological version of Lemma 1.2 above. While completely elementary, it is a basic tool when trying to describe the class $\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\boldsymbol{\mathfrak{Y}})$ generated by a class $\boldsymbol{\mathfrak{Y}}$ of Boolean topological structures.

Lemma 3.1. *Let $\mathcal{Y}$ be a non-empty set of Boolean topological structures of type $\langle G, H, R\rangle$ and let $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$ be a non-empty structure of the same type with a compact Hausdorff topology added.*

(i) *The complete one-element structure $\mathbf{1}$ belongs to $\mathbb{IS}^0_c\mathbb{P}^+(\mathcal{Y})$ if and only if some $\mathbf{Y} \in \mathcal{Y}$ has a substructure isomorphic to $\mathbf{1}$.*

(ii) *Assume that $\mathbf{X}$ is not isomorphic to the complete one-element structure $\mathbf{1}$. Then $\mathbf{X} \in \mathbb{IS}^0_c\mathbb{P}^+(\mathcal{Y})$ if and only if, for all $r \in \{=\} \cup \{\, \mathrm{dom}(h) \mid h \in H\,\} \cup R$ of arity n, and all $x_1, \dots, x_n \in X$ with $(x_1, \dots, x_n) \notin r^{\mathbf{X}}$, there exist $\mathbf{Y} \in \mathcal{Y}$ and a morphism $\psi : \mathbf{X} \to \mathbf{Y}$ such that $(\psi(x_1), \dots, \psi(x_n)) \notin r^{\mathbf{Y}}$.*

Recently, a number of authors have addressed the question of how to describe the topological quasivariety $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ generated by a finite discrete topological structure $\underset{\sim}{\mathbf{M}}$; see Clark, Davey, Haviar, Pitkethly and Talukder [7], Clark, Davey, Freese and Jackson [6], Davey and Talukder [19] and Clark, Davey, Jackson and Pitkethly [8]. A number of powerful techniques have been developed, but we shall restrict our attention to just one that will be particularly useful in the examples considered in Section 7.

Let Σ be a set of universal Horn sentences of type $\langle G, H, R\rangle$. A topological structure $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$ is a *Boolean topological model of* Σ if

- $\mathbf{X}$ is a Boolean topological structure of type $\langle G, H, R\rangle$ and
- the structure $\langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}\rangle$ is a model of Σ.

The class consisting of all non-empty Boolean topological models of Σ is denoted by $\mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$, while $\mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)$ includes the empty structure if G contains no nullaries.

Let $\mathbf{M}$ be a finite structure and consider the topological quasivariety $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ generated by the corresponding discrete topological structure $\underset{\sim}{\mathbf{M}}$. By Theorem 1.1, there is a set Σ of universal Horn sentences with $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$. Perhaps the simplest description of $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ arises when $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)$, that is, when $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is precisely the class of (possibly empty) Boolean topological models of Σ. The papers referred to in the first paragraph of this section give many examples where this is true and many where it fails.

The following important positive result applies to all of our examples in Section 7. In order to state it, we need to make precise what we mean by a quotient of a total structure. Let $\mathbf{A} = \langle A; G, R\rangle$ be a total structure. A congruence θ on the algebraic reduct $\langle A; G\rangle$ of $\mathbf{A}$ determines a quotient structure $\mathbf{A}/\theta$: for each $r \in R$, we have $(a_1/\theta, \dots, a_m/\theta) \in r^{\mathbf{A}/\theta}$ provided there are $b_1, \dots, b_m \in A$ such that $(a_i, b_i) \in \theta$, for $i = 1, 2, \dots, m$, and $(b_1, \dots, b_m) \in r^{\mathbf{A}}$. We say that a class $\mathcal{C}$ of total structures is *closed under finite quotients* if, whenever $\mathbf{A} \in \mathcal{C}$ and θ is a finite-index congruence on the algebraic reduct of $\mathbf{A}$, the quotient structure $\mathbf{A}/\theta$ belongs to $\mathcal{C}$.

Theorem 3.2. [8, 2.13], [6, 4.3 and 6.9] *Let $\mathbf{M} = \langle M; G, R\rangle$ be a finite total structure. Assume that the quasivariety $\mathbb{ISP}(\mathbf{M})$ generated by $\mathbf{M}$ is closed under finite quotients and that the variety generated by the algebraic reduct of $\mathbf{M}$ is congruence distributive.*

(i) *If Σ is a set of universal Horn sentences such that $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$, then $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$ and $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)$.*

(ii) *If Σ is a set of quasi-atomic sentences such that $\mathbb{ISP}(\mathbf{M}) = \mathrm{Mod}(\Sigma)$, then $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathrm{Bt}}(\Sigma)$ and $\mathbb{IS}^0_c\mathbb{P}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)$.*

Remark 3.3. This is a particularly powerful result. It applies, in particular, to every finite algebra $\mathbf{M}$ with a lattice reduct such that $\mathbb{ISP}(\mathbf{M})$ is closed under homomorphic images. For example, when applied to the two-element bounded lattice $\mathbf{2}$, Theorem 3.2(ii) tells us that $\mathbb{IS}_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{2}})$ is the class consisting of all Boolean topological bounded distributive lattices, a result first proved by Numakura [29].

We would also like to be able to derive a first-order axiomatization of the quasivariety $\mathbb{ISP}(\mathbf{M})$ from a non-first-order description of the topological quasivariety $\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. The following simple observation will suffice.

Lemma 3.4. *Let* $\mathbf{M} = \langle M; G, H, R\rangle$ *be a finite structure and let* Σ *be a set of universal Horn sentences of type* $\langle G, H, R\rangle$. *Assume that every finitely generated model of* Σ *is finite and that* $[\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}})]_{\mathrm{fin}} = [\mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma)]_{\mathrm{fin}}$. *Then* $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$.

Proof. It follows from the assumptions that $[\mathbb{ISP}^+(\mathbf{M})]_{\mathrm{fin}} = [\mathrm{Mod}(\Sigma)]_{\mathrm{fin}}$, and hence $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$ by Lemma 1.6. ☐

Remark 3.5. Often we have a description of the topological quasivariety $\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ of the form $\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}^0_{\mathrm{Bt}}(\Sigma_0 \cup \Phi)$, where Φ is some non-first-order topological condition. If we can find some set Σ_1 of universal Horn sentences such that the finite models of $\Sigma_0 \cup \Sigma_1$ are precisely the finite models of $\Sigma_0 \cup \Phi$ and every finitely generated model of $\Sigma_0 \cup \Sigma_1$ is finite, then we have $\mathbb{ISP}^+(\mathbf{M}) = \mathrm{Mod}(\Sigma)$, where $\Sigma := \Sigma_0 \cup \Sigma_1$. For example, $\mathbf{X} \models \Phi$ might be the statement that $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, in which case the natural choice for Σ_1 would be the axioms for an ordered set.

4. Three basic duality theorems

In this section, we present generalisations of the three theorems that have been used to establish most natural dualities: the Second Duality Theorem, the Duality Compactness Theorem and the NU Duality Theorem (see Clark and Davey [5, 2.2.7, 2.2.11 and 2.3.4]). All three theorems are concerned with lifting up a duality from the finite level. Let $\mathbf{M}_1$ be a finite structure, let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ and let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$. If $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}_{\mathrm{fin}}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$, or that $\underset{\sim}{\mathbf{M}}_2$ *yields a duality at the finite level*; we also say that $\underset{\sim}{\mathbf{M}}_2$ *dualises* $\mathbf{M}_1$ *at the finite level*.

Now assume that $\mathbf{M}_1$ is an algebra and define $\boldsymbol{\mathcal{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. The Second Duality Theorem is due to Davey and Werner [20]. It says that, if $\underset{\sim}{\mathbf{M}}_2$ has no partial operations and only a finite number of relations in its type and $\underset{\sim}{\mathbf{M}}_2$ yields a duality at the finite level and is injective in $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$, then $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$ and is injective in $\boldsymbol{\mathcal{X}}$. The Duality Compactness Theorem is due independently to Willard [33] and Zádori [34]. It says that, if $\underset{\sim}{\mathbf{M}}_2$ is of finite type and yields a duality at the finite level, then $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$. The NU Duality Theorem was proved by Davey and Werner [20] and tells us that if $\mathbf{M}_1$ has a $(k+1)$-ary near-unanimity term, then the purely relational alter ego $\underset{\sim}{\mathbf{M}}_2 := \langle M; R, \mathcal{T}\rangle$, where R is the set of all non-empty subuniverses of $\mathbf{M}_1^k$, yields a duality on $\boldsymbol{\mathcal{A}}$.

The proofs of these theorems given in Clark and Davey [5] extend with only the obvious changes (replace *algebra* by *structure*, etc) to the case where $\mathbf{M}_1$ is an arbitrary finite structure, though in the case of the NU Duality Theorem we need to assume that $\mathbf{M}_1$ is a total structure. We state the structure-theoretic versions of the required results from Chapter 2 of [5], and refer to [5] for the proofs. In each case, we indicate the corresponding result in [5] in square brackets at the start of the statement.

Let $\mathbf{M}_1 = \langle M; G_1, H_1, R_1\rangle$ be a finite structure, let $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, H_2, R_2, \mathcal{T}\rangle$ be an alter ego of $\mathbf{M}_1$, and let $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ be the induced dual adjunction between $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ and $\boldsymbol{\mathcal{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. We say that (CLO) *holds*, or more precisely, that $\mathbf{M}_2$ *satisfies* (CLO) *with respect to* $\mathbf{M}_1$ if

(CLO) for each $n \in \mathbb{N}$, every homomorphism $t : \mathbf{M}_2^n \to \mathbf{M}_2$ is a (total) n-ary term function of $\mathbf{M}_1$.

The fact that $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible guarantees that, for every non-empty set S, each total S-ary term function of $\mathbf{M}_1$ is a morphism from $\underset{\sim}{\mathbf{M}}_2^S$ to $\underset{\sim}{\mathbf{M}}_2$. Thus (CLO) says exactly that, for all $n \in \mathbb{N}$, the total n-ary term functions of $\mathbf{M}_1$ and the homomorphisms from $\mathbf{M}_2^n$ to $\mathbf{M}_2$ agree, that is, the structure $\mathbf{M}_2$ determines the clone of total finitary term functions of the structure $\mathbf{M}_1$. The addition of the discrete topology to $\mathbf{M}_2$ extends this to arbitrary non-zero arities.

Theorem 4.1. [5, 2.2.3] *Let* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *be the quasivariety generated by the finite structure* $\mathbf{M}_1$ *and let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$.

(i) *Fix a non-empty set* S *and let* $\mathbf{F} := \mathbf{F}_{\mathbf{M}_1}(S)$. *The map* $e_{\mathbf{F}} : \mathbf{F} \to \mathrm{ED}(\mathbf{F})$ *is an isomorphism if and only if*

$(\mathrm{CLO})_S$ *every morphism* $t : \underset{\sim}{\mathbf{M}}_2^S \to \underset{\sim}{\mathbf{M}}_2$ *is a* (*total*) S*-ary term function of* $\mathbf{M}_1$.

(ii) *The following are equivalent:*
   (1) (CLO) *holds;*
   (2) $(\mathrm{CLO})_S$ *holds, for every non-empty set* S*;*
   (3) $e_{\mathbf{F}} : \mathbf{F} \to \mathrm{ED}(\mathbf{F})$ *is an isomorphism, for every finitely generated* $\boldsymbol{\mathcal{A}}$*-free structure* $\mathbf{F}$*;*
   (4) $e_{\mathbf{F}} : \mathbf{F} \to \mathrm{ED}(\mathbf{F})$ *is an isomorphism, for every* $\boldsymbol{\mathcal{A}}$*-free structure* $\mathbf{F}$.

We note that the First Duality Theorem [5, 2.2.2] holds in the present setting provided, as above, *term function of* $\mathbf{M}_1$ is replaced by *total term function of* $\mathbf{M}_1$. While this theorem has the advantage that it gives necessary and sufficient conditions for $\underset{\sim}{\mathbf{M}}_2$ to yield a duality on $\boldsymbol{\mathcal{A}}$, we will not state it here as it is rarely used in practice. Instead, we state a corollary of the First Duality Theorem that provides sufficient conditions for $\underset{\sim}{\mathbf{M}}_2$ to dualise $\mathbf{M}_1$.

Theorem 4.2. [5, 2.2.2] *Let* $\mathbf{M}_1$ *be a finite structure, let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$ *and define* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathcal{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. *Then* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}$ *provided* (CLO) *holds and* $\underset{\sim}{\mathbf{M}}_2$ *is injective in* $\boldsymbol{\mathcal{X}}$.

By combining (CLO) with the injectivity of $\underset{\sim}{\mathbf{M}}_2$ in $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$, we obtain a natural interpolation condition. We say that $\mathbf{M}_2$ *satisfies the interpolation condition* (IC) *with respect to* $\mathbf{M}_1$ if

(IC) for each $n \in \mathbb{N}$ and each substructure $\mathbf{X}$ of $\mathbf{M}_2^n$, every homomorphism $\alpha : \mathbf{X} \to \mathbf{M}_2$ extends to a total n-ary term function of $\mathbf{M}_1$.

This condition is sufficient to guarantee that $\underset{\sim}{\mathbf{M}}_2$ dualises $\mathbf{M}_1$ at the finite level.

IC Lemma 4.3. [5, 2.2.5] *Let* $\mathbf{M}_1$ *be a finite structure, let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$ *and define* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathcal{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. *The following are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$ *and is injective in* $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$*;*

(2) $\mathbf{M}_2$ *satisfies* (CLO) *with respect to* $\mathbf{M}_1$ *and* $\underset{\sim}{\mathbf{M}}_2$ *is injective in* $\mathcal{X}_{\text{fin}}$*;*
(3) $\mathbf{M}_2$ *satisfies* (IC) *with respect to* $\mathbf{M}_1$*.*

Assume that $\mathbf{M}_2$ satisfies (IC) with respect to $\mathbf{M}_1$. By Theorem 4.2, to show that the duality on $\mathcal{A}_{\text{fin}}$ lifts to a duality on $\mathcal{A}$, we need to know that the injectivity of $\underset{\sim}{\mathbf{M}}_2$ in $\mathcal{X}$ follows from its injectivity in $\mathcal{X}_{\text{fin}}$.

Injectivity Lifting Lemma 4.4. [5, 2.2.7] *Let* $\mathbf{M} = \langle M; G, R\rangle$ *be a finite total structure with* R *finite and define* $\mathcal{X} := \mathbb{IS}^0_{\text{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$*. If* $\underset{\sim}{\mathbf{M}}$ *is injective in* $\mathcal{X}_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}$ *is injective in* $\mathcal{X}$*.*

Combining this result with the previous two yields our first major lift-from-the-finite-level duality theorem.

Second Duality Theorem 4.5. [5, 2.2.7] *Let* $\mathbf{M}_1$ *be a finite structure and let* $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, R_2, \mathcal{T}\rangle$ *be an alter ego of* $\mathbf{M}_1$ *that is a total structure with* R_2 *finite. Define* $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\mathcal{X} := \mathbb{IS}^0_{\text{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*. If* $\mathbf{M}_2$ *satisfies* (IC) *with respect to* $\mathbf{M}_1$*, then* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\mathcal{A}$ *and is injective in* $\mathcal{X}$*.*

We turn now to the Duality Compactness Theorem. The following lemma is proved by an easy application of the fact that the inverse limit of an inverse system of non-empty finite sets is non-empty.

Lemma 4.6. [5, 2.2.9] *Let* $\mathbf{B}$ *be a non-empty substructure of a locally finite structure* $\mathbf{A}$*, let* $\mathbf{D}$ *be a finite structure and let* $h : \mathbf{B} \to \mathbf{D}$ *be a homomorphism. If, for every finite substructure* $\mathbf{F}$ *of* $\mathbf{A}$ *that intersects* $\mathbf{B}$*, there is a homomorphism* $k : \mathbf{F} \to \mathbf{D}$ *that agrees with* h *on* $B \cap F$*, then there is a homomorphism* $g : \mathbf{A} \to \mathbf{D}$ *that extends* h*.*

We state the following immediate corollary more generally than it is stated in [5]. While this corollary is not needed in the proof of the generalised Duality Compactness Theorem, we include it because of the important role that injectivity plays in the theory of natural dualities.

Corollary 4.7. [5, 2.2.10] *Let* $\mathcal{A}$ *be a locally finite class of structures and assume that* $\mathcal{A}$ *is closed under forming substructures. If* $\mathbf{D}$ *is injective in* $\mathcal{A}_{\text{fin}}$*, then* $\mathbf{D}$ *is injective in* $\mathcal{A}$*.*

Let $\mathbf{M}$ be a finite structure and let $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$. By Lemma 1.5, the quasi-variety $\mathcal{A}$ is locally finite. It follows from Corollary 4.7 that if $\mathbf{D}$ is injective in $\mathcal{A}_{\text{fin}}$, then $\mathbf{D}$ is injective in $\mathcal{A}$. Similarly, if $\mathbf{M}$ is a finite total structure and $\mathbf{D}$ is injective in $\mathbb{HSP}(\mathbf{M})_{\text{fin}}$, then $\mathbf{D}$ is injective in $\mathbb{HSP}(\mathbf{M})$.

The conversion of the proof of the Duality Compactness Theorem given in Clark and Davey [5] to the present setting is a minor search-and-replace exercise and is left to the reader. Recall that a structure $\mathbf{M} = \langle M; G, H, R\rangle$ is of *finite type* if $G \cup H \cup R$ is finite.

Duality Compactness Theorem 4.8. [5, 2.2.11] *Let* $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ *be the quasi-variety generated by the finite structure* $\mathbf{M}_1$*. If* $\underset{\sim}{\mathbf{M}}_2$ *is an alter ego of* $\mathbf{M}_1$ *of finite type that yields a duality at the finite level, then* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\mathcal{A}$*.*

This theorem is a special case of Hofmann's Theorem 2.3, [27]. Combining the IC Lemma 4.3 with the Duality Compactness Theorem yields the following immediate corollary.

IC Duality Theorem 4.9. *Let* $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ *be the quasivariety generated by the finite structure* $\mathbf{M}_1$ *and let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$ *of finite type. If* $\mathbf{M}_2$ *satisfies* (IC) *with respect to* $\mathbf{M}_1$*, then* $\underset{\sim}{\mathbf{M}}_2$ *yields a duality on* $\mathcal{A}$*.*

Some care is required when extending other basic results in the theory of natural dualities to this more general setting. For example, in the case that $\mathbf{M}_1$ is an algebra, it is a common practice when seeking a duality to work interchangeably with partial operations and their graphs in the type of the alter ego $\underset{\sim}{\mathbf{M}}_2$. If $\mathbf{M}_1$ includes partial operations or relations in its type, this is no longer possible as the graph of a partial operation can be compatible with $\mathbf{M}_1$ while the partial operation itself is not.

Another common trick that is used in the case that $\mathbf{M}_1$ is an algebra is to take a finite number of homomorphisms $x_1, \dots, x_n : \mathbf{A} \to \mathbf{M}_1$, for some $\mathbf{A} \in \mathbb{ISP}(\mathbf{M}_1)$, and then use image of the product map $x_1 \sqcap \cdots \sqcap x_n : \mathbf{A} \to \mathbf{M}_1^n$, that is, the n-ary relation $\{\, (x_1(a), \dots, x_n(a)) \mid a \in A \,\}$, in an alter ego of $\mathbf{M}_1$. While this trick is still available when $\mathbf{M}_1$ is a total structure, it cannot be used when the type of $\mathbf{M}_1$ includes partial operations, as the image of $x_1 \sqcap \cdots \sqcap x_n$ may not be a substructure of $\mathbf{M}_1^n$. Two important results whose proofs utilise this trick are the Brute Force Duality Theorem [5, 2.3.1] and the NU Duality Theorem [5, 2.3.4]. These theorems continue to hold when $\mathbf{M}_1$ is a total structure.

We close this section with the statement of the NU Duality Theorem as it applies to total structures. For $n \geqslant 3$, a function $t : M^n \to M$ is a *near unanimity function* on the set M if it satisfies $t(a, \dots, a, b) = t(a, \dots, a, b, a) = \cdots = t(b, a, \dots, a) = a$, for all $a, b \in M$.

NU Duality Theorem 4.10. [5, 2.3.4] *Let* $k \geqslant 2$ *and assume that* $\mathbf{M}_1$ *is a finite total structure that has a* $(k{+}1)$*-ary near unanimity term function. Let* $\mathbf{M}_2 = \langle M; R \rangle$*, where* R *is the set of all non-empty subuniverses of* $\mathbf{M}_1^k$*, and define* $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\mathcal{X} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*. Then* $\underset{\sim}{\mathbf{M}}_2$ *satisfies* (IC) *with respect to* $\mathbf{M}_1$*, yields a duality on* $\mathcal{A}$ *and is injective in* $\mathcal{X}$*.*

5. Lifting full duality up from the finite level

The theorems in the previous section give conditions under which a duality for the class $\mathcal{A}_{\text{fin}}$ can be lifted up to a duality for the class $\mathcal{A}$. In this section we turn our attention to finding conditions under which a full duality for $\mathcal{A}_{\text{fin}}$ can be lifted up to a full duality for $\mathcal{A}$. The results are a refinement and simplification, in our restricted setting, of the presentation given by Hofmann [27].

Let $\mathbf{M}_1$ be a finite structure, let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$ and consider the classes $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ and $\mathcal{X} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. If $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathcal{A}_{\text{fin}}$, and $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is an isomorphism, for all $\mathbf{X} \in \mathcal{X}_{\text{fin}}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\mathcal{A}_{\text{fin}}$ *and* $\mathcal{X}_{\text{fin}}$, or simply that $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality at the finite level*, or that $\underset{\sim}{\mathbf{M}}_2$ *fully dualises* $\mathbf{M}_1$ *at the finite level.* In this case, the functors D and E yield a dual category equivalence between the categories $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$. If $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$ and, moreover, $\underset{\sim}{\mathbf{M}}_2$ is injective in $\mathcal{X}_{\text{fin}}$, then we say that $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $\mathcal{A}_{\text{fin}}$ *and* $\mathcal{X}_{\text{fin}}$, or simply that $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality at the finite level*, or that $\underset{\sim}{\mathbf{M}}_2$ *strongly dualises* $\mathbf{M}_1$ *at the finite level.*

The class $\mathcal{X}$ is closed under forming inverse limits. Indeed, let $\mathbf{S} = \langle S; \leqslant \rangle$ be a non-empty directed ordered set and let $\{\, \mathbf{X}_s \mid s \in S \,\}$ be an inverse system in $\mathcal{X}$

with connecting morphisms $\eta_{st} : \mathbf{X}_s \to \mathbf{X}_t$, for all $s \geqslant t$ in $\mathbf{S}$. Then the inverse limit in $\boldsymbol{\mathcal{X}}$ of the system is the closed substructure of $\prod_{s\in S} \mathbf{X}_s$ on the set

$$\{\, x \in \textstyle\prod_{s\in S} X_s \mid (\forall s, t \in S)\ s \geqslant t \implies \eta_{st}(x(s)) = x(t) \,\}$$

and is denoted by $\varprojlim_{s\in S} \mathbf{X}_s$. For a subclass $\boldsymbol{\mathcal{Y}}$ of $\boldsymbol{\mathcal{X}}$, we shall use the notation $\varprojlim \boldsymbol{\mathcal{Y}}$ to denote the full subcategory of $\boldsymbol{\mathcal{X}}$ whose objects are (isomorphic copies of) inverse limits of structures in $\boldsymbol{\mathcal{Y}}$. The following lemma is a simple piece of category theory and can be formulated much more generally (see, for example, Banaschewski [2] and Hofmann [27]).

Lemma 5.1. *Let $\mathbf{M}_1$ be a finite structure and let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$. Define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ and $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ and let $\boldsymbol{\mathcal{Y}}$ be a subclass of $\boldsymbol{\mathcal{X}}$. Assume that $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$ and that $\varepsilon_{\mathbf{Y}} : \mathbf{Y} \to \mathrm{DE}(\mathbf{Y})$ is an isomorphism, for all $\mathbf{Y} \in \boldsymbol{\mathcal{Y}}$. Then $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is an isomorphism, for all $\mathbf{X} \in \varprojlim \boldsymbol{\mathcal{Y}}$.*

Proof. Let $\mathbf{X} \in \varprojlim \boldsymbol{\mathcal{Y}}$. To prove that $\varepsilon_{\mathbf{X}}$ is an isomorphism, it suffices to show that $\mathbf{X} \cong \mathrm{D}(\mathbf{A})$, for some structure $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. Indeed, if $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and $\varphi : \mathbf{X} \to \mathrm{D}(\mathbf{A})$ is an isomorphism, then since $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ is a dual adjunction between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$, we have $\varphi = \mathrm{D}(\mathrm{E}(\varphi) \circ e_{\mathbf{A}}) \circ \varepsilon_{\mathbf{X}}$ (see the triangular commutative diagrams on page 5 of Clark and Davey [5], for example). Since φ and $e_{\mathbf{A}}$ are isomorphisms it follows immediately that $\varepsilon_{\mathbf{X}}$ is also an isomorphism.

We have $\mathbf{X} = \varprojlim_{s\in S} \mathbf{Y}_s$, for some inverse system $\{\, \mathbf{Y}_s \mid s \in S \,\}$ in $\boldsymbol{\mathcal{Y}} \subseteq \boldsymbol{\mathcal{X}}$ with connecting morphisms $\eta_{st} : \mathbf{Y}_s \to \mathbf{Y}_t$. The structures $\mathrm{E}(\mathbf{Y}_s)$ and the connecting maps $\mathrm{E}(\eta_{st})$, with $t \leqslant s$, form a direct system of structures in $\boldsymbol{\mathcal{A}}$. Let $\mathbf{A} := \varinjlim_{s\in S} \mathrm{E}(\mathbf{Y}_s)$ be the direct limit calculated in the quasivariety $\boldsymbol{\mathcal{A}}$. Since $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ is a dual adjunction between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$, the functor D maps direct limits in $\boldsymbol{\mathcal{A}}$ to inverse limits in $\boldsymbol{\mathcal{X}}$ (see Mac Lane [28, V.5]). Thus

$$\mathrm{D}(\mathbf{A}) = \mathrm{D}(\varinjlim_{s\in S} \mathrm{E}(\mathbf{Y}_s)) \cong \varprojlim_{s\in S} \mathrm{DE}(\mathbf{Y}_s) \cong \varprojlim_{s\in S} \mathbf{Y}_s = \mathbf{X},$$

as $\mathbf{Y}_s \in \boldsymbol{\mathcal{Y}}$ and therefore $\mathrm{DE}(\mathbf{Y}_s) \cong \mathbf{Y}_s$. □

We would like to be able to use this observation to lift a full duality at the finite level up to a full duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$. The first step in this process is another general category-theoretic observation that can be stated more generally.

Lifting Full Duality Lemma 5.2. *Let $\mathbf{M}_1$ be a finite structure and let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego of $\mathbf{M}_1$. Define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ and $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ and assume that $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$. Then $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$ if and only if $\underset{\sim}{\mathbf{M}}_2$ yields a full duality at the finite level and $\boldsymbol{\mathcal{X}} = \varprojlim \boldsymbol{\mathcal{X}}_{\mathrm{fin}}$.*

Proof. First assume that the alter ego $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$. Then it certainly yields a full duality between $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$ and $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$. By Lemma 1.4, every structure in $\boldsymbol{\mathcal{A}}$ is the direct limit of its finitely generated substructures. As $\boldsymbol{\mathcal{A}}$ is locally finite (by Lemma 1.5), every structure in $\boldsymbol{\mathcal{A}}$ is therefore a direct limit of structures from $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$. Since the functors D and E give a dual equivalence between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$, it follows that each structure $\mathbf{X}$ in $\boldsymbol{\mathcal{X}}$ is an inverse limit of structures from $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$. Thus, the forward direction holds. The backward direction is an immediate consequence of the previous lemma. □

In order to lift a full duality at the finite level up to a full duality between $\mathcal{A}$ and $\mathcal{X}$, we now need answers to the following two questions.

(I) If the alter ego $\underset{\sim}{\mathbf{M}}_2$ yields a duality between $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$, does it follow that it yields a duality between $\mathcal{A}$ and $\mathcal{X}$?

(II) If the alter ego $\underset{\sim}{\mathbf{M}}_2$ yields a duality between $\mathcal{A}$ and $\mathcal{X}$ and a full duality between $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$, does it follow that $\mathcal{X} = \varprojlim \mathcal{X}_{\text{fin}}$?

The Duality Compactness Theorem 4.8 tells us that the answer to (I) is 'yes', provided the type of $\underset{\sim}{\mathbf{M}}_2$ is finite. A restriction on the alter ego is necessary here. For example, if $\mathbf{I}$ is the two-element implication algebra, then no alter ego yields a duality on the class $\mathcal{A} := \mathbb{ISP}(\mathbf{I})$, yet the alter ego consisting of all finitary relations that are algebraic over $\mathbf{I}$ yields a duality at the finite level (see [5] for details). Clark, Davey, Jackson and Pitkethly [8, Corollary 2.4] prove that the answer to (II) is 'yes' in the case that $\underset{\sim}{\mathbf{M}}_2 = \langle M; G, R, \mathcal{T}\rangle$ is a total structure. Thus we obtain the following result that can be viewed as a limited *Full Duality Compactness Theorem.*

Total Structure Full Duality Theorem 5.3. *Let $\mathcal{A} := \mathbb{ISP}(\mathbf{M}_1)$ be the quasi-variety generated by the finite structure $\mathbf{M}_1$ and let $\underset{\sim}{\mathbf{M}}_2 = \langle M; G, R, \mathcal{T}\rangle$ be an alter ego of $\mathbf{M}_1$ that is a total structure.*

(i) *If $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\mathcal{A}$ and yields a full duality at the finite level, then $\underset{\sim}{\mathbf{M}}_2$ yields a full duality on $\mathcal{A}$.*

(ii) *If $\underset{\sim}{\mathbf{M}}_2$ is of finite type and yields a full duality at the finite level, then $\underset{\sim}{\mathbf{M}}_2$ yields a full duality on $\mathcal{A}$.*

Proof. Part (i) follows immediately from the previous lemma and the fact, proved in [8], that $\mathcal{X} = \varprojlim \mathcal{X}_{\text{fin}}$ provided the type of $\underset{\sim}{\mathbf{M}}_2$ includes no partial operations. The Duality Compactness Theorem 4.8 guarantees that, if the type of $\underset{\sim}{\mathbf{M}}_2$ is finite, then a duality at the finite level lifts to a duality on $\mathcal{A}$. Thus, (ii) follows from (i). □

This is a special case of Theorem 2.5 in Hofmann's paper [27]. Where we have assumed that $\underset{\sim}{\mathbf{M}}_2$ is a total structure, Hofmann assumes that $\mathcal{X}$ has Sur-Inj factorizations. This amounts to assuming that the image of every morphism in $\mathcal{X}$ is a substructure, a condition obviously guaranteed by our assumption that $\underset{\sim}{\mathbf{M}}_2$ is a total structure. In fact, in the case that $\mathbf{M}_1$ is an algebra, Exercise 6.5 of Clark and Davey [5] shows that, in the presence of a full duality, the image of every morphism in $\mathcal{X}$ is a substructure if and only if the alter ego $\underset{\sim}{\mathbf{M}}_2$ is structurally equivalent to a total structure.

As usual, the true role of partial operations in the proof of Theorem 5.3 is somewhat mysterious. The following example shows that, in the presence of partial operations, the answer to (II) can be 'no': it is possible for $\underset{\sim}{\mathbf{M}}_2$ to yield a duality on $\mathbb{ISP}(\mathbf{M}_1)$ and yield a full duality at the finite level and yet satisfy $\varprojlim \mathcal{X}_{\text{fin}} \subsetneq \mathcal{X}$.

Let $\mathbf{3}_L = \langle \{0, a, 1\}; \vee, \wedge, 0, 1\rangle$ be the three-element bounded lattice. Then $\mathcal{D}^{01} := \mathbb{ISP}(\mathbf{3}_L)$ is the class of all bounded distributive lattices. The non-identity endomorphisms of $\mathbf{3}_L$ are f and g, given by

$$f(0) = f(a) = 0,\ f(1) = 1 \ \text{ and } \ g(0) = 0,\ g(a) = g(1) = 1.$$

Davey, Haviar and Priestley [15] proved that $\underset{\sim}{\mathbf{3}}_{fg} := \langle \{0, a, 1\}; f, g, \mathcal{T}\rangle$ yields a duality on the class $\mathcal{D}^{01}$ of bounded distributive lattices. Subsequently, Davey, Haviar and Willard [17] proved that the alter ego $\underset{\sim}{\mathbf{3}}_{fgh} := \langle \{0, a, 1\}; f, g, h, \mathcal{T}\rangle$,

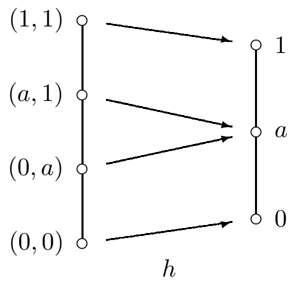

FIGURE 1. The partial operation h

where h is the binary partial operation shown in Figure 1, yields a duality on $\mathfrak{D}^{01}$ that is full at the finite level (but not strong at the finite level).

Example 5.4. *Let* $\mathbf{3}_L = \langle \{0, a, 1\}; \vee, \wedge, 0, 1\rangle$ *and* $\underset{\sim}{\mathbf{3}}_{fgh} = \langle \{0, a, 1\}; f, g, h, \mathcal{T}\rangle$ *be as above and let* $\mathfrak{X} := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{3}}_{fgh})$*. Then*

- (i) $\underset{\sim}{\mathbf{3}}_{fgh}$ *yields a duality on the class* $\mathfrak{D}^{01}$ *of bounded distributive lattices,*
- (ii) $\underset{\sim}{\mathbf{3}}_{fgh}$ *yields a full duality on the class* $\mathfrak{D}^{01}_{\text{fin}}$ *of finite bounded distributive lattices, but*
- (iii) *not every topological structure in* $\mathfrak{X}$ *is an inverse limit of finite structures in* $\mathfrak{X}$*.*

Proof. Since (i) and (ii) are proved in [15] and [17], we turn to (iii). We shall utilise a construction used in [17]. Let $\mathbf{X} \in \mathfrak{X}$ and define $P_{\mathbf{X}} := \text{fix}(f) = \text{fix}(g) \subseteq X$. Endow $P_{\mathbf{X}}$ with the subspace topology and define a binary relation $\preccurlyeq$ on $P_{\mathbf{X}}$ by

$$u \preccurlyeq v \iff (\exists x \in X)\, f(x) = u \;\;\&\;\; g(x) = v.$$

Davey, Haviar and Willard [17] proved that the structure $\mathbf{P_X} := \langle P_{\mathbf{X}}; \preccurlyeq, \mathcal{T}\rangle$ is an ordered Boolean space, that is, $\langle P_{\mathbf{X}}; \preccurlyeq\rangle$ is an ordered set, $\mathcal{T}$ is a Boolean topology on $P_{\mathbf{X}}$ and $\preccurlyeq$ is a closed subset of $P_{\mathbf{X}} \times P_{\mathbf{X}}$. In fact, it is easily seen that $F : \mathbf{X} \mapsto \mathbf{P_X}$ is (the object half of) a functor from $\mathfrak{X}$ to the category $\mathfrak{Z}_{\leqslant}$ of ordered Boolean spaces. Since f (and g) are calculated pointwise in a product of structures from $\mathfrak{X}$ and since inverse limits are calculated pointwise in both $\mathfrak{X}$ and $\mathfrak{Z}_{\leqslant}$, it follows by a simple calculation that F preserves inverse limits. Let $\mathbf{X} = \varprojlim_{s\in S} \mathbf{X}_s$ be an inverse limit in $\mathfrak{X}$ with $\mathbf{X}_s \in \mathfrak{X}_{\text{fin}}$, for all $s \in S$. Then

$$\mathbf{P_X} = F(\mathbf{X}) = F(\varprojlim_{s\in S} \mathbf{X}_s) \cong \varprojlim_{s\in S} F(\mathbf{X}_s) = \varprojlim_{s\in S} \mathbf{P}_{\mathbf{X}_s}.$$

Thus, since an inverse limit of finite ordered sets is a Priestley space, it follows that $\mathbf{P_X}$ is a Priestley space, for all $\mathbf{X} \in \varprojlim \mathfrak{X}_{\text{fin}}$. In [17] an example is given of a structure $\mathbf{Y}$ in $\mathfrak{X}$ for which the ordered Boolean space $\mathbf{P_Y}$ is not a Priestley space. The argument just given shows that $\mathbf{Y}$ does not belong to $\varprojlim \mathfrak{X}_{\text{fin}}$. □

The following observation adds to the mystery. While $H = \varnothing$ is a sufficient condition for $\mathfrak{X} = \varprojlim \mathfrak{X}_{\text{fin}}$, it is certainly not necessary. Let $\mathbf{M}_1$ be a finite, strongly dualisable algebra that is not injective in the quasivariety it generates. For example, let $\mathbf{M}_1$ be the four-element Heyting chain (see Example 7.5 and the discussion preceding it). Let $\underset{\sim}{\mathbf{M}}_2$ be an alter ego that strongly dualises $\mathbf{M}_1$. The general theory tells us that $\underset{\sim}{\mathbf{M}}_2$ *must* have partial operations in its type (see the Total Structure Theorem [5, 6.1.2]). Since $\underset{\sim}{\mathbf{M}}_2$ yields a full duality on $\mathbb{ISP}(\mathbf{M}_1)$, Lemma 5.2 guarantees that $\boldsymbol{\mathcal{X}} = \varprojlim \boldsymbol{\mathcal{X}}_{\text{fin}}$.

We close this section with a result that gives conditions under which a strong duality can be lifted up from the finite level. Note that by using the Second Duality Theorem 4.5 instead of the Duality Compactness Theorem 4.8, we have weakened the assumption that $\underset{\sim}{\mathbf{M}}_2$ is of finite type.

Total Structure Strong Duality Theorem 5.5. *Let* $\mathbf{M}_1$ *be a finite structure and let* $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, R_2, \mathcal{T}\rangle$ *be an alter ego of* $\mathbf{M}_1$ *that is a total structure with* R_2 *finite. Define* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathcal{X}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*. If* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_{\text{fin}}$ *and* $\boldsymbol{\mathcal{X}}_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}$ *and* $\boldsymbol{\mathcal{X}}$*.*

Proof. Assume that $\underset{\sim}{\mathbf{M}}_2$ yields a strong duality at the finite level. By the IC Lemma 4.3, $\mathbf{M}_2$ satisfies (IC) with respect to $\mathbf{M}_1$ and hence $\underset{\sim}{\mathbf{M}}_2$ yields a duality on $\boldsymbol{\mathcal{A}}$ and is injective in $\boldsymbol{\mathcal{X}}$, by the Second Duality Theorem 4.5. It remains to show that this duality is full. But this follows immediately from the Total Structure Full Duality Theorem 5.3(i). □

6. Two-for-one duality theorems

In this section, we investigate when it is possible to remove the topology from $\underset{\sim}{\mathbf{M}}_2$, add it to $\mathbf{M}_1$, and thereby convert a full duality for $\mathbb{ISP}(\mathbf{M}_1)$, induced by the alter ego $\underset{\sim}{\mathbf{M}}_2$, into a full duality for $\mathbb{ISP}(\mathbf{M}_2)$, induced by the alter ego $\underset{\sim}{\mathbf{M}}_1$. When this is possible, we get two dualities for the price of one. The following lemma shows that, in order for this to work, the types of both $\mathbf{M}_1$ and $\mathbf{M}_2$ must include enough nullary operations. First, we need some notation and a definition.

Let $\mathbf{M}$ be a structure. Define C^0 to be the subuniverse of $\mathbf{M}$ consisting of the values of the nullary term functions of $\mathbf{M}$, and define C^1 to be the subuniverse of $\mathbf{M}$ consisting of the values of the constant total unary term functions of $\mathbf{M}$. Obviously we have $C^0 \subseteq C^1$. If every element of M that is the value of a constant total unary term function of $\mathbf{M}$ is the value of a nullary term function of $\mathbf{M}$, that is, if $C^1 = C^0$, then we say that $\mathbf{M}$ *has named constants.* Note that $C^1 = C^0$ if and only if either $C^0 \neq \varnothing$ or $C^1 = \varnothing$.

Now assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible structures. For $i \in \{1, 2\}$, define C^0_i and C^1_i as above, and let K_i be the set consisting of all elements of M that form complete one-element substructures of $\mathbf{M}_i$. Since $\mathbf{M}_1$ and $\mathbf{M}_2$ are compatible, K_1 forms a substructure of $\mathbf{M}_2$ and K_2 forms a substructure of $\mathbf{M}_1$. Moreover, if t is a constant total unary term function of $\mathbf{M}_2$, then t is a constant endomorphism of $\mathbf{M}_1$. Since we have a base assumption that, on $\mathbf{M}_1$, the relations in R_1 and the domains of the partial operations in H_1 are non-empty, it follows that the image of t forms a complete one-element substructure of $\mathbf{M}_1$. So $C^1_2 \subseteq K_1$, and, similarly, $C^1_1 \subseteq K_2$. Thus we have

$$C^0_1 \subseteq C^1_1 \subseteq K_2 \quad \text{and} \quad C^0_2 \subseteq C^1_2 \subseteq K_1.$$

In the presence of a full duality, all but one of these inclusions become equalities.

Lemma 6.1. *Assume that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are compatible structures.*

(i) *If* $\mathbf{M}_2$ *satisfies* $(\mathrm{CLO})_1$ *with respect to* $\mathbf{M}_1$ *(in particular, if* $\underset{\sim}{\mathbf{M}}_2$ *dualises* $\mathbf{M}_1$ *at the finite level), then every element of* M *that forms a complete one-element substructure of* $\mathbf{M}_2$ *is the value of a constant total unary term function of* $\mathbf{M}_1$*, that is,* $C_1^1 = K_2$.

(ii) *If* $\underset{\sim}{\mathbf{M}}_2$ *fully dualises* $\mathbf{M}_1$ *at the finite level, then* $\mathbf{M}_2$ *has named constants.*

(iii) *If* $\underset{\sim}{\mathbf{M}}_2$ *fully dualises* $\mathbf{M}_1$ *at the finite level, then* $C_1^1 = K_2$ *and* $C_2^0 = C_2^1 = K_1$.

Proof. Assume that $\mathbf{M}_2$ satisfies $(\mathrm{CLO})_1$ with respect to $\mathbf{M}_1$ and that $a \in M$ forms a complete one-element substructure of $\mathbf{M}_2$. Then the constant map φ from M to M with value a is a morphism from $\mathbf{M}_2$ to $\mathbf{M}_2$. Since $\mathbf{M}_2$ satisfies $(\mathrm{CLO})_1$ with respect to $\mathbf{M}_1$, the map φ is a constant total unary term function of $\mathbf{M}_1$. This proves (i).

Now define $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ and $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}_{\mathrm{c}}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$, for $i \in \{1, 2\}$, and assume that $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ and $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$. Let $\mathrm{D}_2 : \boldsymbol{\mathcal{A}}_1 \to \boldsymbol{\mathfrak{X}}_2$ and $\mathrm{E}_2 : \boldsymbol{\mathfrak{X}}_2 \to \boldsymbol{\mathcal{A}}_1$ be the functors induced by the alter ego $\underset{\sim}{\mathbf{M}}_2$ of $\mathbf{M}_1$. Suppose that $C_2^0 \neq C_2^1$. Then we must have $C_2^0 = \varnothing$ and $C_2^1 \neq \varnothing$. Let $\mathbf{C}_2^0$ and $\mathbf{C}_2^1$ be the corresponding substructures of $\mathbf{M}_2$. Then, by definition, $\mathrm{E}_2(\mathbf{C}_2^0) = \mathrm{E}_2(\boldsymbol{\varnothing}_2) = \mathbf{1}_1$, where $\boldsymbol{\varnothing}_2$ denotes the empty structure of type $\langle G_2, H_2, R_2\rangle$ and $\mathbf{1}_1$ denotes the complete one-element structure of type $\langle G_1, H_1, R_1\rangle$. The only element of $\mathrm{E}_2(\mathbf{C}_2^1)$ is the inclusion map of C_2^1 into M. An easy calculation, using the fact that, on $\mathbf{M}_1$, each relation in R_1 and the domain of every partial operation in H_1 is non-empty, shows that $\mathrm{E}_2(\mathbf{C}_2^1) \cong \mathbf{1}_1$. So

$$\mathbf{C}_2^0 \cong \mathrm{D}_2\mathrm{E}_2(\mathbf{C}_2^0) \cong \mathrm{D}_2(\mathbf{1}_1) \cong \mathrm{D}_2\mathrm{E}_2(\mathbf{C}_2^1) \cong \mathbf{C}_2^1,$$

a contradiction. Thus, (ii) holds.

We now prove (iii). By (i) and (ii), it remains to prove that $C_2^1 = K_1$. First, we shall prove that $\underset{\sim}{\mathbf{M}}_1$ yields a duality between $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\mathrm{fin}}$. To simplify the notation, our remaining calculations are modulo the obvious addition or removal of the discrete topology. Thus, we regard $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ as a subclass of $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}} \cup \{\mathbf{1}_2\}$. Let $\mathrm{D}_1 : \boldsymbol{\mathcal{A}}_2 \to \boldsymbol{\mathfrak{X}}_1$ and $\mathrm{E}_1 : \boldsymbol{\mathfrak{X}}_1 \to \boldsymbol{\mathcal{A}}_2$ be the functors induced by the alter ego $\underset{\sim}{\mathbf{M}}_1$ of $\mathbf{M}_2$. Let $\mathbf{A} \in (\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$. If $\mathbf{A} \not\cong \mathbf{1}_2$ or if $\mathbf{A} \cong \mathbf{1}_2$ and $\mathbf{1}_2 \in \boldsymbol{\mathfrak{X}}_2$, then $\mathbf{A} \in \boldsymbol{\mathfrak{X}}_2$ and hence $\mathrm{E}_1\mathrm{D}_1(\mathbf{A}) = \mathrm{D}_2\mathrm{E}_2(\mathbf{A}) \cong \mathbf{A}$, as $\underset{\sim}{\mathbf{M}}_2$ fully dualises $\mathbf{M}_1$ at the finite level. Otherwise, $\mathbf{A} \cong \mathbf{1}_2$ and $\mathbf{1}_2 \notin \boldsymbol{\mathfrak{X}}_2$, in which case $\mathrm{D}_1(\mathbf{A}) \cong \mathrm{D}_1(\mathbf{1}_2) = \boldsymbol{\varnothing}_1$ and hence $\mathrm{E}_1\mathrm{D}_1(\mathbf{A}) = \mathrm{E}_1(\boldsymbol{\varnothing}_1) \cong \mathbf{1}_2 \cong \mathbf{A}$. It follows that $\underset{\sim}{\mathbf{M}}_1$ yields a duality between $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\mathrm{fin}}$. An application of (i) gives $C_2^1 = K_1$, as required. □

When can we extend a full duality between $\mathbb{ISP}(\mathbf{M}_1)$ and $\mathbb{IS}_{\mathrm{c}}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ to a full duality between $\mathbb{IS}^0\mathbb{P}(\mathbf{M}_1)$ and $\mathbb{IS}_{\mathrm{c}}^0\mathbb{P}(\underset{\sim}{\mathbf{M}}_2)$? The next lemma answers this question. As adding the complete one-element structure $\mathbf{1}_2$ to $\mathbb{IS}_{\mathrm{c}}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ cannot affect the injectivity of $\underset{\sim}{\mathbf{M}}_2$, the lemma also holds with ‘full’ replaced by ‘strong’.

Lemma 6.2. *Let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of a finite structure* $\mathbf{M}_1$*. Assume that* $\mathbf{M}_1$ *has no constant total unary term functions and that* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\mathbb{ISP}(\mathbf{M}_1)$ *and* $\mathbb{IS}_{\mathrm{c}}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*. Then* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\mathbb{IS}^0\mathbb{P}(\mathbf{M}_1)$ *and* $\mathbb{IS}_{\mathrm{c}}^0\mathbb{P}(\underset{\sim}{\mathbf{M}}_2)$.

Proof. Since $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $\mathbb{ISP}(\mathbf{M}_1)$ and $\mathbb{IS}_{\mathrm{c}}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$, we have $C_1^1 = K_2$, by Lemma 6.1. As C_1^1 is empty, by assumption, we have $K_2 = \varnothing$. Hence

$\mathbf{M}_2$ has no complete one-element substructures and consequently $\mathbf{1}_2 \notin \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$, where $\mathbf{1}_2$ denotes the complete one-element structure of type $\langle G_2, H_2, R_2\rangle$. As the empty structure $\boldsymbol{\varnothing}_1$ in $\mathbb{IS}^0\mathbb{P}(\mathbf{M}_1)$ and the complete one-element structure $\mathbf{1}_2$ in $\mathbb{IS}^0_c\mathbb{P}(\underset{\sim}{\mathbf{M}}_2)$ are dual to each other, the result follows. □

We can now state our two-for-one results. The first, a finite-level two-for-one lemma, is very easy.

Lemma 6.3. *Assume that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are finite compatible structures and define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1, 2\}$*.*

(i) *If* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a duality between* $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$ *that is full provided* $\mathbf{M}_1$ *has named constants.*

(ii) *Assume that both* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *have named constants. Then* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$ *if and only if* $\underset{\sim}{\mathbf{M}}_1$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$*.*

Proof. Ignoring the discrete topology, the only difference between $\mathbb{ISP}(\mathbf{M}_i)_{\text{fin}}$ and $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)_{\text{fin}}$ is the inclusion or exclusion of the empty structure and the complete one-element structure. Assume that $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ and $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$. Then, as in the proof of Lemma 6.1, $\underset{\sim}{\mathbf{M}}_1$ yields a duality on $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$. If $\mathbf{M}_1$ has named constants, then by Lemma 6.1(iii), we have $C_1^0 = C_1^1 = K_2$. Thus $\boldsymbol{\varnothing}_1 \in \boldsymbol{\mathfrak{X}}_1$ implies that $\mathbf{M}_2$ has no complete one-element substructures, whence $\mathrm{D}_1\mathrm{E}_1(\boldsymbol{\varnothing}_1) \cong \mathrm{D}_1(\mathbf{1}_2) = \boldsymbol{\varnothing}_1$. It follows that $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$. Thus (i) holds, and (ii) is an immediate consequence of (i). □

By strengthening the assumptions of this lemma in an asymmetrical way and combining it with the Total Structure Full Duality Theorem 5.3, we obtain the following 'One-and-a-Half-for-a-Half' Full Duality Theorem.

Sesqui Full Duality Theorem 6.4. *Assume that* $\mathbf{M}_1$ *is a finite total structure of finite type that has named constants, let* $\mathbf{M}_2$ *be a structure that is compatible with* $\mathbf{M}_1$ *and define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1, 2\}$*. If* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a full duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*.*

Proof. Assume that $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ and $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$. As $\mathbf{M}_1$ has named constants, it follows from the previous lemma that $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$. Since $\mathbf{M}_1$ is of finite type, $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $\boldsymbol{\mathcal{A}}_2$ and $\boldsymbol{\mathfrak{X}}_1$, by the Total Structure Full Duality Theorem 5.3(ii). □

The following theorem should be compared with the Two-for-One Strong Duality Theorem 3.3.2 in [5]. The theorem here has a stronger assumption, namely that both structures are of finite type, but has the advantage that it separates the fullness of the resulting dualities from considerations of whether they are strong (that is, whether the alter egos are injective in the topological quasivarieties they generate). This is special case of Theorem 2.5 in Hofmann [27].

Two-for-One Full Duality Theorem 6.5. *Assume that* $\mathbf{M}_1 = \langle M; G_1, R_1\rangle$ *and* $\mathbf{M}_2 = \langle M; G_2, R_2\rangle$ *are finite compatible total structures of finite type and that each has named constants. Define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1, 2\}$*. Then the following are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$*;*

(2) $\underset{\sim}{\mathbf{M}}_1$ *yields a full duality between* $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$*;*
(3) $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\boldsymbol{\mathcal{A}}_1$ *and* $\boldsymbol{\mathfrak{X}}_2$*;*
(4) $\underset{\sim}{\mathbf{M}}_1$ *yields a full duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*.*

Proof. Apply the previous theorem twice. □

We turn now to two-for-one strong dualities. The injectivity of $\mathbf{M}_1$ in $\boldsymbol{\mathcal{A}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ in $\boldsymbol{\mathfrak{X}}_2$ are closely linked. The following lemma is proved exactly as it is in the case where $\mathbf{M}_1$ is an algebra.

Injectivity Transfer Lemma 6.6. [5, 3.2.10] *Let* $\mathbf{M}_1$ *be a finite structure, let* $\underset{\sim}{\mathbf{M}}_2$ *be an alter ego of* $\mathbf{M}_1$ *and define* $\boldsymbol{\mathcal{A}}_1 := \mathbb{ISP}(\mathbf{M}_1)$ *and* $\boldsymbol{\mathfrak{X}}_2 := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*. Assume that* $\underset{\sim}{\mathbf{M}}_2$ *yields a full duality between* $\boldsymbol{\mathcal{A}}_1$ *and* $\boldsymbol{\mathfrak{X}}_2$ [*at the finite level*].

(i) *If* $\mathbf{M}_1$ *is a total structure and is injective in* $\boldsymbol{\mathcal{A}}_1$ [*in* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$], *then* $\underset{\sim}{\mathbf{M}}_2$ *is injective in* $\boldsymbol{\mathfrak{X}}_2$ [*in* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$].
(ii) *If* $\underset{\sim}{\mathbf{M}}_2$ *is a total structure and is injective in* $\boldsymbol{\mathfrak{X}}_2$ [*in* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$], *then* $\mathbf{M}_1$ *is injective in* $\boldsymbol{\mathcal{A}}_1$ [*in* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$].

An inspection of the proof of 3.2.10 in [5] shows that, in both parts of this lemma, instead of assuming that the structure is a total structure it suffices to assume that, in the appropriate category, the image of every morphism is a substructure. Our first two-for-one strong duality result is a strong-duality version of Lemma 6.3.

Lemma 6.7. *Assume that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are finite compatible total structures and define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1, 2\}$*.*

(i) *If* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a duality between* $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$ *that is strong provided* $\mathbf{M}_1$ *has named constants.*
(ii) *Assume that both* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *have named constants. Then* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$ *if and only if* $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$*.*

Proof. This follows from Lemma 6.3 and the Injectivity Transfer Lemma 6.6. □

The following is our 'One-and-a-Half-for-a-Half' Strong Duality Theorem.

Sesqui Strong Duality Theorem 6.8. *Let* $\mathbf{M}_1 = \langle M; G_1, R_1\rangle$ *be a finite total structure with* R_1 *finite and assume that* $\mathbf{M}_1$ *has named constants. Let* $\mathbf{M}_2$ *be a structure that is compatible with* $\mathbf{M}_1$ *and, for* $i \in \{1, 2\}$*, define* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*.*

(i) *If* $\mathbf{M}_1$ *is injective in* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $\underset{\sim}{\mathbf{M}}_2$ *yields a full* (*and therefore strong*) *duality between* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*.*
(ii) *If* $\mathbf{M}_2$ *is a total structure and* $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$*, then* $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*.*

Proof. Note that the parenthetic remark in (i) follows from the Injectivity Transfer Lemma 6.6(i). Assume that $\mathbf{M}_1$ is injective $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ and that $\underset{\sim}{\mathbf{M}}_2$ yields a full duality between $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$ and $(\boldsymbol{\mathfrak{X}}_2)_{\text{fin}}$. By Lemma 6.3(i), $\underset{\sim}{\mathbf{M}}_1$ yields a full duality between $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$, and so $\underset{\sim}{\mathbf{M}}_1$ yields a strong duality between $(\boldsymbol{\mathcal{A}}_2)_{\text{fin}}$ and $(\boldsymbol{\mathfrak{X}}_1)_{\text{fin}}$ since $\mathbf{M}_1$ is injective in $(\boldsymbol{\mathcal{A}}_1)_{\text{fin}}$. Thus (i) follows from the Total Structure Strong

Duality Theorem 5.5 with $\mathbf{M}_1$ and $\mathbf{M}_2$ interchanged. Part (ii) follows from part (i) and the Injectivity Transfer Lemma 6.6(ii). $\square$

Combining the Injectivity Lifting Lemma 4.4, the Injectivity Transfer Lemma 6.6 and the Two-for-One Full Duality Theorem 6.5, we see immediately that, under the assumptions of the Two-for-One Full Duality Theorem, if any one of the four dualities listed there is strong, then so are all the others. In fact, by appealing to the Sesqui Strong Duality Theorem 6.8 we can weaken the assumption that $\mathbf{M}_1$ and $\mathbf{M}_2$ are of finite type.

Two-for-One Strong Duality Theorem 6.9. *Assume that* $\mathbf{M}_1 = \langle M; G_1, R_1 \rangle$ *and* $\mathbf{M}_2 = \langle M; G_2, R_2 \rangle$ *are finite compatible total structures with* R_1 *and* R_2 *finite and that each has named constants. Let* $\boldsymbol{\mathcal{A}}_i := \mathbb{ISP}(\mathbf{M}_i)$ *and* $\boldsymbol{\mathfrak{X}}_i := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_i)$*, for* $i \in \{1, 2\}$*. Then the following are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_1$ *and* $\boldsymbol{\mathfrak{X}}_2$*;*
(2) $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $\boldsymbol{\mathcal{A}}_2$ *and* $\boldsymbol{\mathfrak{X}}_1$*;*
(3) $\underset{\sim}{\mathbf{M}}_2$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_1)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_2)_{\mathrm{fin}}$*;*
(4) $\underset{\sim}{\mathbf{M}}_1$ *yields a strong duality between* $(\boldsymbol{\mathcal{A}}_2)_{\mathrm{fin}}$ *and* $(\boldsymbol{\mathfrak{X}}_1)_{\mathrm{fin}}$*;*
(5) *each of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *satisfies* (IC) *with respect to the other.*

Proof. The implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are trivial. By Lemma 6.7, (3) and (4) are equivalent even without the assumption that R_1 and R_2 are finite. The IC Lemma 4.3 says that (3) and (4) together are equivalent to (5). Finally, the implications (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) hold by the previous theorem (or (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) hold by the Total Structure Strong Duality Theorem 5.5). $\square$

The Two-for-One Strong Duality Theorem of Clark and Davey [4] ([5, 3.3.2]) is the special case of this theorem in which $R_1 = R_2 = \varnothing$.

7. Examples

The Two-for-One Strong Duality Theorem 6.9 and, more generally, the Sesqui Strong Duality Theorem 6.8 allow us to convert a large number of known strong dualities into new strong dualities, by simply swapping the topology from one side to the other. Assume that $\mathbf{M}_1$ is a finite algebra that is injective in the quasivariety it generates and for which some alter ego yields a strong duality on $\boldsymbol{\mathcal{A}}_1 := \mathbb{ISP}(\mathbf{M}_1)$. Then the Total Structure Theorem (see Clark and Davey [5, 6.1.2]) tells us that there is an alter ego $\underset{\sim}{\mathbf{M}}_2$ of $\mathbf{M}_1$ that is a total structure and yields a strong duality between $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathfrak{X}}_2 := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$. Provided $\mathbf{M}_1$ has named constants, we conclude immediately from the Sesqui Strong Duality Theorem 6.8 that $\underset{\sim}{\mathbf{M}}_1$ yields a strong duality between the categories $\boldsymbol{\mathcal{A}}_2 := \mathbb{ISP}(\mathbf{M}_2)$ and $\boldsymbol{\mathfrak{X}}_1 := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$. If $\mathbf{M}_1$ is a lattice-based algebra and $\boldsymbol{\mathcal{A}}_1$ is a variety, then an equational description of $\boldsymbol{\mathcal{A}}_1$ yields an equational description of $\boldsymbol{\mathfrak{X}}_1$ via Theorem 3.2. Moreover, if we have a suitable description of $\boldsymbol{\mathfrak{X}}_2$, then we may apply Lemma 3.4 to read off a first-order description of $\boldsymbol{\mathcal{A}}_2$. The first three examples below illustrate these ideas. Our final example is an application of the NU Duality Theorem 4.10 and does not follow by simply swapping the topology on a known duality.

Our first example dates back to Banaschewski [2] in 1976 and is obtained from Priestley duality via the topology-swapping technique described above. It was reproved by Hofmann [27], but with *strong* replaced by *full*.

Example 7.1. *Let* $\mathbf{2}_L := \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ *be the two-element bounded lattice and let* $\mathbf{2}_O := \langle \{0,1\}; \leqslant\rangle$ *be the two-element ordered set with* $0 < 1$. *Then* $\mathcal{P} := \mathbb{IS}^0\mathbb{P}(\mathbf{2}_O)$ *is the category of ordered sets,* $\mathcal{D}^{01}_{\mathrm{Bt}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_L)$ *is the category of Boolean topological bounded distributive lattices, and* $\underset{\sim}{\mathbf{2}}_L$ *yields a strong duality between* $\mathcal{P}$ *and* $\mathcal{D}^{01}_{\mathrm{Bt}}$.

Proof. Priestley duality [31, 32] (see Clark and Davey [5, 4.3.1 and Exercise 4.5]) tells us that $\underset{\sim}{\mathbf{2}}_O$ yields a strong duality between the category $\mathbb{ISP}(\mathbf{2}_L)$ of bounded distributive lattices and the category $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_O)$ of Priestley spaces. By the Two-for-One Strong Duality Theorem 6.9, $\underset{\sim}{\mathbf{2}}_L$ yields a strong duality between $\mathbb{ISP}(\mathbf{2}_O)$ and $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_L)$, and therefore $\underset{\sim}{\mathbf{2}}_L$ yields a strong duality between $\mathbb{IS}^0\mathbb{P}(\mathbf{2}_O)$ and $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_L)$, that is, between $\mathcal{P}$ and $\mathcal{D}^{01}_{\mathrm{Bt}}$, by Lemma 6.2 (taking $\mathbf{M}_1 = \mathbf{2}_O$ and $\mathbf{M}_2 = \mathbf{2}_L$).

It is very easy to show directly that $\mathbb{IS}^0\mathbb{P}(\mathbf{2}_O)$ is the category of ordered sets. (Alternatively, use the fact that the finite Priestley spaces are precisely the finite ordered sets with the discrete topology and apply Lemma 3.4—see Remark 3.5.) Since $\mathbb{ISP}(\mathbf{2}_L)$ is the category of bounded distributive lattices, it follows from Theorem 3.2(ii) that $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_L)$ is the category of Boolean topological bounded distributive lattices—see Remark 3.3. ☐

Remark 7.2. By redefining $\mathbf{2}_L$ and $\mathbf{2}_O$ in the previous example to be $\mathbf{2}_L := \langle \{0,1\}; \vee, \wedge\rangle$ and $\mathbf{2}_O := \langle \{0,1\}; 0, 1, \leqslant\rangle$, respectively, we obtain a strong duality between the category $\mathcal{P}^{01} := \mathbb{ISP}(\mathbf{2}_O)$ of bounded ordered sets and $\mathcal{D}_{\mathrm{Bt}} := \mathbb{IS}^0_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_L)$ of Boolean topological distributive lattices. An elementary proof of this duality, along with applications to canonical extensions of distributive lattices, may be found in Davey, Haviar and Priestley [16].

Our second example is derived from the author's natural duality for Stone algebras [12, 13] (see also Clark and Davey [5, 4.3.6]), again via a topology swap.

Example 7.3. *Let* $\mathbf{3}_S := \langle \{0,a,1\}; \vee, \wedge, {}^*, 0, 1\rangle$, *where* $\langle \{0,a,1\}; \vee, \wedge, 0, 1\rangle$ *is a bounded lattice with* $0 < a < 1$ *and* * *is the pseudocomplementation operation, that is,* $0^* = 1$ *and* $a^* = 1^* = 0$. *Define* $\mathbf{3}_D := \langle \{0,a,1\}; d, \preccurlyeq\rangle$, *where* $\langle \{0,a,1\}; \preccurlyeq\rangle$ *is the ordered set whose only non-trivial relation is* $1 < a$ *and* d *is the unary operation that maps each element to the unique minimal element below it, that is,* $d(0) = 0$ *and* $d(a) = d(1) = 1$. *Define* $\mathcal{A} := \mathbb{IS}^0\mathbb{P}(\mathbf{3}_D)$ *and* $\mathcal{S}_{\mathrm{Bt}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{3}}_S)$.

(i) *A structure* $\langle A; d, \preccurlyeq\rangle$ *belongs to* $\mathcal{A}$ *if and only if* $\langle A; \preccurlyeq\rangle$ *is an ordered set in which each element* a *is above a unique minimal element, namely* $d(a)$.

(ii) $\mathcal{S}_{\mathrm{Bt}}$ *consists of all Boolean topological Stone algebras, that is, Boolean topological bounded distributive latttices that are pseudocomplemented and in which the pseudocomplementation operation is continuous and satisfies* $x^* \vee x^{**} \approx 1$.

(iii) $\underset{\sim}{\mathbf{3}}_S$ *yields a strong duality between* $\mathcal{A}$ *and* $\mathcal{S}_{\mathrm{Bt}}$.

Proof. It is proved in both Davey [12] and Davey [13] (see also [5, 1.4.7]) that a structure $\langle X; d, \preccurlyeq \mathcal{T}\rangle$ belongs to $\mathbb{IS}^0_c\mathbb{P}(\underset{\sim}{\mathbf{3}}_D)$ if and only if $\langle X; \preccurlyeq, \mathcal{T}\rangle$ is a Priestley space in which each element x is above a unique minimal element, namely $d(x)$, and the map d is continuous. Thus (i) follows by an easy application of Lemma 3.4.

It is well known that $\mathbb{ISP}(\mathbf{3}_S)$ is the variety of Stone algebras (see Grätzer [23]), that is, pseudocomplemented distributive lattices satisfying $x^* \vee x^{**} \approx 1$. An application of Theorem 3.2 yields (ii).

Since $\underset{\sim}{\mathbf{3}}_D$ yields a strong duality between $\mathbb{ISP}(\mathbf{3}_S)$ and $\mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{\mathbf{3}}_D)$ [12, 13], it follows by a combination of the Two-for-One Strong Duality Theorem 6.9 and

Lemma 6.2 (with $\mathbf{M}_1 = \mathbf{3}_D$ and $\mathbf{M}_2 = \mathbf{3}_S$) that $\underset{\sim}{\mathbf{3}}_S$ yields a strong duality between $\mathbb{IS}^0\mathbb{P}(\mathbf{3}_D)$ and $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{3}}_S)$, that is, between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{S}}_{\mathrm{Bt}}$. $\square$

Remark 7.4. A version of this duality is proved from first principles by Haviar and Priestley [25]. They replace the topological category $\boldsymbol{\mathcal{S}}_{\mathrm{Bt}}$ with an isomorphic category consisting of doubly algebraic Stone algebras with morphisms that preserve pseudocomplements and arbitrary joins and meets. They apply their version of the duality to show that, in the language of canonical extensions, *Stone algebras are canonical*. Applications of natural dualities for structures to the canonicity of other classes of algebras will appear in a paper by Davey, Gehrke and Priestley [14].

The simple topology-swapping technique illustrated in the two examples above can be applied to many lattice-based algebras for which we have a well-behaved strong duality. For example, it can be applied to the known strong dualities for

- double Stone algebras (Davey [13] and [5, 4.3.13 and 4.3.14]),
- Kleene algebras (Davey and Werner [20] and [5, 4.3.9 and 4.3.10]), and
- De Morgan algebras (Cornish and Fowler [10] and [5, 4.3.16]).

This technique can also be applied when $\mathbf{M}_1$ is not injective in $\mathbb{ISP}(\mathbf{M}_1)$, but in this case we must use the Sesqui Full Duality Theorem 6.4. For example, consider the Heyting chain $\mathbf{C}_H = \langle\{0, a, b, 1\}; \vee, \wedge, \rightarrow, 0, 1\rangle$, with $0 < a < b < 1$. While a natural duality for $\mathbb{ISP}(\mathbf{C}_H)$ dates back to Davey [11] in 1976, a strong duality was discovered only in 1995 by Clark and Davey [4]. Let $\mathbf{C}_D := \langle\{0, a, b, 1\}; g, h\rangle$, where g is the endomorphism of $\mathbf{C}_H$ given by $g(0) = 0$, $g(a) = b$ and $g(b) = g(1) = 1$, and h is the partial endomorphism of $\mathbf{C}_H$ with domain $\{0, b, 1\}$ given by $h(0) = 0$, $h(b) = a$ and $h(1) = 1$. For an explanation of why $\underset{\sim}{\mathbf{C}}_D$ yields a strong duality on $\mathbb{ISP}(\mathbf{C}_H)$ and a proof of the following axiomatization of $\mathbb{IS}_c^0\mathbb{P}(\underset{\sim}{\mathbf{C}}_D)$, see Davey and Talukder [19]. A Boolean topological structure $\mathbf{X} = \langle X; g, h, \mathcal{T}\rangle$, with g a total unary operation and h a partial unary operation, belongs to $\mathbb{IS}_c^0\mathbb{P}(\underset{\sim}{\mathbf{C}}_D)$ if and only if $\mathbf{X}$ satisfies the following axioms:

(S_1) $ggg(x) = gg(x)$,
(S_2) $x \in \mathrm{dom}(h) \iff gg(x) = g(x)$,
(S_3) $g(x) = x \iff \big(x \in \mathrm{dom}(h)\ \&\ h(x) = x\big)$,
(S_4) $x \in \mathrm{dom}(h) \implies gh(x) = x$,
(S_5) $g(x) \in \mathrm{dom}(h)$.

We shall now see that an application of the Sesqui Full Duality Theorem 6.4 and Lemma 6.2 yields a full but not strong duality for the quasivariety generated by the partial algebra $\mathbf{C}_D$. Recall that the *Full versus Strong Problem* [5, 3.2.7] asks whether every full duality for the quasivariety $\mathbb{ISP}(\mathbf{M})$ generated by a finite *total* algebra $\mathbf{M}$ is necessarily strong. The example below shows that if we allow $\mathbf{M}$ to include partial operations in its type, then the answer is 'no'. (Since this paper was written, the problem has been solved in the negative by Clark, Davey and Willard [9].)

Example 7.5. *Let $\mathbf{C}_H$ be the four-element Heyting chain and let $\mathbf{C}_D$ be the four-element partial algebra described above. Let $\boldsymbol{\mathcal{A}} := \mathbb{IS}^0\mathbb{P}(\mathbf{C}_D)$ and $\mathcal{C} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{C}}_H)$.*

(i) *A structure $\langle A; g, h\rangle$ belongs to $\boldsymbol{\mathcal{A}}$ if and only if it satisfies axioms* (S_1)–(S_5).
(ii) *$\mathcal{C}$ consists of all Boolean topological Heyting algebras satisfying the identity* $(x_0 \rightarrow x_1) \vee (x_1 \rightarrow x_2) \vee (x_2 \rightarrow x_3) \vee (x_3 \rightarrow x_4) \approx 1$.
(iii) *$\underset{\sim}{\mathbf{C}}_H$ yields a full but not strong duality between $\boldsymbol{\mathcal{A}}$ and $\mathcal{C}$.*

Proof. Let $\Sigma := \{(\mathrm{S}_1), \dots, (\mathrm{S}_5)\}$. It is easy to check directly that a 1-generated model of Σ has at most 5 elements. Indeed, if $\mathbf{A}$ is a model of Σ then the substructure generated by $a \in A$ consists of the elements a, $g(a)$, $gg(a)$ and $hg(a)$, and $h(a)$ if it is defined; see Table 1. Hence, as the type is unary, an n-generated model of Σ has at most $5n$ elements. Since Σ describes $\mathbb{I}\mathbb{S}^0_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{C}}_D)$, part (i) follows from Lemma 3.4.

	a	$g(a)$	$gg(a)$	$hg(a)$	$h(a)$
g	$g(a)$	$gg(a)$	$gg(a)$	$g(a)$	a
h	$h(a)$	$hg(a)$	$gg(a)$	$g(a)$	a

TABLE 1. The substructure generated by a

Hecht and Katriňák [26] proved that a Heyting algebra belongs to $\mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{C}_H)$ if and only if it satisfies the identity given in (ii). Thus (ii) follows at once from Theorem 3.2. As $\underset{\sim}{\mathbf{C}}_D$ yields a full duality between $\mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{C}_H)$ and $\mathbb{I}\mathbb{S}^0_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{C}}_D)$, the Sesqui Full Duality Theorem 6.4 and Lemma 6.2 (with $\mathbf{M}_1 = \mathbf{C}_H$ and $\mathbf{M}_2 = \mathbf{C}_D$) show that $\underset{\sim}{\mathbf{C}}_H$ yields a full duality between $\boldsymbol{\mathcal{A}}$ and $\mathfrak{C}$. Finally, $\underset{\sim}{\mathbf{C}}_H$ is not injective in $\mathfrak{C}$ as the homomorphism h does not extend to an endomorphism of $\mathbf{C}_H$. Thus (iii) holds. □

We close the paper with an application of the NU Duality Theorem 4.10. Let $\mathbf{2}_b := \langle\{0,1\}; \vee, \wedge, b\rangle$, where $\langle\{0,1\}; \vee, \wedge\rangle$ is the two-element lattice and $\langle\{0,1\}; b\rangle$ is the directed graph with one edge pointing from 0 to 1. Let $\mathbf{2}_{ud} := \langle\{0,1\}; u, d, \leqslant\rangle$, where u (for 'up') is the partial operation with domain $\{0\}$ and $u(0) = 1$, the partial operation d (for 'down') has domain $\{1\}$ and $d(1) = 0$, and $\leqslant$ is the order on $\{0,1\}$ with $0 < 1$. We shall see that, in a certain way, the quasivariety $\mathfrak{D}^b := \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{2}_b)$ may be thought of as the disjoint union of the classes $\mathfrak{D}^{01}$ and $\mathfrak{D}$ of bounded distributive lattices and distributive lattices, and likewise, $\mathfrak{X} := \mathbb{I}\mathbb{S}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{ud})$ may be thought of as the disjoint union of the classes of Priestley spaces and bounded Priestley spaces.

Example 7.6. *Let* $\mathbf{2}_b := \langle\{0,1\}; \vee, \wedge, b\rangle$ *and* $\mathbf{2}_{ud} = \langle\{0,1\}; u, d, \leqslant\rangle$ *be as given above, and define* $\mathfrak{D}^b := \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{2}_b)$ *and* $\mathfrak{X} := \mathbb{I}\mathbb{S}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{ud})$.

(i) *Let* $\mathbf{A} = \langle A; \vee, \wedge, b\rangle$ *be a structure of the same type as* $\mathbf{2}_b$. *The following are equivalent:*

  (a) $\mathbf{A}$ *belongs to* $\mathfrak{D}^b$*;*

  (b) $\langle A; \vee, \wedge\rangle$ *is a distributive lattice and* $\mathbf{A}$ *satisfies the quasi-equations* $(x, y) \in b \implies x \leqslant z \leqslant y$*;*

  (c) $\mathbf{A}$ *is either a distributive lattice with bounds* 0 *and* 1 *and with an edge pointing from* 0 *to* 1*, or* $\mathbf{A}$ *is a distributive lattice with no edges.*

(ii) *Let* $\mathbf{X} = \langle X; u, d, \leqslant, \mathfrak{T}\rangle$ *be a Boolean topological structure of the same type as* $\mathbf{2}_{ud}$. *The following are equivalent:*

  (a) $\mathbf{X}$ *belongs to* $\mathfrak{X}$*;*

  (b) $\langle X; \leqslant, \mathfrak{T}\rangle$ *is a Priestley space and* $\mathbf{X}$ *satisfies the quasi-equations*

$$x \in \mathrm{dom}(u) \implies x \leqslant y \ \ \&\ \ u(x) \in \mathrm{dom}(d),$$
$$x \in \mathrm{dom}(d) \implies x \geqslant y \ \ \&\ \ d(x) \in \mathrm{dom}(u);$$

(c) $\mathbf{X}$ *is either a Priestley space with both* dom(u) *and* dom(d) *empty, or* $\mathbf{X}$ *is a bounded Priestley space in which* u *maps the bottom to the top and* d *maps the top to the bottom.*

(iii) $\underset{\sim}{\mathbf{2}}_{ud}$ *yields a strong duality between* $\mathcal{D}^b$ *and* $\mathcal{X}$.

Proof. The equivalences in parts (i) and (ii) follow by standard arguments concerning distributive lattices and Priestley spaces using Lemmas 1.2 and 3.1. Thus, $\mathcal{D}^b$ is essentially the disjoint union of $\mathcal{D}^{01}$ and $\mathcal{D}$. The morphism class of $\mathcal{D}^b$ is just the disjoint union of the morphisms in $\mathcal{D}^{01}$, the morphisms in $\mathcal{D}$ and the class of all lattice homomorphisms from lattices in $\mathcal{D}$ to lattices in $\mathcal{D}^{01}$. Similarly, $\mathcal{X}$ can be thought of as the disjoint union of the categories of Priestley spaces and bounded Priestley spaces with the continuous order-preserving maps from Priestley spaces into bounded Priestley spaces added as additional morphisms.

The set of all subuniverses of $(\mathbf{2}_b)^2$ is

$$\{\leqslant, \geqslant, \Delta_{\{0,1\}}\} \cup \{\, A \times B \mid A, B \subseteq \{0,1\}\,\}.$$

Since u and d entail their domains, the structure $\underset{\sim}{\mathbf{2}}_{ud}$ entails every non-empty subuniverse of $(\mathbf{2}_b)^2$ (see [5, 2.4.5]). The NU Duality Theorem 4.10 now tells us that $\underset{\sim}{\mathbf{2}}_{ud}$ yields a duality between $\mathcal{D}^b$ and $\mathcal{X}$, and that $\underset{\sim}{\mathbf{2}}_{ud}$ is injective in $\mathcal{X}$. The fact that the duality is full follows from the fact that both the duality between bounded distributive lattices and Priestley spaces and the duality between (not necessarily bounded) distributive lattices and bounded Priestley spaces are full. □

This duality is also a candidate for a topology swap. By the Sesqui Strong Duality Theorem 6.8(i) and Lemma 6.2, the alter ego $\underset{\sim}{\mathbf{2}}_b = \langle\{0,1\}; \vee, \wedge, b, \mathcal{T}\rangle$ of the structure $\mathbf{2}_{ud} = \langle\{0,1\}; u, d, \leqslant\rangle$ yields a strong duality between $\mathcal{A} := \mathbb{IS}^0\mathbb{P}(\mathbf{2}_{ud})$ and $\mathcal{D}^b_{\mathrm{Bt}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{2}}_b)$. By Theorem 3.4, a structure $\mathbf{A} = \langle A; u, d, \leqslant\rangle$ belongs to $\mathcal{A}$ if and only if $\langle A; \leqslant\rangle$ is an ordered set and $\mathbf{A}$ satisfies the quasi-equations in (ii) above. Since $\mathbb{ISP}(\mathbf{2}_b)$ is closed under (finite) quotients, Theorem 3.2 tells us that $\mathbf{X} = \langle X; \vee, \wedge, b, \mathcal{T}\rangle$ belongs to $\mathcal{D}^b_{\mathrm{Bt}}$ if and only if $\langle X; \vee, \wedge, \mathcal{T}\rangle$ is a Boolean topological distributive lattice and $\mathbf{X}$ satisfies satisfies the quasi-equations in (i) above.

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AN EXAMPLE OF A CONGRUENCE DISTRIBUTIVE VARIETY HAVING NO NEAR-UNANIMITY TERM

I. Chajda and R. Halaš

Abstract. By a nearlattice is meant a join-semilattice whose every principal filter is a lattice with respect to the induced order. Every nearlattice can be described as an algebra with one ternary operation satisfying eight simple identities. This algebra is called a nearlattice-algebra. Hence, nearlattice-algebras form a variety $\mathcal{N}$. We shall show that the variety $\mathcal{N}$ is congruence distributive but $\mathcal{N}$ has not a near-unanimity term.

By a *nearlattice* we mean a semilattice $\mathcal{S} = (A; \vee)$ where for each $a \in A$ the principal filter $[a) = \{x \in A; a \leq x\}$ is a lattice with respect to the induced order $\leq$ of $\mathcal{S}$.

Nearlattices were studied by M. Scholander [3,4] under a different name. The term "nearlattice" was firstly used by A. S. Noor and W. H. Cornish [2].

Obviously, the operation $x \wedge y$ (meet) in $\mathcal{S}$ is defined if and only if the elements x, y have a common lower bound. This implies that for $x, y, z \in A$ the operation

$$m(x, y, z) = (x \vee z) \wedge (y \vee z)$$

is everywhere defined. Moreover, $m(x, y, z)$ satisfies the identities (P1) – (P8):

(P1) $m(x, y, x) = x$;
(P2) $m(x, x, y) = m(y, y, x)$;
(P3) $m(m(x, x, y), m(x, x, y), z) = m(x, x, m(y, y, z))$;
(P4) $m(x, y, p) = m(y, x, p)$;
(P5) $m(m(x, y, p), z, p) = m(x, m(y, z, p), p)$;
(P6) $m(x, m(y, y, x), p) = m(x, x, p)$;
(P7) $m(m(x, x, p), m(x, x, p), m(y, x, p)) = m(x, x, p)$;
(P8) $m(m(x, x, z), m(y, y, z), z) = m(x, y, z)$.

It was shown in [1] that there is a one-to-one correspondence between nearlattices and algebras $\mathcal{A} = (A; m)$ of type (3) satisfying the identities (P1) – (P8). Hence, an algebra $\mathcal{A} = (A; m)$ of type (3) satisfying (P1) – (P8) will be called a *nearlattice-algebra*.

One can easily see that for nearlattice-algebras the relation $\leq$ defined by

$$x \leq y \quad \text{iff} \quad m(x, x, y) = y$$

is an order and $(A; \leq)$ is a semilattice where $x \vee y = m(x, x, y)$ and for each $z \in A$ the filter $[z)$ is a lattice with $x \wedge y = m(x, y, z)$. Hence, $m(x, y, z) = (x \vee z) \wedge (y \vee z)$. Now, we can state

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Theorem 1. The variety $\mathcal{N}$ of all nearlattice-algebras is congruence distributive.

Proof. By Jónsson's Theorem, we only need to verify the existence of ternary Jónsson terms. We take $n = 4$, and

$$\begin{aligned} p_0(x,y,z) &= x; \\ p_1(x,y,z) &= m(z,y,x); \\ p_2(x,y,z) &= m(x,x,z); \\ p_3(x,y,z) &= m(x,y,z); \\ p_4(x,y,z) &= z. \end{aligned}$$

Then we need to show that:

$$p_0(x,y,z) = x; \quad p_4(x,y,z) = z;$$

$$p_0(x,y,x) = p_1(x,y,x) = p_2(x,y,x) = p_3(x,y,x) = p_4(x,y,x) = x;$$

$$p_0(x,x,y) = p_1(x,x,y); \quad p_2(x,x,y) = p_3(x,x,y);$$

$$p_1(x,y,y) = p_2(x,y,y); \quad p_3(x,y,y) = p_4(x,y,y).$$

Evidently $p_0(x,y,z) = x$ and $p_4(x,y,z) = z$ hold.
Clearly

$$m(x,y,x) \overset{(P1)}{=} x, \quad m(x,x,x) \overset{(P1)}{=} x,$$

so we obtain $p_0(x,y,x) = x$, $p_1(x,y,x) = m(x,y,x) = x$, $p_2(x,y,x) = m(x,x,x) = x$, $p_3(x,y,x) = m(x,y,x) = x$, $p_4(x,y,x) = x$.

Further,

$$p_0(x,x,y) = x \overset{(P1)}{=} m(x,y,x) \overset{(P4)}{=} m(y,x,x) = p_1(x,x,y),$$

$$p_2(x,x,y) = m(x,x,y) = p_3(x,x,y).$$

Finally,

$$p_1(x,y,y) = m(y,y,x) \overset{(P2)}{=} m(x,x,y) = p_2(x,y,y)$$

$$p_3(x,y,y) = m(x,y,y) \overset{(P4)}{=} m(y,x,y) \overset{(P1)}{=} y = p_4(x,y,y). \quad \square$$

To prove that $\mathcal{N}$ has not a near-unanimity term, we introduce the following concept.

Let $p(x_1,\ldots,x_n)$ be a term of the variety $\mathcal{N}$. By induction of term complexity, we define: a variable x_i is called the *right-most* of p if

(i) $p(x_1,\ldots,x_n)$ is a projection and $p(x_1,\ldots,x_n) = x_i$ or
(ii) $p(x_1,\ldots,x_n) = m(p_1,p_2,p_3)$ where p_1,p_2,p_3 are subterms of p and x_i is the right-most variable of p_3.

Theorem 2. Let $\mathcal{A} = (A; m) \in \mathcal{N}$ has a greatest element 1 and $p(x_1, \dots, x_n)$ be an n-ary term of $\mathcal{N}$. If x_i is the right-most variable of p and $a_1, \dots, a_n \in A$ then $p(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 1$.

Proof. We proceed by induction of complexity of $p(x_1, \dots, x_n)$. If p is projection, the proof is trivial. Hence, suppose $p(x_1, \dots, x_n) = m(p_1, p_2, p_3)$ for some subterms p_1, p_2, p_3 and let x_i be the right-most variable of p. Then x_i is the right-most variable of p_3 and, by the induction hypothesis,

$$p_3(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 1.$$

Thus

$$p(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = (p_1 \vee p_3) \wedge (p_2 \vee p_3) = \\ = (p_1 \vee 1) \wedge (p_2 \vee 1) = 1 \wedge 1 = 1. \quad \square$$

Example. Let $\mathcal{I} = (\{0, 1\}, m)$ be a two-element nearlattice-algebra. Then it is the two-element chain but there is no near-unanimity term-function on $\mathcal{I}$ since $u(0, \dots, 0, 1, 0, \dots, 0) = 1$ (i-th position) whenever x_i is the right-most variable of u. On the contrary, there exists a near-unanimity polynomial on $\mathcal{I}$ which is e.g.

$$t(x, y, z) = m(m(x, y, z), m(x, x, y), 0).$$

It is in fact the majority term $(x \vee z) \wedge (y \vee z) \wedge (x \vee y)$. However, this polynomial exists only on nearlattices having the least element 0 which need not be the case.

Corollary 1. The variety $\mathcal{N}$ is congruence distributive but it has not a near-unanimity term.

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SYMMETRIC CUBIC GRAPHS OF GIRTH AT MOST 7

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Abstract. By a symmetric graph we mean a graph X which automorphism group acts transitively on the arcs of X. A graph is *s-regular* if its automorphism group acts regularly on the set of its s-arcs. Tutte [31, 32] showed that every finite symmetric cubic graph is s-regular for some $s \leq 5$. It is well-known that there are precisely five symmetric cubic graphs of girth less than 6. All these graphs can be represented as one-skeletons of regular polyhedra in the plane, projective plane or in torus. With the exception of $K_{3,3}$, we can find an associated regular polyhedron such that the girth of the graph coincide with the face-size.

In this paper we show that with three more exceptions the symmetric cubic graphs of girth $g \leq 7$ are one-skeletons of trivalent regular maps with face-size g. All the symmetric cubic graphs of girth 6 except the generalised Petersen graphs $GP(8,3)$ and $GP(10,3)$ are one-skeletons of toroidal regular maps of type $\{6,3\}$. We give a simple numerical criterium to determine the degree s of s-regularity of these graphs and derive the presentations of the automorphism groups. As concerns girth 7, the only exceptional graph is the well-known Coxeter graph on 28 vertices. We prove that all the other symmetric cubic graphs of girth 7 are underlying graphs of regular maps of type $\{7,3\}$ which are known as Hurwitz maps. Some more results on symmetric cubic graphs with exactly two girth cycles passing through an edge are proved

1 Introduction

Throughout this paper a graph means an undirected finite graph, without loops or multiple edges. For a graph X, we denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its automorphism group, respectively. For further group- and graph-theoretic notation and terminology, we refer the reader to [15] and [17].

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for every $1 \leq i < s$; in other words, a directed walk of length s which never includes the reverse of an arc just crossed. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of all s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. An arc-transitive graph X is said to be *s-regular* if for any two s-arcs in X, there is a unique automorphism of X mapping one to the other. An s-regular graph $(s \geq 1)$

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is a union of isomorphic s-regular connected graphs and isolated vertices. Thus, in what follows we consider only non-trivial connected graphs. Tutte [31, 32] showed that every finite symmetric cubic graph is s-regular for some $s \leq 5$. Depending $s = 1, 2, 3, 4, 5$ the vertex-stabisers of the respective groups acting s-regularly on a (connected) cubic graph are respectively isomorphic to the cyclic group Z_3, to the symmetric group S_3, to the dihedral group D_{12} of order 12, to the symmetric group S_4 and to the direct product $S_4 \times Z_2$. For $s = 2$ and $s = 4$ there are two different possibilities for the edge-stabilisers. Taking into the account possible vertex- and edge-stabilisers we have 7 sorts of arc-transitive actions of a group onto a cubic graph. These 7 sorts of actions give rise to 7 universal groups acting arc-transitively on the infinite cubic tree (see [12, 14]). Presentations of the seven groups were found by Conder and Lorimer in [6]. It follows that the automorphism group of a symmetric cubic graph is an epimorphic image of one of the 7 groups. The corresponding seven families of graphs were proved to be infinite.

In the present paper we consider symmetric cubic graphs with girth constraints. In particular, we will be interested in symmetric cubic graphs of girth at most 7. It is well-known that there are five connected symmetric cubic graphs with girth less than 6, namely the tetrahedral graph, the complete bipartite graph $K_{3,3}$, the 3-dimensional cube, the Petersen graph and the dodecahedral graph. This can easily be shown by case to case analysis with respect to girth 3, 4 or 5. Three of the graphs are one-skeletons of the 3-valent Platonic solids. The Petersen graph has a highly symmetric 5-gonal embedding into the Projective plane while $K_{3,3}$ has a symmetrical 6-gonal embedding into the torus. In all these geometrical representations girth of the graph is equal to the face size, except for the embedding of $K_{3,3}$ into torus. Automorphism groups of symmetric cubic graphs of girth 6 were studied by Miller in [24]. He proved that all but finitely many of them can be defined as double coset graphs from a family of 2-generator groups

$$G(s,t,k) = \langle x, y | x^3 = y^2 = (xy)^6 = [x,y]^{sk} = (xyx^{-1}y)^{st}(x^{-1}yxy)^{-s} = 1\rangle,$$

where $k > 0$, $0 < 2t \leq k+1$ and $t^2 - t + 1 \equiv 0 (\text{mod } k)$. Further, Morton [25] characterised 4-arc-transitive cubic graphs up to girth 13. It follows that the automorphism group of such a graph is an epimorphic image of the triangle group $\Delta^+(12,3,2) = \langle x, y | x^3 = y^2 = (xy)^{12} = 1\rangle$ or it is one of the nine exceptional graphs. Conder in [5] constructed an infinite family of 4-arc-transitive cubic graphs of girth 12.

In the present paper we deal with the family of symmetric cubic graphs of girth at most 7 in detail. In Section 5 we prove a classification theorem (Theorem 6.2) showing that with two exceptions all the symmetric cubic graphs of girth 6 are one skeletons of toroidal regular maps of type $\{6,3\}$, a popular family of 3-valent hexagonal maps (see Coxeter-Moser [9]). The two exceptional graphs are well-known, these are the generalised Petersen graphs $GP(8,3)$ and $GP(10,3)$. Using this geometric characterisation we describe the automorphism groups of the symmetric graphs of girth 6 by means of group presentations.

Similarly, we prove that except the Coxeter graph all the symmetric cubic graphs of girth 7 are underlying graphs of regular, or orientably regular maps of type $\{7,3\}$ (see Theorem 5.2). These maps are in a correspondence with compact Riemann surfaces of Euler characteristic χ with the maximum possible number of symmetries reaching the Hurwitz bound -84χ. Thus the maps of type $\{7,3\}$ are sometimes called Hurwitz maps. As concerns girth 7, to give a list of presentations for the corresponding groups is a difficult task. It follows from our characterisation that this problem is equivalent with the problem of classification of normal subgroups of finite index of the triangle group $\Delta^+(7,3,2)$ and of the extended triangle group $\Delta(7,3,2)$. Even if we are restricted to simple non-abelian quotients of $\Delta(7,3,2)$ or of $\Delta^+(7,3,2)$, a complete list of such groups is not known (see Section 6 for more details). Some general results on the family of symmetric cubic graphs such that there are precisely two girth cycles passing through an edge are proved in Sections 4 and 6. In particular, Propositions 7.1 and 7.2 give existence results of 2- and 1-regular cubic graphs belonging to the family.

2 Maps and groups acting on maps

The aim of this section is to survey some known facts on regular maps with the emphasis to trivalent regular maps. The proofs of the results mentioned here one can find in [3, 13, 19]. A *map* is a cellular decomposition of a closed surface into 0-cells called *vertices*, 1-cells called *edges* and 2-cells called *faces*. The vertices and edges of a map form its *underlying graph*. A map is said to be *orientable* if the supporting surface is orientable, and is *oriented* if one of two possible orientations of the surface has been specified; otherwise, a map is *unoriented*. Every map can be described in a purely combinatorial way as follows: Let F be the set of mutually incident triples of the form vertex-edge-face which we shall call flags of a map $\mathcal{M}$. There are three fixed-point-free permutations ρ, λ and τ associated with $\mathcal{M}$, ρ interchanges flags sharing the same vertex and face, λ interchanges flags sharing the same face and edge. Finally, τ interchanges flags sharing the same vertex and edge. It follows that $(\lambda\tau)^2 = 1$. We shall write $\mathcal{M} = (F, \rho, \lambda, \tau)$. On the other hand, given set F of (abstract) flags and three involutions acting freely and transitively on F such that two of them commute we can reconstruct the associated topological map. The vertices, edges and faces are in correspondence with the orbits of $\langle \rho, \tau \rangle$, $\langle \lambda, \tau \rangle$ and $\langle \rho, \lambda \rangle$, respectively. The incidence relation between vertices, edges and faces is determined by the (non-empty) intersection of the respective orbits. Given map $\mathcal{M} = (F, \rho, \lambda, \tau)$ the map $(F, \rho, \lambda\tau, \tau)$ will be called the Petrie dual of $\mathcal{M}$. The underlying graph of the Petrie dual and of the original map are the same. The following well-known result determines the topological structure of the surface associated with a map (F, ρ, λ, τ).

Lemma 2.1. Let $M = (F, \rho, \lambda, \tau)$ be a combinatorial map. Denote by $G^+ = \langle \rho\tau, \lambda\tau \rangle$, and by v, e and f the respective numbers of orbits of $\langle \rho, \tau \rangle$, $\langle \lambda, \tau \rangle$ and $\langle \rho, \lambda \rangle$ in the action of $G = \langle \rho, \lambda, \tau \rangle$ on F.

Then the Euler characteristic of the underlying surface S is $v - e + f$ and S is orientable if and only if $G^+ < G$ is index two subgroup of G.

A permutation φ of flags is an automorphism of $\mathcal{M} = (F, \rho, \lambda, \tau)$ if it commutes with ρ, λ, τ. Every map automorphism act on vertices of the underlying graph and preserve the incidence relation between edges and vertices. If the graph is simple and the valency of every vertex is at least 3 we have $\mathrm{Aut}(\mathcal{M}) \leq \mathrm{Aut}(X)$. Thus every map automorphism can be viewed as a graph automorphism. Generally, the action of $\mathrm{Aut}(\mathcal{M})$ on flags is semi-regular so $|\mathrm{Aut}(\mathcal{M})| \leq |F| = 4|E(X)|$. If the equality holds, the action is regular on flags and the map itself will be called *regular*. For regular maps we have the following.

Proposition 2.2. [3] Let $\mathcal{M} = (F, \rho, \lambda, \tau)$ be a map. The following three statements are equivalent:

(1) $\mathcal{M}$ is regular,

(2) $\mathrm{Aut}(\mathcal{M}) \cong \langle \rho, \lambda, \tau \rangle$,

(3) $\mathrm{Aut}(\mathcal{M})$ contains three involutory automorphisms mapping a fixed flag x respectively, onto $\rho(x)$, $\lambda(x)$ and $\tau(x)$.

Hence if M is regular the action of $\mathrm{Aut}(\mathcal{M})$ is arc-transitive with dihedral vertex-stabiliser and with edge-stabiliser isomorphic to the Klein's group $Z_2 \times Z_2$. The backward implication (see [13]) holds true as well. Whenever we have a group G of automorphisms of a graph X satisfying the above assumptions then we can construct a regular map $\mathcal{M}$ with the underlying graph X such that $\mathrm{Aut}(\mathcal{M}) = G$. In particular, if X is a cubic graph we have

Proposition 2.3. [13] Let X be a (simple) cubic graph. Then there is a regular map with the underlying graph X and automorphism group G if and only if $G \leq \mathrm{Aut}(X)$ acts 2-arc-transitively with edge-stabiliser $Z_2 \times Z_2$.
Moreover, the map $\mathcal{M}$ is uniquely determined by G up to Petrie duality.

Assume the underlying surface is orientable, i.e. there exists a subgroup $G^+ = \langle \rho\tau, \lambda\tau \rangle = \langle R, L \rangle \leq$ of G with index 2. A permutation of arcs of the map will be called an orientation preserving automorphism of M if it commutes with R and L. The group of orientation preserving automorphisms $\mathrm{Aut}^+(M)$ acts semi-regularly on arcs of M and if the action is regular then M is called *orientably regular.* If the underlying graph X of a map is simple of valency at least 3, we have a faithful action of both groups on vertices so that $\mathrm{Aut}^+(M) \leq \mathrm{Aut}(M) \leq \mathrm{Aut}(X)$.

If a surface S is orientable we can fix one of the two global rotations. In such case we can describe a map on S by means of rotation and arc- reversing involution acting on the set of arcs of the map. More precisely, by an *oriented map* we mean a triple $\mathcal{M} = (D; R, L)$, where D is the set of arcs, $\langle R, L \rangle$ is a transitive group of permutations of D with L being involutory and R being the rotation. A permutation ψ of D is called a map automorphisms if it commutes with both R and L. The map $\mathcal{M}^{-1} = (D; R^{-1}, L)$ is called the mirror image of $\mathcal{M}$. An oriented map $\mathcal{M}$ is called regular if $\mathrm{Aut}(\mathcal{M})$ acts regularly on D. Similarly as in the non-oriented case we have the following characterisation of oriented regular maps.

Proposition 2.4. [19] Let $\mathcal{M} = (D; R, L)$ be an oriented map. The following three statements are equivalent:

(1) $\mathcal{M}$ is (oriented) regular,

(2) $\mathrm{Aut}(\mathcal{M}) \cong \langle R, L \rangle$,

(3) given edge $e = uv$ the automorphism group $\mathrm{Aut}(\mathcal{M})$ contains two automorphisms, one fixes v and cyclically permutes the incident arcs with v following the local action of R at v, the other rotates the map round the center of uv by 180 degrees interchanging the two arcs associated with uv.

In particular we have

Proposition 2.5. [13] Let X be a (simple) cubic graph. Then there is an oriented regular map with the underlying graph X and with $\mathrm{Aut}(\mathcal{M}) = G$ if and only if $G \leq \mathrm{Aut}(X)$ acts regularly on arcs.
Moreover, the oriented map $\mathcal{M}$ is uniquely determined by G up to mirror image.

Assume that each vertex of $\mathcal{M} = (F, \rho, \lambda, \tau)$ has the same valency k and each face of $\mathcal{M}$ is m-gonal, for some integers k, $m \geq 3$. Then $(\rho\tau)^k = (\rho\lambda)^m = 1$. In this case we say that $\mathcal{M}$ is a map of type $\{m, k\}$. It follows that G is a quotient of the *extended triangle group* with presentation

$$\Delta(m, k, 2) = \langle x, y, z | x^2 = y^2 = z^2 = (xy)^k = (xz)^m = (yz)^2 = 1 \rangle.$$

The kernel N of canonical epimorphism $x \mapsto \rho$, $y \mapsto \tau$ and $z \mapsto \lambda$ is a normal torsion free subgroup of $\Delta(m, k, 2)$. The group $\Delta(m, k, 2)$ is the automorphism group of a *k-valent m-gonal tesselation* $\mathcal{U}(m, k)$ of a hyperbolic plane. Hence every regular map arises as a quotient $\mathcal{U}(m, k)/N$, where N is a normal torsion free subgroup of $\Delta(m, k, 2)$ of finite index. The vertices of $\mathcal{U}(m, k)/N$ are the orbits of the action of N on the vertices and two orbits A, B are adjacent if there are vertices $u \in A$ and $v \in B$ such that uv is an edge in $\mathcal{U}(m, k)$. Namely, we have the following statement.

Proposition 2.6. [13] Let X be a cubic graph. Let G be a 2-regular group of automorphisms of X such that an edge stabiliser is isomorphic to $Z_2 \times Z_2$. Then there exists m and a torsion free normal subgroup $N \trianglelefteq \Delta(m, 3, 2)$ such that $G \cong \Delta(m, 3, 2)/N$ and $X \cong \mathcal{U}(m, 3)/N$.

Similarly, setting

$$\Delta^+(m, k, 2) = \langle x, y | x^k = (xy)^m = y^2 = 1 \rangle,$$

one can establish a correspondence between the normal torsion free subgroups of finite index of the *triangle group* $\Delta^+(m, k, 2)$ and maps of type $\{m, k\}$. In particular, we have

Proposition 2.7. [13] Let X be a cubic graph. Let G be a 1-regular group of automorphisms of X. Then there exists m and a torsion free normal subgroup $N \trianglelefteq \Delta^+(m, 3, 2)$ such that $G = \Delta^+(m, 3, 2)/N$ and $X \cong \mathcal{U}(m, 3)/N$.

The universal graph $\mathcal{U}(m, 3)$ is 2-regular of girth m. A question arises under what condition a finite quotient $\mathcal{U}(m, 3)/N$ by some normal torsion free subgroup shares the same properties. The following proposition suggests that a combinatorial condition on the number of girth cycles passing through an edge is important.

Proposition 2.8. Let X be a symmetric cubic graph. Then the number of girth cycles c passing through an edge is even. If $c < 2^t$ then X is at most t-regular, for $t = 2, 3, 4, 5$. In particular, if $c = 2$ then either X is 1-regular or it is 2-regular.

Proof. Let m be the number of girth cycles through a vertex. Then $3c = 2m$. Assuming that X is $t + 1$-arc-transitive, and considering the automorphisms permuting the $t + 1$-arcs based at a fixed arc x we create at least 2^t different girth cycles passing through x. □

3 Five exceptional graphs

In this section we define five exceptional cubic graphs which will play a key role in the following text and mention some of their properties. The first four are of girth 6, the last one is the Coxeter graph, the smallest symmetric cubic graph of girth 7.

Let $n \geq 3$ and $k \in \mathbb{Z}_n \backslash \{0\}$. The *generalised Petersen graph* $GP(n, k)$ is a graph with vertex set $\{x_i, y_i | \, i \in \mathbb{Z}_n\}$ and edge set $\{x_i x_{i+1}, \; x_i y_i, \; y_i y_{i+k}; i \in \mathbb{Z}_n\}$.

The generalised Petersen graph $GP(8,3)$**.** The graph is a double cover of $GP(4,3)$ which is the 3-dimensional cube. Hence its group is isomorphic to a semidirect product $(S_4 \times Z_2) : Z_2$.

$$\mathrm{Aut}(GP(8,3)) = \langle h, a, p | h^3 = a^2 = p^2 = (ap)^2 = 1, php = h^{-1}, (ha)^3(h^{-1}a)^3 = 1\rangle$$

The graph is 2-regular, the action of the group determines an octagonal embedding of the graph into the double torus giving rise to a regular map of genus 2 (see [9 (p. 29, Fig. 3.6c)]). One can easily check that girth of the graph is 6 and there are six 6-cycles passing through an edge. However, the graph is only 2-regular showing that the implication in Proposition 2.8 cannot be reversed. More information on this graph one can find in [22].

The generalised Petersen graph $GP(10,3)$**.** Since $GP(10,3)$ is the canonical double cover of the Petersen graph its automorphism group is $\mathrm{Aut}(GP(5,2)) \times Z_2 = S_5 \times Z_2$. By [7] it has presentation

$$\mathrm{Aut}(GP(10,3)) = \langle h, a, p, q | h^3 = a^2 = p^2 = q^2 = 1, qp = pq,$$
$$h^{-1}ph = p, qhq = h^{-1}, apa = q, pq(h^{-1}a)^2(ha)^2(h^{-1}a)^2 = 1\rangle.$$

The automorphism group has 240 elements, and consequently, the graph is 3-regular. Since $\mathrm{Aut}(GP(10,3))$ contains no 1-regular subgroup, $GP(10,3)$ has no regular embedding into an orientable surface. Since it admits a subgroup acting 2-regularly with an edge stabiliser $Z_2 \times Z_2$, it is the underlying graph of a non-orientable regular map. There are two such maps both of type $\{10,3\}$. The maps are Petrie duals of each other.

The Pappus graph 9_3 is the incidence graph of Pappus configuration

$$\{123, 456, 789, 147, 258, 369, 158, 348, 267\},$$

which is a union of the three parallel classes of lines in the affine geometry $AG(2,3)$ (exactly one set of three parallel lines is missing). Consequently, the automorphism group is a semidirect product $(Z_3 \times Z_3) : Z_2$ of a group consisting of 9 translations extended by a point-line duality. The vertex-stabiliser is isomorphic to the dihedral group D_{12}. A presentation of the automorphism group reads by [7] as follows:

$$\mathrm{Aut}(9_3) = \langle h, a, p, q | h^3 = a^2 = p^2 = q^2 = 1, qp = pq,$$
$$h^{-1}ph = p, qhq = h^{-1}, apa = q, (h^{-1}a)^6 = 1\rangle$$

Consequently, the graph 9_3 is 3-regular. Another remarkable property of 9_3 is that it has a hexagonal embedding in the torus giving rise to a self-Petri regular map, the map $\{6,3\}_{3,0}$ in the notation of Coxeter and Moser (see Figure 11).

The Heawood graph is the incidence graph of the Fano plane

$$\mathcal{P} = \{123, 345, 156, 147, 257, 367, 246\}.$$

It follows that the automorphism group of the Heawood graph is $PSL(3,2).2 \cong PGL(2,7)$. In [7] it has presentation:

$$\mathrm{Aut}(He) = \langle h, a, p, q, r | h^3 = a^2 = p^2 = q^2 = r^2 = 1, pq = qp, pr = rp,$$

$$rq = pqr, h^{-1}ph = q, h^{-1}qh = pq, rhr = h^{-1}, apa = p, aqa = r, p(ha)^3(h^{-1}a)^3 = 1\rangle$$

The graph is 4-regular and it admits a 1-regular action. There is a well-known hexagonal embedding of the Heawood graph giving rise to an irreflexible oriented regular map, the map $\{6,3\}_{2,1}$ in the Coxeter-Moser notation, see Figure 12.

The Coxeter graph. Vertices are antiflags of the Fano plane $\mathcal{P}$, i.e. $\gamma \in V$ if and only if $\gamma = (p, \ell)$ for some line ℓ and a point not incident to ℓ. Two vertices $\gamma = (p, \ell)$ and $\delta = (q, m)$ are adjacent if $\mathcal{P} = \ell \cup m \cup \{p, q\}$. The group $PGL(2,7)$ has a natural action on the 28 vertices of the Coxeter graph, indeed by [4 (Theorem 12.3.1)] the automorphism group has 336 elements and it is isomorphic to $PGL(3,2).2 \cong PGL(2,7)$. A presentation of the group given in [7] reads as follows

$$\mathrm{Aut}(Cox) = \langle h, a, p, q | h^3 = a^2 = p^2 = q^2 = 1, qp = pq,$$

$$h^{-1}ph = p, qhq = h^{-1}, apa = q, pha(h^{-1}a)^2(ha)^2(h^{-1}a)^2 = 1\rangle.$$

Consequently, the Coxeter graph is 3-regular. The automorphism group of the Coxeter graph contains no 1-regular subgroup. It contains 2-regular subgroups but the edge-stabiliser of the respective action is isomorphic to $\mathbb{Z}_4$. Hence the Coxeter graph has no regular embedding into a surface and it is the smallest symmetric cubic graph with this property. There are some other remarkable properties of this graph, see [10, 33, 11] for more information.

4 Graphs with more than two girth cycles passing through an edge

In this section we classify symmetric cubic graphs of girth 6 and 7 such that the number of girth cycles passing through an edge is greater than 2 (Lemma 4.2 and 4.3). It transpires that there are exactly five such graphs. The following Proposition proved in [26] will be useful.

Proposition 4.1. Let X be an arc-transitive cubic graph of girth g. Then X has no cycle separating edge-cut of size $< g$. In particular, X has no edge-cut consisting of $< g$ independent edges.

Let X be a symmetric cubic graph of girth g. Fix a vertex $v \in V(X)$ and denote by $V_i = V_i(v) = \{u \in V(X) | \, d(u,v) = i\}$ the set of vertices at distance i from u. Denote by E_i^{i+1} the edge-set formed by the edges xy, $x \in V_i$ and $y \in V_{i+1}$. Note that E_i^{i+1} is an edge-cut provided it is non-empty. Furthermore, for $j \geq i$ we denote by $V_i^j = V_i \cup V_{i+1} \cup \ldots V_j$, and by $[V_i^j]$ the subgraph induced by V_i^j. In what follows we denote by α a fixed element of order 3 in the vertex-stabiliser G_v of v in G, where G is the automorphism group $\mathrm{Aut}(X)$ of X. By the *quotient* $\bar{X} = X/\langle\alpha\rangle$ we mean a graph which vertices are orbits of the action of α on the vertex set of X, two orbits $[v]$ and $[u]$ being adjacent if there exist vertices $v' \in [v]$ and $u' \in [u]$ such that $v'u'$ is an edge in X. We say that an edge xy in X is of *type* AB, if $x \in A$ and $y \in B$, where A, B are orbits of α. The notion of the type of an edge naturally extends to walks in X. In particular, every walk in X projects to a walk in $\bar{X}$ and this fact is expressed by saying that it has some type. There may be double adjacency between two 3-orbits meaning that the corresponding induced subgraph is a 6-cycle. Orbits of length three will be denoted by capital letters while we use the same small letter for a fixed point as well as for the respective orbit of length 1. Since we assume $g \geq 6$ there are no edges joining vertices belonging to the same 3-orbit. Hence the mapping $X \to \bar{X}$ taking $v \mapsto [v]$ restricted to the union of 3-orbits defines a true regular covering between subgraphs.

Lemma 4.2. Let X be a symmetric connected cubic graph with girth 6 and let c be the number of 6-cycles passing through an edge in X. Then, $c = 2, 4, 6$, or 8. If $c > 2$ then X is isomorphic to one of the following four graphs: Heawood graph, Pappus graph 9_3, generalised Petersen graph $GP(8, 3)$, and generalised Petersen graph $GP(10, 3)$.

Proof. By Proposition 2.8, c is even. There are 8 vertices at distance 2 from a given edge e and edges joining these 8 vertices are in 1-1 correspondence with 6-cycles going through e. Since X has valency 3, we have $c = 2, 4, 6$, or 8.

The proof of the statement is done by a case-to-case analysis. Firstly observe that α acts freely on the 6 elements of V_2 and it fixes at most two vertices in V_3. Indeed, a fixed vertex u in V_2 in the action of α implies an existence of a 4-cycle going through v and u contradicting the assumption. Hence V_2 splits into two α-orbits of length 3, say B and C. If a vertex u in V_3 is fixed by α then either it is adjacent to all the vertices in B, or to all the vertices in C. Assume there are three vertices in V_3 fixed by α. Then two of them share the same vertices in their neighbourhood which gives rise to a 4-cycle, a contradiction.

Note that there are at least six different 6-cycles passing through v and there is no 4-cycle in X. Then one of the following cases happens:

(1) there are two vertices fixed by α in V_3, and they are adjacent to different 3-orbits in V_2;
(2) there is one vertex fixed by α in V_3 and at least one 3-orbit of which each vertex is joined to at least two vertices in V_2;
(3) the action of α is free on V_3, there are exactly two 3-orbits in V_3 and E_2^3 is a union of cycles.

In what follows we denote by A the unique 3-orbit in V_1 and by B and C the two 3-orbits in V_2.

Case 1. Denote by u the fixed vertex adjacent to B and by w the fixed vertex adjacent to C. Let E, F be the 3-orbits adjacent to B and C in V_3, respectively. Assume $E \neq F$. Consider the set of 6-cycles passing through an edge e of type AC. The subgraph $[V_0^3]$ contains exactly two 6-cycles passing through e. They are both of type $(vACwCA)$. The only possibility to create another 6-cycle passing through e is to extend the unique path of type $EBACF$ onto a 6-cycle. But there can be at most one such 6-cycle. Hence we have at most three 6-cycles passing through e, a contradiction. Hence $E = F$. Note that $[V_0^3]$ is uniquely determined. By Proposition 4.1 X cannot contain an independent 3-edge-cut, hence the 3-orbit $E = F$ is adjacent to a fixed vertex in V_4. In this way, we have constructed a unique graph on 16 vertices, namely $GP(8, 3)$ (see Fig. 1).

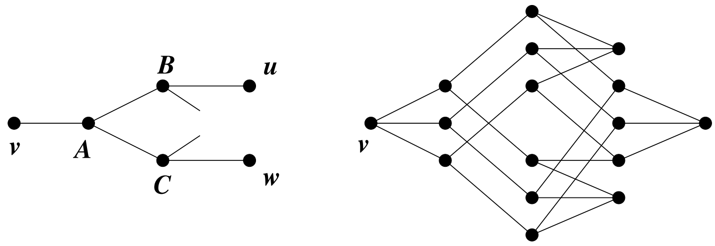

FIGURE 1. A quotient in Case 1 and the generalized Petersen graph $GP(8,3)$

Case 2. Let there be exactly one vertex u fixed by α in V_3 and one 3-orbit D which vertices are joined to at least two vertices in V_2. We may assume that u is adjacent to B.

Subcase 2.1: D is adjacent to both B and C (see Fig. 2). Denote by $E \subseteq V_3$ the third 3-orbit adjacent to C. Clearly, the orbit E is not formed by a fixed point, otherwise we are in Case 1. Assume $E \neq D$. Let e be an edge of type AC. By the assumption there are at least four 6-cycles passing through e. There are two 6-cycles passing through e in the subgraph $[V_0^3]$, one of type $(vABDCA)$, the other one of type $(uBDCAB)$. The only possibility to create another 6-cycle passing through e is to extend the unique path of type $DBACE$ containing e onto a 6-cycle. In this way only one more 6-cycle passing through e can be constructed, a contradiction. Hence $E = D$ and there is a double adjacency between C and D. The graph has 14 vertices and it is isomorphic to the Heawood graph.

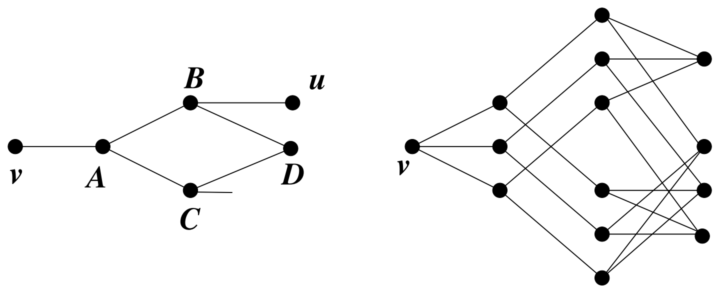

FIGURE 2. A quotient in Case 2.1 and the Heawood graph

Subcase 2.2: D is doubly adjacent to C (see Fig. 3). There is a 3-orbit E in V_3 adjacent to B. We may assume that $E \neq D$, otherwise we get the previously discussed subcase. As above take an edge e of type AC. There are exactly two 6-cycles in $[V_0^3]$ both of type $(vACDCA)$. To create another 6-cycle we need to extend the unique path of length 3 containing e which is of type $EBAC$. Since no 4-cycle exists, at most one such 6-cycle is constructed, a contradiction.

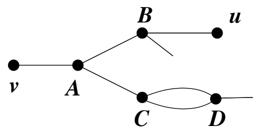

FIGURE 3. A quotient in Case 2.2

Case 3. We distinguish two subcases.

Subcase 3.1: There are two orbits D, E in V_3, both adjacent to both B and C (see Fig. 4).

Note that every 3-arc based at v extents to a 6-cycle. By the vertex-transitivity it holds true for any 3-arc. Consider a 3-arc of type $ECDB$. Then a 6-cycle passing through it contains an edge of type BE, as well. Hence, two vertices in E are connected by a path of length 2. This is possible only if the orbit adjacent to E in V_4 is a fixed vertex. Similarly, there is a fixed vertex adjacent to D. In this way a unique graph on 18 vertices is constructed, namely the Pappus graph 9_3.

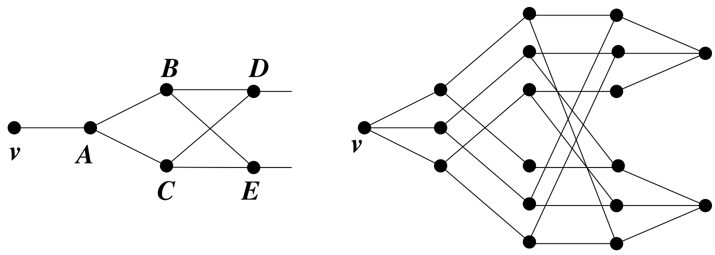

FIGURE 4. A quotient in Subcase 3.1 and the Pappus graph 9_3

Subcase 3.2: There are two orbits D, E in V_3, doubly adjacent to B and C, respectively (see Fig. 5).

As above every 3-arc extends to a 6-cycle. Consider a 3-arc of type $ECAB$. There is no 6-cycle in $[V_0^3]$ passing through such a 3-arc. Hence there is a 3-orbit F in V_4 adjacent to both D and E. Since we cannot have an independent 3-edge-cut, the orbit adjacent to F in V_5 is a fixed point. In this way a unique cubic graph with 20 vertices is constructed, namely $GP(10,3)$.

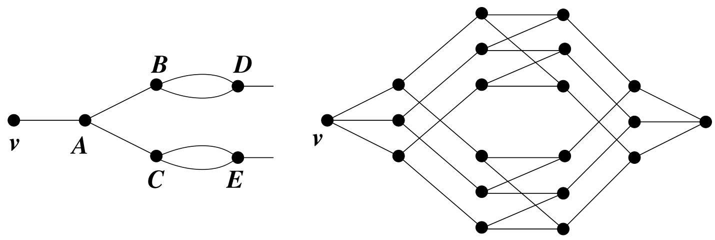

FIGURE 5. A quotient in Subcase 3.2 and the generalized Petersen graph $GP(10,3)$

Lemma 4.3. Let X be a symmetric cubic graph of girth 7 such that there are more than two 7-cycles passing through an edge of X. Then X is the Coxeter graph.

Proof. Since X has girth 7, there are two 3-orbits in V_2 denoted by B and C, and four 3-orbits in V_3 denoted by D, E, F and G. We may assume that D and E, F and G are adjacent to B, C respectively. Let A be the 3-orbit in V_1. Then A is adjacent to B, C and to v. Clearly, α acts freely on V_i for $i = 1, 2$ or 3. Recall that we use c to denote the number of 7-cycles passing through an edge in X. The proof splits into two claims.

Claim 1. Let $c \geq 4$. Then $c = 4$ and $[V_3]$ is a perfect matching consisting of 6 edges. Furthermore, every 3-arc is included in precisely one 7-cycle.

Let m be the number of edges in V_3. Since α acts freely on V_3 and $c \geq 4$, $m = 6$, 9 or 12. If $m = 12$ then X has 22 vertices. But there is no symmetric cubic graph of girth 7 with 22 vertices. If $m = 9$ we have only 6 edges in E_3^4. By Proposition 4.1 E_3^4 cannot separate cycles. It follows that the complement $\bar{V}_0^3$ has at most 4 vertices, and consequently, the whole graph has at most 26 vertices. However, the least symmetric cubic graph of girth 7 is the Coxeter graph having 28 vertices [7], a contradiction. Thus $m = 6$. Since each edge in V_3 corresponds a 7-cycle passing through v, we have $c = 4$. Since the girth is 7, two 3-orbits in V_3 cannot be doubly adjacent.

Suppose that there is a 3-orbit in $[V_3]$, say E, adjacent to two 3-orbits in V_3. We may assume that E is adjacent to F and one of D and G (see Fig. 6). Since X is vertex-transitive, for any arc v_1v_2 there is a 3-arc $v_1v_2v_3v_4$ and there are two 7-cycles passing through the 3-arc such that the number of the common vertices of the two 7-cycles is 4 because it holds true for each arc of type vA. But this is not true for an arc of type AC. In fact, for a 3-arc of type $ACFE$ or $ACGE$ there is only one 7-cycle passing through the 3-arc, which is of type $(ACFEBAv)$ or $(ACGEBAv)$. For a 3-arc of type $ACFH$ or $ACGH$, where H is an orbit in V_4, any two 7-cycles passing through such a 3-arc have at least 5 vertices in common, because the two 7-cycles pass through a 4-arc of type $BACFH$ or $BACGH$ as well.

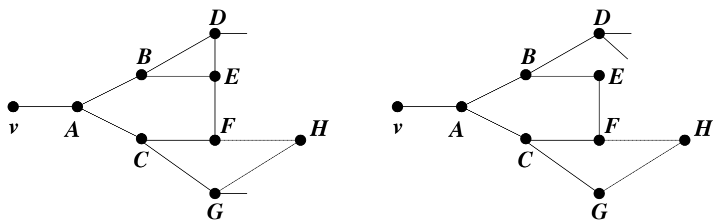

FIGURE 6. Quotients in Claim 1

Thus, $[V_3]$ consists of a perfect matching of 6 edges, implying that each 3-arc based at v extends to a unique 7-cycle and by the transitivity of the action on vertices this property holds true for any 3-arc in X. As concerns the matching between the four 3-orbits in V_3, there are two cases to consider: we have either edges of type DE and FG, or DF and EG.

Claim 2 The graph X has at most 44 vertices.

We shall distinguish two cases.

Case 1. There are some fixed points in V_4.

We may assume that a fixed point u is adjacent to D. If there is a matching of type DE, FG in V_3 then we get a 5-cycle of type $uDEBD$ passing through u, a contradiction. Thus, we assume that there are edges of type DF, EG in V_3. If there is another fixed point, say w, in V_4 then $[V_0^3 \cup \{u, w\}]$ is separated by a 6-edge cut, which implies that X has at most 28 vertices. Thus, we may assume that there is only one fixed point in V_4, that is u. Since E_3^4 has 12 edges, there are one, two or three 3-orbits in V_4. If there is one 3-orbit in V_4 then X has 26 vertices. If there are two 3-orbits in V_4 then one of them, say H, is adjacent to two 3-orbits in V_3, then $[V_0^3 \cup H \cup \{u\}]$ is separated by a 6-edge cut forcing that X has at most 30 vertices. Thus, we may assume that there are three 3-orbits in V_4, say H, I and J with adjacences to E, F and G respectively (see Fig. 7). By the existence of 7-cycle passing through a 3-arc of type $ABEH$, $ACFI$ or $ACGJ$ (Claim 1), there are at least 6 edges in the induced subgraph $[H \cup I \cup J]$. It follows that X has 32 vertices or $[V_0^4]$ is separated by a 6-edge cut. In the latter case, X has at most 36 vertices.

Case 2. The action of α is free on V_4.

We distinguish three subcases.

Subcase 2.1. There are four 3-orbits H, I, J, K in V_4, say we have edges of type DH, EI, FJ and GK (see Fig. 7.

By Claim 1, every 3-arc is included in precisely one 7-cycle. Considering the 3-arcs of type $ABDH$, $ABEI$, $ACFJ$ and $ACGK$, we derive that each 3-orbit in V_4 is adjacent to at least one 3-orbit in V_4. Furthermore, H and I are adjacent to one of J or K. It follows that $[V_4]$ has 6, 9 or 12 edges. If $[V_4]$ has 12 or 9 edges then X has 34 vertices or X has a 6-edge cut. For the later, X has at most 38 vertices. Thus, we may assume that $[V_4]$ has 6 edges, that is $[V_4]$ consists of a perfect matching of 6-edges, which are of type HJ and IK or of type HK and IJ.

If there is a matching of type DE, FG in V_3 then the existence of 7-cycle passing through a 3-arc of type $HDEB$ implies that there is a fixed vertex in V_5 that is adjacent to H. Similarly, there are another three fixed vertices in V_5 that are adjacent to I, J and K, respectively. It follows that X has 38 vertices. Thus, we assume the matching in V_3 is of type DF, EG (Fig. 7). Consider the 3-arcs of type $HDFC$ and $IEGC$, and the existence of 7-cycles passing through these 3-arcs implies that there are two 3-orbit L and M in V_5 such that L is adjacent to H and K, and M is adjacent to I and J. It follows that either M is adjacent to L, or $[V_0^5]$ is separated by a 6-edge-cut, implying that X has 40 vertices or at most 44 vertices.

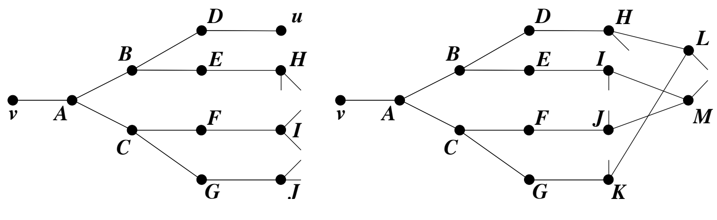

FIGURE 7. Two quotients in Case 1 and Subcase 2.1

Subcase 2.2. There are three 3-orbits in V_4, say H, I and J (see Fig. 8).

Since E_3^4 has 12 edges, one of 3-orbits in V_4, say I, is adjacent to two 3-orbits in V_3 and so H and J are adjacent to one 3-orbit in V_3. As above considering a 3-arc starting at A and terminating at H or J, we derive that H and J is adjacent to some orbits in V_4. Thus, H and J must be adjacent. If $[V_4]$ has 6 edges then the induced subgraph $[V_0^4]$ is separated by a 3-edge cut and hence X has 32 vertices. We may assume that $[V_4]$ has 3 edges and so E_4^5 has 9 edges. If there is a fixed point in V_5, say u, then the induced subgraph $[V_0^4 \cup \{u\}]$ is separated by a 6-edge cut and so X has most 36 vertices. Thus, we assume that there is one, two or three 3-orbits in V_5. If there is one 3-orbit in V_5 then the graph X has 34 vertices. If there are two 3-orbits in V_5 then a 3-orbit in V_5, say K, is adjacent to two 3-orbits in V_4. This implies that the induced subgraph $[V_0^4 \cup K]$ is separated by a 6-edge cut and so X has at most 38 vertices. Now, we assume that there are three 3-orbits in V_5, say K, L and M that are adjacent to H, I and J respectively (Fig. 8). Considering a 3-arc starting at V_2 and terminating at V_5, Claim 1 implies that $[V_5]$ has at least 6 edges. It follows that $[V_5]$ has 6 or 9 edges. For the later, X has 40 vertices. Let $[V_5]$ have 6 edges. Then $[V_0^5]$ is separated by a 6-edge cut and so X has at most 44 vertices.

Subcase 2.3. There are two 3-orbits in V_4, say H and I (see Fig. 8).

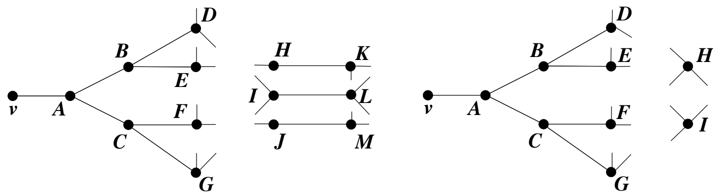

FIGURE 8. Two quotients in Subcases 2.2 and 2.3

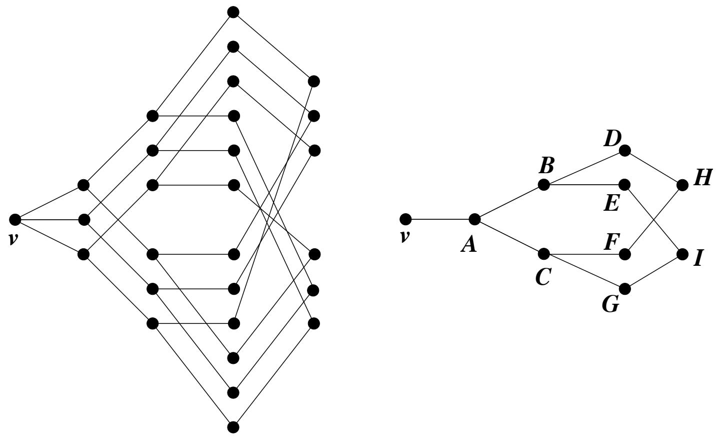

FIGURE 9. The Coxeter graph and its quotient

If one 3-orbit in V_4, say H, is adjacent to three 3-orbits in V_3 then $[V_0^3 \cup H]$ is separated by a 3-edge cut. This implies that there is a fixed point in V_4, which is discussed in Case 1. Thus, we may assume that both H and I are adjacent to two 3-orbits in V_3. If H and I are not adjacent then $[V_0^4]$ is separated by a 6-edge cut. Hence X has at most 32 vertices. If H and I are adjacent then X has 28 vertices. In this case, the Coxeter graph appears (see Fig. 9). The proof of Claim 2 is complete.

Checking the list of arc-transitive cubic graphs in [7] we see that there are only 17 arc-transitive graphs with at most 44 vertices. Out of these 17 graphs only one, the Coxeter graph has girth 7 and it satisfies the property that there are exactly four 7-cycles passing through an edge in it. $\square$

5 Symmetric cubic graphs with exactly two girth cycles passing through an edge

Theorem 5.1. Let X be a symmetric connected cubic graph of girth g such that there are exactly two girth cycles passing through an edge. Then one of the following three cases happen:

(1) X is 2-regular, and $\mathrm{Aut}(X)$ is a quotient of $\Delta(g,3,2) = \langle x,y,z | x^2 = y^2 = z^2 = (xy)^g = (yz)^3 = (xz)^2 = 1\rangle$ by some normal torsion free subgroup,

(2) X is 1-regular, and $\mathrm{Aut}(X)$ is a quotient of $\Delta^+(g,3,2) = \langle x,y | x^3 = y^2 = (xy)^g = 1\rangle$ by some normal torsion free subgroup,

(3) X is 1-regular and g is even, there exists $m > g$ such that $\mathrm{Aut}(X)$ is a quotient of $\Delta^+(m,3,2;g/2) = \langle x,y | x^3 = y^2 = (xy)^m = [x,y]^{g/2} = 1\rangle$ by some normal torsion free subgroup.

Proof. By Proposition 2.8 X is either 1-regular or 2-regular. Let $G = \mathrm{Aut}(X)$ and let $e = vu$ be an edge in X. Let $\{2,4,u\}$ and $\{1,3,v\}$ be the neighbors of v and u, respectively. Since there are exactly two girth cycles passing through an edge there is at most one girth cycle passing through a 2-arc. Thus, we may assume that two girth cycles, say C and C', pass through the 3-arcs $1uv2$ and $3uv4$, respectively. Then, C and C' are the only two girth cycles passing through e.

Assume X is 2-regular. Since $G_v \cong S_3$, there are two involutions y and z in G_v such that y interchanges 2 and u, and z interchanges 2 and 4. Then, $G_v = \langle y,z\rangle$ and yz has order 3. Let $x \in G$ interchange u and v. We claim that x is an involution. Otherwise, x permutes the neighbors of e as (1234) or (1432) and hence x sends the 3-arc $1uv2$ onto $2vu3$ or $4vu1$. This is impossible because there is no girth cycle passing through $2vu3$ or $4vu1$. Thus, x is an involution and $G_e = \langle x,z\rangle \cong Z_2 \times Z_2$. One can easily check that either x or xz preserve the cycle C, say it is x. Thus, x interchanges u and v, 1 and 2, and 3 and 4. Furthermore, xy fixes C and maps the 2-arc $1uv$ onto $uv2$. By the 2-regularity of X, $|xy| = g$ and by the symmetry and connectivity, $G = \langle G_v, x\rangle = \langle x,y,z\rangle$.

Assume that X is 1-regular. Let $x \in G_v$ permute the neighbors of v as $(4u2)$ and let y be the involution in G interchanging u and v. Then, x takes C' onto C and y permutes the neighbors of e as (12)(34) or (14)(23), which corresponds y fixing C and C', or interchanging C and C'. If y interchanges C and C' then yx fixes C and takes $1u$ onto uv. Thus, $|yx| = |xy| = g$ and we have the Case (2). If y fixes C and C' then $[x,y] = xyx^{-1}y^{-1}$ fixes C' and takes $4v$ onto $u3$, that is $[x,y]$ rotates C' with step two. If g is odd then both $[x,y]^{\frac{g+1}{2}}$ and xy take $4v$ onto vu. By the 1-regularity, $[x,y]^{\frac{g+1}{2}} = xy$ and hence $[x,y]^{\frac{g-1}{2}} = yx$. It follows that $yx[x,y] = xy$ and so $x^2y = yx^2$. This implies x and y commute because x has order 3. Clearly, it is impossible. Thus, g must be even and $[x,y]^{\frac{g}{2}} = 1$. Now, we show that if $(xy)^m = 1$ then $m > g$. In fact, xy sends $4v$ onto vu, vu onto $u1$. By considering the action of xy on the arc $4v$, we can get a cycle containing the 3-arc $4vu1$. Since no girth cycle passing through $4vu1$, xy has order more than g. □

Theorem 5.2. Let X be a symmetric cubic graph of girth 7. Then the following statements hold true:

(0) there is no 4-arc-transitive cubic graph of girth 7,
(1) X is 3-regular if and only if it is the Coxeter graph,
(2) X is 2-regular if and only if $X = \mathcal{U}(7,3)/N$, where $N \lhd \Delta(7,3,2)$ is a proper normal subgroup of finite index, $\mathrm{Aut}(X) \cong \Delta(7,3,2)/N$.
(3) X is 1-regular if and only if $X = \mathcal{U}(7,3)/N$, where $N \lhd \Delta^+(7,3,2)$ is a normal subgroup of finite index but $N \ntriangleleft \Delta(7,3,2)$, $\mathrm{Aut}(X) \cong \Delta^+(7,3,2)/N$.

Moreover, $\mathcal{U}(7,3)/N$ is of girth 7 for any non-trivial normal subgroup $N \lhd \Delta^+(7,3,2)$ of finite index.

Proof. By Lemma 4.3, if X is not the Coxeter graph it has exactly two girth cycles passing through an edge. Consequently, X is either 1- or 2-regular. Applying Theorem 5.1 (1) we get that if X is 2-regular then $X \cong \mathcal{U}(7,3)/N$ for some normal subgroup $N \lhd \Delta(7,3,2)$ of finite index. Let X be 1-regular. Since the girth is odd Case (2) of Theorem 5.1 applies proving $X = \mathcal{U}(7,3)/N$ for some $N \lhd \Delta^+(7,3,2)$ but $N \ntriangleleft \Delta(7,3,2)$. To see the opposite direction observe that $\Delta(7,3,2)/N$ acts 2-regularly on $X = \mathcal{U}(7,3)/N$ provided $N \lhd \Delta(7,3,2)$. Similarly, $\Delta^+(7,3,2)/N$ acts 1-regularly on $X = \mathcal{U}(7,3)/N$ if $N \lhd \Delta^+(7,3,2)$. Finally, observe that a nontrivial $N \lhd \Delta^+(7,3,2)$ is torsion free. Hence, the respective quotient $X = \mathcal{U}(7,3)/N$ is a symmetric cubic graph of girth at most 7. Since 7 divides the number of vertices of $X = \mathcal{U}(7,3)/N$, if X is exceptional of girth ≤ 6 then X is the Heawood graph, which cannot be since X has cycles of length 7. Thus assuming that the girth of X is at most 6 we get that $\mathrm{Aut}(X)$ is either isomorphic to $\Delta(6,3,2)/K$ or $\Delta^+(6,3,2)/K$ for some normal subgroup $K \lhd \Delta^+(6,3,2)$. Since $\Delta^+(6,3,2) \cong (Z \times Z) : Z_6$ is solvable (see the next section) $\mathrm{Aut}(X)$ is solvable as well. However, a non-trivial finite quotient of $\Delta(7,3,2)$, or of $\Delta^+(7,3,2)$ is insolvable. Hence the girth of $X = \mathcal{U}(7,3)/N$ is 7. □

6 Quotients of the tesselation $\mathcal{U}(6,3)$ and graphs of girth 6

Let us consider the hexagonal infinite tessellation $\mathcal{U}(6,3)$ of the Euclidean plane $\mathbb{E}_2$. Assume $\mathbb{E}_2$ is endowed with the standard Cartesian coordinate system. Let $\vec{i} = (1,0)$ and $\vec{j} = (1/2, \sqrt{3}/2)$ be two vectors based at $(0,0)$. Note that $\vec{j}$ arises by counterclockwise rotation of $\vec{i}$ by 60 degrees. Without loss of generality we may identify the centers of the hexagons of the tesselation with points of the plane with coordinates $m\vec{i} + n\vec{j}$, where m and n are integers. Hence we can identify the center of every hexagon with a couple (m,n) of integers. Let us denote by $\psi_{m,n} = m\vec{i} + n\vec{j}$ the translation of $\mathbb{E}_2$ shifting the points by the vector $m\vec{i} + n\vec{j}$, hence $\vec{x}^{\psi_{m,n}} = \vec{x} + m\vec{i} + n\vec{j}$. Let ρ be the counterclockwise rotation of the plane by 60 degrees around the point $(0,0)$. See Figure 10.

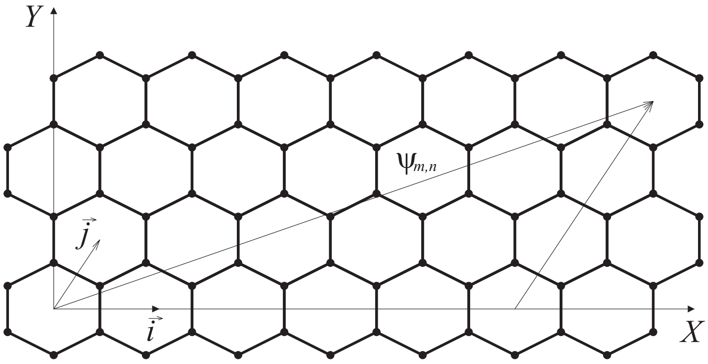

FIGURE 10. Construction of the $\mathcal{G}(m,n)$

Let $N = N(m,n) = \langle \psi_{m,n}, \rho^{-1}\psi_{m,n}\rho\rangle = \langle \psi_{m,n}, \psi_{-n,m+n}\rangle$. It follows that the group of translations $N(1,0) = \langle \psi_{1,0}, \psi_{0,1}\rangle$ acts regularly on the set of centers of all hexagons and it forms a semidirect product $G = N(1,0) \rtimes \langle \rho \rangle$. Observe that G acts 1-regularly on arcs of $\mathcal{U}(6,3)$, hence it is isomorphic to $\Delta^+(6,3,2)$. Indeed, it is easy to check that the assignment $x \mapsto \rho^2\psi_{1,0}$ and $y \mapsto \rho^3\psi_{1,0}$ extends to an isomorphism $\Delta^+(6,3,2) = \langle x,y | x^3 = y^2 = (xy)^6 = 1\rangle \to G$. This shows that $\Delta^+(6,3,2) \cong N(1,0) \rtimes \langle \rho \rangle$. Since $\Delta(6,3,2)$ is an 2-extension of $\Delta^+(6,3,2)$ by an element of order 2 taking $x \mapsto x^{-1}$ and $y \mapsto y$ we have $\Delta(6,3,2) \cong \Delta^+(6,3,2) \rtimes Z_2$. Using the above interpretation of the triangle group $\Delta^+(6,3,2)$ in the group of isometries of the Euclidean plane one can prove (see [9]) that $N(m,n) \triangleleft \Delta^+(6,3,2)$ is a torsion-free normal subgroup of $\Delta^+(6,3,2)$ for any two non-negative integers m, n, $m+n \neq 0$ and any torsion free normal subgroup of the triangle group $\Delta^+(6,3,2)$ is of this form. Moreover, every torsion free normal subgroup of the extended triangle group $\Delta(6,3,2)$ is $N(m,n)$ for some m, n satisfying $mn = 0$ or $m = n$. It follows that $\mathcal{U}(6,3)/N(m,n)$ is an (oriented) regular map of type $(6,3)$ in the torus, and the respective arc-transitive graph will be denoted by $\mathcal{G}(m,n)$. It is easy to get a picture of the graph $\mathcal{G}(m,n)$ by identifying the parallel sides of the fundamental region (a connected sector of the plane containing representatives of the orbits in the action of $N(m,n)$). In this particular case, the fundamental region forms a parallelogram which corners coincide with the centers of four hexagons with coordinates $(0,0)$, (m,n), $(-n,m+n)$, $(m-n,m+2n)$ in the coordinate system defined by the unit vectors $\vec{i}$, $\vec{j}$.

The automorphism group of $\mathcal{G}(m,n)$ contains a 1-regular subgroup isomorphic to $\Delta^+(6,3,2)/N(m,n)$ and it contains a 2-regular subgroup of the form $\Delta(6,3,2)/N(m,n)$ if and only if $mn(m-n) = 0$ (see [9] (p. 107)). Since $\mathcal{G}(m,n) \cong \mathcal{G}(n,m)$ in what follows we will assume $m \leq n$.

Lemma 6.1. Let $G = \langle x,t |\ x^3 = t^2 = [x,t]^3 = 1\rangle$. Then $G \leq \Delta(6,3,2)$ is an index 2 subgroup and every normal torsion free subgroup $N \triangleleft G$ of finite index is a normal torsion free subgroup of $\Delta(6,3,2)$ as well. In particular, $N \cong N(m,m)$ or $N \cong N(0,m)$ for some positive integer m.

Proof. Identifying x with an element of order 3 in the stabiliser of a vertex v of $\mathcal{U}(6,3)$ and t with a reflection taking v onto one of its neighbours we get a 1-regular action of G on $\mathcal{U}(6,3)$. Since $\Delta(6,3,2) \cong \mathrm{Aut}(\mathcal{U}(6,3))$ and it acts 2-regularly,

$G \leq \Delta(6,3,2)$ and it is an index 2 subgroup. Moreover, $G^+ = \Delta^+(6,3,2) \cap G$ consists of elements expressed as words in terms of the generators containing even number of appearances of t. Hence G^+ is an index 2 subgroup of G. If $N \triangleleft G$ is a normal torsion free subgroup of finite index, then $N^+ = N \cap G^+ \leq \Delta^+(6,3,2)$ is such a group as well. Hence $N^+ = N(m,n)$ for some integers m, n. Since it is normal in G, N^+ is invariant under the conjugation by the reflection t. Hence $N^+ \triangleleft \Delta(6,3,2)$ as well. Either $N = N^+$, or $N = \langle N^+, t\rangle$. However, the latter case is excluded since we assume that N is torsion free. Hence $N = N^+$ and we are done. □

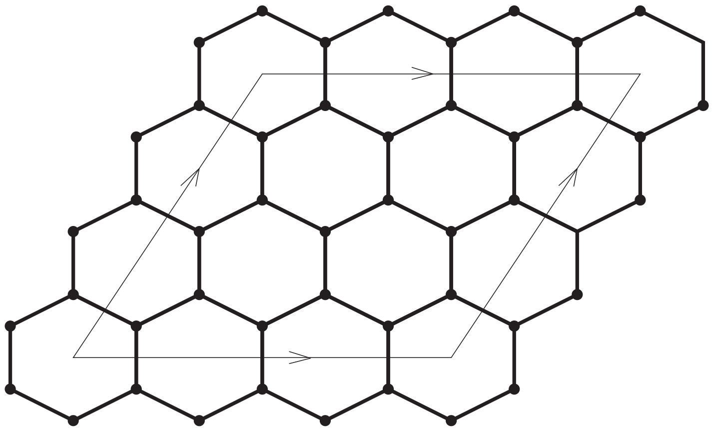

FIGURE 11. Pappus graph represented as $\mathcal{G}(3,0)$

Now we are ready to prove the following classification theorem.

Theorem 6.2. Let X be a symmetric cubic graph of girth 6. Then the following statements hold:

(0) there is no 5-regular cubic graph of girth 6,
(1) X is 4-regular if and only if it is the Heawood graph,
(2) X is 3-regular if and only if it is the Pappus graph, or the generalized Petersen graph $GP(10,3)$,
(3) X is 2-regular if and only if X is the generalized Petersen graph $GP(8,3)$ or $X \cong \mathcal{G}(m,n) = \mathcal{U}(6,3)/N(m,n)$ with $\mathrm{Aut}(X) \cong \Delta(6,3,2)/N(m,n)$, where $0 < m = n$ or $0 = m < n$, and (m,n) is different from $(0,1)$, $(1,1)$, $(0,2)$, $(0,3)$,
(4) X is 1-regular if and only if $X \cong \mathcal{G}(m,n) = \mathcal{U}(6,3)/N(m,n)$, with $\mathrm{Aut}(X) \cong \Delta^+(6,3,2)/N(m,n)$ for some integers m,n satisfying $0 < m < n$, and $(m,n) \neq (1,2)$.

Proof. Assume the number of 6-cycles passing through an edge is grater than 2. By Lemma 4.2 X is one of the four exceptional graphs. Note that 3-arc-transitivity of X implies that X has more than two 6-cycles passing through an edge and thus if X is 3-arc-transitive it is one of the exceptional graphs. Checking the symmetries of the exceptional graphs we get the first three items of the statement. The graph $GP(8,3)$ is 2-regular although it has more than two girth cycles passing through an edge. In what follows we assume that there are exactly 2 girth cycles passing through an edge in which case X is at most 2-regular.

If X is 2-regular Theorem 5.1 (1) applies. Since every normal torsion free subgroup of $\Delta(6,3,2)$ of finite index is either $N(m,m)$ or $N(0,m)$ for some positive integer m we have proved item (3). The exceptional groups $N(0,1)$, $N(1,1)$, $N(0,2)$ and $N(0,3)$ give rise, respectively, to a graph with multiple edges, to the complete bipartite graph $K_{3,3}$ of girth 4, to the cube Q_3 of girth 4 and to the Pappus graph which is known to be 3-regular, see Fig. 11. From the remaining cubic graphs which are 2-regular of girth at most 5 the graphs K_4, $GP(5,2)$ and the dodecahedron are not isomorphic to $\mathcal{G}(m,m)$, or to $\mathcal{G}(0,m)$. An easy argument to see it comes from the fact that the number of vertices of $\mathcal{G}(m,m)$ is $6m^2$, while the number of vertices of $\mathcal{G}(0,m)$ is $2m^2$ (see [9 (p. 107)]).

If X is 1-regular then by Lemma 6.1 case (3) of Theorem 5.1 cannot happen. Hence Theorem 5.1 (2) applies. Consequently, X is one of $\mathcal{G}(m,n) = \mathcal{U}(6,3)/N(m,n)$ for some integers $0 < m < n$. The exceptional group $N(2,1)$ gives rise to the Heawood graph which is known to be 4-regular, see Fig. 12. □

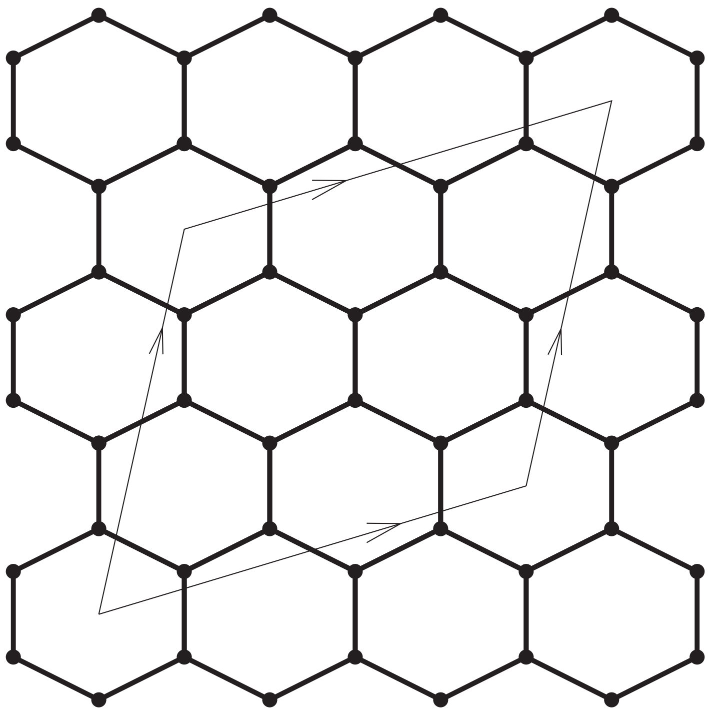

FIGURE 12. The Heawood graph as $\mathcal{G}(2,1)$

Corollary 6.3. Let X be a symmetric cubic graph of girth 6. Then $X \cong \mathcal{G}(m,n)$ for some integers $m \leq n$, $m+n > 1$, except X is the generalized Petersen graph $GP(8,3)$ or $GP(10,3)$. In particular, all symmetric cubic graphs of girth 6 are bipartite.

Proof. First part follows from Theorem 6.2 The biparticity of $GP(8,3)$ and $GP(10,3)$

can be verified directly from the definition. As concerns the graphs $\mathcal{G}(m,n)$, observe that the group of translations $N(1,0)$ acts on the vertices of $\mathcal{U}(6,3)$ with two orbits forming a bipartition of the vertex set. It follows that the vertex-orbits of $N(1,0)/N(m,n) \leq \mathrm{Aut}(\mathcal{G}(m,n))$ form a bipartition of $\mathcal{G}(m,n) = \mathcal{U}(6,3)/N(m,n)$. □

Theorem 6.4. Let $X = \mathcal{G}(m,n)$, where $0 \leq m \leq n$ and

$$(m,n) \notin \{(0,0),(0,1),(1,1),(0,2),(1,2),(0,3)\}.$$

Then either

(1) $0 < m < n$ and $\mathrm{Aut}(X) = \langle x,y |\, x^3 = y^2 = (xy)^6 = 1, (x^{-1}yxy)^m(xyx^{-1}y)^n = 1\rangle$;

or

(2) $\mathrm{Aut}(X) = \langle t,u,z |\, t^2 = u^2 = z^2 = (tu)^3 = (uz)^2 = (tz)^6 = 1, (utuztz)^m(tzutuz)^n = 1\rangle$, where $mn(m-n) = 0$.

Proof. By the assumptions X is 1-regular or 2-regular. By Theorem 6.2 $\mathrm{Aut}(X) \cong \Delta^+(6,3,2)/N(m,n)$ provided $0 < m < n$. Hence $\mathrm{Aut}X = \langle x,y |\, x^3 = y^2 = (xy)^6 = 1, \ldots\rangle$ is a quotient by $N(m,n) = \langle \psi_{m,n}, \psi_{-n,m+n}\rangle \cong Z \times Z$. Without loss of generality we may identify x with the 120 degree counterclockwise rotation of $\mathcal{U}(6,3)$ around the point $(0,0)$ and y with the 180 degree turn round the center of the common edge of the hexagons with centers $(0,0)$ and $(1,0)$. This identification establishes an embedding of $\Delta^+(6,3,2) = \langle x,y |\, x^3 = y^2 = (xy)^6 = 1\rangle$ in the group of isometries of the Euclidean plane. Direct computation of images of the points $(0,0)$ and $(1,0)$ shows that $\psi_{2,-1} = x^{-1}yxy$ and $\psi_{1,1} = xyx^{-1}y$. Hence the relation $(x^{-1}yxy)^m(xyx^{-1}y)^n = 1$ transforms to $\psi^m_{2,-1}\psi^n_{1,1} = \psi_{2m+n,n-m} = 1$. Since $\psi_{2m+n,n-m} \in N(m,n)$ and $N(n,m) = Cl\langle\psi_{2m+n,n-m}\rangle$ is the normal closure of $\langle\psi_{2m+n,n-m}\rangle$ in $\Delta^+(6,3,2)$, we are done.

Let $mn(m-n) = 0$. In this case X is 2-regular and the automorphism group has presentation of the form $\mathrm{Aut}(X) = \langle t,u,z |\, t^2 = u^2 = z^2 = (tu)^3 = (uz)^2 = (tz)^6 = 1, \ldots\rangle$. In this case t, u generate the vertex stabiliser and we may assume that $x = tu$ and $y = uz = zu$. The automorphism group contains as an index two subgroup a 1-regular subgroup generated by x and y and satisfying $(x^{-1}yxy)^m(xyx^{-1}y)^n = 1$. Putting $x = tu$ and $y = uz = zu$ we get the required relation. The equality $mn(m-n) = 0$ guarantees then $N(m,n) = Cl\langle\psi_{2m+n,n-m}\rangle \triangleleft \Delta^+(6,3,2)$ is normal in the extended triangle group $\Delta(6,3,2)$ as well. □

7 Graphs with two girth cycles passing through an edge, existence problems

It follows from Theorem 5.1 that a symmetric cubic graph X of girth g such that there are two girth cycles passing through an edge is either 2-regular, or 1-regular and in the 1-regular case there are two sorts of the action on the compact surface S arising by gluing 2-cells to each girth cycle. It is clear, that every graph automorphism extends to a self-homeomorphism of S and hence the embedding of X into S gives rise to a regular map, or to an oriented regular map which automorphism group coincide with the full automorphism group of the graph, or it is the Petrie dual of an oriented (chiral) regular map. In this section, we shall discuss the existence of such graphs and maps for $g \geq 7$. In a correspondence with Theorem 5.1 we distinguish the above mentioned three cases.

Case 1: X is 2-regular.

The following statement holds true.

Proposition 7.1. For every $g \geq 6$ there are infinitely many 2-regular cubic graphs X with girth g. Moreover, each such X is a quotient $X \cong \mathcal{U}(g,3)/N$, where N is an appropriate normal torsion-free subgroup of $\Delta(g,3,2)$.

The history of the proof of this statement is quite long. In the context of permutation groups it was proved in 1902 in [23], later rediscovered, reproved and improved by many other authors (see [27] for more information). A general argument to see the existence of infinitely many finite quotients of $\Delta(g,3,2)$ uses the residual finiteness of $\Delta(g,3,2)$. A group G is called *residually finite* if for any finite set $A \subseteq G$ of elements of G and for any $x \in A$, $x \neq 1$ there exists a normal subgroup $N \trianglelefteq G$ of finite index such that $x \notin N$. The idea of the proof of Proposition 7.1 is to take A to be the set of all elements in $\Delta(g,3,2)$ expressible in terms of the three involutory generators by all words of length at most d for some d. By residual finiteness of $\Delta(g,3,2)$, for every d there is a normal subgroup $N \trianglelefteq \Delta(g,3,2)$ of finite index such that a part of the universal graph $\mathcal{U}(g,3)$ formed by the images of a particularly chosen arc under the elements of A is mapped isomorphically into $\mathcal{U}(g,3)/N$. Thus $\mathcal{U}(g,3)/N$ is a 2-regular cubic graph of girth g. Taking different d we can construct an infinite family of non-isomorphic graphs satisfying the required properties.

A standard argument to see the residual finiteness of triangle groups is by using a deep theorem of Malcev [20] saying that any finitely generated matrix group is residually finite. Concrete matrix representations of $\Delta(g,3,2)$ one can find in [29, 30]. Proofs based on permutation representation of some quotients of $\Delta(g,3,2)$ can be found for instance in [18, 27].

Case 2. X is 1-regular, and $\mathrm{Aut}(X)$ is a quotient of $\Delta^+(g,3,2) = \langle x,y | x^3 = y^2 = (xy)^g = 1\rangle$ by some normal torsion free subgroup.

To prove that there are infinitely many 1-regular cubic graphs of girth $g \geq 7$ of this sort it is not enough to argue by the residual finiteness of the triangle group $\Delta^+(g,3,2)$. We need an additional argument to guarantee that an appropriate normal subgroup $N \triangleleft \Delta^+(g,3,2)$ of finite index used to produce the graph $\mathcal{U}(g,3)/N$ is not normal in $\Delta(g,3,2)$. In general, this is not easy. In what follows we show that it is true for any g divisible by 6.

Proposition 7.2. For every g divisible by 6 there are infinitely many 1-regular cubic graphs X of girth g such that $X \cong \mathcal{U}(g,3)/N$, for some normal torsion free subgroup N of $\Delta^+(g,3,2)$.

Proof. By Theorem 6.2 the graphs $\mathcal{G}(m,1)$ are for any $m \geq 3$ one-regular quotients of $\mathcal{U}(6,3)$ of girth 6. The least 2-regular cubic graph of girth 6 covering $X = \mathcal{G}(m,1)$ is of the form $Y = \mathcal{U}(6,3)/K$ for some normal subgroup $K \triangleleft \Delta(6,3,2)$. It follows that $K \leq N(m,1) \cap N(1,m)$. Since $N(m,1) \cap N(1,m) \triangleleft \Delta(6,3,2)$ we get $K = N(m,1) \cap N(1,m)$. Let κ be the index of the covering $Y \to X$. By the isomorphism theorem κ is equal to the index of the covering $X \to Z = \mathcal{U}(6,3)/(N(m,1)N(1,m))$. An easy calculation shows that the product $N(m,1)N(1,m) = N(1,0)$. Hence Z is a 2-vertex graph. Since X has $2(m^2+m+1)$ vertices we have $\kappa = m^2+m+1$.

Let $X(g) = \mathcal{U}(g,3)/N$ be a symmetric cubic graph of girth g for some normal subgroup $N \triangleleft \Delta^+(g,3,2)$. Since $6|g$ we have a natural epimorphism $\phi : \Delta^+(g,3,2) \to$

$\Delta^+(6,3,2)$. Choose m such that $m > |\text{Aut}^+(X)| = |\Delta^+(g,3,2)/N|$. We claim that the graph $W = \mathcal{U}(g,3)/N \cap \phi^{-1}(N(m,1))$ is 1-regular of girth g.

Firstly, since W contains a 1-regular subgroup $\Delta^+(g,3,2)/N \cap \phi^{-1}(N(m,1))$ isomorphic to a subgroup of $\Delta^+(g,3,2)/N \times \Delta^+(g,3,2)/\phi^{-1}(N(m,1)) \cong \text{Aut}^+(X(g)) \times \text{Aut}(\mathcal{G}(m,1))$ (see [34,1] for more details) the number of arcs of W cannot exceed $6m(m^2+m+1)$. On the other hand, assuming that W is 2-regular we get a graph $\phi(W) = \mathcal{U}(6,3)/\phi(N) \cap N(m,1)$, where $\phi(N) \cap N(m,1) \triangleleft \Delta(6,3,2)$. Hence, it is a 2-regular graph of girth 6 covering X. Consequently, $\phi(W)$ has at least $\kappa 6(m^2+m+1) = 6(m^2+m+1)^2$ arcs, a contradiction. Since W covers $X(g)$, its girth is at least g, but since it is a quotient of $\mathcal{U}(g,3)$ its girth is at most g. Choosing different values for m we create an infinite family of graphs satisfying the required properties. □

Let us note that the above proof employs some general ideas on the 'chirality' of oriented regular maps and hypermaps developed in [1] and later generalised in [2].

Problem 1. Prove that for every $g \geq 6$ there are infinitely many 1-regular cubic graphs with girth g of the form $\mathcal{U}(g,3)/N$ for some $N \triangleleft \Delta^+(g,3,2)$.

Case 3. As concerns the existence of 1-regular cubic graphs of girth g for some even $g \geq 8$ such that the girth cycles come from the relation $[x,y]^{g/2} = 1$, see Theorem 5.1 (3) we cannot say too much. The following example shows that for girth $g = 8$ this case happens.

Example. Checking the list of all symmetric cubic graphs up to 768 vertices [7] we see that there exists a 1-regular graph X of girth 8 on 400 vertices which group has presentation

$$G = < h, a | h^3 = a^2 = [h,a]^4 = (ha)^{12} = 1, (ha)^5 h^{-1} a (ha)^2 (h^{-1}a)^2 ha (h^{-1}a)^5 = 1 > .$$

It follows X admits a 1-regular action of G of the third type.

As concerns the existence of some other examples of this sort, the following statement proved in [16] supports a conjecture that most probably there are infinitely many such graphs for any even $g \geq 8$.

Theorem 7.3. [16] With possible exceptions of $(m,q) = (13,4)$ and $(7,11)$ the group $\Delta^+(m,3,2;q) = \langle x,y | x^3 = y^2 = (xy)^m = [x,y]^q = 1\rangle$ is infinite if and only if m and q satisfy one of the following conditions: $m = 7$, $q \geq 9$; $m = 8$ or $m = 9$ and $q \geq 6$; $m = 10$ or $m = 11$ and $q \geq 5$; $m \geq 12$ and $q \geq 4$.

Inspired by the above result we give the following problem.

Problem 2. Prove that for every even $g \geq 8$ there are infinitely many 1-regular cubic graphs of girth g which automorphism group is an epimorfic image of $\langle x,y | x^3 = y^2 = [x,y]^{g/2} = 1\rangle$.

Girth 7. A particular instance of Proposition 7.1 for $g = 7$ establishing the existence of infinitely many 2-regular cubic graphs of girth 7 (Case (2) of Theorem 5.2) is a consequence the theorem of McBeath [21] showing that there are infinitely many Hurwitz maps. A regular, or an oriented regular map of type $\{7,3\}$ is called a *Hurwitz map*. It follows from Theorem 5.2 that the family of symmetric cubic graphs of girth 7 coincide with the exception of the Coxeter graph with the family of underlying graphs of Hurwitz maps.

By Theorem 5.2 a 1-regular cubic graph of girth 7 is a quotient $\mathcal{U}(7,3)/N$ by some nontrivial subgroup $N \triangleleft \Delta^+(7,3,2) = \langle x, y | x^3 = y^2 = (xy)^7 = 1\rangle$ of finite index such that the mapping $x \mapsto x^{-1}$, $y \mapsto y$ does not extend to a group automorphism. The residual finiteness of $\Delta^+(7,3,2)$ is not sufficient to see that there are infinitely many such normal subgroups N. This can be done by presenting some finite groups by means of two generators x, y satisfying $x^3 = (xy)^7 = y^2 = 1$ and using an argument to show a non-existence of a group automorphism taking $x \mapsto x^{-1}$ and $y \mapsto y$. In particular, the Ree group $G = Re(3^f)$, for odd $f > 1$, is a simple epimorphic image of the triangle group $\Delta^+(2,3,7)$, with generators x and y of orders 3 and 2. As shown in [28], x is not inverted by any automorphism of G.

Proposition 7.4. There are infinitely many 1-regular cubic graphs of girth 7.

As concerns the problem to classify symmetric cubic graphs of girth 7 it seems to be difficult if possible at all. The core of the problem consists in the fact that the structure of normal subgroups of the triangle groups $\Delta^+(7,3,2)$ $(\Delta(7,3,2))$ of finite index is too complex. Infinite families of simple nonabelian groups appear as epimorphic images of these groups. In fact, one can show that the automorphism group of every symmetric cubic graph of girth 7 is insolvable, which is in a clear contrast to the situation for the graphs of girth 6 in which case the automorphism groups are solvable with a well-understandable structure, or the graph is one of the exceptional graphs discussed in Section 3.

Remark Recently main results of this paper were generalized by Conder and Nedela [8] by proving that a symmetric cubic graphs of girth at most 9 is either 1-regular or $2'$-regular or it belongs to a small family of 15 exceptional graphs. In contrast to the approach used in this paper, the proof is computer-assisted.

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USING TRACE TO IDENTIFY IRREDUCIBLE POLYNOMIALS

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Abstract. We prove a criterion to check whether a polynomial is irreducible. This criterion is related to trace map computations. It may be effectively used to detect irreducibility of polynomials of prime degree over their base field.

1 Introduction

Motivation for our paper is to provide a new way to check if a polynomial with coefficients in a finite field is irreducible. In computer science as well as experimental mathematics, this is a crucial problem to solve in order to generate an explicit finite field.

The context is as follows. Let F be a finite field of cardinality q, and a polynomial $f(x)$ of degree n over F. For any $m \geq 1$ one can define $\boldsymbol{F}$-linear trace map

$$\mathrm{Tr}_m : y \mapsto y^{q^{m-1}} + \ldots + y^q + y$$

that maps $\boldsymbol{F}[x]$ to itself. It induces an $\boldsymbol{F}$-linear map on $\boldsymbol{F}[x]/(f)$, which we denote by $\mathrm{Tr}_{m,f}$.

If f is irreducible, then $\boldsymbol{E} := \boldsymbol{F}[x]/(f)$ is a field and in fact $\boldsymbol{E}/\boldsymbol{F}$ is a cyclic Galois extension of degree n. Its Galois group is generated by the Frobenius map $F : x \mapsto x^q$. For any element $e \in \boldsymbol{E}$ the sum

$$e + F(e) + F^2(e) + \ldots + F^{n-1}(e) = \mathrm{Tr}_{n,f}(e)$$

is clearly invariant under the Frobenius F and thus belongs to $\boldsymbol{F}$. In fact, the image of $\mathrm{Tr}_{n,f}$ is precisely $\boldsymbol{F}$. All this holds *if* the polynomial f is irreducible. (see e.g. [3 (VI, §5, Theorem 5.2, p. 286)], or [2 (Chaper 12)] for basic properties of finite fields).

In this paper we investigate whether a converse holds with the intention of producing a criterion to check irreducibility of f. This paper builds upon our previous paper [4] where we studied irreducibility of quadratic polynomials. Here we deal with polynomials of arbitrary degree. We note that our main result, Theorem 5, essentially proves Conjecture 3 from [4].

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2 Trace maps

It is well known that $x^{q^n} - x$ is the product of all monic irreducible polynomials of degree dividing n with coefficients in a finite field of cardinality q [3 (V, §6, exercise 22, p. 254)]. The following is a less known, but closely related fact.

Lemma 1. For any element a in $\boldsymbol{F}$, the polynomial $g_{a,m}(x) := \mathrm{Tr}_m(x) - a$ has no repeated roots, and its divisors are only the irreducible polynomials of degree dividing m.

Proof. Since the derivative of $g_{a,m}(x)$ is 1, it clearly has no repeated roots. Now we proceed to prove the rest of the lemma.

Suppose $h(x)$ is an irreducible polynomial of degree k. Then $\mathrm{Tr}_{k,h}(x)$ is a constant, in fact if

$$h(x) = a_k x^k + a_{k-1}x^{k-1} + \ldots + a_0,$$

then $\mathrm{Tr}_{k,h}(x) = -a_{k-1}/a_k$. Moreover, for any mutiple of k we have

$$\mathrm{Tr}_{kj,h}(x) = j\,\mathrm{Tr}_{k,h}(x) = -ja_{k-1}/a_k.$$

It follows that $h(x)$ divides $\mathrm{Tr}_{kj,h}(x) + ja_{k-1}/a_k$.

Consider the product P of all irreducible monic polynomials of degree dividing m. By the above reasoning

$$P | \prod_{a\in\boldsymbol{F}} (\mathrm{Tr}_m(x) - a)$$

On the other hand, the product P is known to equal to

$$P = x^{q^m} - x$$

Since each polynomial $\mathrm{Tr}_m(x) - a$ is monic of degree q^{m-1}, it follows that

$$q^m = \deg P = \deg \prod_{a\in\boldsymbol{F}} (\mathrm{Tr}_m(x) - a) = q^m$$

and thus

$$P = \prod_{a\in\boldsymbol{F}} (\mathrm{Tr}_m(x) - a)$$

and the lemma is proved. □

Corollary 2. If $\mathrm{Tr}_{n,f}(x)$ is a constant in $\boldsymbol{F}[x]/(f)$, then f has no repeated roots.

Proof. To say that $\mathrm{Tr}_{n,f}(x)$ is a constant is to say that f divides $\mathrm{Tr}_n(x) - a$ for some a in $\boldsymbol{F}$. But $\mathrm{Tr}_n(x) - a$ is squarefree by the above lemma.

3 Key lemma

Lemma 3. Let p be a prime and n an integer ≥ 1. Denote $M_{n,p}$ the set of positive integers k dividing n such that $(p, n/k) = 1$. If $f(x)$ is a monic irreducible polynomial of degree d in $M_{n,p}$ over a finite field $\boldsymbol{F}$ of characteristic p, then knowing $\mathrm{Tr}_{n,f}(x^i)$ for $i = 1, \ldots, 2n-1$ uniquely determines $f(x)$ among all irreducible monic polynomials of degree from $M_{n,p}$. If $\mathrm{char}(\boldsymbol{F}) > n$, then it is sufficient to know $\mathrm{Tr}_{n,f}(x^i)$ for $i = 1, \dots, n$.

Proof. For brevity, let us denote $S_i = \mathrm{Tr}_{d,f}(x^i)$ and write $f(x) = \sum_k a_k x^k$. Well known Newton identities state

$$\begin{aligned} a_{d-1} + a_d S_1 &= 0 \\ 2a_{d-2} + a_{d-1}S_1 + a_d S_2 &= 0 \\ &\vdots \\ da_0 + a_1 S_1 + \ldots + a_{d-1}S_{d-1} + a_d S_d &= 0 \end{aligned}$$

For $k = 1, 2, 3, \dots$

$$a_0 S_k + a_1 S_{k+1} + \ldots a_{d-1} S_{k+d-1} + a_d S_{k+d} = 0 \tag{1}$$

Consider now the matrix

$$A := \begin{pmatrix} S_d & S_{d-1} & \ldots & S_0 \\ S_{d+1} & S_d & \ldots & S_1 \\ \ldots & & & \\ S_{2d-1} & S_{2d-2} & \ldots & S_{d-1} \end{pmatrix}$$

We claim that A has rank d.

In fact the determinant of the minor gotten from A by leaving out the first column is nonzero. It is the discriminant of the trace form which is equal to [3 (VI, §Ex, exercise 32, pg. 325)] the discriminant of f, which is nonzero, because f is irreducible. Thus the right nullspace W of A is a rank 1 vector space over $\boldsymbol{F}$. An obvious element of W is the column vector $(a_d, a_{d-1}, \dots, a_0)$. It is the only element of W whose first coordiate equals to 1. It follows that the only element of W whose first coordinate is 0 is the zero vector.

Let us return to traces $\mathrm{Tr}_{n,f}(x^i) = (n/d)S_i$. Suppose there was another polynomial $f'(x) = \sum a'_k x^k$ of degree $d' \geq d$ such that

$$\mathrm{Tr}_{n,f'}(x^i) = \mathrm{Tr}_{n,f}(x^i)$$

Then we would have for $k \geq 0$

$$a'_0 S_k + a'_1 S_{k+1} + \ldots a'_{d'-1} S_{k+d'-1} + a'_{d'} S_{k+d'} = 0 \tag{2}$$

Let us write

$$f'(x) = f(x)g(x) + h(x), \quad \deg h(x) < d$$

where

$$g(x) = \sum_k b_k x^k$$
$$h(x) = \sum_k c_k x^k$$

We can substract a linear combination of shifted relations (1) from (2) to arrive at

$$c_0 S_k + c_1 S_{k+1} + \ldots c_{d-1} S_{k+d-1} = 0, \qquad k \geq 0$$

Vector $(0, b_{d-1}, \dots, b_0)$ belongs to W, thus by above analysis, it has to be the zero vector. It follows that $f'(x)$ is divisible by $f(x)$.

If $\mathrm{char}(\boldsymbol{F}) > n$, then one can use Newton formulae to recursively compute $a_{d-1}, \dots, a_0$. $\square$

Example 4. Note that over field of three elements $\boldsymbol{F} = \boldsymbol{Z}/3\boldsymbol{Z}$, the polynomials $f_1(x) = (x^4 + x^3 + 2)$ and $f_2(x) = (x^4 + x^3 + 2x + 1)$ have identical matrix of trace form. Thus knowing the trace quadratic form by itself does not determine the underlying monic irreducible polynomial uniquely. In particular it implies that knowing $\mathrm{Tr}_{n,f}(x^i)$ for $i \leq 2n - 2$ is not sufficient to determine a monic irreducible polynomial.

4 Main result

Now we can prove our main result.

Theorem 5. Polynomial $f(x)$ of degree n over a finite field $\boldsymbol{F}$ of cardinality q is irreducible, if and only if the image of the trace map $\mathrm{Tr}_{n,f}$ are precisely the constants.

Proof. If $f(x)$ is irreducible, then any element of $\boldsymbol{F}[x]/(f)$ can viewed as an element of the splitting field of f, and its trace is necessarily constant. Since the trace form is nondegenerate, the image of trace map cannot consists of only 0. This proves the "if" part.

Suppose now that $\mathrm{Tr}_{n,f}$ consists only of constants. By Corollary 2, $f(x)$ is a squarefree polynomial. Let $f = f_1 \cdots f_r$ be its factorization over $\boldsymbol{F}$. Then

$$\boldsymbol{F}[x]/(f) \approx \boldsymbol{F}[x]/(f_1) \oplus \cdots \oplus \boldsymbol{F}[x]/(f_r)$$

and $\mathrm{Tr}_{n,f} = \mathrm{Tr}_{n,f_1} \oplus \cdots \oplus \mathrm{Tr}_{n,f_r}$. The constants in $\boldsymbol{F}[x]/(f)$ are precisely elements $(a, a, \ldots, a)$ with a in $\boldsymbol{F}$, the so called Berlekamp subalgebra. From Lemma 1 it follows that $\deg f_i$ divides n for $i = 1, \ldots, n$. Since the image of $\mathrm{Tr}_{n,f}$ does not consists of only zero, the same is true for Tr_{n,f_i}. Therefore for all i, $n/\deg(f_i)$ are not divisible by p. But it follows from Lemma 3 that this implies that all f_i are equal. Since $f(x)$ is squarefree, it follows that $f(x)$ is irreducible.

5 Applications

In [1 (Section 5], an algorithm is presented that computes the trace map $\mathrm{Tr}_{n,f}$ using $\tilde{O}(n^{(\omega+1)/2} + n \log q)$ and tests irreducibility of degree n polynomial with the same complexity. Here ω denotes the complexity of the algorithm used for multiplying two $n \times n$ matrices (one can choose $\omega < 2.376$, while standard algorithm uses $\omega = 3$), and $g = \tilde{O}(h)$ means that $g = O(h(\log h)^k)$ for some constant k.

Our main result, Theorem 5, implies an algorithm to test irreducibility of $f(x)$. Namely, compute trace values $\mathrm{Tr}_{n,f}(x^i)$ for $i = 1, \ldots (n-1)$ and the polynomial is irreducible if and only if they are all constants. However, complexity of this algorithm is $\tilde{O}(n(n^{(\omega+1)/2} + n \log q))$ steps, which is worse than known algorithms, e.g. above, if n is large.

It would be nice if it were sufficient to check whether a single $\mathrm{Tr}_{n,f}(x^i)$ is a constant. This is not true however.

Example 6. We can construct an example from polynomials shown in Example 4. Consider $f(x) = (x^4 + x^3 + 2)(x^4 + x^3 + 2x + 1)$ over the field of cardinality three. Then $\mathrm{Tr}_{8,f}(x^i)$ is constant for $i = 1, \ldots, 6$. It is only $\mathrm{Tr}_{8,f}(x^7)$ that is not constant.

But there is one special case, when our algorithm is equally fast, because it is sufficient to test whether *single* $\mathrm{Tr}_{n,f}(x)$ is constant.

Lemma 7. If the degree of $f(x)$ is prime and not divisible by char($\boldsymbol{F}$), then $f(x)$ is irreducible if and only if $\mathrm{Tr}_{n,f}(x)$ is a constant in $\boldsymbol{F}[x]/(f)$.

Proof. If $f(x)$ is irreducible, then $\mathrm{Tr}_{n,f}(x)$ is clearly constant. In fact it is the minus of coefficient of x^{n-1} of $f(x)$.

Suppose now $\mathrm{Tr}_{n,f}(x)$ is a constant. From Lemma 1 it follows that either $f(x)$ is irreducible, or that $f(x)$ is the product of distinct linear factors $(x-a_1)\cdots(x-a_n)$. In the latter case the trace $\mathrm{Tr}_{n,f}(x)$ is then $n(a_1, \ldots, a_n)$ in

$$\boldsymbol{F}[x] \approx \boldsymbol{F}[x]/(x-a_1) \oplus \cdots \oplus \boldsymbol{F}[x]/(x-a_n)$$

which cannot be constant if p does not divide n. □

6 Errata

In our previous paper [4], in the proof of Proposition 1, we incorrectly stated that $f(x)$ is irreducible if and only if $P_q(x, y) = 0$. In fact $f(x)$ is irreducible if and only if $P_q(x, y) = -1$. The rest of proof stands as written. The author would like to thank Ms. Soontharanon from Thailand for pointing this out.

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