Editorial

Seminar on Fuzzy Relations

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The Seminar on Fuzzy Relations was organized by the Department of Mathematics and Descriptive Geometry of the Faculty of Civil Engineering of the Slovak University of Technology in Bratislava, Slovakia. The following collection of six papers is the result of work done at the seminar. The papers by J. Recasens and by D. Hliněná & P. Vojtáš, were presented at the seminar. The papers by V. Janiš & S. Montes Rodriguez, by Z. Havranová & M. Kalina and by J. Špirková are enlarged versions of papers presented at the seminar or papers which were written as the result of discussions after the seminar. The paper by S. Bodjanova was written afterwards, but its topic is related to papers presented there. Altogether, they present only a part of the broad variety of topics which are studied nowadays and are related to fuzzy relations.

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T_L AND S_L EVALUATORS: AGGREGATION AND MODIFICATION

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Abstract. T_L and S_L evaluators were introduced in [1] and their basic properties were studied. In this paper we discuss which aggregation of T_L (S_L) evaluators yields a T_L (S_L) evaluator and which modification of an evaluator will change it into a T_L (S_L) evaluator. Duality of evaluators is also studied.

1. Evaluators and aggregation operators

We will consider a complete lattice $(L, \leq, \perp, \top)$ with the least and the greatest elements $\perp$ and $\top$, respectively. Normalized scalar evaluators of elements from L were characterized in [3] by a function $\varphi : L \to [0, 1]$ satisfying properties

(i) $\varphi(\perp) = 0$, $\varphi(\top) = 1$,
(ii) for all $a, b \in L$, if $a \leq b$ then $\varphi(a) \leq \varphi(b)$.

Evaluator φ is called existentional if for $a \in L$,

$$\varphi(a) = 0 \Rightarrow a = \perp. \tag{1}$$

Evaluator φ is called universal if for $a \in L$,

$$\varphi(a) = 1 \Rightarrow a = \top. \tag{2}$$

In applications, different properties of the same object are evaluated by different evaluators. For comparison of two or more objects, an aggregation of evaluations is needed. An aggregation operator [2, 6, 9] is a function $\boldsymbol{A}\colon \bigcup_{n \in N} [0, 1]^n \to [0, 1]$ such that

(i) $\boldsymbol{A}\ (x_1, \dots, x_n) \leq \boldsymbol{A}\ (y_1, \dots, y_n)$ whenever
$$x_i \leq y_i \text{ for all } i \in \{1, \dots, n\},$$
(ii) $\boldsymbol{A}\ (x) = x$ for all $x \in [0, 1]$,
(iii) $\boldsymbol{A}\ (0, \dots, 0) = 0$ and $\boldsymbol{A}\ (1, \dots, 1) = 1$.

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Each aggregation operator $\boldsymbol{A}$ can be canonically represented by a family $(A_n)_{n\in N}$ of n-ary operations, e.g., functions $A_n : [0,1]^n \to [0,1]$ given by

$$A_n(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_n). \tag{3}$$

Function A_n is an evaluator on the lattice $([0,1]^n, \leq, \perp, \top)$, where $\perp = (0,\ldots,0)$ and $\top = (1,\ldots,1)$. If $\mathbf{A}(x_1,\ldots,x_n) = 0$ implies that $x_i = 0$ for $i = 1,\ldots,n$, we say that aggregation operator $\mathbf{A}$ does not have zero divisors. In this case, function A_n is an existentional evaluator and $\mathbf{A}$ is an existentional aggregator. If $\mathbf{A}(x_1,\ldots,x_n) = 1$ implies that $x_i = 1$ for $i = 1,\ldots,n$, function A_n is a universal evaluator and $\mathbf{A}$ is a universal aggregator.

Proposition 1. *Let $\Phi = \{\varphi_1,\ldots,\varphi_n\}$ be a set of evaluators on a complete lattice $(L,\leq,\perp,\top)$ and let $\mathbf{A}$ be an aggregation operator. Then function $A_\Phi : L \to [0,1]$ defined for all $a \in L$ by*

$$A_\Phi(a) = \mathbf{A}(\varphi_1(a),\ldots,\varphi_n(a)) \tag{4}$$

is an evaluator on L.

Obviously, aggregation of existentional evaluators by an existentional aggregator yields an existentional evaluator and aggregation of universal evaluators by a universal evaluator yields a universal evaluator.

Frequently used aggregation operators are averaging operators. We will consider only arithmetic mean

$$\mathbf{M}(x_1,\ldots,x_n) = (x_1 + \ldots + x_n)/n.$$

Aggregator $\mathbf{M}$ is existentional as well as universal.

Arithmetic mean belongs to the family of ordered weighted averaging operators (OWA operators) introduced in [10].

$$\mathbf{OWA}(x_1,\ldots,x_n) = \sum_{j=1}^{n} w_j y_j,$$

where y_j is the j^{th} largest value of x_i, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. More about OWA operators can be found in [4, 9].

A special class of aggregation operators is the class of triangular norms (t-norms) and triangular conorms (t-conorms). For more details refer to [7, 8, 9]. The four basic t-norms are:
the minimum $T_M(x,y) = \min(x,y)$,
the product $T_P(x,y) = x.y$,
the Łukasiewicz t-norm $T_L(x,y) = \max(x+y-1, 0)$,
and the drastic product

$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

The four basic t-conorms are:
the maximum $S_M(x,y) = \max(x,y)$,
the probabilistic sum $S_P(x,y) = x + y - x.y$,
the Łukasiewicz t-conorm $S_L(x,y) = \min(x+y,1)$,
and the drastic sum

$$S_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in]0,1]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

It can be shown that T_M, T_P, T_L and T_D are universal aggregators while S_M, S_P, S_L and S_D are existentional aggregators.

The relationship between evaluators and Łukasiewicz t-norm and t-conorm was studied in [1], where the notions of T_L and S_L evaluators were introduced.

Definition 1. *Consider a complete lattice $(L, \leq, \perp, \top)$. A normalized evaluator φ on L is called a T_L evaluator if and only if for all $a, b \in L$*

$$T_L(\varphi(a), \varphi(b)) \leq \varphi(a \wedge b), \tag{5}$$

and it is called an S_L evaluator if and only if

$$S_L(\varphi(a), \varphi(b)) \geq \varphi(a \vee b). \tag{6}$$

Proposition 2. *Consider a complete lattice $(L, \leq, \perp, \top)$.
A normalized evaluator φ is a T_L evaluator if and only if for all $a, b \in L$*

$$\varphi(a \wedge b) \geq \varphi(a) + \varphi(b) - 1, \tag{7}$$

and it is an S_L evaluator if and only if

$$\varphi(a \vee b) \leq \varphi(a) + \varphi(b). \tag{8}$$

Example 1. Let $\mathcal{F}(X)$ denote the family of all fuzzy sets on a universal finite set X. For $A, B \in \mathcal{F}(X)$, $A \leq B$ means that $A(x) \leq B(x)$ for all $x \in X$, $(A \vee B)(x) = \max\{A(x), B(x)\}$ and $(A \wedge B)(x) = \min\{A(x), B(x)\}$. Obviously, $(\mathcal{F}(X), \leq)$ is a complete lattice with $\perp = \emptyset$ and $\top = X$. Some normalized scalar evaluators on $\mathcal{F}(X)$ are height ht, plinth pl and relative cardinality RC defined for $A \in \mathcal{F}(X)$ by (9), (10) and (11), respectively.

$$ht(A) = \max_{x \in X} A(x), \tag{9}$$

$$pl(A) = \min_{x \in X} A(x), \tag{10}$$

$$RC(A) = \frac{|A|}{|X|}, \tag{11}$$

where $|.|$ denotes cardinality. Because X is a finite set,

$$|A| = \sum_{x \in X} A(x). \tag{12}$$

Evaluator ht is an S_L evaluator, pl is a T_L evaluator and RC is both S_L and T_L evaluator.

Example 2. All fuzzy measures defined on the power set of a nonempty crisp set X are evaluators on the complete lattice $(2^X, \subseteq, \emptyset, X)$. Well known examples of fuzzy measures are probability measure Pr, possibility measure Pos and necessity measure Nec. Then Pos is anS_L evaluator, Nec is a T_L evaluator, and Pr is an S_L as well as a T_L evaluator.

2. Aggregation of T_L and S_L evaluators

We already know that aggregation of evaluators yields an evaluator (Proposition 1). Now we will focus on aggregation of T_L and S_L evaluators. We will consider as possible aggregation operators some OWA operators (arithmetic mean), some t-norms (T_M and T_L) and some t-conorms (S_M and S_L). We would like to know what aggregation of T_L (S_L) evaluators yields a T_L (S_L) evaluator.

Proposition 3. *Arithmetic mean of T_L (S_L) evaluators is a T_L (S_L) evaluator.*

Proof: Let $\Phi = \{\varphi_1, \dots, \varphi_n\}$ be a set of T_L evaluators on a complete lattice $(L, \leq, \perp, \top)$. Then for all $a, b \in L$ we have

$$\varphi_i(a \wedge b) \geq \varphi_i(a) + \varphi_i(b) - 1,$$

for all $i \in \{1, \dots, n\}$. Therefore

$$\sum_{i=1}^{n} \varphi_i(a \wedge b) \geq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b) - n.$$

Then

$$\begin{aligned} M_\Phi(a \wedge b) &= \mathbf{M}(\varphi_1(a \wedge b), \dots, \varphi_n(a \wedge b)) \\ &= \left(\sum_{i=1}^{n} \varphi_i(a \wedge b) \right) /n \\ &\geq \left(\sum_{i=1}^{n} \varphi_i(a) \right) /n + \left(\sum_{i=1}^{n} \varphi_i(b) \right) /n - 1 \\ &= M_\Phi(a) + M_\Phi(b) - 1, \end{aligned}$$

and therefore M_Φ is a T_L evaluator.
Analogously we can prove that arithmetic mean of S_L evaluators yields an S_L evaluator.

Corollary 1. *Let $\Phi = \{\varphi_1, \dots, \varphi_n\}$ be a set of $T_L(S_L)$ evaluators on a complete lattice $(L, \leq, \perp, \top)$. Let for all $a \in L$, $\varphi_1(a) \leq \dots \leq \varphi_n(a)$. Then $OWA_\Phi = \mathbf{OWA}(\varphi_1, \dots, \varphi_n)$ is a T_L (S_L) evaluator on L.*

Proposition 4. *Aggregation of T_L evaluators by t-norm T_M yields a T_L evaluator. Aggregation of S_L evaluators by t-conorm S_M yields an S_L evaluator.*

Proof. Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of T_L evaluators on a complete lattice $(L, \leq, \perp, \top)$. Then for all $a, b \in L$

$$\varphi_i(a \wedge b) \geq \varphi_i(a) + \varphi_i(b) - 1,$$

for all $i \in \{1, \ldots, n\}$. Let $\min(\varphi_1(a \wedge b), \ldots, \varphi_n(a \wedge b)) = \varphi_r(a \wedge b),\ r \in \{1, \ldots, n\}$. Then

$$\begin{aligned} T_{M_\Phi}(a \wedge b) &= \varphi_r(a \wedge b) \geq \varphi_r(a) + \varphi_r(b) - 1 \\ &\geq \min(\varphi_1(a), \ldots, \varphi_n(a)) + \min(\varphi_1(b, \ldots, \varphi_n(b)) - 1 \\ &= T_{M_\Phi}(a) + T_{M_\Phi}(b) - 1, \end{aligned}$$

and therefore T_{M_Φ} is a T_L evaluator.
Analogously we can prove that aggregation of S_L evaluators by t-conorm S_M yields an S_L evaluator. □

Proposition 5. *Aggregation of T_L evaluators by Łukasiewicz t-norm yields a T_L evaluator. Aggregation of S_L evaluators by Łukasiewicz t-conorm yields an S_L evaluator.*

Proof. Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of T_L evaluators on a complete lattice $(L, \leq, \perp, \top)$. Then for all $a, b \in L$ we have

$$\varphi_i(a \wedge b) \geq \varphi_i(a) + \varphi_i(b) - 1,$$

for all $i \in \{1, \ldots, n\}$. For $a, b \in L$ we obtain

$$\begin{aligned} T_{L_\Phi}(a \wedge b) &= T_L(\varphi_1(a \wedge b), \ldots, \varphi_n(a \wedge b)) \\ &= \max\left(\sum_{i=1}^{n} \varphi_i(a \wedge b) - (n-1), 0\right). \end{aligned}$$

Then, because

$$\sum_{i=1}^{n} \varphi_i(a \wedge b) \geq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b) - n,$$

we have that

$$
\begin{aligned}
T_{L_\Phi}(a \wedge b) &\geq \sum_{i=1}^{n} \varphi_i(a \wedge b) - (n-1) \geq \\
&\geq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b) - n - (n-1) = \\
&= \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b) - n - (n-1) + 1 - 1 = \\
&= \sum_{i=1}^{n} \varphi_i(a) - (n-1) + \sum_{i=1}^{n} \varphi_i(b) - (n-1) - 1.
\end{aligned}
$$

Because $\max\left(\sum_{i=1}^{n} \varphi_i(a \wedge b) - (n-1), 0\right) \geq 0$, we can rewrite the inequality above as follows:

$$
\begin{aligned}
T_{L_\Phi}(a \wedge b) = \max &\left(\sum_{i=1}^{n} \varphi_i(a \wedge b) - (n-1), 0\right) \geq \\
&\geq \max\left(\sum_{i=1}^{n} \varphi_i(a) - (n-1), 0\right) + \\
&+ \max\left(\sum_{i=1}^{n} \varphi_i(b) - (n-1), 0\right) - 1 = \\
&= T_{L_\Phi}(a) + T_{L_\Phi}(b) - 1,
\end{aligned}
$$

which shows that T_{L_Φ} is a T_L evaluator.

Now we will prove that aggregation of S_L evaluators by Łukasiewicz t-conorm results in an S_L evaluator.

Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of S_L evaluators on a complete lattice $(L, \leq, \perp, \top)$. Then for all $a, b \in L$ we have

$$\varphi_i(a \vee b) \leq \varphi_i(a) + \varphi_i(b),$$

for all $i \in \{1, \ldots, n\}$. For $a, b \in L$ we obtain

$$
\begin{aligned}
S_{L_\Phi}(a \vee b) &= S_L(\varphi_1(a \vee b), \ldots, \varphi_n(a \vee b)) \\
&= \min\left(\sum_{i=1}^{n} \varphi_i(a \vee b), 1\right).
\end{aligned}
$$

Then, because

$$\sum_{i=1}^{n} \varphi_i(a \vee b) \leq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b),$$

we have that

$S_{L_\Phi}(a \vee b) \leq \sum_{i=1}^{n} \varphi_i(a \vee b) \leq \sum_{i=1}^{n} \varphi_i(a) + \sum_{i=1}^{n} \varphi_i(b)$.

Because $\min\left(\sum_{i=1}^{n} \varphi_i(a \vee b), 1\right) \leq 1$, we can rewrite the inequality above as follows:

$$\begin{aligned} S_{L_\Phi}(a \vee b) &= \min\left(\sum_{i=1}^{n} \varphi_i(a \vee b), 1\right) \leq \\ &\leq \min\left(\sum_{i=1}^{n} \varphi_i(a), 1\right) + \min\left(\sum_{i=1}^{n} \varphi_i(b), 1\right) = \\ &= S_{L_\Phi}(a) + S_{L_\Phi}(b), \end{aligned}$$

which shows that S_{L_Φ} is an S_L evaluator.

In the following example we will show that aggregation of T_L evaluators by Łukasiewicz t-conorm does not need to result in a T_L evaluator. □

Example 3. Let $(\mathcal{F}(X), \leq)$ be the lattice from Example 1, where $X = \{x_1, x_2, x_3\}$. Consider $A, B \in \mathcal{F}(X)$ defined by membership functions $A = 0.6/x_1 + 1/x_2 + 0.6/x_3$ and $B = 0.9/x_1 + 0.2/x_2 + 0.1/x_3$, respectively. We will use the following set of T_L evaluators: $\Phi = \{pl, RC\}$, where pl is plinth and RC is relative cardinality. Then $pl(A) = 0.6$, $RC(A) = 0.73$ and $pl(B) = 0.1$, $RC(B) = 0.4$. For $A \wedge B = 0.6/x_1 + 0.2/x_2 + 0.1/x_3$ we have
$pl(A \wedge B) = 0.1$, $RC(A \wedge B) = 0.3$). Then
$S_{L_\Phi}(A \wedge B) = S_L(pl(A \wedge B), RC(A \wedge B)) = S_L(0.1, 0.3) = \min(0.1 + 0.3, 1) = 0.4$.
On the other hand,
$S_{L_\Phi}(A) = S_L(0.6, 0.73) = \min(0.6 + 0.73, 1) = 1$ and
$S_{L_\Phi}(B) = S_L(0.1, 0.4) = \min(0.1 + 0.4, 1) = 0.5$.
Obviously,
$0.4 = S_{L_\Phi}(A \wedge B) < S_{L_\Phi}(A) + S_{L_\Phi}(B) - 1 = 0.5$,
and therefore S_{L_Φ} is not a T_L evaluator.

Analogously, aggregation of S_L evaluators by Łukasiewicz t-norm does not need to be an S_L evaluator.

Example 4. Let $(\mathcal{F}(X), \leq)$ be the lattice from Example 1, where $X = \{x_1, x_2, x_3\}$. Consider $A, B \in \mathcal{F}(X)$ defined by membership functions $A = 0.4/x_1 + 0/x_2 + 0.9/x_3$ and $B = 0.1/x_1 + 0.8/x_2 + 0.2/x_3$, respectively. We will use the following set of S_L evaluators: $\Phi = \{ht, RC\}$, where ht is height and RC is relative cardinality. Then $ht(A) = 0.9$, $RC(A) = 0.433$ and $ht(B) = 0.8$, $RC(B) = 0.367$. For $A \vee B = 0.4/x_1 + 0.8/x_2 + 0.9/x_3$ we have
$ht(A \vee B) = 0.9$, $RC(A \vee B) = 0.7$. Then
$T_{L_\Phi}(A \vee B) = T_L(ht(A \vee B), RC(A \vee B)) = T_L(0.9, 0.7) = \max(0.9 + 0.7 - 1, 0) = 0.6$.
On the other hand,
$T_{L_\Phi}(A) = T_L(0.9, 0.433) = \max(0.9 + 0.433 - 1, 0) = 0.333$ and

$T_{L_\Phi}(B) = T_L(0.8, 0.367) = \max(0.8 + 0.367 - 1, 0) = 0.167$. Obviously,
$0.6 = T_{L_\Phi}(A \vee B) > T_{L_\Phi}(A) + T_{L_\Phi}(B) = 0.5$,
and therefore T_{L_Φ} is not an S_L evaluator.

3. Composition of evaluators

Properties of evaluators can be changed by appropriate modifications of evaluators. We will explore how evaluators on a complete lattice $(L, \leq, \perp, \top)$ can be transformed (modified) into T_L and S_L evaluators. In this section we will discuss modification of evaluators on L by a composition with evaluators on the lattice $([0,1], \leq, 0, 1)$.

Proposition 6. *Consider a lattice $(L, \leq, \perp, \top)$. Function $\varphi : L \to [0,1]$ is an evaluator on L if and only if there exists an evaluator ψ on L and an evaluator f on $([0,1], \leq, 0, 1)$ such that*

$$\varphi = f \circ \psi. \tag{13}$$

If ψ and f are universal (existentional) evaluators then φ is a universal (existentional) evaluator.

Proof. If φ is an evaluator on L, we can choose $\psi = \varphi$ and $f = id$ (identity). Then for all $a \in L$,

$$\varphi(a) = (id \circ \psi)(a) = id(\psi(a)) = id(\varphi(a)) = \varphi(a).$$

Now we will show that $\varphi = f \circ \psi$ is an evaluator on L.
(i) $\varphi(\perp) = (f \circ \psi)(\perp) = f(\psi(\perp)) = f(0) = 0$,
and $\varphi(\top) = (f \circ \psi)(\top) = f(\psi(\top)) = f(1) = 1$.
(ii) For all $a, b \in L$, if $a \leq b$ then $\varphi(a) = (f \circ \psi)(a)$
$= f(\psi(a)) \leq f(\psi(b)) = (f \circ \psi)(b) = \varphi(b)$.
Therefore φ is an evaluator on L.

If f and ψ are universal, then for all $a \in L$,
$\varphi(a) = f(\psi(a)) = 0 \Rightarrow \psi(a) = 0 \Rightarrow a = \perp$, and therefore φ is universal.
If f and ψ are existentional, then for all $a \in L$,
$\varphi(a) = f(\psi(a)) = 1 \Rightarrow \psi(a) = 1 \Rightarrow a = \top$, and therefore φ is existentional.

For an evaluator φ on L we want to find an evaluator f on $[0,1]$ such that $f \circ \varphi$ is an T_L (S_L) evaluator. □

Proposition 7. *Let f be an evaluator on $([0,1], \leq, 0, 1)$ such that for all $x \in [0,1[$, $f(x) \in [0, 0.5]$. Let φ be a universal evaluator on $(L, \leq, \perp, \top)$. Then $\varphi_f = f \circ \varphi$ is a universal T_L evaluator on L.*

Proof. Because f is a universal evaluator, from Proposition 6 it follows that φ_f is a universal evaluator on L.

Now we will show that for all $a, b \in L$,

$$\varphi_f(a \wedge b) \geq \varphi_f(a) + \varphi_f(b) - 1. \tag{14}$$

If $a = \top$ or $b = \top$ then $\varphi_f(a \wedge b) = \min\{\varphi_f(a), \varphi_f(b)\}$, and $\varphi_f(a) + \varphi_f(b) = \min\{\varphi_f(a), \varphi_f(b)\} + 1$. Therefore $\varphi_f(a) + \varphi_f(b) - 1 = \min\{\varphi_f(a), \varphi_f(b)\} = \varphi_f(a \wedge b)$ and inequality (14) holds.

Let $a \neq \top$ and $b \neq \top$. Because φ is universal, $\varphi(a) < 1$, $\varphi(b) < 1$ and $f(\varphi(a)) \leq 0.5$, $f(\varphi(b)) \leq 0.5$. Hence $\varphi_f(a) + \varphi_f(b) = f(\varphi(a)) + f(\varphi(b)) \leq 1$. Therefore $\varphi_f(a) + \varphi_f(b) - 1 \leq 0 \leq \varphi_f(a \wedge b)$, which proves inequality (14). □

Some examples of a function f satisfying properties from Proposition 7 are:

1) For $x \in [0, 1]$ and $\alpha \in [0, 0.5]$,

$$f_1(x) = \begin{cases} 1 & \text{if } x = 1, \\ \alpha & \text{if } \alpha \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Evaluator φ on L modified by composition $f_1 \circ \varphi$ is the alpha-lower levelization of φ discussed in [1]. For $\alpha = 0$, composition $f_1 \circ \varphi$ is the trivial universal evaluator

$$\varphi_U(a) = \begin{cases} 1 & \text{if } a = \top, \\ 0 & \text{otherwise.} \end{cases}$$

2) For $x \in [0, 1]$ and $\alpha \in [0, 0.5]$,

$$f_2(x) = \begin{cases} 1 & \text{if } x = 1, \\ \min(\alpha, x) & \text{otherwise.} \end{cases}$$

For $\alpha = 0, f_2 \circ \varphi = \varphi_U$.

3) For $x \in [0, 1]$,

$$f_3(x) = \begin{cases} 1 & \text{if } x = 1, \\ 1 - 0.5^x & \text{otherwise.} \end{cases}$$

Proposition 8. *Let g be an evaluator on $([0, 1], \leq, 0, 1)$ such that for all $x \in]0, 1]$, $f(x) \in [0.5, 1]$. Let φ be an existentional evaluator on $(L, \leq, \bot, \top)$. Then $\varphi_g = g \circ \varphi$ is an existentional S_L evaluator on L.*

Proof. Because g is an existentional evaluator, from Proposition 6 it follows that φ_g is an existentional evaluator on L. Now we will show that for all $a, b \in L$,

$$\varphi_g(a \vee b) \leq \varphi_g(a) + \varphi_g(b). \tag{15}$$

If $a = \bot$ or $b = \bot$ then $\varphi_g(a \vee b) = \max\{\varphi_g(a), \varphi_g(b)\}$, and $\varphi_g(a) + \varphi_g(b) = \max\{\varphi_g(a), \varphi_g(b)\} + 0$. Therefore $\varphi_g(a) + \varphi_g(b) = \max\{\varphi_g(a), \varphi_g(b)\} = \varphi_g(a \vee b)$ and inequality (15) holds.

Let $a \neq \bot$ and $b \neq \bot$. Because φ is existentional, $\varphi(a) > 0$, $\varphi(b) > 0$ and $g(\varphi(a)) \geq 0.5$, $g(\varphi(b)) \geq 0.5$ Hence $\varphi_g(a) + \varphi_g(b) = g(\varphi(a)) + g(\varphi(b)) \geq 1$. Therefore $\varphi_g(a) + \varphi_g(b) \geq \varphi_g(a \vee b)$, which proves inequality (15). □

Some examples of a function g satisfying properties from Proposition 8 are:

1) For $x \in [0,1]$ and $\alpha \in [0.5,1]$,

$$g_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ \alpha & \text{if } 0 < x \le \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

Evaluator φ on L modified by composition $g_1 \circ \varphi$ is the alpha-upper levelization of φ discussed in [1]. For $\alpha = 1$, composition $g_1 \circ \varphi$ is the trivial existentional evaluator

$$\varphi_E(a) = \begin{cases} 0 & \text{if } \ a = \perp, \\ 1 & \text{otherwise.} \end{cases}$$

2) For $x \in [0,1]$ and $\alpha \in [0.5,1]$,

$$g_2(x) = \begin{cases} 0 & \text{if } x = 0, \\ \max(\alpha, x) & \text{otherwise.} \end{cases}$$

For $\alpha = 1, g_2 \circ \varphi = \varphi_E$.

3) For $x \in [0,1]$,

$$g_3(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.5^{1-x} & \text{otherwise.} \end{cases}$$

One can recognize duality between Proposition 7 and Proposition 8 and also between functions $(f_i, g_i), i = 1,2,3$. We will discuss duality of evaluators in the next section.

4. Duality of evaluators

We will consider a complemented lattice $(L, \le, \ ', \perp, \top)$, where for each $a \in L$, there is complement $a' \in L$ such that $a \wedge a' = \perp$ and $a \vee a' = \top$. For $a, b \in L$,

$$a \wedge b = (a' \vee b')'. \tag{16}$$

The proof of the following proposition is trivial.

Proposition 9. *Let φ be an existentional (universal) evaluator on a complemented lattice $(L, \le, \ ', \perp, \top)$. Then function $\bar{\varphi} : L \to [0,1]$ defined for all $a \in L$ by*

$$\bar{\varphi}(a) = 1 - \varphi(a') \tag{17}$$

is a universal (existentional) evaluator on L.

Evaluator $\bar{\varphi}$ given by (17) will be called the dual evaluator to φ.

Assume the lattice of fuzzy sets from Example 1. The standard complement of a fuzzy set A is fuzzy set A', where $A'(x) = 1 - A(x)$ for all $x \in X$. Evaluators height (ht) and plinth (pt) of fuzzy sets are dual to each other. Evaluator relative cardinality (RC) is dual to itself.

Assume the lattice of crisp sets from Example 2. For each crisp set $A \in 2^X$, complement is defined by $A' = X - A$. Fuzzy measures necessity (Nec) and possibility (Pos) are dual evaluators on the lattice $(2^X, \subseteq, {}', \emptyset, X)$. Fuzzy measure probability (Pr) is dual to itself.

Proposition 10. *Let $(L, \leq, {}', \perp, \top)$ be a complemented lattice and let φ be a T_L and also S_L evaluator on L. Then for all $a \in L$,*

$$\varphi(a) + \varphi(a') = 1. \tag{18}$$

Proof: If φ is a T_L evaluator on L, then $0 = \varphi(a \wedge a') \geq \varphi(a) + \varphi(a') - 1$, and therefore

$$\varphi(a) + \varphi(a') \leq 1. \tag{19}$$

If φ is an S_L evaluator on L, then $1 = \varphi(a \vee a') \leq \varphi(a) + \varphi(a')$, and therefore

$$\varphi(a) + \varphi(a') \geq 1. \tag{20}$$

From (19) and (20) it follows that $\varphi(a) + \varphi(a') = 1$.

Note that one can find an evaluator φ on L such that for all $a \in L$, $\varphi(a) + \varphi(a') = 1$, but φ is neither T_L nor S_L evaluator.

Example 5 Let $X = \{x_1, x_2, x_3, x_4\}$. Consider lattice $(2^X, \subseteq, {}', \emptyset, X)$. Let fuzzy measure $m : 2^X \to [0,1]$ be given as follows: $m(\emptyset) = 0$, $m(X) = 1$, $m(x_1) = m(x_2) = m(x_3) = 0.2$, $m(x_4) = 0.5$, $m(x_1, x_2) = m(x_2, x_3) = m(x_1, x_3) = 0.2$, $m(x_3, x_4) = m(x_1, x_4) = m(x_2, x_4) = 0.8$, $m(x_2, x_3, x_4) = m(x_1, x_3, x_4) = m(x_2, x_1, x_4) = 0.8$, $m(x_1, x_2, x_3) = 0.5$. Function m is an evaluator on 2^X such that for each $A \in 2^X$, $m(A) + m(A') = 1$.
However, $m((x_1, x_2) \cup (x_2, x_3)) = m(x_1, x_2, x_3) = 0.5$ is greater than $m(x_1, x_2) + m(x_2, x_3) = 0.2 + 0.2 = 0.4$, and therefore m is not an S_L evaluator. We also obtain that $m((x_3, x_4) \cap (x_1, x_4)) = m(x_1) = 0.5$ is less than $\max\{m(x_3, x_4) + m(x_1, x_4) - 1, 0\} = \max\{0.8 + 0.8 - 1, 0\} = 0.6$, and therefore m is not a T_L evaluator.

Proposition 11. *Let φ be a T_L (S_L) evaluator on a complemented lattice $(L, \leq, {}', \perp, \top)$. Then dual evaluator of φ is an S_L (T_L) evaluator on L.*

Proof. Let φ be a T_L evaluator. Then for all $a, b \in L$, $\varphi(a \wedge b) \geq \varphi(a) + \varphi(b) - 1$. Because of (16), $(a \vee b)' = a' \wedge b'$ and we obtain:
$\bar{\varphi}(a \vee b) = 1 - \varphi((a \vee b)') = 1 - \varphi(a' \wedge b') \leq 1 - (\varphi(a') + \varphi(b') - 1) = 1 - \varphi(a') + 1 - \varphi(b') = \bar{\varphi}(a) + \bar{\varphi}(b)$, and therefore $\bar{\varphi}$ is an S_L evaluator.
Let φ be an S_L evaluator. Then for all $a, b \in L$, $\varphi(a \vee b) \leq \varphi(a) + \varphi(b)$. Because of (16), $(a \wedge b)' = a' \vee b'$ and we obtain:
$\bar{\varphi}(a \wedge b) = 1 - \varphi((a \wedge b)') = 1 - \varphi(a' \vee b') \geq 1 - (\varphi(a') + \varphi(b')) = 1 - \varphi(a') + 1 - \varphi(b') - 1 = \bar{\varphi}(a) + \bar{\varphi}(b) - 1$, and therefore $\bar{\varphi}$ is a T_L evaluator. □

Corollary 2. *Assume evaluator* ψ *on* $(L, \leq, ', \perp, \top)$ *and evaluator* f *on*$([0,1], \leq, ', 0, 1)$. *Let* $f \circ \psi$ *be a* T_L (S_L) *evaluator on* L. *Then* $\bar{f} \circ \bar{\psi}$ *is an* S_L (T_L) *evaluator on* L.

Proof: It is enough to show that $\bar{f} \circ \bar{\psi}$ is dual evaluator of $f \circ \psi$. For all $a \in L$ we obtain:
$\overline{f \circ \psi}(a) = 1-(f \circ \psi)(a') = 1-f(\psi(a')) = \bar{f}([(\psi(a')]') = \bar{f}(1-\psi(a')) = \bar{f}(\bar{\psi}(a)) = \bar{f} \circ \bar{\psi}(a)$.

5. Conclusion

We have shown that aggregation of evaluators by an aggregation operator yields an evaluator. Aggregation of T_L evaluators by arithmetic mean, t-norm T_M or by Łukasiewicz t-norm results in a T_L evaluator. Aggregation of S_L evaluators by arithmetic mean, t-conorm S_M or by Łukasiewicz t-conorm results in an S_L evaluator. A normalized evaluator on a complete lattice can be transformed into a T_L or S_L evaluator by composition with an appropriate evaluator on $[0,1]$. Dual evaluator of a T_L (S_L) evaluator is an S_L (T_L) evaluator. An evaluator which is T_L and also S_L is dual of itself. However, not every evaluator which is dual of itself is a T_L and S_L evaluator. Successful applications of T_L (S_L) evaluators were already reported in [5]. More applications will be presented in our future paper.

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T-FILTERS AND T-IDEALS

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Abstract. This paper is devoted to generalizing of fuzzy filters and fuzzy ideals and to studying the relationship between maximal T-filters (i.e. maximal elements of the lattice of all T-filters) and T-ultrafilters (which are so-called T-and S-evaluators).

1. Introduction and basic definitions

Filters are broadly used in topology and in set-theoretical constructions (ultraproducts). Since a couple of years the notion of filters has been fuzzified (as stated below) to generalized filters and to Łukasiewicz filters. The main importance of Łukasiewicz filters lies in preserving of T_L-transitivity when constructing a fuzzy relation by aggregating some partial T_L-transitive fuzzy relations. More the reader can find in [9].

For the purposes of this paper we will use the following definition of a (proper) filter on a non-empty set X:

Definition 1. *Let $X \neq \emptyset$. A function $F : 2^X \to \{0, 1\}$ is said to be a filter on X iff the following is satisfied:*

- $F(X) = 1,\ F(\emptyset) = 0$
- *for $A, B \subseteq X$ if $A \subset B$, then $F(A) \le F(B)$*
- *for $A, B \subseteq X$ we have $F(A \cap B) \ge F(A) \cdot F(B)$.*

As a complementary notion to filters we have a (proper) ideal on the set $X \neq \emptyset$ (more precisely, on the Boolean lattice of subsets of X, equipped with union and intersection):

Definition 2. *Let $X \neq \emptyset$. A function $I : 2^X \to \{0, 1\}$ is said to be an ideal on X iff the following is satisfied:*

- $I(X) = 0,\ I(\emptyset) = 1$
- *for $A, B \subseteq X$ if $A \subset B$, then $I(A) \ge I(B)$*
- *for $A, B \subseteq X$ we have $I(A \cup B) \ge I(A) \cdot I(B)$.*

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The relationship between a filter on X and an ideal on X gives the following lemma:

Lemma 1. *Let $X \neq \emptyset$. $F : 2^X \to \{0, 1\}$ is a filter on X if and only if $I : 2^X \to \{0, 1\}$, defined by $I(A) = F(A^c)$ for each $A \in 2^X$, is an ideal on X, where $A^c = X \setminus A$.*

An important notion is that of an ultrafilter on X:

Definition 3. *Let $X \neq \emptyset$. A function $U : 2^X \to \{0, 1\}$ is said to be an ultrafilter on X iff U is a filter on X and moreover if for each $A \subseteq X$ either $U(A) = 1$ or $U(A^c) = 1$.*

The following assertions may be used as alternative definitions of ultrafilters on X:

Proposition 1. *Let us denote $\Psi(X)$ the system of all filters on X. Then $(\Psi(X), \wedge, \vee)$ is a lattice with*

$$F_0(A) = \begin{cases} 1, & \text{if } A = X \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

as its bottom element. Ultrafilters on X are its maximal elements.

Proposition 2. *Let $X \neq \emptyset$ and $F : 2^X \to \{0, 1\}$ be a filter on X. Then F is an ultrafilter on X if and only if $I = 1 - F$ is an ideal on X.*

As Proposition 2 states, we have two possibilities how to define ideals via an ultrafilter U on X: $I_1(A) = U(A^c)$, $I_2(A) = 1 - U(A)$. An easy consideration gives $I_1 = I_2$.

To avoid confusion, filters, ultrafilters and ideals on X will be called crisp filters on X, crisp ultrafilters on X and crisp ideals on X, respectively.

Filters were already fuzzified to so-called generalized filters in [2, 3, 5, 6] in the following way:

Definition 4. *Let $X \neq \emptyset$. A function $G : 2^X \to [0, 1]$ is said to be a generalized filter on X iff the following is satisfied:*

- $G(X) = 1,\ G(\emptyset) = 0$
- *for $A, B \subseteq X$ if $A \subset B$, then $G(A) \leq G(B)$*
- *for $A, B \subseteq X$ we have $G(A \cap B) \geq \min\{G(A), G(B)\}$.*

Before proceeding, we give the definition of a t-norm, which will be a very important notion for us (for details on t-norms an their duals, t-conorms, see [12]):

Definition 5. *$T : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm iff the following is satisfied:*

- *for each $y \in [0,1]$ $T(1, y) = y$*

- *for all* $x, y_1, y_2 \in [0,1]$ *if* $y_1 \le y_2$ *then* $T(x, y_1) \le T(x, y_2)$
- *for all* $x, y \in [0,1]$ $T(x,y) = T(y,x)$
- *for all* $x, y, z \in [0,1]$ $T(x, T(y,z)) = T(T(x,y), z)$.

There are the following four basic t-norms:

(1) minimum t-norm, $T_M(x,y) = \min\{x,\ y\}$
(2) product t-norm, $T_P(x,y) = x \cdot y$
(3) Łukasiewicz t-norm, $T_L(x,y) = \max\{0, x+y-1\}$
(4) drastic product,

$$T_D(x,y) = \begin{cases} 0, & \text{if } \max\{x,y\} < 1 \\ \min\{x,y\}, & \text{if } \max\{x,y\} = 1 \end{cases}$$

To each t-norm $T : [0,1] \times [0,1] \to [0,1]$ we may define its dual t-conorm $S : [0,1] \times [0,1] \to [0,1]$ by

$$S(x,y) = 1 - T(1-x, 1-y)$$

i.e. to each of the basic four t-norms we have a t-conorm respectively:

(1) maximum t-conorm, $S_M(x,y) = \max\{x,\ y\}$
(2) probabilistic sum, $S_P(x,y) = x + y - xy$
(3) Łukasiewicz t-conorm, $S_L(x,y) = \min\{1, x+y\}$
(4) drastic sum,

$$S_D(x,y) = \begin{cases} 1, & \text{if } \min\{x,y\} > 0 \\ \max\{x,y\}, & \text{if } \min\{x,y\} = 0 \end{cases}$$

If we replace in Definition 4 min by the Łukasiewicz t-norm T_L, we get the Łukasiewicz filter, which was proposed in [10]. In papers [7, 8, 11] the properties of Łukasiewicz filters were studied.

Definition 6. *Let* $X \neq \emptyset$*. A function* $\mathcal{F} : 2^X \to [0,\ 1]$ *is said to be a Łukasiewicz filter on* X *iff the following is satisfied:*

- $\mathcal{F}(X) = 1,\ \mathcal{F}(\emptyset) = 0$
- *for* $A,\ B \subseteq X$ *if* $A \subset B$*, then* $\mathcal{F}(A) \le \mathcal{F}(B)$
- *for* $A,\ B \subseteq X$ *we have*

$$\mathcal{F}(A \cap B) \ge T_L\{\mathcal{F}(A),\ \mathcal{F}(B)\}. \tag{2}$$

Some useful properties of Łukasiewicz filters, when constructing fuzzy preference relations, were shown in [9]. Łukasiewicz ideals were introduced in [8] and their connections to Łukasiewicz ultrafilters and fuzzy preference relations were studied in [9].

Definition 7. *Let* $X \neq \emptyset$*. A function* $\mathcal{I} : 2^X \to [0,\ 1]$ *is said to be a Łukasiewicz ideal on* X *iff the following is satisfied:*

- $\mathcal{I}(X) = 0,\ \mathcal{I}(\emptyset) = 1$

- *for A, $B \subseteq X$ if $A \subset B$, then $\mathcal{I}(A) \geq \mathcal{I}(B)$*
- *for A, $B \subseteq X$ we have*

$$\mathcal{I}(A \cup B) \geq T_L\{\mathcal{I}(A), \mathcal{I}(B)\}. \tag{3}$$

Similarly to crisp filters, each Łukasiewicz filter $\mathcal{F}$ defines a Łukasiewicz ideal $\mathcal{I}$ by $\mathcal{I}(A) = \mathcal{F}(A^c)$.

2. Łukasiewicz ultrafilters

In the whole paper by X will be denoted a fixed non-empty set.

As it was already stated above, there are at least three possible characterizations of crisp ultrafilters U on X:

- ultrafilters are maximal elements of the lattice $(\Psi(X), \wedge, \vee)$
- ultrafilters are such filters that for each $A \subseteq X$ $U(A) + U(A^c) = 1$
- a filter U is an ultrafilter on X if $1 - U$ is an ideal on X.

In [4] evaluators were characterized. In [1] so-called T_L and S_L evaluators were proposed:

Definition 8. *Let $(L, \wedge, \vee, \perp, \top)$ be a lattice with its bottom and top elements $\perp$ and $\top$, respectively. Then $\varphi : L \to [0, 1]$ is a normalized evaluator if*

- *$\varphi(\perp) = 0$, $\varphi(\top) = 1$*
- *for a, $b \in L$ $a \leq b$ implies $\varphi(a) \leq \varphi(b)$.*

A normalized evaluator φ is said to be a T_L evaluator if

- *for a, $b \in L$ $\varphi(a \wedge b) \geq T_L(\varphi(a), \varphi(b))$.*

A normalized evaluator φ is said to be an S_L evaluator if

- *for a, $b \in L$ $\varphi(a \vee b) \leq S_L(\varphi(a), \varphi(b))$.*

Theorem 1 ([1])**.** *Let us have the lattice $(2^X, \cap, \cup, \emptyset, X)$. Then $\varphi : 2^X \to [0, 1]$ is a T_L evaluator iff it is a Łukasiewicz filter. $\psi : 2^X \to [0, 1]$ is an S_L evaluator iff $1 - \psi$ is a Łukasiewicz ideal.*

As a direct corollary to the definitions of Łukasiewicz t-norm T_L and t-conorm S_L and to Theorem 1 we get the following

Lemma 2 ([1])**.** *$\varphi : 2^X \to [0, 1]$ is a T_L and S_L evaluator iff for each $A \subseteq X$*

$$\varphi(A) + \varphi(A^c) = 1$$

Denote $\Phi(X, T_L)$ the system of all Łukasiewicz filters on X. Theorem 1 and Lemma 2 imply

Theorem 2. *Let $\mathcal{F} \in \Phi(X, T_L)$. Then the following are equivalent:*

1. *for each $A \subseteq X$ $\mathcal{F}(A) + \mathcal{F}(A^c) = 1$*
2. *$1 - \mathcal{F}$ is a Łukasiewicz ideal*
3. *$\mathcal{F}$ is a maximal element of the lattice $(\Phi(X, T_L), \wedge, \vee)$.*

Since property 2 plays an important role in construction of fuzzy preference relations (particularly, in decision wether there is some incomparability or not, see [9]) we define Łukasiewicz ultrafilters by the following:

Definition 9. *$\mathcal{U} \in \Phi(X, T_L)$ is a Łukasiewicz ultrafilter iff $1-\mathcal{U}$ is a Łukasiewicz ideal.*

As Theorem 2 states, from the algebraic point of view Łukasiewicz ultrafilters behave exactly as crisp ultrafilters.

3. T-filters and T-ideals

If we replace in formulae (2) and (3) the Łukasiewicz t-norm by some other t-norm T, we get the definition of a T-filter and T-ideal, respectively. Let us denote $\Phi(X, T)$ the system of all T-filters on X.

Definition 10. *$\mathcal{U} \in \Phi(X, T)$ is a T-ultrafilter iff $1 - \mathcal{U}$ is a T-ideal.*

Obviously, if $T_1 \geq T_2$ are some t-norms, then $\Phi(X, T_1) \leq \Phi(X, T_2)$, and since each T-filter defines some T-ideal, the same inequality holds also for systems of T-ideals. As a result we get

Lemma 3. *Let $T_1 \geq T_2$ be arbitrary t-norms. Then, if $\mathcal{U}_1$ is a T_1-ultrafilter, then it is also a T_2-ultrafilter.*

The definition of T-ultrafilters implies that each T-ultrafilter $\mathcal{U}$ defines two T-ideals on X:

$$\mathcal{I}_1(A) = \mathcal{U}(A^c), \quad \mathcal{I}_2(A) = 1 - \mathcal{U}(A) \tag{4}$$

As we will see later on, unlike crisp ultrafilters and Łukasiewicz ultrafilters, for a general t-norm T we may get $\mathcal{I}_1 \neq \mathcal{I}_2$.

By definitions of a T-ultrafilter and T-ideal we get the following for each T-ultrafilter $\mathcal{U}$ on X and each $A \subseteq X$:

$$\begin{aligned} \mathcal{U}(A \cap A^c) &\geq T(\mathcal{U}(A), \mathcal{U}(A^c)) \\ 1 - \mathcal{U}(A \cup A^c) &\geq T(1 - \mathcal{U}(A), 1 - \mathcal{U}(A^c)) = 1 - S(\mathcal{U}(A), \mathcal{U}(A^c)) \end{aligned}$$

hence we get the following system of equations:

$$\begin{aligned} T(\mathcal{U}(A), \mathcal{U}(A^c)) = 0 \\ S(\mathcal{U}(A), \mathcal{U}(A^c)) = 1 \end{aligned} \tag{5}$$

Now, we will distinguish a couple of types of t-norms T. For each of the type we will study the structure of the system of T-ultrafilters:

3.1. **T-norms with no 0-divisors.** A t-norm T has no 0-divisors iff

$$T(x,y)=0 \quad \Leftrightarrow \quad \min\{x,y\}=0$$

The above condition gives the following for each $\mathcal{F}\in\Phi(X,T)$:

$$(\forall A\subseteq X)\,\mathcal{F}(A)>0 \;\Rightarrow\; \mathcal{F}(A^c)=0$$

Hence we get that only crisp ultrafilters are T-ultrafilters and moreover crisp ultrafilters are the only maximal elements of $(\Phi(X,T),\wedge,\vee)$.

3.2. **Left-continuous T-norms $T>T_L$ with 0-divisors.** We split this paragraph into two parts:

(1) Let us consider t-norms T such that

$$T(x,y)=0\,\&\,0<x<1 \quad \Rightarrow \quad x+y<1$$

As an example of such a t-norm is the Yager t-norm

$$T_Y(x,y)=\max\left\{0,\,1-\sqrt{(1-x)^2+(1-y)^2}\right\}.$$

Let $T>T_L$ be an arbitrary t-norm with 0 divisors. Then for the dual t-conorm S we get

$$S(x,y)=1\,\&\,0<x<1 \quad \Rightarrow \quad x+y>1$$

Hence we get that only crisp ultrafilters are T-ultrafilters. Since T is left-continuous, there exists

$$z=\max\{x;T(x,x)=0\}.$$

If we put

$$\mathcal{F}(A)=\begin{cases}1, & \text{if } A=X\\ 0, & \text{if } A=\emptyset\\ z, & \text{otherwise,}\end{cases}$$

then $\mathcal{F}$ is a maximal element of the lattice $(\Phi(X,T),\wedge,\vee)$. I.e., in this case the system of T-ultrafilters does not coincide with the system of maximal elements of $(\Phi(X,T),\wedge,\vee)$.

(2) Let T_N be the nilpotent minimum, which means the following t-norm:

$$T_N(x,y)=\begin{cases}0, & \text{if } x+y\le 1\\ \min\{x,\,y\}, & \text{otherwise.}\end{cases}$$

Then the dual t-conorm S_N is the following:

$$S_N(x,y)=\begin{cases}1, & \text{if } x+y\ge 1\\ \max\{x,\,y\}, & \text{otherwise.}\end{cases}$$

The system of equations (5) has the following solution for each T_N-ultrafilter $\mathcal{U}$:

$$\forall\,A\subseteq X \quad \mathcal{U}(A)+\mathcal{U}(A^c)=1$$

We get the following result:

Theorem 3. *Let $\mathcal{F} \in \Phi(X, T_N)$. Then the following are equivalent:*
(a) *for each $A \subseteq X$ $\mathcal{F}(A) + \mathcal{F}(A^c) = 1$*
(b) *$1 - \mathcal{F}$ is a T_N-ideal*
(c) *$\mathcal{F}$ is a maximal element of the lattice $(\Phi(X, T_N), \wedge, \vee)$.*

The following is an example of a T_N-ultrafilter and of a Łukasiewicz ultrafilter, which is not a T_N-ultrafilter:

Example 1. Let $X = \{a, b, c\}$. The following table defines a T_N-ultrafilter on X:

A	X	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$
$\mathcal{U}(A)$	1	0	0.1	0.2	0.8	0.2	0.8	0.9

The next example is that of a Łukasiewicz ultrafilter on X, which is not a T_N-ultrafilter (nor a T_N-filter):

A	X	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$
$\mathcal{U}(A)$	1	0	0.1	0.1	0.8	0.2	0.9	0.9

3.3. **Left-continuous t-norms $T < T_L$.** Left-continuous t-norms $T < T_L$ have the following property:

$$0 < x < 1 \ \& \ z = \max_y \{x, y\} = 0 \quad \Rightarrow \quad x + z > 1.$$

As an example for such t-norms we can take again a Yager t-norm

$$T_Y(x, y) = \max\left\{0, \, x + y - 1 - 2\sqrt{(1-x)(1-y)}\right\}.$$

Evidently, Łukasiewicz ultrafilters are not maximal elements of $(\Phi(X, T), \wedge, \vee)$, where $T < T_L$ is an arbitrary left-continuous t-norm, however they are T-ultrafilters, since the system of T-ultrafilters is antitone with respect to t-norms (as it was already stated above).

If we take the, just defined Yager t-norm T_Y, we get the following example:

Example 2. Let $X \neq \emptyset$. We have the following T_Y-ultrafilter $\mathcal{U}$ on X:

$$\mathcal{U}(A) = \begin{cases} 1, & \text{if } A = X \\ 0, & \text{if } A = \emptyset \\ \frac{3}{4}, & \text{otherwise} \end{cases}$$

The ultrafilter $\mathcal{U}$ defines two different T_Y-ideals:

$$\mathcal{I}_1(A) = \begin{cases} 0, & \text{if } A = X \\ 1, & \text{if } A = \emptyset \\ \frac{3}{4}, & \text{otherwise} \end{cases} \qquad \mathcal{I}_2(A) = \begin{cases} 0, & \text{if } A = X \\ 1, & \text{if } A = \emptyset \\ \frac{1}{4}, & \text{otherwise} \end{cases}$$

where $\mathcal{I}_1(A) = \mathcal{U}(A^c)$, $\mathcal{I}_2(A) = 1 - \mathcal{U}(A)$.

We can formulate the following characterization of T-ultrafilters and T-ideals:

Theorem 4. *Let $T < T_L$ be an arbitrry left-continuous t-norm. Each maximal element of $(\Phi(X,T), \wedge, \vee)$ is a T-ultrafilter on X. There are ultrafilters on X which are not maximal elements of $(\Phi(X,T), \wedge, \vee)$. Let $\mathcal{U}$ be a T-ultrafilter on X. Then T-ideals $\mathcal{I}_1(A) = \mathcal{U}(A^c)$ and $\mathcal{I}_2(A) = 1 - \mathcal{U}(A)$ may be different. $\mathcal{I}_1 = \mathcal{I}_2$ if and only if $\mathcal{U}$ is a Łukasiewicz ultrafilter.*

3.4. **Drastic product t-norm T_D.** This t-norm is not left-continuous. This implies that the only maximal elements of $(\Phi(X,T_D), \wedge, \vee)$ are crisp ultrafilters on X. However, by definition of T_D and S_D we get that a T_D-ultrafilter is each crisp ultrafilter and each monotonic function $\mathcal{F} : 2^X \to [0,1]$ such that

$$\begin{aligned} \mathcal{F}(\emptyset) &= 0, \\ \mathcal{F}(X) &= 1, \\ \mathcal{F}(A) &\in \;]0,1[\quad \text{for } A \notin \{X, \emptyset\}. \end{aligned}$$

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A NOTE ON AN EXAMPLE OF USE OF FUZZY PREFERENCE STRUCTURES

DANA HLINĚNÁ AND PETER VOJTÁŠ

ABSTRACT. In this paper we present a problem of multicriterial optimization and different models to solve it. We illustrate various alternatives on a practical example.

1. Motivation Example

In this paper we illustrate several aspects of a multicriterial problem that we try to approach from different perspectives – deductive, inductive, and different formal models – Choquet integrals, fuzzy preference structures.

Example 1. *(Michel Grabisch, Marc Roubens* [3]*)*
In [3] the authors consider the problem of the evaluation of trainees learning to drive military vehicles. The instructors evalueted the trainees according to 4 criteria:

- *C1. Firing precision:* The percentage of success during the exercise is computed.
- *C2. Target detection rapidity:* The elapsed time between the appearance of the target and the detection is measured in tu (time unit).
- *C3. Driving:* In order to go from one point to another, the trainee has to choose a suitable trajectory, or to follow a given one as precisely as possible. A qualitive score is given by the instructor, ranging from A (excelent) to E (hopeless).
- *C4. Communication:* The trainee is supposed to belong to some unit, and thus he should understand and obey orders, and also report actions. As for the driving criterion, a qualitative score is given by the instructor, ranging from A (perfect) to E (incontrollable).

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Table 1. Performances of the different trainees, cf. [3].

name	*precision (%)*	*rapidity (tu)*	*driving*	*communication*
Arthur	90	2	B	D
Lancelot	80	4	B	B
Yvain	95	5	C	A
Perceval	60	6	B	B
Erec	65	2	C	B

Table 2. Scores on the different criteria, cf. [3]

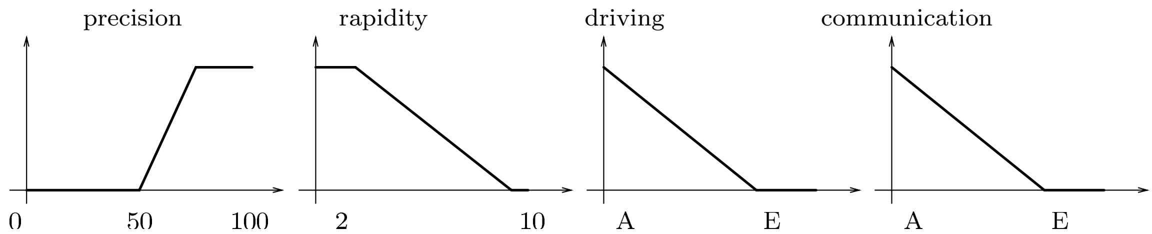

Table 3. Numerical scores on criteria, cf. [3].

name	*precision*	*rapidity*	*driving*	*communication*
Arthur	1.000	1.000	0.750	0.250
Lancelot	0.750	0.750	0.750	0.750
Yvain	1.000	0.625	0.500	1.000
Perceval	0.250	0.500	0.750	0.750
Erec	0.375	1.000	0.500	0.750

In this example, Grabisch and Roubens consider 5 trainees, whose names and performances on each criterion are given in Table 1.

Instructor's comments:

C.1 (precision): over 90% of success is perfect, below 50% is totally unacceptable.

C.2 (rapidity): below 2 tu is perfect, over 10 tu is totally unacceptable.

C.3 and C.4: these criteria are already expressed in the form of an equidistant numerical score.

This permits us to draw utility functions which give the following numerical scores for the trainees in Tables 2, 3.

Looking at the performances of the different trainees, the instructor is able to rank the trainees, as given in Table 4. There are three predetermined classes,

TABLE 4. Ranking of the five trainees, cf. [3].

name	*class*	*rank in the class*
Arthur	bad	2
Lancelot	good	1
Yvain	good	2
Perceval	bad	1
Erec	average	1

TABLE 5. Mapping from class and rank to $[0, 1]$, cf. [3].

class	*interval for the global score*
good	[0.75, 1.0]
average	[0.4, 0.75]
bad	[0.0, 0.4]

called good, average, bad. In each class, a ranking is done, labelling by 1 the best in the class, by 2 the second best, etc.

Inductive task. Now we are in a multicriterial situation. In [3] the authors solve the inductive problem, given a global evaluation, how to learn an objective function which explains global ranking from particular attributes. This is the point where different models have different representations of a utility function.

In [3] an approach is taken, where the global ranking is represented as Choquet integral, and we have to learn the measure. The condition for learning is either;

1. approach by the minimization of the quadratic error,

or

2. approach based on constraint satisfaction.

For this, [3] assignes intervals to classes as in Table 5.

Grabish and Roubens [2] present an algorithm which specifies a measure such that the **Choquet integral** mimics the global evaluation. The idea of the first approach (minimization of squared errors) is to identify the fuzzy measure in a Choquet integral: We suppose that the decision maker is able to assess a numerical score for each act and each criterion, and also a numerical global score for each act. So we want to find the fuzzy measure which minimizes the total squared error of the model.

In the second approach (**constrained satisfaction**) we assume that we have an expert who is able to tell the relative importance of criteria and kind of interaction between them, if any. All this information can be transformed in terms of linear equalities or inequalities linking the unknown weights. These

TABLE 6. Numerical data on criteria and global performance, cf. [3].

name	*precision*	*rapidity*	*driving*	*communication*	global 1st
Arthur	1.000	1.000	0.750	0.250	0.133
Lancelot	0.750	0.750	0.750	0.750	0.917
Yvain	1.000	0.625	0.500	1.000	0.833
Perceval	0.250	0.500	0.750	0.750	0.276
Erec	0.375	1.000	0.500	0.750	0.575

name	*precision*	*rapidity*	*driving*	*communication*	global 2nd
Arthur	1.000	1.000	0.750	0.250	0.3
Lancelot	0.750	0.750	0.750	0.750	0.75
Yvain	1.000	0.625	0.500	1.000	0.7
Perceval	0.250	0.500	0.750	0.750	0.35
Erec	0.375	1.000	0.500	0.750	0.5

methods are in fact not comparable, since they do not take exactly the same input, nor provide the same kind of output.

The results of both aproaches are given in Table 6.

Our approach is based on conenction between fuzzy and annotated logic programs [5] and an inductive logic programming method for learning rules of annotated programs [4]. In this approach we start from an instructors evaluation expressed in a lineary ordered scale, here it can be

$$bad2 < bad1 < average < good2 < good1$$

or any order preserving mapping into $[0, 1]$ (here understood as an ordinal scale). Then the task has a possible input as in the following Table 7.

TABLE 7. Linear ranking of the five trainees

name	*global rank*
Arthur	0.125
Lancelot	0.875
Yvain	0.75
Perceval	0.375
Erec	0.625

This global numerical rank gives a partial function f from $[0, 1]^4$ into $[0, 1]$, as depicted in Table 8. This function can be extended to F on whole $[0, 1]^4$ preserving monotonicity, in the following sense.

TABLE 8. Function on attributes

name	*precision*	*rapidity*	*driving*	*communication*	*global rank*
Arthur	1.000	1.000	0.750	0.250	0.125
Lancelot	0.750	0.750	0.750	0.750	0.875
Yvain	1.000	0.625	0.500	1.000	0.75
Perceval	0.250	0.500	0.750	0.750	0.375
Erec	0.375	1.000	0.500	0.750	0.625

Denote Lancelot's attribute scores as $x^L = (x_1^L, x_2^L, x_3^L, x_4^L)$ and Perceval's attribute scores as $x^P = (x_1^P, x_2^P, x_3^P, x_4^P)$. Note that $x_i^P \leq x_i^L$ for $i = 1, ..., 4$ and $f(x^P) \leq f(x^L)$. Hence global score of trainees does not violate monotonicity. A straitforward way to prolongate it the whole $[0,1]^4$ is the definition

$$F(y_1, y_2, y_3, y_4) = \max\{f(x^T) : T \in \{A, E, L, P, Y\} \text{ and } (\forall i) x_i^T \leq y_i\}$$

Note that $\max \emptyset = 0$. Our method from [4] is able to learn such a monotonic extension of any function given in a multirelational set of data and different preferences. This si especialy interesting on bigger data and intervals asigned to global score violating colinearity (Choqeut integral is able to represent only colinear functions).

Deductive task. In a similar setting, having trainees and their achievements (same data) we can assume that from previous experiments we already have a utility function. Now the problem is about efficient algorithms to find the best trainee, assuming we have a huge set of data, possibly distributed, and so the question of efficiency becomes crucial.

In this paper we describe the problem setting which is a common starting point for different approaches.

2. Introduction to preference structures and fuzzy preference structures

The preference structure is a basic step of preference modeling. Given two alternatives, decision maker defines three binary relation-preference, indifference and incomparability.

A preference structure is a basic concept of preference modelling. In a classical preference structure (PS) a decision-maker makes three decission for any par (a, b) from the set $\boldsymbol{A}$ of all alternatives. His decision define a triplet P, I, J of a crisp binary relations on $\boldsymbol{A}$:

1) a is prefered to $b \Leftrightarrow (a, b) \in P$ (strict preference).
2) a and b are indifferent $\Leftrightarrow (a, b) \in I$ (indifference).
3) a and b are incomparable $\Leftrightarrow (a, b) \in J$ (incomparability).

A preference structure (PS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

(ps1) I is reflexive, P and J are irreflexive.
(ps2) P is asymmetric, I and J are symmetric.
(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$.
(ps4) $P \cup I \cup J \cup P^t = A \times A$ where $P^t(x, y) = P(y, x)$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the large preference relation. The relation R can be interpreted as

$$(a, b) \in R \Leftrightarrow a \text{ is prefered to } b \text{ or } a \text{ and } b \text{ are indifferent.}$$

It can be easily proved that

$$co(R) = P^t \cup J$$

where $coR(a, b) = 1 - R(a, b)$ and

$$P = R \cap co(R^t), I = R \cap R^t, J = co(R) \cap co(R^t).$$

It allows us to construct a preference structure (P, I, J) from a reflexive binary operation R only.

Decision-makers are often uncertain even inconsistent in their judgements. The restriction on two-valued relations have been an important drawback to their practical use. A natural demand led researchers to the introduction of a fuzzy preference structure (FPS). The original idea of using numbers between zero and one to describe the strenth of links between two alternatives goes back to Menger. The introduction of fuzzy relations allowed to express degrees of preference, indifference and incomparability. Of course, the attempts simply to replace the notion used in the definition of (PS) by their fuzzy equivalents have met some problems.

To define (FPS) it is necessary to consider some fuzzy connectives. We shall consider a continuous De Morgan triple (T, S, N) consisting of a continuous t-norm T, continuous t-conorm S and a strong negator N such that $T(x, y) = N(S(N(x), N(y)))$. The main problem consists in the fact that the completeness condition (ps4) can be written in many forms, e.g.:

$$co(P \cup P^t) = I \cup J, P = co(P^t \cup I \cup J), P \cup I = co(P^t \cup J).$$

Let (T,S,N) be De Morgan triplet. A fuzzy preference structure (FPS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary fuzzy relations on $\boldsymbol{A}$ such that

(f1) I is reflexive, P and J are irreflexive. $I(a, a) = 1, P(a, a) = J(a, a) = 0$
(f2) P is T-asymmetrical, I and J are symmetrical. $T(P(a, b), P(b, a)) = 0$
(f3) $T(P, I) = T(P, J) = T(I, J) = 0.$ for all pair of alternatives
(f4) $(\forall (a, b) \in A^2) S(P, P^t, I, J) = 1$ or $N(S(P, I)) = S(P^t, J)$ or another completeness conditions.

Note that it was proved [1, 7] that reasonable constructions of fuzzy preference structure (FPS) should use a nilpotent t-norm only. Since any nilpotent t-norm (t-conorm) is isomorphic to the Lukasiewicz t-norm (t-conorm), it is enough to restrict our attention to De Morgan triple $(T_L, S_L, 1-x)$.

3. Preference structures and fuzzy preference structures and their applications

Let us turn our attention to motivation example. We denote by $M = \{A, E, L, P, Y\}$ the set of all trainees. We are able to construct the large preference relations R_P, R_R, R_D and R_C derived from orderings in our four criteria (precision, rapidity, driving, communication):

R_P	A	E	L	P	Y
A	1	1	1	1	1
E	0	1	0	1	0
L	0	1	1	1	0
P	0	0	0	1	0
Y	1	1	1	1	1

$R_P = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,Y],[Y,A],[A,L],[A,E],[A,P],[Y,L],[Y,E],[Y,P],[L,E],[L,P],[E,P]\}$

R_R	A	E	L	P	Y
A	1	1	1	1	1
E	1	1	1	1	1
L	0	0	1	1	1
P	0	0	0	1	0
Y	0	0	0	1	1

$R_R = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,E],[E,A],[A,L],[A,Y],[A,P],[E,L],[E,Y],[E,P],[L,Y],[L,P],[Y,P]\}$

R_D	A	E	L	P	Y
A	1	1	1	1	1
E	0	1	0	0	1
L	1	1	1	1	1
P	1	1	1	1	1
Y	0	1	0	0	1

$R_D = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,L],[L,A],[A,P],[P,A],[L,P],[P,L],[A,E],[A,Y],[L,E],[L,Y],[P,E],[P,Y],[E,Y],[Y,E]\}$

R_C	A	E	L	P	Y
A	1	0	0	0	0
E	1	1	1	1	0
L	1	1	1	1	0
P	1	1	1	1	0
Y	1	1	1	1	1

$R_C = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[Y,L],[Y,P],[Y,E],[Y,A],[L,P],[P,L],[L,E],[E,L],[P,E],[E,P],[L,A],[P,A],[E,A]\}$

And we are able to construct large preference relation R_I which is derived from instructor's global ordering, too:

R_I	A	E	L	P	Y
A	1	0	0	1	0
E	1	1	0	1	0
L	1	1	1	1	1
P	1	0	0	1	0
Y	1	1	1	1	1

$R_I = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[L,Y],[Y,L],[L,E],[L,A],[L,P],[Y,E],[Y,A],[Y,P],[E,A],[E,P],[A,P],[P,A]\}$

The relation R_I is not linear order set. For global evaluation we will modify this ordering to linear ordering. First, we need order the criteria.

The first idea is; we can pairwise compare the relations R_P, R_R, R_D and R_C with respect to relation R_I by the following rule:

$$X > Y \iff \frac{|R_X \cap R_I|}{|R_X \triangle R_I|} > \frac{|R_Y \cap R_I|}{|R_Y \triangle R_I|}, \tag{1}$$

where $X, Y \in \{P, R, D, C\}$. The idea is; the more R_X is similar to R_I, the more important criterion are X is. This method gives the following ordering of criteria: communication >precision >rapidity > driving. Note that this method is not the only one possible, and investigation of other possibilities is subject of ongoing research.

Generaly speaking, we obtain the ordering of criteria from the relation preference which is given by $P = R \cap co(R^t)$. However, in this example we have got the same ordering of criteria via both the preference relations and the large relations (with respect to previous method (1) for comparing the relations).

P_P	A	E	L	P	Y
A	0	1	1	1	0
E	0	0	0	1	0
L	0	1	0	1	0
P	0	0	0	0	0
Y	0	1	1	1	0

$P_P = \{[A, L], [A, E], [A, P], [Y, L], [Y, E], [Y, P], [L, E], [L, P], [E, P]\}$

P_R	A	E	L	P	Y
A	0	0	1	1	1
E	0	0	1	1	1
L	0	0	0	1	1
P	0	0	0	0	0
Y	0	0	0	1	0

$P_R = \{[A, L], [A, Y], [A, P], [E, L], [E, Y], [E, P], [L, Y], [L, P], [Y, P]\}$

P_D	A	E	L	P	Y
A	0	1	0	0	1
E	0	0	0	0	0
L	0	1	0	0	1
P	0	1	0	0	1
Y	0	0	0	0	0

$P_D = \{[A, E], [A, Y], [L, E], [L, Y], [P, E], [P, Y]\}$

P_C	A	E	L	P	Y
A	0	0	0	0	0
E	1	0	0	0	0
L	1	0	0	0	0
P	1	0	0	0	0
Y	1	1	1	1	0

$P_C = \{[Y, L], [Y, P], [Y, E], [Y, A], [L, A], [P, A], [E, A]\}$

P_I	A	E	L	P	Y
A	0	0	0	0	0
E	1	0	0	1	0
L	1	1	0	1	0
P	0	0	0	0	0
Y	1	1	0	1	0

$P_I = \{[L, E], [L, A], [L, P], [Y, E], [Y, A], [Y, P], [E, A], [E, P]\}$

Fuzzification. For better expresion of reality, we can use fuzzy preference structures. Natural way of fuzzification of preference relations P_P, P_R, P_D and P_C from our motivation example is given as follows:

The value of fuzzy preference in precision (FP_P) for Arthur and Erec, we compute from Table 3 as $FP_P(A, E) = \max\{x_1^A - x_1^E, 0\}$, where x_1^A and x_1^E are Arthur's and Erec's precision score in Table 3, etc. The fuzzification of preference relation P_I is given in the last table and it is derived from Tables 6 and 7.

FP_P	A	E	L	P	Y
A	0	0.625	0.25	0.75	0
E	0	0	0	0.125	0
L	0	0.375	0	0.5	0
P	0	0	0	0	0
Y	0	0.625	0.25	0.75	0

FP_R	A	E	L	P	Y
A	0	0	0.25	0.5	0.375
E	0	0	0.25	0.5	0.375
L	0	0	0	0.25	0.125
P	0	0	0	0	0
Y	0	0	0	0.125	0

FP_D	A	E	L	P	Y
A	0	0.25	0	0	0.25
E	0	0	0	0	0
L	0	0.25	0	0	0.25
P	0	0.25	0	0	0.25
Y	0	0	0	0	0

FP_C	A	E	L	P	Y
A	0	0	0	0	0
E	0.5	0	0	0	0
L	0.5	0	0	0	0
P	0.5	0	0	0	0
Y	0.75	0.25	0.25	0.25	0

FP_I	A	E	L	P	Y
A	0	0	0	0	0
E	0.5	0	0	0.25	0
L	0.75	0.25	0	0.5	0.125
P	0.125	0	0	0	0
Y	0.75	0.25	0	0.5	0

Using the formula for comparing the relations (compare to (1))

$$(2) \qquad X \succ Y \iff \frac{\sum\limits_{i,j} |\{fp_{x_{ij}}\} \cap \{fp_{I_{ij}}\}|}{\sum\limits_{i,j} |\{fp_{x_{ij}}\} \triangle \{fp_{I_{ij}}\}|} > \frac{\sum\limits_{i,j} |\{fp_{y_{ij}}\} \cap \{fp_{I_{ij}}\}|}{\sum\limits_{i,j} |\{fp_{y_{ij}}\} \triangle \{fp_{I_{ij}}\}|},$$

where X, Y are our criteria, m is a number of alternatives, $fp_{x_{ij}}, fp_{y_{ij}}$ are values of fuzzy preference structures of criterion X, Y, $fp_{I_{ij}}$ are values of fuzzy preference relation which is based on expert's global score and $i, j \in \{1, 2 \dots m\}$, we obtain the following ordering $\succ$ of our criteria:

$$\text{communication} \succ \text{precision} = \text{driving} \succ \text{rapidity}.$$

Note that our intersection $\cap$ and symmetric difference $\triangle$ are ordinary. We can see, this ordering is different from ordering, which we obtain via strict preference structure. Note, that another fuzzification leeds to different ordering of criteria and subsequently to different ordering of trainees.

Simple deduction. Let us imagine the next situation: We have ordering $\succ$, and now look at another trainee Bruno with scores given in the Table 9.

TABLE 9. Function on attributes

name	*precision*	*rapidity*	*driving*	*communication*
Arthur	1.000	1.000	0.750	0.250
Lancelot	0.750	0.750	0.750	0.750
Yvain	1.000	0.625	0.500	1.000
Perceval	0.250	0.500	0.750	0.750
Erec	0.375	1.000	0.500	0.750
Bruno	0.400	0.750	0.600	0.750

Our task is to compare Bruno with others. Denote Bruno's attribute scores as $x^B = (x_1^B, x_2^B, x_3^B, x_4^B)$ and Erec's attribute scores as $x^E = (x_1^E, x_2^E, x_3^E, x_4^E)$. We can see that $x_i^E \le x_i^B$ for $i = 1, 3, 4$. Since there is a tie in the most important criterion (communication) so we decide based on the next criteria (precision and driving) in our ordering $\succ$. This results in Bruno's superiority over Erec. The final ordering of trainees is: Yvain $>$ Lancelot $>$ Bruno $>$ Erec $>$ Perceval $>$ Arthur.

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DISTANCE BETWEEN FUZZY SETS AS A FUZZY QUANTITY

VLADIMÍR JANIŠ AND SUSANA MONTES

Abstract. The traditional methods of comparing images, like using the Hamming distance, may sometimes fail, especially if we do not insist on careful checking all the details of the images, but compare them just broadly. An n-dimensional image with various grades of grey colours can be represented by a fuzzy set. We introduce a method of estimating the difference between such images by a fuzzy set, which corresponds to various levels of identifying close parts of the given images, or, in other words, to the grade of accuracy, with which the images are observed. Examples and some properties of such a distance are shown.

1. Introduction

A fuzzy subset of a space X can be interpreted as a model for the image on X containing various shapes of gray colour. The membership degrees then correspond to grades of darkness, when 0 can be assigned to white and 1 to black colour (or vice versa). Conversely, a gray image can be represented by a corresponding fuzzy set.

A natural question is to estimate the grade of similarity of two such images, which is analogical to estimating the distance between two fuzzy sets. There are several attitudes to this problem, which can be divided into two groups. The first one works with differences between membership values at particular points of X. Another one is based on differences between cuts at particular levels (see e.g. Cabrelli et al. in [2], [3] and [4]).

However, both mentioned concepts can lead to unsatisfactory results from the applications point of view. The examples of such cases can be found in [8], where Lowen and Peeters also show the way how to avoid such problems. They suggest a way to estimate the distance between two fuzzy sets accounting both differences between membership values and between cuts. However, the result is a single real number, which may in some applications mean the loss of information. Our aim

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is to develop the results of Lowen and Peeters so that we obtain a fuzzy quantity which reflects the difference between two given fuzzy sets.

Another problem, which may appear for example in pattern recognition is, that not all the points in the space X may have the same importance. The noise at the edge of a screen can be sometimes considered not so important as the noise in its centre. Although this is just a technical problem, we also incorporate it in our consideration.

2. Concepts of measuring differences between fuzzy sets

There are many different attitudes to comparing fuzzy sets that can be found in the literature. Generally they are based on one of the two principles, which we shortly describe below.

In many occasions the comparison of two fuzzy sets is done by quantifying the degree of similarity or equality between them (see, for instance [5], [11] or [12]), but there hardly are references related to the degree of inequality or difference between them.

In [1] the authors proposed a measure of similarity between fuzzy sets and also a measure of dissimilarity. Thus, they defined a μ-measure of dissimilarity on X as a function $S : F(X) \times F(X) \to [0; 1]$ such that

$$S(m, n) = F_S(\mu(m \cap n), \mu(n - m), \mu(m - n)),$$

where μ is a measure on X and $F_S : [0; \infty)^3 \to [0; 1]$ is a function independent of the first coordinate, increasing in the other two and such that $F(x, 0, 0) = 0$ for all $x \in [0; \infty)$.

The most frequent definitions of classical distances between fuzzy sets m, n in a universe X are:

- The Hamming distance:

$$d(m, n) = \sum_{x \in X} |m(x) - n(x)|.$$

- A generalization of the Hamming distance proposed by Kacprzyk in [6]

$$d(m, n) = \sum_{x \in X} |m(x) - n(x)|^2.$$

- The generalization of the previous ones, using the Minkowski distance (see e.g. [7])

$$d(m, n) = \left(\sum_{x \in X} |m(x) - n(x)|^r \right)^{\frac{1}{r}}, \ r \geq 1.$$

This class of distances includes, as a particular case, the supremum distance, used to compare fuzzy sets among others by Nowakowska in [10] and Wenstøp in [13]. Its definition is

$$d(m,n) = \sup_{x \in X} |m(x) - n(x)|.$$

All these distances are particular cases of the dissimilarity measures defined in [1].

In relation to dissimilarities Montes et al. introduced in [9] the definition of divergence measure as a map $D : F(X)^2 \to R$ such that for all $m, n, \rho \in F(X)$ the following conditions are satisfied:

(1) $D(m,n) = D(n,m)$,
(2) $D(m,m) = 0$,
(3) $\max\{D(m \cup \rho, n \cup \rho), D(m \cap \rho, n \cap \rho)\} \leq D(m,n)$.

This definition generalizes, except for the symmetry property (that could be excluded from the set of axioms in some particular cases) the concept of dissimilarity measures previously proposed. Moreover, local divergencies are distances between fuzzy sets according to the definition proposed in [14], which will be recalled in Definition 3.

All these measures were applied in different fields, but they are not too appropriate for some very natural circumstances as we will explain in the following.

3. Distance function and distance

Suppose (X,d) is a pseudometric space, let $F(X)$ denote the system of all its fuzzy subsets. Let $m, n \in F(X)$. For each $x \in X$ we assign a nonincreasing function f_x such that $f_x : [0,1] \to [0;\infty]$. To be compatible with [8] we may call f_x a tolerance function for x. The shape of this function depends on the importance of the point x in the image (for better understanding see the examples later in this paper).

First we define a distance function at a point.

Definition 1. *Let $S(x,r)$ be the closed neighborhood of x with diameter r, let $m, n \in F(X)$. If $x \in X$, then the mapping $g_x^{m,n} : [0;1] \to [0;1]$ such that*

$$g_x^{m,n}(\alpha) = \inf\{|m(z) - n(y)|; z, y \in S(x, f_x(\alpha))\}$$

is called the distance function at a point x.

Here we follow the idea of a tolerance introduced by Lowen and Peeters in [8], but in our attitude the tolerance is not constant.

It is easy to see that any distance function is nondecreasing. The purpose of such a function is to model the grade of accuracy with which the image is observed. The value $\alpha = 0$ corresponds to the "least careful" view of the image, while the value of $\alpha = 1$ models the "most detailed" look at it. A good example has been given in [8], namely two chessboards with a very large number of rows and columns, inverse to each other. At a close look we see that they are totally different, but from a large distance we do not distinguish small squares, but observe two identical large (gray) squares.

The distance function at a point enables us to define the main notion of this work.

Definition 2. *Let for each $x \in X$ be $g_x^{m,n}$ its distance function. The distance between the fuzzy sets m and n is then given by the fuzzy set $g^{m,n} : [0,1] \to [0,1]$ defined for $\alpha \in [0,1]$ as follows:*

$$g^{m,n}(\alpha) = \sup\{g_x^{m,n}(\alpha), x \in X\}.$$

Thus we obtain a fuzzy quantity which gives us more information about two fuzzy sets than a single number, as it can be seen from examples in the following section.

The distance defined above has properties similar to some of the distance measure, as it was introduced in [14]. We recall its definition:

Definition 3. *Let $F(X)$ be the system of al fuzzy sets on a universe X. A function $\delta : F(X)^2 \to [0, \infty[$ is called a distance measure if it satisfies the following properties:*

(1) $\delta(A,B) = \delta(B,A)$ *for all* $A, B \in F(X)$,
(2) $\delta(A,A) = 0$ *for all* $A \in F(X)$,
(3) $\delta(D, X \setminus D) = \max_{A,B \in F(X)} \delta(A,B)$ *for all crisp subsets* D *of* X,
(4) *if* $A \subseteq B \subseteq C$, *then* $\delta(A,B) \leq \delta(A,C)$ *and* $\delta(B,C) \leq \delta(A,C)$ *for all* $A, B, C \in F(X)$.

Clearly the distance as we have defined it, cannot be a distance measure, as its values are not real numbers, but fuzzy quantities. However, it has some similar properties, which are formulated in the following propositions.

Proposition 1. *If $m, n \in F(X)$, then $g^{m,n} = g^{n,m}$.*

Proposition 2. *If $m \in F(X)$, then $g^{m,m}$ is a zero function.*

Both propositions follow directly from the definition of the distance. The transitivity for our definition is preserved by means of the fact, that for $m, n, p \in F(X), m \leq n \leq p$ the difference $n - m$ differs less from the zero function, than the difference $p - m$.

Proposition 3. *If $m, n, p \in F(X), m \leq n \leq p$, then $g^{0,n-m} \leq g^{0,p-m}$.*

Proof. Clearly it is sufficient to prove the statement for $r, s \in F(X), r \leq s$, as $n - m \leq p - m$. But if $r \leq s$, then

$$\inf\{r(y), y \in S(x, f_x(\alpha))\} \leq \inf\{s(z), z \in S(x, f_x(\alpha))\}$$

for all $x \in X, \alpha \in [0,1]$. This means that $g_x^{0,r} \leq g_x^{0,s}$ for all $x \in X$. Using suprema to get the distance functions and the fact that the inequality remains also for them, we conclude $g^{0,r} \leq g^{0,s}$. Putting $r = n - m, s = p - m$ we finish the proof.

The only property of distance measure, which cannot be mechanically transferred for our distance, is the third one, stating that any crisp set and its complement have the maximal possible distance measure. This is no surprise, as our attitude is based on the principle that (using the language of pattern recognition) considers the white patterns with small pieces of black color as a kind of fuzzy sets. However, for sets, that are "crisp enough" a kind of a similar property holds.

Proposition 4. *Let D be a (crisp) subset of X. Then*

$$g^{D,X\setminus D} = \max\{g^{m,n}; m, n \in F(X)\}$$

if and only if there is an $x_0 \in D$ such that $S(x_0, f_{x_0}(0)) \subseteq D$.

Proof. Let D be a (crisp) subset of X. For the convenience we will denote by the same letter its characteristic function, as well as for its complement. Suppose there is an $x_0 \in D$ such that $S(x_0, f_{x_0}(0)) \subseteq D$. Then for the distance function at x_0 we have

$$g_{x_0}^{D,X\setminus D}(0) = \inf\{d(D(y), (X \setminus D)(z)); y, z \in S(x_0, f_{x_0}(0))\} = 1$$

due to the assumption which asserts that $D(y) = 1$ and $(X \setminus D)(z) = 0$ for any $y, z \in S(x_0, f_{x_0}(0))$.

As any distance function is nondecreasing, we have $g_{x_0}^{D,X\setminus D}(\alpha) = 1$ for all $\alpha \in [0,1]$, hence also $g^{D,X\setminus D}(\alpha) = 1$ for all $\alpha \in [0,1]$. Evidently this is the maximal possible distance for any pair of fuzzy sets in $F(X)$.

To show the reverse implication let us assume that for all $x \in D$ there is

$$S(x, f_x(0)) \cap (X \setminus D) \neq \emptyset.$$

Then for any $x \in D$ we have

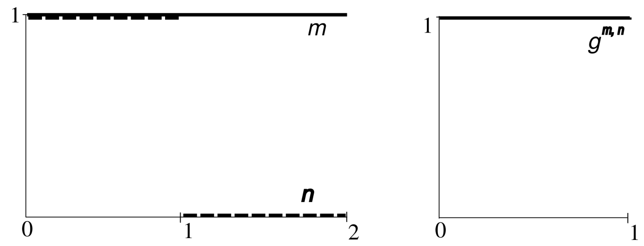
$$g_x^{D,X\setminus D}(0) = \inf\{d(D(y), (X \setminus D)(z)); y, z \in S(x, f_x(0))\} = 0$$

as in each $S(x, f_x(0))$ there is a point belonging to D and also a point in its complement.

4. Examples

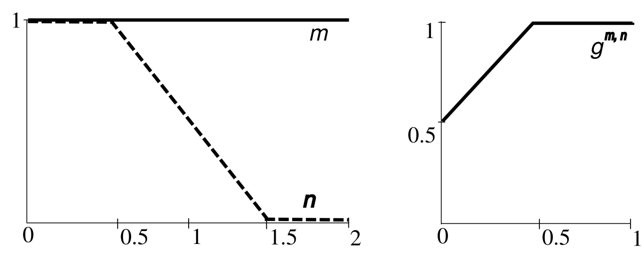
In the following we present a series of simple examples which demonstrate some properties of the distance. In all of them the space X will be the interval $[0, 2]$ with the usual metric. The tolerance function for all the points of X in Examples 1 – 4 will be $f_x(\alpha) = 1 - \alpha$. In each example we present the graphs of m, n and their difference $g^{m,n}$. For better understanding it is good to think of the fuzzy sets used in the examples as of image representations, where 1 represents black and 0 white color and the values between correspond to degrees of grey color. In all the graphs m is sketched in a full line, n in a dashed one.

Example 1. $m(x) = 1$, $n(x) = 1$ for $x \in [0, 1]$, otherwise $n(x) = 0$.



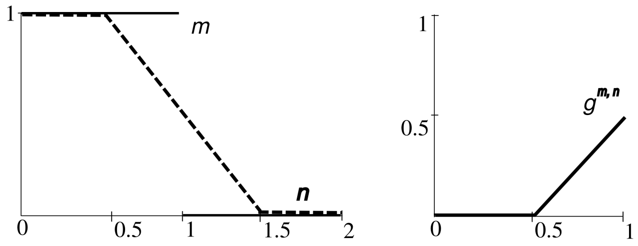
Here the fuzzy set n is in the sense of Proposition 4 sufficiently crisp to have the maximal possible distance from m.

Example 2. $m(x) = 1$, $n(x) = 1$ for $x \in [0, 0.5]$, $n(x) = 1.5 - x$ for $x \in]0.5, 1.5[$, $n(x) = 0$ otherwise.



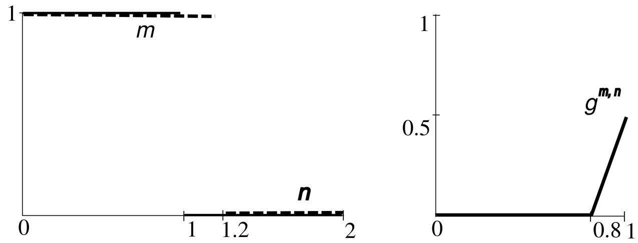
As we see, considering only the membership degrees close to edge values (not mentioning the shades of gray), both sets are far from each other. If we consider also degrees of gray color, they are closer than in the previous example.

Example 3. $m(x) = 1$ for $x \in [0, 1]$, $m(x) = 0$ otherwise, n is the same as in the previous example.



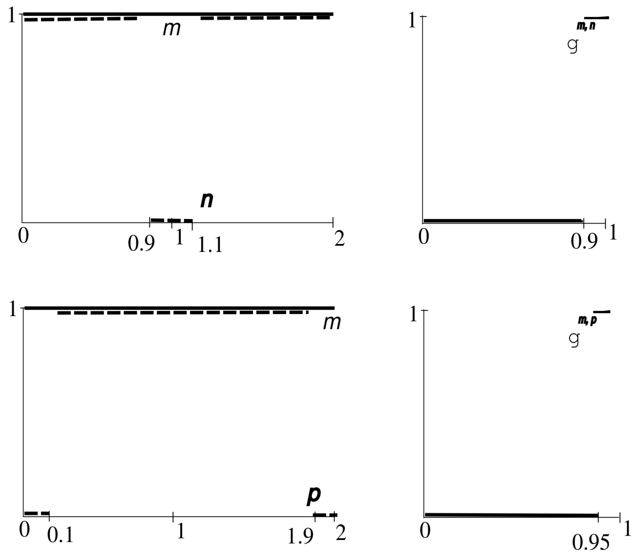
In this example we see that the more attention is paid to the grey colors, the closer are the images.

Example 4. $m(x) = 1$ for $x \in [0, 1], m(x) = 0$ otherwise, Let n be the crisp set $[0; 1.2]$. In the graph of the distance function we see that these sets are closer to each other than the pair in the previous example. Although the difference in colours in the middle of the image is sharp, it is just on a set with a small measure and in the broader view these sets tend to coincide.



The following example shows the possibility to assign various importance to different parts of the underlying space. We see that although the measure of the space where noise (white color in a black image) is present is the same, the distance is bigger if the noise appears in the center of the space X.

Example 5. Let for $x \in [0.5, 1.5]$ the tolerance function be $f_x(\alpha) = 1 - \alpha$, for the remaining points in X let $f_x(\alpha) = 2 - 2\alpha$. Let m be the crisp set $[0, 2]$, let n be the crisp set $[0, 0.9] \cup [1.1, 2]$, let p be the crisp set $[0.1, 1.9]$.



Here we see that due to the smaller importance of the points closer to the endpoints, the distance between m and n is larger than the distance between m and p.

5. Concluding remarks

We have defined a distance of a pair of fuzzy sets expressed by a mapping which enables us to estimate the similarity of given fuzzy sets depending on the level, on which we identify points close to each other. As we have shown, the properties of such a distance have much in common with the properties of the distance measure from [14]. Moreover, it has the following properties, which are quite obvious:

If f_x is a zero function for all x, then our distance is equivalent to the supremum distance usually denoted by d_∞, (the distance used in [10] or [13]). This means that in such case $g^{m,n}$ is a constant function with its value $d_\infty(m, n)$.

If f_x are all equal to the same constant τ, then our distance is equivalent to that introduced in [8].

If $f_x(1) > 0$, then the noise in the singleton $\{x\}$ is ignored. Moreover, if $f_x(1) \geq c$ for all $x \in X$, then also the noise on sets with diameter not exceeding c is ignored.

By putting $s^{m,n} = 1 - g^{m,n}$ we obtain s with properties similar to similarity measure as was introduced in [14].

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A GEOMETRICAL APPROACH TO AGGREGATION

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Abstract. Considering the family F of contour curves $F = \{h(x, y) = k\}$ of an (idempotent) aggregation operator h in two variables as a one-parametric family of curves, the differential equation $y' = f(x, y)$ having F as general solution is associated to h. Properties of h have then be translated to properties of its differential equation. Reciprocally, for a differential equation fulfilling some easy properties its general solution can be seen as the contour curves of an idempotent aggregation operator so that properties of the equation can have their counterpart in the ones of the aggregation operator.

1. Introduction

It is well known that the orthogonal projection of a point $P = (a, b)$ of the plane on the line l with equation $y = x$ is the point $(\frac{a+b}{2}, \frac{a+b}{2})$, so that the coordinates of the projection of P are the arithmetic mean of the coordinates of P. This gives a nice geometrical interpretation to the arithmetic mean of two numbers. In a similar way, if we project the point P following the direction given by the vector $(-q, p)$ (with $p, q > 0$ and $p + q = 1$) the projection on l is the point with coordinates $(pa + qb, pa + qb)$ obtaining the weighted arithmetic mean in, again, a geometrical way.

Going back to the arithmetic mean, two points lying in a line perpendicular to l will have the same orthogonal projection onto l and therefore their coordinates will have the same arithmetic mean. In this way, we have a one-parametric family of lines $F = \{x + y = k\}$ with the property that all points of a given line of F have the same arithmetic mean.

A similar situation occurs for weighted arithmetic means where, given the weights $p, q > 0$, $p + q = 1$, the one-parametric family $\{px + qy = k\}$ plays the same role as F.

It seems therefore interesting to study what happens if we permit points to move toward l without the constraint of following a straight line, but allowing more general curves. This paper is devoted to this study.

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Next Section will determine which conditions a family of curves must fulfill to be associated to an (idempotent) aggregation operator and which ones are related to desirable properties of these operators.

Of course, if an aggregation operator should be obtained from a family of curves, they must satisfy at least the following two conditions:

- There must pass a curve of the family through every point of the plane (or some restricted domain).
- No two curves can pass through the same point.

These are necessary conditions for the existence of an ordinary differential equation having the family as solution. If the curves are "smooth enough", then the aggregation operator will have associated this differential equation as well.

Two very easy examples are the arithmetic mean and weighted arithmetic means, the family associated to the first one being the solution of the differential equation $y' = -1$ while the latter to the equation $y' = -\frac{p}{q}$.

Reciprocally, a differential equation fulfilling some conditions will generate an aggregation operator and the properties of this operator can be translated to the equation.

Section 3 will study the relation with aggregation operators, one-parametric families of curves and differential equations.

As particular cases, the one-parametric family of curves and differential equations of the most popular aggregation operators will be given; namely: means, quasi-arithmetic means and OWA operators.

2. Contour curves

Definition 2.1. *An aggregation operator (in two variables) is a map $h : X \times X \to X$ where X is some subset of the real line satisfying*

(1) $Min(x, y) \leq h(x, y) \leq Max(x, y)\ \forall x, y \in X$
(2) $h(x_1, y_1) \leq h(x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ $\forall x_1, x_2, y_1, y_2 \in X$ *(monotonicity)*

h is symmetric if and only if

$$h(x, y) = h(y, x)\ \forall x, y \in X$$

It is straightforward to prove that aggregation operators are *idempotent*, i.e.: they satisfy

$$h(x, x) = x\ \forall x \in X$$

Throughout the paper we will assume that all aggregation operators are continuous.

Given an aggregation operator h, we can consider its contour curves, i.e. the sets of points (x, y) in the domain of h with $h(x, y) = k$ where k is a given constant. In this way we associate a continuous one-parametric family F of curves to h with the particularity that the coordinates of all points P of the same curve have the same aggregation with h, which geometrically are the coordinates of the intersecting point (p, p) of this curve with l. (We will write in short that the point (p, p) is the aggregation of P). Some of the properties of h can be translated to F and visualized by its behavior.

Reciprocally, a continuous one-parametric family F of curves of the plane satisfying certain conditions can be seen as the contour curves of h.

Let us consider a family F of continuous parameterized curves $c_k(t) = (x_k(t), y_k(t))$, $k \in X \subset R$ of the plane such that all of them intersect the line l of equation $y = x$ in a single point and let (p_k, p_k) be the intersection of the curve c_k with l. If we want that (p_k, p_k) could be considered an aggregation of any point (a, b) of $c_k(t)$, some restrictions should be imposed to $c_k(t)$.

First, p_k must be between a and b for any point $(a, b) \in c_k(t)$ $(Min(a, b) \leq p_k \leq Max(a, b))$ which means that (p_k, p_k) must lay between the intersections of the lines $x = a$ and $y = b$ with l.

Proposition 2.2. *Let $F = \{c_k\}$ be a one-parametric family of continuous curves such that each curve c_k of F intersects the line l in a point (p_k, p_k). If F is the family of contour curves of an aggregation operator, then for all points (a_k, b_k) of the curve c_k (p_k, p_k) must lay between the intersections of the lines $x = a_k$ and $y = b_k$ with l.*

For example, the curve partly represented in Figure 1 could be a member of a family F generated by an aggregation operator, while the one of Figure 2 could not.

Next proposition provides a geometric translation of monotonicity.

Proposition 2.3. *Let $F = \{c_k\}$ be the contour family of a map h satisfying 2.1.1. h is non-monotonic if and only if there exists a curve c of F with $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ two points of c with $x_0 \leq x_1$ and $y_0 < y_1$.*

Proof. $\Leftarrow$)

In this case, there would be a point $R = (x_2, y_1)$ with $x_2 < x_1$ in the region of points of the plane above c belonging to another curve c' of the family. Since $c \cap c' = \emptyset$, the aggregation of (x_2, y_1) is greater than the aggregation of (x_1, y_1) and therefore the aggregation operator is non-monotonic. (See Figure 3).

$\Rightarrow$)

Trivial.

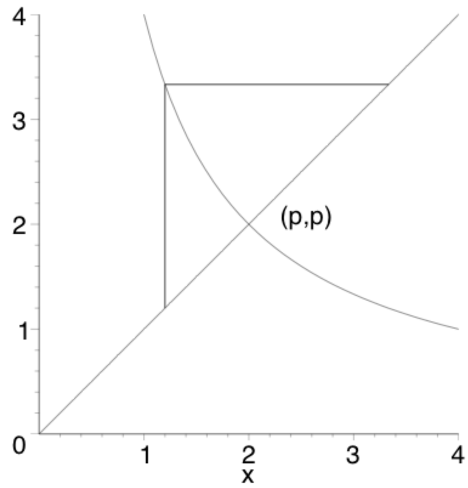

FIGURE 1

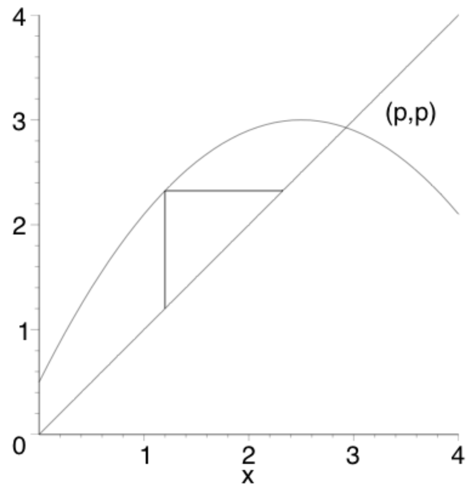

FIGURE 2

In the case that the curves c_k of F are functional, i.e. we can describe c_k with a map $y = f_k(x)$, Proposition 2.3 simply means that the associated map h is monotonic if and only if f_k are non-increasing monotonic maps. $\square$

Symmetry of an aggregation operator can also be easily translated to the behavior of their contour curves.

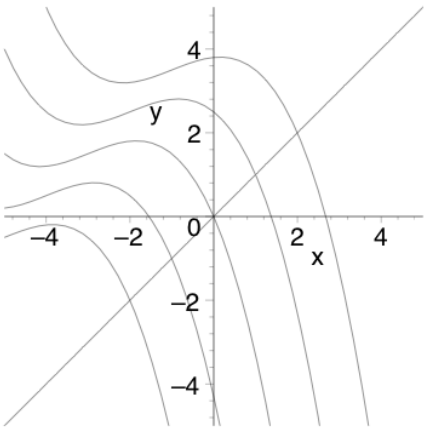

FIGURE 3

Proposition 2.4. *Let $F = \{c_k\}$ be the contour family of an aggregation operator h. h is symmetric if and only if all curves of F are symmetric with respect to the line l (i.e., if (a, b) is a point of curve c_k, then (b, a) is a point of c_k as well).*

If the curves of F are functional ($y = f_k(x)$) then the associated aggregation operator is symmetric if and only if the maps f_k are strictly decreasing with $f_k = f_k^{-1}$ $\forall k$.

3. Differential equations

In the previous Section we have seen the relation between the properties of an aggregation operator h and its family of contour curves $F = \{h(x, y) = k\}$. Let us now suppose that the aggregation has nice differential properties, namely that in some region of the plane there exist $h_x = \frac{\partial h}{\partial x}$ and $h_y = \frac{\partial h}{\partial x} \neq 0$. In this situation, the family F is determined by the differential equation $y' = -\frac{h_x}{h_y}$.

For example, the family F of contour curves of the arithmetic mean $h(x, y) = \frac{x+y}{2}$ is $F = \{x+y = k\}$ that are the solutions of the differential equation $y' = -1$.

Reciprocally, a differential equation $y' = f(x, y)$ satisfying certain conditions will have as solution the family of contour curves of an aggregation operator h. We will say in this case that $y' = f(x, y)$ is the differential equation associated to h.

This Section will study which conditions a differential equation must satisfy for having as solution a family of contour curves of an aggregation operator h and how the properties of h are transfered to the differential equation.

Proposition 3.1. *Let $y' = f(x, y)$ be a differential equation. If $f(x, y) \leq 0\ \forall x, y \in R$ then it is associated to an aggregation operator.*

Proof. If $y' \leq 0$, the curves solution of the equation will not satisfy the hypothesis of Proposition 2.3. □

Proposition 3.2. *The differential equation $y' = f(x, y)$ represents a symmetric aggregation operator iff $f(x, y) \cdot f(y, x) = 1$.*

Proof. This implies that the curves are symmetric with respect to the line $y = x$. □

Example 3.3. Table 1 *shows the most popular symmetric aggregation operators with their respective families of one-parametrized curves F and the corresponding differential equation.*

TABLE 1

	h	F	y'
arithmetic mean	$\frac{x+y}{2}$	$x + y = k$	$y' = -1$
geometric mean	$\sqrt{xy}$	$xy = k$	$y' = -\frac{y}{x}$
harmonic mean	$\frac{xy}{x+y}$	$2xy = k(x + y)$	$y' = -\frac{y^2}{x^2}$
generalized means	$(\frac{x^\alpha+y^\alpha}{2})^{\frac{1}{\alpha}}$	$x^\alpha + y^\alpha = k$	$y' = -\frac{x^{\alpha-1}}{y^{\alpha-1}}$
quasi − arithmetic means	$f^{-1}(\frac{f(x)+f(y)}{2})$	$f(x) + f(y) = k$	$y' = -\frac{f'(x)}{f'(y)}$
OWA operators	$pMax(x, y)$ $+qMin(x, y)$	$pMax(x, y)$ $+qMin(x, y) = k$	$y' = \begin{cases} -\frac{p}{q} & \text{if } x < y \\ -\frac{q}{p} & \text{if } x > y \end{cases}$

Example 3.4. Table 2 *displays the most popular non-symmetric aggregation operators with their respective families of one-parametrized curves F and the corresponding differential equation.*

If $F = \{y = f(x, k)\}$ is a one-parametric family of curves such that for all k $\frac{\partial f}{\partial x}$ is between -1 and 1, then rotating F -45^{o} with respect to the origin $(0, 0)$, we obtain a family of contour curves of an aggregation operator.

TABLE 2

	h	F	y'
weighted arithmeticmean	$px+qy$	$px+qy=k$	$y'=-\frac{p}{q}$
weighted geometric mean	x^py^q	$x^py^q=k$	$y'=-\frac{py}{qx}$
weighted harmonic mean	$\frac{xy}{qx+py}$	$xy=k(qx+py)$	$y'=-\frac{qx^2}{py^2}$
weighted generalized means	$(px^\alpha+qy^\alpha)^{\frac{1}{\alpha}}$	$px^\alpha+qy^\alpha=k$	$y'=-\frac{px^{\alpha-1}}{qy^{\alpha-1}}$
weighted quasi arithmetic means	$f^{-1}(pf(x)+qf(y))$	$pf(x)+qf(y)=k$	$y'=-\frac{pf'(x)}{qf'(y)}$

A nice easy way to generate aggregation operators is therefore to start with a map $y=f(x)$ with $-1\leq f'(x)\leq 1$ $\forall x$ and rotate -45^o the family $F=\{y=f(x)+k\}$ with respect to the origin $(0,0)$.

Definition 3.5. *Let $y=f(x)$ be a map with $-1\leq f'(x)\leq 1$ $\forall x$. m_f will be the aggregation operator whose contour curves are the family obtained form the curves $y=f(x)+k$ rotated -45^o with respect to the origin $(0,0)$.*

The differential equation fulfilled by this family is

$$y'=\frac{f'(\frac{x-y}{\sqrt{2}})-1}{f'(\frac{x-y}{\sqrt{2}})+1}$$

The aggregation m_f is

$$m_f(x,y)=\frac{x+y-\sqrt{2}f(\frac{x-y}{\sqrt{2}})+\sqrt{2}f(0)}{2}$$

Example 3.6. (1) *From $y=\alpha cos(x)$ with $0\leq\alpha\leq 1$ we get the family solution of*

$$y'=\frac{\alpha sin(\frac{x-y}{\sqrt{2}})+1}{\alpha sin(\frac{x-y}{\sqrt{2}})-1}$$

and with aggregation operator

$$m(x,y)=\frac{x+y-\sqrt{2}\alpha cos(\frac{x-y}{\sqrt{2}})+\sqrt{2}\alpha}{2}$$

(See Figure 4).

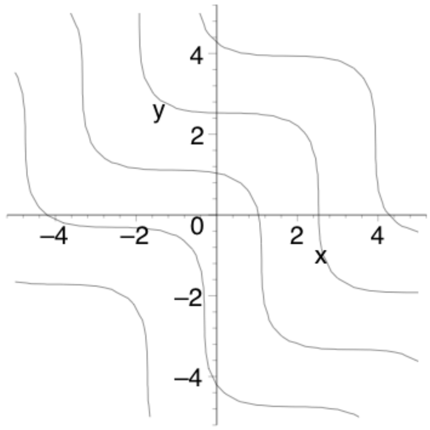

FIGURE 4

(2) *From* $y = \alpha e^{-x^2}$ *with* $0 \leq \alpha \leq \sqrt{\frac{e}{2}}$ *we get the family solution of*

$$y' = \frac{-a\sqrt{2}(x-y)e^{-\frac{(x-y)^2}{2})} - 1}{-a\sqrt{2}(x-y)e^{-\frac{(x-y)^2}{2})} + 1}$$

and with aggregation operator

$$m(x,y) = \frac{x + y - \sqrt{2}\alpha e^{-\frac{(x-y)^2}{2}} + \sqrt{2}\alpha}{2}$$

(See Figure 5).

The following result is straightforward.

Proposition 3.7. m_f *is symmetric if and only if* f *is an even function (i.e.: f(x)=f(-x)).*

4. Concluding Remarks

This paper has provided a first attempt to relate aggregation operators with differential equations. This have been achieved assigning to every (idempotent) aggregation operator in two variables the differential equation which has its contour curves as general solution.

Some properties of the aggregation operator have been translated to properties of the associated differential equation and vice versa.

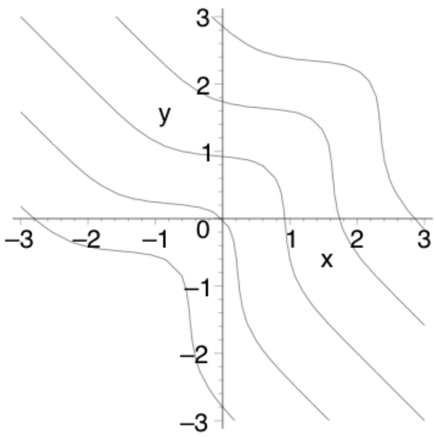

Figure 5

The author will try to extend the results of this paper to more than two variables in forthcoming works.

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EQUALITY OF SPECIAL MIXTURE OPERATORS AND QUASI-ARITHMETIC MEANS

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Abstract. In our paper we introduce the problem of equality of special mixture operators and quasi-arithmetic means. From equality of mixture operators and quasi-arithmetic means we get as solutions the arithmetic, harmonic, geometric means and more special types of aggregation operators belonging simultaneously to both discussed classes.

1. Introduction

Let $I \subset R$ be any interval. Let $\varphi : I \to R$ be a continuous strictly increasing function. For any weighting function $f : I \to]0, \infty[$, φ and f induces a quasi-mixture operator $M_{\varphi}^{f} : \bigcup_{n \in N} I^n \to I$,

$$M_f^{\varphi}(x_1, x_2, \ldots, x_n) = \varphi^{-1}\left(\frac{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum_{i=1}^{n} f(x_i)}\right).$$

For details see [4], [5].

In special case, if transformation function is $\varphi(x) = x$, the quasi-mixture operator induces the mixture operator

$$M_f(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{n} f(x_i) \cdot x_i}{\sum_{i=1}^{n} f(x_i)}.$$

More informations about mixture operators can be found in [3], [6], [8].

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If weighting function $f(x) = const$, quasi-mixture operator goes to the quasi-arithmetic mean

$$M^{\varphi}(x_1, x_2 \dots, x_n) = \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right).$$

See e.g. [1], Section 5.3 in [3].

In our paper we recall the problem of the equality of two quasi-mixture operators, which was solved in [2], [4], [5]. We modify these solutions to solve a related problem of the equality of a mixture operator M_g and a quasi-arithmetic mean M^{η}.

The paper is organized as follows. In the next section we summarize the solutions of the equality of two quasi-mixture operators from [2], [4], [5] described in a transformed way as an equality of a quasi-mixture operator and the arithmetic mean. In the third section we solve the equality problem $M_g = M^{\eta}$ based separately on all introduced solutions from Section 2. Finally, some conclusions are given.

2. Equality of mixture and quasi-mixture operators

In paper [2] Bajraktarević solved the functional equation

$$(1)\qquad \Phi^{-1}\left(\frac{\sum_{i=1}^{n}\Phi(x_i)F(x_i)}{\sum_{i=1}^{n}F(x_i)}\right) = \Psi^{-1}\left(\frac{\sum_{i=1}^{n}\Psi(x_i)G(x_i)}{\sum_{i=1}^{n}G(x_i)}\right), (x_1, \dots, x_n \in I)$$

where $\Phi, \Psi : I \to R$ are the strictly monotonic and continuous functions and $F, G : I \to]0, \infty[$ are weighting functions. He supposed for a fixed $n \geq 3$ and that the functions Φ, Ψ, F and G are twice differentiable and proved that there are constants a, b, c, $d \in R$ such that

$$(c^2 + d^2) \cdot (ad - bc) \neq 0$$

and

$$\Psi(x) = \frac{a\Phi(x) + b}{c\Phi(x) + d} \qquad G(x) = F(x) \cdot (c\Phi(x) + d),$$

what attends to the arithmetic mean.

In paper [5] Losonczi solved the two - variable equality problem of the quasi-mixture operators

$$(2)\qquad \Phi^{-1}\left(\frac{\Phi(X)F(X) + \Phi(Y)F(Y)}{F(X) + F(Y)}\right) = \Psi^{-1}\left(\frac{\Psi(X)G(X) + \Psi(Y)G(Y)}{G(X) + G(Y)}\right)$$

that holds all $X, Y \in I$. He supposed six times differentiability of the functions involved and got 32 new families of solutions.

Daróczy et al. in [4] solved the equality of two quasi-mixture operators $M_F^\Phi = M_G^\Psi$ without differentiability conditions. Authors used the substitutions $x = \Psi(X)$, $y = \Psi(Y)$ and with the definitions $J = \Psi(I)$, $g = G \circ \Psi^{-1}$, $f = F \circ \Psi^{-1}$, $\varphi = \Phi \circ \Psi^{-1}$ the equation (2) was rewritten into

$$\frac{\varphi(x)f(x) + \varphi(y)f(y)}{f(x) + f(y)} = \varphi\left(\frac{g(x)x + g(y)y}{g(x) + g(y)}\right),$$

where $x, y \in J$. They supposed that G is a constant, thus g is a constant too, and they got

$$\varphi^{-1}\left(\frac{\varphi(x)f(x) + \varphi(y)f(y)}{f(x) + f(y)}\right) = \frac{x + y}{2}, \qquad (x, y \in J). \tag{3}$$

The solution of the equality (3) is writen in a regularity theorem in [4], where the pair (φ, f) is a solution on J if and only if it has one of the following forms

	$\varphi(x)$	$f(x)$
(1)	$Ax + D$	E
(2)	$\dfrac{A}{x + C} + D$	$E(x + C)$
(3)	$Atanh(Bx + C) + D$	$Ecosh(Bx + C)$
(4)	$Acoth(Bx + C) + D$	$Esinh(Bx + C)$
(5)	$Atan(Bx + C) + D$	$Ecos(Bx + C)$
(6)	$Aexp(-2Bx) + D$	$Eexp(Bx)$

for all $x \in J$ and for some constants $A, B, C, D \in R$ such that $ABE \neq 0$ and $f(x) > 0$.

Daróczy et al. in [4] for arbitrary g and for recalled couples (φ, f) got the solution of equality of quasi-mixture and mixture operators

$$\varphi^{-1}\left(\frac{\sum\limits_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum\limits_{i=1}^{n} f(x_i)}\right) = \frac{\sum\limits_{i=1}^{n} g(x_i) \cdot x_i}{\sum\limits_{i=1}^{n} g(x_i)}. \tag{4}$$

In the next we remark and analyze separately the equality of quasi-mixture and mixture operators for the couples 1.-6. (φ, f) and arbitrary g.

(1) Function $\varphi(x) = Ax + D$ from the first couple (f, φ) acts the same as $id \equiv x$. For $g(x) = const$ left side and right side of equation (4) give us the arithmetic mean.

(2) The function $\varphi(x) = \frac{A}{x+C} + D$ has the inverse function $\varphi^{-1}(x) = \frac{A}{x-D} - C$ and weighting function is $f(x) = E(x + C)$. The equality (4) can be rewrite as

$$\frac{A}{\frac{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum_{i=1}^{n} f(x_i)} - D} - C = \frac{\sum_{i=1}^{n} g(x_i) \cdot x_i}{\sum_{i=1}^{n} g(x_i)}.$$

For $A = 1$ and $D = 0$ we get

$$\frac{\sum_{i=1}^{n} f(x_i)}{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)} = \frac{\sum_{i=1}^{n} g(x_i) \cdot (x_i + C)}{\sum_{i=1}^{n} g(x_i)}.$$

From the last equality we see for arbitrary f weighting function g is given by

$$g(x) = f(x) \cdot \varphi(x).$$

For arbitrary g we get strictly monotonic function

$$f(x) = g(x) \cdot (x + C) = \frac{g(x)}{\varphi(x)}.$$

For $\varphi(x) = \frac{1}{x+C}$ is satisfied the identity $M_f^\varphi = M_{f \cdot \varphi}$.

- Specially for $C = 0$, $f = const(= 1)$ we get $\varphi(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{x}$. We get quasi-mixture and mixture operator as a harmonic mean:

$$\varphi^{-1}\left(\frac{\sum_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum_{i=1}^{n} f(x_i)}\right) = \frac{\sum_{i=1}^{n} f(x_i)}{\sum_{i=1}^{n} f(x_i) \cdot \varphi(x_i)} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

  and

$$\frac{\sum_{i=1}^{n} g(x_i) \cdot x_i}{\sum_{i=1}^{n} g(x_i)} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}.$$

- For $\varphi(x) = \dfrac{1}{x+C}$ we have

$$M^{\varphi} = \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right) = \frac{n}{\sum\limits_{i=1}^{n}\varphi(x_i)} - C.$$

When $n = 2$ and $C = 0$ we get harmonic mean $M^{\varphi} = \dfrac{2}{\frac{1}{x}+\frac{1}{y}}$.
Similarly mixture operator for $n = 2$ is the same harmonic mean $M_{\varphi} = \dfrac{2}{\frac{1}{x}+\frac{1}{y}}$, so $M^{\varphi} = M_{\varphi}$. So quasi-mixture operator and mixture operator are the same for $\varphi(x) = \dfrac{1}{x+C}$.

(3) For $\varphi(x) = Atanh(Bx + C)$ the inverse function is given by $\varphi^{-1}(x) = \dfrac{arctanh\frac{x}{A} - C}{B}$ and weighting function $f(x) = Ecosh(Bx+C)$. If $A = 1$, $E \neq 0$, $B = 1$, $C = 0$ we get quasi-mixture operator as arithmetic mean $M_f^{\varphi} = \dfrac{x+y}{2}$. For $g(x) = const$, $n = 2$ we get mixture operator as arithmetic mean too $M_f^{\varphi} = \dfrac{x+y}{2}$.

(4) In fourth case the function $\varphi(x) = Acotanh(Bx + C)$ has the inverse function $\varphi^{-1}(x) = \dfrac{arccotanh\frac{x}{A} - C}{B}$ and $f(x) = Esinh(Bx + C)$. If $A = 1$, $E \neq 0$, $B = 1$, $C = 0$ we get quasi-mixture operator as arithmetic mean $M_f^{\varphi} = \dfrac{x+y}{2}$ and for $g(x) = const$, $n = 2$ we get mixture operator again as arithmetic mean $M_g = \dfrac{x+y}{2}$.

(5) For the couple $\varphi(x) = Atan(Bx + C)$,$f(x) = Ecos(Bx + C)$ we have $\varphi^{-1}(x) = \dfrac{arctan\frac{x}{A} - C}{B}$. The equality (4) for this pair and for $A = 1$, $E \neq 0$ is given by

$$\frac{arctan\left(\frac{\sum_{i=1}^{n}\varphi(x_i)\cdot f(x_i)}{\sum_{i=1}^{n} f(x_i)}\right) - C}{B} = \frac{\sum\limits_{i=1}^{n} g(x_i)\cdot x_i}{\sum\limits_{i=1}^{n} g(x_i)}.$$

After some processing we get

$$arctan\left(\frac{\sum_{i=1}^{n}\varphi(x_i)\cdot f(x_i)}{\sum_{i=1}^{n}f(x_i)}\right)=\frac{\sum_{i=1}^{n}g(x_i)\cdot(Bx_i+C)}{\sum_{i=1}^{n}g(x_i)}.$$

For $n=2$, $B=1$, $C=0$ we get arithmetic mean $M_f^{\varphi}=\frac{x+y}{2}$ and once more for $g(x)=const$ again we get $M_g=\frac{x+y}{2}$.

(6) For the last couple $\varphi(x)=Aexp(-2Bx)$, $f(x)=Eexp(Bx)$, $\varphi^{-1}(x)=-\frac{1}{2B}ln\frac{x}{A}$ the equation (4) for $A=1$, $E\neq 0$ we rewrite as

$$-\frac{1}{2B}ln\left(\frac{\sum_{i=1}^{n}\varphi(x_i)f(x_i)}{\sum_{i=1}^{n}f(x_i)}\right)=\frac{\sum_{i=1}^{n}g(x_i)\cdot x_i}{\sum_{i=1}^{n}g(x_i)}.$$

For $n=2$ we get the equation

$$\frac{1}{2B}ln\frac{exp(Bx)+exp(By)}{\frac{exp(Bx)+exp(By)}{exp(Bx)\cdot exp(By)}}=\frac{\sum_{i=1}^{2}g(x_i)\cdot x_i}{\sum_{i=1}^{2}g(x_i)}.$$

For $B=const$ we get quasi-mixture operator as the arithmetic mean $M_f^{\varphi}=\frac{x+y}{2}$. And once more if $g(x)=const$, we get mixture operator as the arithmetic mean $M_g=\frac{x+y}{2}$.

Remark 2.1. For the first two couples (φ, f) the equation (3) is satisfied also if we reformulate it for $n\geq 2$, while for the other couples (φ, f) the equation (3) is satisfied only for $n=2$.

Remark 2.2. Note that for all A, D from R, $A\neq 0$, $E>0$, and for all φ, f it holds $M_f^{\varphi}=M_{Ef}^{A\varphi}+D$. Therefore, when solving the problem of equality of different types of quasi-mixture operators, it is enough to assume $A=E=1$ and $D=0$ when ever this is convenient.

3. Equality of special mixture operators and quasi-arithmetic means

Each quasi-mixture operator M_f^φ on an interval I can be identified with the couple (φ, f). The equality of two quasi-mixture operators $M_f^\varphi = M_g^\eta$ alows to introduce an equivalence $(\varphi, f) \approx (\eta, g)$ for the corresponding pairs of generating and weighting functions.

Proposition 3.1. *Let $I \subset R$. Let φ, η: $I \to R$ be continuous strictly monotone functions and f, g: $I \to]0, \infty[$ be weighting functions.*
Let $\tau : J \to I$ be an increasing bijection and $I = \tau(J)$. Then the equivalence $(\varphi, f) \approx (\eta, g)$ (on interval I), holds if and only if the equivalence $(\varphi \circ \tau, f \circ \tau) \approx (\eta \circ \tau, g \circ \tau)$ (on interval J) is true.

Proof. Suppose that $(\varphi, f) \approx (\eta, g)$, i. e., for all $(x_1, \dots, x_n) \in I^n$ it holds

$$\varphi^{-1}\left(\frac{\sum\limits_{i=1}^{n} \varphi(x_i) \cdot f(x_i)}{\sum\limits_{i=1}^{n} f(x_i)}\right) = \eta^{-1}\left(\frac{\sum\limits_{i=1}^{n} \eta(x_i) \cdot g(x_i)}{\sum\limits_{i=1}^{n} g(x_i)}\right). \tag{5}$$

We have to show the equality

$$\begin{aligned} (\varphi \circ \tau)^{-1}\left(\frac{\sum\limits_{i=1}^{n} \varphi \circ \tau(u_i) \cdot f \circ \tau(u_i)}{\sum\limits_{i=1}^{n} f \circ \tau(u_i)}\right) &= \\ = (\eta \circ \tau)^{-1}\left(\frac{\sum\limits_{i=1}^{n} \eta \circ \tau(u_i) \cdot g \circ \tau(u_i)}{\sum\limits_{i=1}^{n} g \circ \tau(u_i)}\right) & \end{aligned} \tag{6}$$

for all $(u_1, \dots, u_n) \in J^n$.
Recall that $(\varphi \circ \tau)^{-1} = \tau^{-1} \circ \varphi^{-1}$, and thus the equality (6) can be rewritten into

$$\varphi^{-1}\left(\frac{\sum\limits_{i=1}^{n} \varphi \circ \tau(u_i) \cdot f \circ \tau(u_i)}{\sum\limits_{i=1}^{n} f \circ \tau(u_i)}\right) = \eta^{-1}\left(\frac{\sum\limits_{i=1}^{n} \eta \circ \tau(u_i) \cdot g \circ \tau(u_i)}{\sum\limits_{i=1}^{n} g \circ \tau(u_i)}\right).$$

Now, it is enough to put $\tau(u_i) = x_i$ and apply the equality (6).
The opposite implication is immediate. □

Our aim is to find the solutions of the equivalence problem $(id, g) \approx (\eta, const)$. Recall that in the Section 2 we have summarized the results from [2], [4], [5] solving the equivalence problem $(\varphi, f) \approx (id, const)$. Based on Proposition 3.1.

and putting $\tau = \varphi^{-1}$, we see that we can transform the solutions of $(\varphi, f) \approx (\eta, const)$ into

$$(id, f \circ \varphi^{-1}) \approx (\varphi^{-1}, const). \tag{7}$$

Now, it is enough to put $g = f \circ \varphi^{-1}$ and $\eta = \varphi^{-1}$ to get the desired solutions of the equality of the mixture operators and the quasi-arithmetic means. Now we will analyze all cases 1 - 6 summarized in Section 2.

(1) In this case we have only the trivial solution

$$(id, const) \approx (Aid + D, const)$$

yielding the arithmetic mean M, independently of the interval $I \subset R$ and for the arbitrary $n \in N$.

(2) Due to the Remark 2.2., we can assume $A = E = 1$. Then for φ given by $\varphi(x) = \dfrac{1}{x+C} + D$ (necessarily defined on a subinterval of $]-\infty, C[$ or $]C, \infty[$) we have $\varphi^{-1}(x) = \dfrac{1}{x-D} - C$. Applying the equivalence (6), we can define g by $g(x) = \dfrac{1}{x-D}$, and to ensure the positiveness of g, necessarily it should be defined on a subinterval J of $]D, \infty[$. Moreover, taking into account Remark 2.2., we can put $\eta = g$. Hence the operator $H_D :]D, \infty[^n \to]D, \infty[$ given for any $n \in N$ and any $(x_1, \ldots, x_n) \in]D, \infty[^n$ by

$$H_D(x_1, \ldots, x_n) = \frac{n}{\sum\limits_{i=1}^{n} \dfrac{1}{x_i - D}} + D = \frac{\sum\limits_{i=1}^{n} \dfrac{x_i}{x_i - D}}{\sum\limits_{i=1}^{n} \dfrac{1}{x_i - D}}$$

is both a mixture operator and a quasi-arithmetic mean. Observe that for $D = 0$ we recover the standard harmonic mean H and that

$$H_D(x_1, \ldots, x_n) = H_0(x_1 - D, \ldots, x_n - D) + D.$$

(3) For the third couple (φ, f), similarly as in the previous case, we can assume $A = E = 1$. The function φ is given by $\varphi(x) = tanh(Bx + C) + D$ is defined on R and its inverse function $\varphi^{-1}(x) = \dfrac{arctanh(x-D) - C}{B}$ is defined on the interval $]-1+D, 1+D[$. Recall that the corresponding weighting function f is given by $f(x) = cosh(Bx + C)$. Applying our couple on the equivalence (7) we get the weighting function $g :]-1+D, 1+D[\to]0, \infty[$ given by $g(x) = cosh(arctanh(x - D))$. Denote by $M_D^{(3)}$ the mixture operator M_g which is also a quasi-arithmetic mean M^{η},

$\eta = \varphi^{-1}$.This equality is true only for $n = 2$ and $M_D^{(3)}$ is given by

$$M_D^{(3)}(x,y) = \frac{x \cdot cosh(arctanh(x-D) + y \cdot cosh(arctanh(y-D))}{cosh(arctanh(x-D) + cosh(arctanh(y-D))} =$$

$$= tanh\left[\frac{1}{2}\left(arctanh(x-D) + arctanh(y-D)\right)\right] + D$$

is mixture operator and quasi- arithmetic mean. After some proccesing for $D = 0$ we can rewrite our mixture operator and quasi-arithmetic mean by

$$M_0^{(3)}(x,y) = \frac{x+y}{1 + xy + \sqrt{(1-x^2)\cdot(1-y^2)}}.$$

Recall that also in this case $M_D^{(3)}(x,y) = M_0^{(3)}(x-D, y-D) + D$.

(4) The function $\varphi(x) = cotanh(Bx+C)+D$ is defined on the subinterval $]-\infty, -\frac{C}{B}[$ or $]-\frac{C}{B}, \infty[$ and its inverse function $\varphi^{-1}(x) = \dfrac{arccotanhx - D - C}{B}$, necessarily it should be defined on a subinterval J of $]-\infty, B[\cap]-\infty, -1+D[$ or $]B, \infty[\cap]1+D, \infty[$. The corresponding weighting function f is given by $f(x) = sinh(Bx+C)$. To ensure positivness of $f \circ \varphi^{-1}$ in equivalence (6) we get the weighting function $g :]1+D, \infty[\to]0, \infty[$ given by $g(x) = sinh(arccotanh(x-D))$.

Denote by $M_D^{(4)}$ the mixture operator M_g which is also a quasi-arithmetic mean M^η, $\eta = \varphi^{-1}$. This equality is true only for $n = 2$ and $M_D^{(4)}$ is given by

$$M_D^{(4)}(x,y) = \frac{x \cdot sinh(arccotanh(x-D) + y \cdot sinh(arccotanh(y-D))}{sinh(arccotanh(x-D) + sinh(arccotanh(y-D))} =$$

$$= cotanh\left[\frac{1}{2}\left(arccotanh(x-D) + arccotanh(y-D)\right)\right] + D,$$

which is mixture operator and quasi- arithmetic mean. After some proccesing for $D = 0$ we can rewrite our mixture operator and quasi-arithmetic mean by

$$M_0^{(4)}(x,y) = \frac{x+y}{1 + xy - \sqrt{(x^2-1)\cdot(y^2-1)}}.$$

Recall that also in this case $M_D^{(4)}(x,y) = M_0^{(4)}(x-D, y-D) + D$.

(5) The function $\varphi(x) = tan(Bx+C) + D$ is defined on a subinterval of $]\dfrac{\frac{-\pi}{2} - C}{B}, \dfrac{\frac{\pi}{2} - C}{B}[$ and its inverse function $\varphi^{-1}(x) = \dfrac{arctan(x-D) - C}{B}$ necessarily it should be defined on a subinterval J of $]-\infty, B[$ or $]B, \infty[$.

Note, that weighting function g from this couple is given by $g(x) = cos\,(arctan(x-D)) = \dfrac{1}{\sqrt{1+(x-D)^2}}$ and is positive for $x \in R$. Denote by $M_D^{(5)}$ the mixture operator M_g which is also a quasi-arithmetic mean M^η, $\eta = \varphi^{-1}$. This equality is true only for $n = 2$ and $M_D^{(5)}$ is given by

$$M_D^{(5)}(x,y) = \frac{x \cdot cos(arctan(x-D) + y \cdot cos(arctan(y-D))}{cos(arctan(x-D) + cos(arctan(y-D))} =$$

$$= tan\left[\frac{1}{2}\left(arctan(x-D) + arctan(y-D)\right)\right] + D$$

is both a mixture operator and a quasi-arithmetic mean. For $D = 0$ and using some proccesing we get

$$M_0^{(5)}(x,y) = \frac{x+y}{1 - xy + \sqrt{(1+x^2)\cdot(1+y^2)}}.$$

Recall that also in this case $M_D^{(5)}(x,y) = M_0^{(5)}(x-D, y-D) + D$.

(6) For the sixth couple (φ, f) we have $\varphi(x) = exp(-2Bx) + D$ defined on the interval R. Due inverse function $\varphi^{-1}(x) = -\dfrac{1}{2B} ln(x-D)$ is defined on a subinterval J of $]-\infty, B[\, \cap \,]D, \infty[$ or $]B, \infty[\, \cap \,]D, \infty[$. Weighting function $f \circ \varphi^{-1}$ from the equivalence (6) should be positive, so necessarily is defined on the subinterval J of $]D, \infty[$ and we get the weighting function $g :]D, \infty[\to]0, \infty[$ given by $g(x) = \frac{1}{x-D}$. Denote by G_D the mixture operator M_g which is also a quasi-arithmetic mean M^η, $\eta = \varphi^{-1}$. This equality is true only for $n = 2$ and G_D is given by

$$G_D(x,y) = \frac{\dfrac{x}{x-D} + \dfrac{y}{y-D}}{\dfrac{1}{x-D} + \dfrac{1}{y-D}} = \sqrt{(x-D)(y-D)} + D.$$

For $D = 0$ we recover the standard geometric mean G and that

$$G_D(x,y) = G_0(x-D, y-D) + D.$$

4. Concluding remarks

We have discussed special equality of quasi-mixture, mixture operators and quasi-arithmetic means with special stress on their identity. By solving our equation (6) for different pairs (φ, f) we can conclude, that the intersection of special mixture operators and quasi-arithmetic means includes the arithmetic mean, the harmonic mean, the geometric mean and special type of mixture operators.

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