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DISTANCE BETWEEN FUZZY SETS AS A FUZZY QUANTITY

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Abstract. The traditional methods of comparing images, like using the Hamming distance, may sometimes fail, especially if we do not insist on careful checking all the details of the images, but compare them just broadly. An n-dimensional image with various grades of grey colours can be represented by a fuzzy set. We introduce a method of estimating the difference between such images by a fuzzy set, which corresponds to various levels of identifying close parts of the given images, or, in other words, to the grade of accuracy, with which the images are observed. Examples and some properties of such a distance are shown.

1. Introduction

A fuzzy subset of a space X can be interpreted as a model for the image on X containing various shapes of gray colour. The membership degrees then correspond to grades of darkness, when 0 can be assigned to white and 1 to black colour (or vice versa). Conversely, a gray image can be represented by a corresponding fuzzy set.

A natural question is to estimate the grade of similarity of two such images, which is analogical to estimating the distance between two fuzzy sets. There are several attitudes to this problem, which can be divided into two groups. The first one works with differences between membership values at particular points of X. Another one is based on differences between cuts at particular levels (see e.g. Cabrelli et al. in [2], [3] and [4]).

However, both mentioned concepts can lead to unsatisfactory results from the applications point of view. The examples of such cases can be found in [8], where Lowen and Peeters also show the way how to avoid such problems. They suggest a way to estimate the distance between two fuzzy sets accounting both differences between membership values and between cuts. However, the result is a single real number, which may in some applications mean the loss of information. Our aim

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is to develop the results of Lowen and Peeters so that we obtain a fuzzy quantity which reflects the difference between two given fuzzy sets.

Another problem, which may appear for example in pattern recognition is, that not all the points in the space X may have the same importance. The noise at the edge of a screen can be sometimes considered not so important as the noise in its centre. Although this is just a technical problem, we also incorporate it in our consideration.

2. Concepts of measuring differences between fuzzy sets

There are many different attitudes to comparing fuzzy sets that can be found in the literature. Generally they are based on one of the two principles, which we shortly describe below.

In many occasions the comparison of two fuzzy sets is done by quantifying the degree of similarity or equality between them (see, for instance [5], [11] or [12]), but there hardly are references related to the degree of inequality or difference between them.

In [1] the authors proposed a measure of similarity between fuzzy sets and also a measure of dissimilarity. Thus, they defined a μ-measure of dissimilarity on X as a function $S : F(X) \times F(X) \to [0; 1]$ such that

$$S(m, n) = F_S(\mu(m \cap n), \mu(n - m), \mu(m - n)),$$

where μ is a measure on X and $F_S : [0; \infty)^3 \to [0; 1]$ is a function independent of the first coordinate, increasing in the other two and such that $F(x, 0, 0) = 0$ for all $x \in [0; \infty)$.

The most frequent definitions of classical distances between fuzzy sets m, n in a universe X are:

- The Hamming distance:

$$d(m, n) = \sum_{x \in X} |m(x) - n(x)|.$$

- A generalization of the Hamming distance proposed by Kacprzyk in [6]

$$d(m, n) = \sum_{x \in X} |m(x) - n(x)|^2.$$

- The generalization of the previous ones, using the Minkowski distance (see e.g. [7])

$$d(m, n) = \left(\sum_{x \in X} |m(x) - n(x)|^r \right)^{\frac{1}{r}}, \ r \geq 1.$$

This class of distances includes, as a particular case, the supremum distance, used to compare fuzzy sets among others by Nowakowska in [10] and Wenstøp in [13]. Its definition is

$$d(m,n) = \sup_{x \in X} |m(x) - n(x)|.$$

All these distances are particular cases of the dissimilarity measures defined in [1].

In relation to dissimilarities Montes et al. introduced in [9] the definition of divergence measure as a map $D : F(X)^2 \to R$ such that for all $m, n, \rho \in F(X)$ the following conditions are satisfied:

(1) $D(m,n) = D(n,m)$,
(2) $D(m,m) = 0$,
(3) $\max\{D(m \cup \rho, n \cup \rho), D(m \cap \rho, n \cap \rho)\} \leq D(m,n)$.

This definition generalizes, except for the symmetry property (that could be excluded from the set of axioms in some particular cases) the concept of dissimilarity measures previously proposed. Moreover, local divergencies are distances between fuzzy sets according to the definition proposed in [14], which will be recalled in Definition 3.

All these measures were applied in different fields, but they are not too appropriate for some very natural circumstances as we will explain in the following.

3. Distance function and distance

Suppose (X,d) is a pseudometric space, let $F(X)$ denote the system of all its fuzzy subsets. Let $m, n \in F(X)$. For each $x \in X$ we assign a nonincreasing function f_x such that $f_x : [0,1] \to [0;\infty]$. To be compatible with [8] we may call f_x a tolerance function for x. The shape of this function depends on the importance of the point x in the image (for better understanding see the examples later in this paper).

First we define a distance function at a point.

Definition 1. *Let $S(x,r)$ be the closed neighborhood of x with diameter r, let $m, n \in F(X)$. If $x \in X$, then the mapping $g_x^{m,n} : [0;1] \to [0;1]$ such that*

$$g_x^{m,n}(\alpha) = \inf\{|m(z) - n(y)|; z, y \in S(x, f_x(\alpha))\}$$

is called the distance function at a point x.

Here we follow the idea of a tolerance introduced by Lowen and Peeters in [8], but in our attitude the tolerance is not constant.

It is easy to see that any distance function is nondecreasing. The purpose of such a function is to model the grade of accuracy with which the image is observed. The value $\alpha = 0$ corresponds to the "least careful" view of the image, while the value of $\alpha = 1$ models the "most detailed" look at it. A good example has been given in [8], namely two chessboards with a very large number of rows and columns, inverse to each other. At a close look we see that they are totally different, but from a large distance we do not distinguish small squares, but observe two identical large (gray) squares.

The distance function at a point enables us to define the main notion of this work.

Definition 2. *Let for each $x \in X$ be $g_x^{m,n}$ its distance function. The distance between the fuzzy sets m and n is then given by the fuzzy set $g^{m,n} : [0,1] \to [0,1]$ defined for $\alpha \in [0,1]$ as follows:*

$$g^{m,n}(\alpha) = \sup\{g_x^{m,n}(\alpha), x \in X\}.$$

Thus we obtain a fuzzy quantity which gives us more information about two fuzzy sets than a single number, as it can be seen from examples in the following section.

The distance defined above has properties similar to some of the distance measure, as it was introduced in [14]. We recall its definition:

Definition 3. *Let $F(X)$ be the system of al fuzzy sets on a universe X. A function $\delta : F(X)^2 \to [0, \infty[$ is called a distance measure if it satisfies the following properties:*

(1) $\delta(A,B) = \delta(B,A)$ *for all* $A, B \in F(X)$,
(2) $\delta(A,A) = 0$ *for all* $A \in F(X)$,
(3) $\delta(D, X \setminus D) = \max_{A,B \in F(X)} \delta(A,B)$ *for all crisp subsets D of X,*
(4) *if $A \subseteq B \subseteq C$, then $\delta(A,B) \leq \delta(A,C)$ and $\delta(B,C) \leq \delta(A,C)$ for all $A, B, C \in F(X)$.*

Clearly the distance as we have defined it, cannot be a distance measure, as its values are not real numbers, but fuzzy quantities. However, it has some similar properties, which are formulated in the following propositions.

Proposition 1. *If $m, n \in F(X)$, then $g^{m,n} = g^{n,m}$.*

Proposition 2. *If $m \in F(X)$, then $g^{m,m}$ is a zero function.*

Both propositions follow directly from the definition of the distance. The transitivity for our definition is preserved by means of the fact, that for $m, n, p \in F(X), m \leq n \leq p$ the difference $n - m$ differs less from the zero function, than the difference $p - m$.

Proposition 3. *If $m, n, p \in F(X), m \leq n \leq p$, then $g^{0,n-m} \leq g^{0,p-m}$.*

Proof. Clearly it is sufficient to prove the statement for $r, s \in F(X), r \leq s$, as $n - m \leq p - m$. But if $r \leq s$, then

$$\inf\{r(y), y \in S(x, f_x(\alpha))\} \leq \inf\{s(z), z \in S(x, f_x(\alpha))\}$$

for all $x \in X, \alpha \in [0, 1]$. This means that $g_x^{0,r} \leq g_x^{0,s}$ for all $x \in X$. Using suprema to get the distance functions and the fact that the inequality remains also for them, we conclude $g^{0,r} \leq g^{0,s}$. Putting $r = n - m, s = p - m$ we finish the proof.

The only property of distance measure, which cannot be mechanically transferred for our distance, is the third one, stating that any crisp set and its complement have the maximal possible distance measure. This is no surprise, as our attitude is based on the principle that (using the language of pattern recognition) considers the white patterns with small pieces of black color as a kind of fuzzy sets. However, for sets, that are "crisp enough" a kind of a similar property holds.

Proposition 4. *Let D be a (crisp) subset of X. Then*

$$g^{D,X\setminus D} = \max\{g^{m,n}; m, n \in F(X)\}$$

if and only if there is an $x_0 \in D$ such that $S(x_0, f_{x_0}(0)) \subseteq D$.

Proof. Let D be a (crisp) subset of X. For the convenience we will denote by the same letter its characteristic function, as well as for its complement. Suppose there is an $x_0 \in D$ such that $S(x_0, f_{x_0}(0)) \subseteq D$. Then for the distance function at x_0 we have

$$g_{x_0}^{D,X\setminus D}(0) = \inf\{d(D(y), (X \setminus D)(z)); y, z \in S(x_0, f_{x_0}(0))\} = 1$$

due to the assumption which asserts that $D(y) = 1$ and $(X \setminus D)(z) = 0$ for any $y, z \in S(x_0, f_{x_0}(0))$.

As any distance function is nondecreasing, we have $g_{x_0}^{D,X\setminus D}(\alpha) = 1$ for all $\alpha \in [0, 1]$, hence also $g^{D,X\setminus D}(\alpha) = 1$ for all $\alpha \in [0, 1]$. Evidently this is the maximal possible distance for any pair of fuzzy sets in $F(X)$.

To show the reverse implication let us assume that for all $x \in D$ there is

$$S(x, f_x(0)) \cap (X \setminus D) \neq \emptyset.$$

Then for any $x \in D$ we have

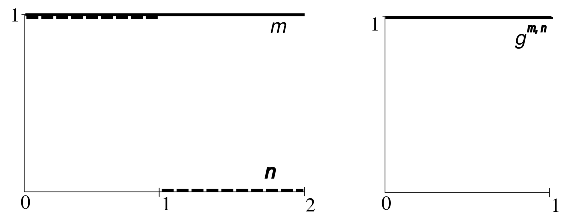
$$g_x^{D,X\setminus D}(0) = \inf\{d(D(y), (X \setminus D)(z)); y, z \in S(x, f_x(0))\} = 0$$

as in each $S(x, f_x(0))$ there is a point belonging to D and also a point in its complement.

4. Examples

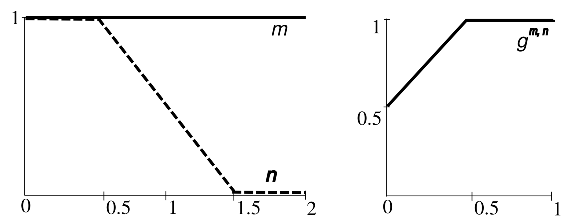
In the following we present a series of simple examples which demonstrate some properties of the distance. In all of them the space X will be the interval $[0, 2]$ with the usual metric. The tolerance function for all the points of X in Examples 1 – 4 will be $f_x(\alpha) = 1 - \alpha$. In each example we present the graphs of m, n and their difference $g^{m,n}$. For better understanding it is good to think of the fuzzy sets used in the examples as of image representations, where 1 represents black and 0 white color and the values between correspond to degrees of grey color. In all the graphs m is sketched in a full line, n in a dashed one.

Example 1. $m(x) = 1$, $n(x) = 1$ for $x \in [0, 1]$, otherwise $n(x) = 0$.



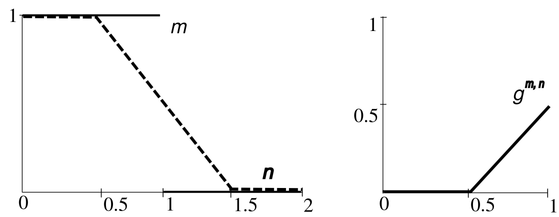
Here the fuzzy set n is in the sense of Proposition 4 sufficiently crisp to have the maximal possible distance from m.

Example 2. $m(x) = 1$, $n(x) = 1$ for $x \in [0, 0.5]$, $n(x) = 1.5 - x$ for $x \in]0.5, 1.5[$, $n(x) = 0$ otherwise.



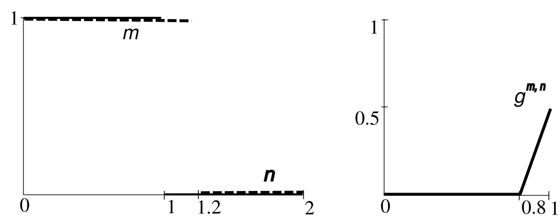
As we see, considering only the membership degrees close to edge values (not mentioning the shades of gray), both sets are far from each other. If we consider also degrees of gray color, they are closer than in the previous example.

Example 3. $m(x) = 1$ for $x \in [0, 1]$, $m(x) = 0$ otherwise, n is the same as in the previous example.



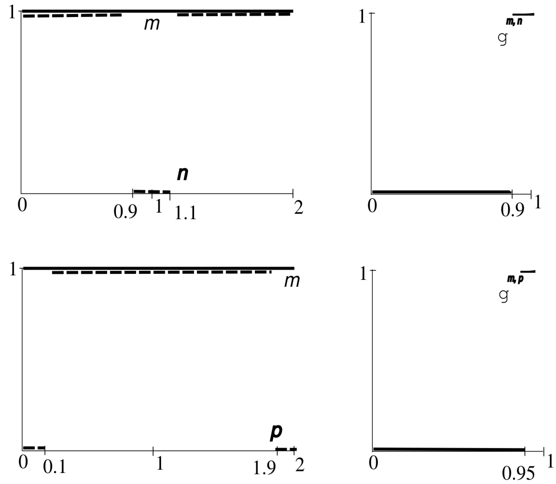
In this example we see that the more attention is paid to the grey colors, the closer are the images.

Example 4. $m(x) = 1$ for $x \in [0, 1], m(x) = 0$ otherwise, Let n be the crisp set $[0; 1.2]$. In the graph of the distance function we see that these sets are closer to each other than the pair in the previous example. Although the difference in colours in the middle of the image is sharp, it is just on a set with a small measure and in the broader view these sets tend to coincide.



The following example shows the possibility to assign various importance to different parts of the underlying space. We see that although the measure of the space where noise (white color in a black image) is present is the same, the distance is bigger if the noise appears in the center of the space X.

Example 5. Let for $x \in [0.5, 1.5]$ the tolerance function be $f_x(\alpha) = 1 - \alpha$, for the remaining points in X let $f_x(\alpha) = 2 - 2\alpha$. Let m be the crisp set $[0, 2]$, let n be the crisp set $[0, 0.9] \cup [1.1, 2]$, let p be the crisp set $[0.1, 1.9]$.



Here we see that due to the smaller importance of the points closer to the end-points, the distance between m and n is larger than the distance between m and p.

5. Concluding remarks

We have defined a distance of a pair of fuzzy sets expressed by a mapping which enables us to estimate the similarity of given fuzzy sets depending on the level, on which we identify points close to each other. As we have shown, the properties of such a distance have much in common with the properties of the distance measure from [14]. Moreover, it has the following properties, which are quite obvious:

If f_x is a zero function for all x, then our distance is equivalent to the supremum distance usually denoted by d_∞, (the distance used in [10] or [13]). This means that in such case $g^{m,n}$ is a constant function with its value $d_\infty(m,n)$.

If f_x are all equal to the same constant τ, then our distance is equivalent to that introduced in [8].

If $f_x(1) > 0$, then the noise in the singleton $\{x\}$ is ignored. Moreover, if $f_x(1) \geq c$ for all $x \in X$, then also the noise on sets with diameter not exceeding c is ignored.

By putting $s^{m,n} = 1 - g^{m,n}$ we obtain s with properties similar to similarity measure as was introduced in [14].

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