Special issue dedicated to 70th birthday of
Alfonz Haviar

edited by

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Editorial

Fonzo Haviar is seventy this year

Alfonz Haviar was born on February 24, 1939, in Poluvsie, a small village in Central Slovakia. After attending a secondary school in Prievidza in 1954-57, he entered Faculty of Natural Sciences at Pedagogic School in Bratislava where he studied teaching combination mathematics-physics. After graduation in 1961, he started to work as a teacher of mathematics and physics at secondary school in Stropkov, a small town in Eastern Slovakia. Here he nurtured and trained talented students of which many were successful in mathematical and physics olympiads.

Fonzo Haviar, one of the most emblematic figures of Banská Bystrica mathematics.

In 1965 he moved to Banská Bystrica where he was interviewed and subsequently awarded a position of an assistant professor at Mathematics Department of Pedagogic Faculty. Fonzo remained faithful to this department for his entire professional life. The department, originally belonging to Pedagogic Faculty, became a part of newly-established Matej Bel University in 1992.

Since 1970 until 1992 he regularly attended algebraic seminars in Bratislava headed by the legendary professor Milan Kolibiar. The seminar was usually held every two weeks during the semester and according to seminar's program, each of the participants gave 3–4 talks on new results in universal algebra, lattice theory or posets. Usually papers from research journals or submitted manuscripts were read, whose copies were obtained by different canals from the western world. Sometimes these were parts of monographs which were often still in preparation. Occasionally, own results of the participants were presented. Many of the seminar members have nice memories also on traditional (at the time) algebra winter schools.

In Bratislava Fonzo was awarded a degree RNDr. (Rerum Naturalium Doctor) in 1973 and a PhD degree (Candidate of Sciences in those times) in 1981 under the supervision of prof. Milan Kolibiar. Since 1984 until now he has been holding a position of an associate professor (called also Reader or Dozent) at the Department of Mathematics. He was the head of this department for two periods, 1998-2001 and 2004-2006.

In 1992, after Matej Bel University was established and prof. Ján Findra became its first rector, he chose Fonzo to the position of the vice-rector responsible for the further development of the university. Their main goal was that the university should first of all become a research and cultural center of the Central Slovakian region. The idea of promoting an international research scarred many members of the academic community whose ideas of university research were associated with certain regional explorations.

Fonzo's main merit is that, in conditions of a regional higher education institution, he recognized the importance of a research on an international level and its necessity for further development of this institution. He was one of few who advocated this idea while for many others the research was either a kind of a hobby of few "odd fellows" or a synonym of writing papers in obscure regional publications (just in order their list of papers is long enough so that they be eligible for becoming full professors). In that period of time Fonzo played a crucial role in discussions about the future routing of the university. Also thanks to him it was the goal of establishing a research university which finally won (at least apparently), and not the strategy of a university aimed at producing new 'professors' by means of decreasing requirements for the title of "Professor" and doing business at the expense of doing research. However, Fonzo's service in the position of a vice-rector ended in 1993 with the appointment of prof. Findra as the head of the Presidential office in Bratislava. Unfortunately, this happened earlier than the tendency towards a research university could become irreversible. Otherwise, science at Matej Bel University would almost certainly be in a stronger position today. Fortunately, Fonzo's strong influence sustained at least at the Department of Mathematics of Faculty of Natural Sciences. No matter whether he was a head of the department or just a member of it, he always emphasized the necessity of a research, which also affected the personal policy at the department. It is therefore also his merit that today Banská Bystrica belongs to important mathematical centers in Slovakia.

After 1990 Fonzo also had various duties in academic senates; in 1994-95 he was the principal of the academic senate of the Faculty of Humanities and Natural Sciences. He was a member of the scientific board of Matej Bel University in 1994-2000 and the statutory ambassador of the Foundation Matthias Belius for supporting university science and research in 1995-96. In 1993 he established the journal Acta Univ. M. Belii, ser. Mathematics and until 2000 was its editor-in-chief. Fonzo served the academic community also as a member of Board ŠVOČ (students' research competition), a head-teacher at different levels, a member of the scientific boards of the faculty and the university. His sharp thinking, strong will and organizational skills were always beneficial for the academic community in Banská Bystrica.

Fonzo also served the academic and in particular mathematical communities at regional and national levels. He was an advisor of the regional pedagogic institute for modernizing teaching of mathematics at secondary schools in 1972-75, a member of the Czechoslovak committee for teaching mathematics and drawing at primary schools in 1973-84 and a member of the committee for doctoral studies in didactics of mathematics in 1998-2005. He was also a member of different committees and boards within the Union of Czech and Slovak Mathematicians and Physicists (JČSMF), and later within the Union of Slovak Mathematicians and Physicists (JSMF).

Now, after his 45-year pedagogic work at the Department of Mathematics in Banská Bystrica, he says he taught all subjects in future teacher training except probability and statistics. However, in his pedagogic work he focused mainly on algebra and number theory. Algebra has always been the field in which he specialized, and was also awarded his PhD and became an associate professor.

Fonzo with his wife Milka and the elder son Alfonz in 1965 when they moved to Banská Bystrica (the first author of this article is in mum's womb).

In his research he focused, besides algebra, on graph theory. He published 9 research papers in journals abroad, of which the most cited is the paper *All trees of diameter five are graceful* published in *Discrete Mathematics* in 2001 jointly with P. Hrnčiar. In this paper they proved the result in the title regarding the famous Ringel-Kotzig conjecture on graceful labelings of trees from early 1960s and this result is still the best worldwide in this direction. He also published results on varieties of graphs and orgraphs (both in *Discuss. Math. Graph Theory*, the former with R. Nedela, the latter with G. Monoszová) and on varieties of posets (in *Order*, with J. Lihová). For *Order* he also wrote a paper with P. Hrnčiar on the dimension of orthomodular posets constructed by pasting Boolean algebras. In Slovak journals he published 13 papers (among them 6 joint papers with colleagues from the department) and in local proceedings *Acta Fac. Paed. B. Bystrica* he published 9 papers between 1972 and 1989. Moreover, he coauthored 8 lecture notes or textbooks, of which *Algebra and theoretical arithmetic 2* from 1986 published in *Alfa* Bratislava became a national textbook. His full List of publications (which is attached) comprises moreover 6 articles in teachers' vocational journals and 7 other articles. He was a member of several grant projects of Slovak grant agencies VEGA, KEGA, APVT and APVV.

Fonzo Haviar received Award for excellent pedagogic work at anniversary conference of Union of Czech and Slovak Mathematicians and Physicists in Prague

(1987), became an honorary member of the Union (2002) and was awarded a Great medal and Silver medal of Matej Bel University by its rector (1999 and 2004, respectively). Probably the most important award would be given to him by his colleagues for his great character and personality, for suppressing his own ambitions to serve the others.

List of publications of Alfonz Haviar

Research papers in mathematical journals abroad

(1) The dimension of orthomodular posets constructed by pasting Boolean algebras, *Order* **10** (1993), 183 – 197 (with P. Hrnčiar).
(2) V-lattices of varieties of algebras of different types, *Czechoslovak Math. J.* **46** (1993), 419 – 428.
(3) Some characteristics of the edge distance between graphs, *Czechoslovak Math. J.* **46** (1996), 665 – 675 (with P. Hrnčiar and G. Monoszová).
(4) A metric on a system of ordered sets, *Math. Bohem.* **121** (1996), 123 – 131 (with P. Klenovčan).
(5) On varieties of graphs, *Discuss. Math. Graph Theory* **18** (1998), 209 – 223 (with R. Nedela).
(6) All trees of diameter five are graceful, *Discrete Math.* **233** (2001), 133 – 150 (with P. Hrnčiar).
(7) Varieties of orgraphs, *Discuss. Math. Graph Theory* **21** (2001), 207 – 221 (with G. Monoszová).
(8) Constructions of cell algebras, *Math. Bohem.* **130** (2005), 89 – 100 (with G. Monoszová).
(9) Varieties of posets, *Order* **22** (2005), 343 – 356 (with J. Lihová).

Research papers in Slovak mathematical journals

(10) N-Schrägverbände und Quasiordnungen, *Mat. Časopis Sloven. Akad. Vied* **23** (1973), 240 – 248.
(11) On a generalized distributivity in modular lattices, *Acta Fac. Rerum Natur. Univ. Comenian. Math. Publ.* **29** (1974), 35 – 42.
(12) On G-lattices, *Math. Slovaca* **29** (1979), 17 – 24.
(13) Notes on the congruence lattices of algebras, *Acta Univ. M. Belii, ser. Mathematics* **1** (1993), 7 – 14.
(14) The lattice of order varieties, *Acta Univ. M. Belii, ser. Mathematics* **1** (1993), 15 – 20 (with P. Konôpka).
(15) On congruence lattice representations, *Acta Univ. M. Belii, ser. Mathematics* **2** (1994), 9 – 16.
(16) Metrics on systems of finite algebra, *Acta Univ. M. Belii, ser. Mathematics* **3** (1995), 9 – 16.

(17) Valuations and metrics on a poset, *Acta Univ. M. Belii, ser. Mathematics* **4** (1996), 25 - 38 (with G. Monoszová).

(18) Minimal eccentric sequences with least eccentricity three, *Acta Univ. M. Belii, ser. Mathematics* **5** (1997), 27 – 50 (with P. Hrnčiar and G. Monoszová).

(19) The Dimension of orthomodular posets constructed by pasting Boolean algebras II, *Acta Univ. M. Belii, ser. Mathematics* **7** (1999), 63 – 70 (with P. Hrnčiar).

(20) The lattice of varieties of graphs, *Acta Univ. M. Belii, ser. Mathematics* **8** (2000), 11 – 19.

(21) The lattice of varieties of orgraphs, *Acta Univ. M. Belii, ser. Mathematics* **9** (2001), 43 – 50 (with G. Monoszová).

(22) Eccentric sequences and cycles in graphs, *Acta Univ. M. Belii, ser. Mathematics* **11** (2004), 7 – 25 (with P. Hrnčiar and G. Monoszová).

Research papers in local proceedings (in Slovak)

(23) Úplné N-šikmé zväzy, *Acta Fac. Paed. B. Bystrica, Matematika* **I** (1972), 97 – 103.

(24) O distributívnosti a doplnkoch zväzu, *Acta Fac. Paed. B. Bystrica, Matematika* **I** (1972), 59 – 75.

(25) O distributívnych G-zväzoch, *Fac. Paed. B. Bystrica, Matematika* **II** (1979), 67 – 81.

(26) O N-zväzoch, *Acta Fac. Paed. B. Bystrica, Prírodné vedy* **II** (1980), 191 – 213.

(27) O kongruenciách a varietách N-zväzov, *Acta Fac. Paed. B. Bystrica, Prírodné vedy* **III** (1982), 291 – 314.

(28) Variety G-zväzov, *Acta Fac. Paed. B. Bystrica, Prírodné vedy* **IV** (1983), 497 – 513.

(29) Konštrukcia unárnej algebry s dvoma operáciami, *Acta Fac. Paed. B. Bystrica, Prírodné vedy* **IX** (1989), 15 – 21.

(30) Dimenzia usporiadaných množín, *Acta Fac. Paed. B. Bystrica, Prírodné vedy* **X** (1989), 35-48 (with P. Hrnčiar and P. Konôpka).

(31) Usporiadané množiny s operáciou komplementu, *Acta Fac. Paed. B. Bystrica, Prírodné vedy* **X** (1989), 17 – 33 (with P. Klenovčan).

Vocational articles (in Slovak)

(32) Poznatky z prijímacích pohovorov z matematiky na SVŠ v Stropkove, *Pedagogický obzor*, Bardejov, 1965 (with V. Smolko).

(33) Použitie zvyškových tried pri riešení lineárnych diofantických rovníc, *Matematicko-fyz. rozhledy* **58** (1980), 342 – 345.

(34) Filozofické aspekty niektorých problémov teórie množín, *Zborník príspevkov RŠ-4-03*, PF B. Bystrica, 1988, 137 – 143.

(35) O niektorých filozofických problémoch matematiky, *Acta Fac. Paed. B. Bystrica, Prírodné vedy* **IX** (1991), 281 – 290.
(36) Vzdelávanie bez "predmetárov" je nebezpečná ilúzia, *Pedagogická revue* **2** (2008).
(37) Poznámky k tvorbe učiteľských kompetencií a spôsobilostí, *Pedagogické rozhľady* **2** (2008), 16 – 17.

Other articles (in Slovak)

(38) K životnému jubileu Ľudmily Berackovej, *Pokroky matematiky, fyziky a astronómie*, 1981.
(39) Doc. Ondrej Gábor šesťdesiatpäťročný, *Pokroky matematiky, fyziky a astronómie*, 1987.
(40) O činnosti pobočky JSMF Zvolen, *Zjazdový zborník JČSMF*, 1987.
(41) K životnému jubileu Ľ. Berackovej, *Obzory matematiky, fyziky a informatiky*, 1995.
(42) Osemročné gymnáziá a pedagogická prax študentov, *Učiteľské noviny*, roč. 44, č. 27, 1994 (with J. Klincková).
(43) Nadácia Matthias Belius pri UMB, *Spravodaj UMB* 2, č. 1, 1996.
(44) Spomienky pri príležitosti dvoch jubileí, *Obzory matematiky, fyziky a informatiky* 31, č. 4, 2002.

Textbooks (in Slovak)

(45) Algebra pre poslucháčov pedagogických fakúlt, *Pedagogická fakulta*, B. Bystrica, 1972 (with Ľ. Beracková and Š. Fekiač).
(46) Operácie a algebrické štruktúry, *SPN*, Bratislava, 1973 (with Ľ. Beracková).
(47) Algebrické štruktúry, *SPN*, Bratislava, 1977 (with A. Legéň).
(48) Algebra a teoretická aritmetika 2, *Alfa*, Bratislava, 1986 (with T. Šalát, T. Hecht and T. Katriňák).
(49) Metodický materiál pre činnosť v matematických záujmových útvaroch žiakov 5. ročníka ZŠ, *KDPM*, B. Bystrica, 1986 (with J. Gombalová).
(50) Zbierka náročnejších úloh z matematiky pre žiakov 5. roč. ZŠ, *KPÚ*, B.Bystrica, 1990 (with Ž. Sobôtková).
(51) Algebra I, *Pedagogická fakulta*, B. Bystrica, 1991 (with P. Hrnčiar and P. Klenovčan).
(52) Úvod do štúdia matematiky, *Pedagogická fakulta UMB*, B. Bystrica, 1996 (with P. Klenovčan and M. Haviar).

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DIRECT DECOMPOSITIONS OF BASIC ALGEBRAS AND THEIR IDEMPOTENT MODIFICATIONS

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Dedicated to the 70th birthday of Alfonz Haviar

ABSTRACT. We get a necessary and sufficient condition under which a given basic algebra $\mathcal{A}$ is isomorphic to a direct product of non-trivial basic algebras $\mathcal{A}_1, \mathcal{A}_2$ which are in fact interval subalgebras of $\mathcal{A}$. Further, we prove that the idempotent modification of $\mathcal{A}$ is directly indecomposable whenever $\mathcal{A}$ has at least one element which is not boolean.

1. INTRODUCTION

It is well-known that a bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ is directly decomposable into lattices $\mathcal{L}_1, \mathcal{L}_2$ isomorphic to the intervals $[a, 1], [b, 1]$ of $\mathcal{L}$ if b is a complement of a and a, b are standard elements. Since every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ induces a lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ which is bounded by 0 and $1 = \neg 0$, we can ask if also $\mathcal{A}$ is directly decomposable whenever there exists a complemented and standard element of $\mathcal{L}(A)$. In what follows we show that the condition concerning this element must be enlarged due to the fact that the operations $\oplus$ and $\neg$ cannot be derived by means of the lattice operations of $\mathcal{L}(A)$. However, we set up a natural necessary and sufficient condition for the direct decomposability of $\mathcal{A}$.

By a **basic algebra** (see e.g. [1, 2]) is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following four axioms

(BA1) $x \oplus 0 = x$;
(BA2) $\neg\neg x = x$;
(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;
(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

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As usual, we will write 1 instead of $\neg 0$. We say that a basic algebra $\mathcal{A}$ is **non-trivial** if $0 \neq 1$ (i.e. $|A| > 1$).

Having a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, one can introduce the **induced order** $\leq$ on $\mathcal{A}$ as follows

$$x \leq y \qquad \text{if and only if} \qquad \neg x \oplus y = 1.$$

It is an easy exercise to verify that $\leq$ is really an order on A and $0 \leq x \leq 1$ for each $x \in A$. Moreover, $(A; \leq)$ is a bounded lattice in which

$$x \vee y = \neg(\neg x \oplus y) \oplus y \qquad \text{and} \qquad x \wedge y = \neg(\neg x \vee \neg y).$$

For some details, the reader is referred to [1]. The lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ will be called the **induced lattice** of $\mathcal{A}$. In particular for each $a \in A$ there exists an antitone involution $x \mapsto x^a$ on the interval $[a, 1]$ (called a **section**) where $x^a = \neg x \oplus a$.

It is well-known (see e.g. [1,3]) that also conversely, if $(A; \vee, \wedge, ({}^a)_{a\in A}, 0, 1)$ is a bounded lattice with section antitone involutions, we are able to construct a basic algebra using the operations

$$x \oplus y = (x^0 \vee y)^y \qquad \text{and} \qquad \neg x = x^0. \tag{1}$$

Lemma 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\leq$ the induced order and $a \in A$. Define the polynomial operations $\neg_a$ and $\oplus_a$ on the interval $[a, 1]$ as follows*

$$\neg_a x = \neg x \oplus a \qquad \text{and} \qquad x \oplus_a y = \neg(\neg x \oplus a) \oplus y.$$

Then $([a, 1]; \oplus_a, \neg_a, a)$ is a basic algebra.

Proof. We use the facts that $y \leq x \oplus y$, $0 \oplus x = x$ and $\neg x \oplus x = 1$ hold in each basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ (for more details see e.g. [1]). If $x, y \in [a, 1]$ then $a \leq y \leq \neg(\neg x \oplus a) \oplus y = x \oplus_a y$ thus $\oplus_a$ is really a binary operation on $[a, 1]$. Since $a \leq \neg x \oplus a$, $\neg_a x$ is a unary operation on $[a, 1]$. Moreover, $\neg_a a = \neg a \oplus a = 1$ and $\neg_a 1 = \neg 1 \oplus a = 0 \oplus a = a$. We must check the axioms (BA1)–(BA4).
(BA1) and (BA2): For $x \in [a, 1]$ we have $x \oplus_a a = \neg(\neg x \oplus a) \oplus a = x \vee a = x$ and, analogously, $\neg_a \neg_a x = \neg(\neg x \oplus a) \oplus a = x \vee a = x$.
(BA3): Assume that $x, y \in [a, 1]$. Since $y \leq \neg x \oplus y$, then also $a \leq \neg x \oplus y$. Further, we have $\neg_a x \oplus_a y = \neg x \oplus y$. Hence, we compute

$$\neg_a(\neg_a x \oplus_a y) \oplus_a y = \neg(\neg x \oplus y) \oplus y = x \vee y$$

and, by symmetry, also

$$\neg_a(\neg_a y \oplus_a x) \oplus_a x = y \vee x = x \vee y.$$

(BA4): Let $x, y, z \in [a, 1]$. Since $a \leq x \oplus_a y$, $a \leq y \leq \neg(x \oplus_a y) \oplus y$ and $a \leq z \leq \neg(\neg(x \oplus_a y) \oplus y) \oplus z$, we obtain

$$\begin{aligned}\neg_a(\neg_a(\neg_a(x \oplus_a y) \oplus_a y) \oplus_a z) \oplus_a (x \oplus_a z) &= \\ &= \neg_a(\neg(\neg(x \oplus_a y) \oplus y) \oplus z) \oplus_a (x \oplus_a z) = \\ &= \neg(\neg(\neg(x \oplus_a y) \oplus y) \oplus z) \oplus (x \oplus_a z) = \\ &= \neg(\neg(\neg(w \oplus y) \oplus y) \oplus z) \oplus (w \oplus z) = 1,\end{aligned}$$

where $w = \neg(\neg x \oplus a)$. □

The basic algebra $([a, 1]; \oplus_a, \neg_a, a)$ where the operations $\oplus_a$, $\neg_a$ are defined as in Lemma 1 will be called an **interval basic algebra**. Our motivation for introducing the operations $\oplus_a$ and $\neg_a$ in this way is inspired by (1), where we only replace x^0 by x^a due to the fact that a is the bottom element of the section $[a, 1]$. Since $x \in [a, 1]$, by (1) we have

$$\neg_a x = x^a = (x \vee a)^a = \neg x \oplus a$$

and then for $x, y \in [a, 1]$

$$x \oplus_a y = (\neg_a x \vee y)^y = ((\neg x \oplus a) \vee y)^y = \neg(\neg x \oplus a) \oplus y.$$

Due to the fact that every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is in a one-to-one correspondence with the enriched lattice $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a \in A}, 0, 1)$ as mentioned above (see also [1]), the interval basic algebra $([a, 1]; \oplus_a, \neg_a, a)$ is in the same correspondence with the interval enriched lattice $([a, 1]; \vee, \wedge, (^b)_{b \in [a,1]}, a, 1)$ where $\vee, \wedge, ^b$ are the same as in $\mathcal{L}(A)$. Hence, our interval basic algebra is quite a natural "cut" of the original one.

Lemma 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $a, b, c \in A$. Then*

$$(b \wedge c) \oplus a = (b \oplus a) \wedge (c \oplus a).$$

Proof. We compute by (1)

$$\begin{aligned}(b \wedge c) \oplus a &= (\neg(b \wedge c) \vee a)^a = ((\neg b \vee \neg c) \vee a)^a = ((\neg b \vee a) \vee (\neg c \vee a))^a = \\ &= (\neg b \vee a)^a \wedge (\neg c \vee a)^a = (b \oplus a) \wedge (c \oplus a)\end{aligned}$$

since $\neg b \vee a \in [a, 1]$ and $\neg c \vee a \in [a, 1]$. □

2. Direct decomposibility of basic algebras

Now, we will set up the conditions under which a basic algebra $\mathcal{A}$ can be directly decomposed. First, we define several concepts.

Definition 1. *An element a of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called **strong** if*

(a) $x \oplus a = x \vee a$ *and* $x \oplus \neg a = x \vee \neg a$
for every $x \in A$.

A strong element a *of* $\mathcal{A}$ *is called a* ***decomposing element*** *if it moreover satisfies*

(b) $(x \oplus y) \oplus a = x \oplus (y \oplus a)$, $(x \oplus y) \oplus \neg a = x \oplus (y \oplus \neg a)$
and $x \oplus a = a \oplus x$, $x \oplus \neg a = \neg a \oplus x$
for all $x, y \in A$.

Let us note that 0 and 1 are decomposing elements for every basic algebra $\mathcal{A}$.

Recall (see [4]) that the element a of a lattice $(L; \vee, \wedge)$ is called **distributive** if for all $x, y \in L$

$$(x \wedge y) \vee a = (x \vee a) \wedge (y \vee a)$$

and the element a of a lattice $(L; \vee, \wedge)$ is called **standard** if for all $x, y \in L$

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y).$$

Further, recall that if $(L; \vee, \wedge)$ is a lattice and $a \in L$ then the following two conditions are equivalent:

(α) a is standard

(β) a is distributive and, for $x, y \in L$,

$$a \wedge x = a \wedge y \quad \text{and} \quad a \vee x = a \vee y \qquad \text{imply that} \qquad x = y$$

(for more details see [4]).

Lemma 3. *Let* a *be a strong element of a basic algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$. *Then*

(i) *a is boolean (i.e.* $a \vee \neg a = 1$, $a \wedge \neg a = 0$*);*

(ii) *a and $\neg a$ are distributive elements.*

Proof. (i) By Definition 1 we have $a \oplus a = a \vee a = a$ and $\neg a \oplus \neg a = \neg a \vee \neg a = \neg a$ thus both a and $\neg a$ are $\oplus$-idempotents. Then $\neg a \vee a = \neg a \oplus a = 1$ and dually (by De Morgan law) also $\neg a \wedge a = 0$.
(ii) Follows directly by Lemma 2 and the condition (a) of Definition 1. ☐

Lemma 4. *Let* a *be a strong element of a basic algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$ *and* $\neg a$ *be a standard element of the induced lattice* $\mathcal{L}(A) = (A; \vee, \wedge)$. *Then the mapping* $\varphi_a(x) = (x \vee a, x \vee \neg a)$ *is a lattice isomorphism of* $\mathcal{L}(A)$ *onto the direct product of lattices* $([a, 1]; \vee, \wedge) \times ([\neg a, 1]; \vee, \wedge)$.

Proof. The proof is only a slight modification of the proof of Theorem 1.4. (p. 200) in [4]. Namely, if $\varphi_a(x) = \varphi_a(y)$ then $x \vee a = y \vee a$ and $x \vee \neg a = y \vee \neg a$, i.e.

$$\neg x \wedge \neg a = \neg y \wedge \neg a \tag{2}$$

by the first equality and $\neg x \wedge a = \neg y \wedge a$ by the second one, i.e. also

$$(\neg x \wedge a) \vee \neg a = (\neg y \wedge a) \vee \neg a$$

thus, by Lemma 3, also

$$\neg x \vee \neg a = \neg y \vee \neg a. \tag{3}$$

Since $\neg a$ is a standard element of $\mathcal{L}(A) = (A; \vee, \wedge)$, (2) and (3) yields $\neg x = \neg y$, i.e. $x = \neg\neg x = \neg\neg y = y$. Hence, φ_a is injective. If $\langle c, d\rangle \in [a, 1] \times [\neg a, 1]$ then $a \leq c$, $\neg a \leq d$, i.e. $d \vee a \geq \neg a \vee a = 1$, $c \vee \neg a \geq a \vee \neg a = 1$ and for $c \wedge d \in A$ we have

$$\begin{aligned}
\varphi_a(c \wedge d) &= ((c \wedge d) \vee a, (c \wedge d) \vee \neg a) = \\
&= ((c \vee a) \wedge (d \vee a), (c \vee \neg a) \wedge (d \vee \neg a)) = \\
&= ((c \vee a) \wedge 1, 1 \wedge (d \vee \neg a)) = (c, d)
\end{aligned}$$

hence φ_a is also surjective. Therefore it is a bijection from A to $[a, 1] \times [\neg a, 1]$. Further,

$$\begin{aligned}
\varphi_a(x \vee y) &= ((x \vee y) \vee a, (x \vee y) \vee \neg a) = \\
&= (x \vee a, x \vee \neg a) \vee (y \vee a, y \vee \neg a) = \varphi_a(x) \vee \varphi_a(y)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_a(x \wedge y) &= ((x \wedge y) \vee a, (x \wedge y) \vee \neg a) = \\
&= ((x \vee a) \wedge (y \vee a), (x \vee \neg a) \wedge (y \vee \neg a)) = \varphi_a(x) \wedge \varphi_a(y)
\end{aligned}$$

thus φ_a is a lattice isomorphism of $\mathcal{L}(A)$ onto $([a, 1]; \vee, \wedge) \times ([\neg a, 1]; \vee, \wedge)$. $\square$

Theorem 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Then $\mathcal{A}$ is isomorphic to a direct product of non-trivial basic algebras $\mathcal{B}_1, \mathcal{B}_2$ if and only if there exists a decomposing element $a \in A$, $0 \neq a \neq 1$ such that $\neg a$ is standard in the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$. If it is the case then $\mathcal{A}$ is isomorphic to the direct product of interval basic algebras $([a, 1]; \oplus_a, \neg_a, a)$ and $([\neg a, 1]; \oplus_{\neg a}, \neg_{\neg a}, \neg a)$.*

Proof. Due to Lemma 4, we must only show that φ_a preserves the operations $\oplus$ and $\neg$. Denote by $\widehat{\oplus}$ and $\widehat{\neg}$ the corresponding operations on the direct product of the interval algebras. Then, since a and $\neg a$ are strong elements, we have

$$\varphi_a(\neg x) = (\neg x \vee a, \neg x \vee \neg a) = (\neg x \oplus a, \neg x \oplus \neg a) = (\neg_a x, \neg_{\neg a} x) = \widehat{\neg}\varphi_a(x).$$

Since a is a decomposing element, we derive also

$$\varphi_a(x \oplus y) = ((x \oplus y) \oplus a, (x \oplus y) \oplus \neg a)$$

and, by Lemma 2, we compute

$$
\begin{aligned}
\varphi_a(x)\widehat{\oplus}\varphi_a(y) &= (x \vee a, x \vee \neg a)\widehat{\oplus}(y \vee a, y \vee \neg a) = \\
&= (x \oplus a, x \oplus \neg a)\widehat{\oplus}(y \oplus a, y \oplus \neg a) = \\
&= ((x \oplus a) \oplus_a (y \oplus a), (x \oplus \neg a) \oplus_{\neg a} (y \oplus \neg a)) = \\
&= (\neg(\neg(x \oplus a) \oplus a) \oplus (y \oplus a), \neg(\neg(x \oplus \neg a) \oplus \neg a) \oplus (y \oplus \neg a)) = \\
&= (\neg(\neg x \vee a) \oplus (y \oplus a), \neg(\neg x \vee \neg a) \oplus (y \oplus \neg a)) = \\
&= ((x \wedge \neg a) \oplus (y \oplus a), (x \wedge a) \oplus (y \oplus \neg a)) = \\
&= ((x \oplus (y \oplus a)) \wedge (\neg a \oplus (y \oplus a)), (x \oplus (y \oplus \neg a)) \wedge (a \oplus (y \oplus \neg a))) = \\
&= ((x \oplus (y \oplus a)) \wedge (\neg a \vee y \vee a), (x \oplus (y \oplus \neg a)) \wedge (a \vee y \vee \neg a)) = \\
&= ((x \oplus (y \oplus a)) \wedge 1, (x \oplus (y \oplus \neg a)) \wedge 1) = \\
&= (x \oplus (y \oplus a), x \oplus (y \oplus \neg a)).
\end{aligned}
$$

Due to (b) of Definition 1, we conclude

$$\varphi_a(x \oplus y) = \varphi_a(x)\widehat{\oplus}\varphi_a(y)$$

thus φ_a preserves $\oplus$ and $\neg$ and hence it is an isomorphism of $\mathcal{A}$ onto the direct product $([a,1]; \oplus_a, \neg_a, a) \times ([\neg a, 1], \oplus_{\neg a}, \neg_{\neg a}, \neg a)$.

Conversely, assume that a basic algebra $\mathcal{A}$ is isomorphic to a direct product $\mathcal{B}_1 \times \mathcal{B}_2$ of non-trivial basic algebras $\mathcal{B}_1 = (B_1; \oplus_1, \neg_1, 0_1)$ and $\mathcal{B}_2 = (B_2; \oplus_2, \neg_2, 0_2)$. It is an easy exercise to show that $(0_1, 1_2)$ (where $1_2 = \neg_2 0_2$) is a decomposing element of $\mathcal{B}_1 \times \mathcal{B}_2$ and hence $h^{-1}((0_1, 1_2))$ is a decomposing element of $\mathcal{A}$ (where h is the isomorphism of $\mathcal{A}$ onto $\mathcal{B}_1 \times \mathcal{B}_2$). $\square$

Example 1. Consider the lattice drawn in Fig. 1.

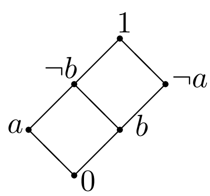

Fig. 1

We can define the operations $\oplus^1$ and $\oplus^2$ as follows

$\oplus^1$	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	$\neg a$	$\neg b$	1	1
b	b	$\neg b$	$\neg b$	1	$\neg a$	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
$\neg a$	$\neg a$	1	$\neg b$	1	$\neg a$	1
1	1	1	1	1	1	1

$\oplus^2$	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	$\neg b$	$\neg b$	1	1
b	b	$\neg b$	$\neg a$	1	$\neg a$	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
$\neg a$	$\neg a$	1	$\neg a$	1	$\neg a$	1
1	1	1	1	1	1	1

Then for $A = \{0, a, b, \neg b, \neg a, 1\}$ we have that $\mathcal{A}_1 = (A; \oplus^1, \neg, 0)$ and $\mathcal{A}_2 = (A; \oplus^2, \neg, 0)$ are basic algebras (where $\mathcal{A}_2$ is even an MV-algebra but $\mathcal{A}_1$ is not). In the both cases a is a strong element, but in $\mathcal{A}_1$ a is not a decomposing element, since for $x = b$ we have

$$a \oplus b = \neg a \neq \neg b = b \oplus a$$

which contradicts (b) of Definition 1. On the other hand, one can check by a direct computation that a is a decomposing element of $\mathcal{A}_2$. ♢

3. Idempotent modification of basic algebras

The concept of idempotent modification of an algebra was introduced by J. Ježek [6] as follows.

Definition 2. *An **idempotent modification** of an algebra $\mathcal{A} = (A; F)$ is an algebra $\mathcal{A}_I = (A; F_I)$ with the same underlying set A, where $|F| = |F_I|$ and for every $f \in F$ the corresponding operation $f_I \in F_I$ is defined as follows*

(i) *if f is at most unary then $f_I = f$;*
(ii) *if f is n-ary with $n > 1$ and $a_1, \ldots, a_n \in A$ then*

$$f_I(a_1, \ldots, a_n) = \begin{cases} a_1 & \text{if} \quad a_1 = a_2 = \cdots = a_n \\ f(a_1, \ldots, a_n) & \text{otherwise.} \end{cases}$$

The main result of [6] is that for any group G its idempotent modification G_I is subdirectly irreducible.

In what follows we will treat direct decomposability of an idempotent modification of a basic algebra.

For this we slightly modify our definition of basic algebra. As mentioned above, every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ has induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ where $\vee$ and $\wedge$ are term operations of $\mathcal{A}$. Hence, inserting $\vee$ and $\wedge$ into the type of $\mathcal{A}$, we obtain an algebra with the same clone of term operations and hence term equivalent to $\mathcal{A}$. From now on, by a basic algebra we will understand an algebra $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ where the term operations $\vee$ and $\wedge$ are defined by $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$.

The reason of this insertion is that when an idempotent modification of $(A; \oplus, \neg, 0)$ is considered, the resulting algebra does not have the lattice structure. However, if $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ is treated then the lattice structure for $\mathcal{A}_I$ is preserved because both $\vee$ and $\wedge$ are idempotent operations on A.

Theorem 2. *Let $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ be a basic algebra whose at least one element is not boolean. Then its idempotent modification $\mathcal{A}_I = (A; \oplus_I, \neg, 0, \vee, \wedge)$ is not directly decomposable.*

Proof. At first we show that if $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ is a basic algebra and its idempotent modification $\mathcal{A}_I$ is directly decomposable into non-trivial algebras $\mathcal{B}_1 = (B_1; \oplus_1, \neg_1, 0_1, \vee, \wedge)$ and $\mathcal{B}_2 = (B_2; \oplus_2, \neg_2, 0_2, \vee, \wedge)$ then also $\mathcal{A}$ is directly decomposable. Denote by $1_1 = \neg_1 0_1$ and $1_2 = \neg_2 0_2$. Let φ be an isomorphism of $\mathcal{A}_I$ onto $\mathcal{B}_1 \times \mathcal{B}_2$. For $x \in A$ let $\varphi(x) = (x_1, x_2)$. Define new operations $\oplus^1, \oplus^2$ on B_1, B_2, respectively as follows. If $y_1, z_1 \in B_1$ and $y_1 \neq z_1$ then $y_1 \oplus^1 z_1 = y_1 \oplus_1 z_1$, if $y_2, z_2 \in B_2$ and $y_2 \neq z_2$ then $y_2 \oplus^2 z_2 = y_2 \oplus_2 z_2$. If $x_1 \in B_1$, denote by $\overline{x_1} = \varphi^{-1}((x_1, 1_2))$ and if $x_2 \in B_2$, denote by $\overline{x_2} = \varphi^{-1}((1_1, x_2))$. Now we define

$$x_1 \oplus^1 x_1 = \mathrm{pr}_1(\varphi(\overline{x_1} \oplus \overline{x_1}))$$

and

$$x_2 \oplus^2 x_2 = \mathrm{pr}_2(\varphi(\overline{x_2} \oplus \overline{x_2})).$$

Since $\varphi(x) = (x_1, x_2) = (x_1, 1_2) \wedge (1_1, x_2) = \varphi(\overline{x_1}) \wedge \varphi(\overline{x_2})$ and since φ and also φ^{-1} preserve the lattice operations, we have $x = \overline{x_1} \wedge \overline{x_2}$. This yields that $\oplus^1, \oplus^2$ are correctly defined (i.e. the result $x_1 \oplus^1 x_1$ in the first coordinate does not depend on the second coordinate and vice versa), i.e.

$$\varphi(x \oplus x) = (x_1, x_2) \overline{\oplus} (x_1, x_2) = (x_1 \oplus^1 x_1, x_2 \oplus^2 x_2),$$

where $\overline{\oplus}$ is the binary operation provided coordinatewise on the Cartesian product $B_1 \times B_2$.

It is obvious that $\mathcal{A}^1 = (B_1; \oplus^1, \neg_1, 0_1, \vee, \wedge)$ and $\mathcal{A}^2 = (B_2; \oplus^2, \neg_2, 0_2, \vee, \wedge)$ are basic algebras and φ is also an isomorphism of $\mathcal{A}$ onto $\mathcal{A}_1 \times \mathcal{A}_2$. Moreover, we see that $\mathcal{B}_1 = \mathcal{A}^1_I$ and $\mathcal{B}_2 = \mathcal{A}^2_I$. Hence, if $\mathcal{A}_I$ is directly decomposable then also $\mathcal{A}$ has this property.

Assume now that $x \in A$ is not boolean and that $\mathcal{A}_I$ is directly decomposable. We can apply the reasoning used by J. Jakubík [5]. Let $\varphi(x) = (x_1, x_2)$. Then also $\varphi(x)$ is not boolean, i.e. at least one of x_1, x_2 is not boolean. Without loss of generality, suppose that x_1 is not boolean. Then $x_1 \oplus^1 x_1 \neq x_1$. Since $|B_2| > 1$, there exists $y_2 \in B_2$ such that $x_2 \neq y_2$. Let $y = \varphi^{-1}(x_1, y_2)$. Then $x \neq y$ and $x \oplus y = x \oplus_I y$, hence $\varphi(x \oplus y) = \varphi(x \oplus_I y)$. However, $\varphi(x \oplus y) = (x_1 \oplus^1 x_1, x_2 \oplus^2 y_2)$ and $\varphi(x \oplus_I y) = (x_1 \oplus_1 x_1, x_2 \oplus_2 y_2) = (x_1, x_2 \oplus^2 y_2)$, which is a contradiction. Thus $\mathcal{A}_I$ is not directly decomposable. □

Call a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ **distributive** if the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ is distributive. For example, if $\mathcal{A}$ is commutative then $\mathcal{A}$ is distributive (but not vice versa, see Example 1) see e.g. [1]. For distributive basic algebras, we can modify our result as follows

Corollary. *Let $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ be a distributive basic algebra with $|A| > 2$. Then its idempotent modification is directly indecomposable if and only if $\mathcal{A}$ contains an element which is not boolean.*

Proof. If all elements of $\mathcal{A}$ are boolean then, due to the fact that $\mathcal{L}(A)$ is distributive, its idempotent modification $\mathcal{A}_I$ is in fact a Boolean algebra (where $\oplus$ coincides with $\vee$). Hence, $\mathcal{A}_I$ is directly decomposable since $|A| > 2$.

Conversely, if $\mathcal{A}$ contains an element which is not boolean then $\mathcal{A}_I$ is not directly decomposable by Theorem 2. □

References

1. Chajda I., Halaš R., Kühr J.: *Semilattice Structures*, Heldermann Verlag (Lemgo, Germany), 2007.
2. Chajda I., Kolařík M.: *Independence of axiom system of basic algebras*, Soft Computing **13**, 1 (2009), 41–43.
3. Chajda I., Kolařík M.: *Interval basic algebras*, Novi Sad Journal of Mathematics, to appear.
4. Grätzer G.: *General lattice theory*, 2^{nd} ed. Birkäuser Verlag, Basel – Boston – Berlin, 2003.
5. Jakubík J.: *On idempotent modification of generalized MV-algebras*, Math. Slovaca, to appear.
6. Ježek J.: *A note on idempotent modification of groups*, Czechoslovak Math. J. **54** (2004), 229–231.

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SUPERPRIMES AND GENERALIZED DIRICHLET THEOREM

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Dedicated to the 70th birthday of Alfonz Haviar

ABSTRACT. A concept of a superprime meaning a prime number whose all digits are prime numbers is introduced and a question whether there is an infinite number of superprimes is raised. A positive answer to this and a few related questions is conjectured and supported by several observations and computations via Mathematica. Among the conjectures is a generalized version of Dirichlet's Theorem on primes which implies certain conjectures presented here as well as the famous conjectures about the infinite number of Mersenne and Fermat primes.

1. THE MAIN PROBLEM

There are several different proofs of the fact that there is an infinite number of primes [1], the best known being likely the one due to Euclid. In this note we introduce a more specific notion of a superprime and ask if there is still an infinite number of superprimes.

Definition 1.1. By a *superprime* we mean a prime number whose all digits (in its decimal representation) are prime numbers.

We note that instead of the decimal representation one can consider base m positional notation for $m \geq 4$. (The case $m = 3$ is not interesting as it gives us only one prime digit 2.)

Example 1.2. The numbers 2, 3, 5, 7, 23, 37, 53, 73, 223, 227, 233, 257, 277, 337, 353, 373, 523, 557, 577, 727, 733, 757 and 773 are all superprimes among the natural numbers up to one thousand.

Problem 1. *Is there an infinite number of superprimes?*

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Example 1.3. A simple way to generate (and print) all superprimes having at most r digits is to use, within *Mathematica*, the following command:

$$p[0]=2;p[1]=3;p[2]=5;p[3]=7;$$
$$\mathrm{Do}[number=0;\mathrm{Do}[m=n;q=0;\mathrm{Do}[z=\mathrm{Mod}[m,4];m=\mathrm{Floor}[m/4];$$
$$q=p[z]*10^i+q,\{i,0,k-1\}];\mathrm{If}[\mathrm{PrimeQ}[q],number{+}{+};\mathrm{Print}[\{k,q\}]],$$
$$\{n,0,4^k-1\}];\mathrm{Print}[number],\{k,1,r\}]$$

Here are the four-digit superprimes obtained:
2237, 2273, 2333, 2357, 2377, 2557, 2753, 2777, 3253, 3257, 3323, 3373, 3527, 3533, 3557, 3727, 3733, 5227, 5233, 5237, 5273, 5323, 5333, 5527, 5557, 5573, 5737, 7237, 7253, 7333, 7523, 7537, 7573, 7577, 7723, 7727, 7753, 7757.

In the table below, P_k is the number of k-digit superprimes for $1 \le k \le 15$. From this table one can conjecture that $P_k > 3^k$ for $k \ge 10$.

k	P_k	$\sqrt[k]{P_k}$	k	P_k	$\sqrt[k]{P_k}$	k	P_k	$\sqrt[k]{P_k}$
1	4	4.000000000	6	389	2.701831538	11	214432	3.052549327
2	4	2.000000000	7	1325	2.792742150	12	781471	3.097961899
3	15	2.466212074	8	4643	2.873094002	13	2884201	3.139966685
4	38	2.482823796	9	16623	2.944202734	14	10687480	3.177331457
5	128	2.639015822	10	59241	3.000974037	15	39838489	3.211344203

Based on the computations above we now state the following two conjectures:

Conjecture 1. There is an infinite number of superprimes.

Conjecture 2. For any integer $k > 0$ there is a k-digit superprime.

Remark 1. We note that it would be interesting to find the limit $L := \lim_{k\to\infty} \sqrt[k]{P_k}$. It is likely that $L > 3$ and one cannot refute that $L = 4$. For the limit L we have the asymptotic inequality $P_k > (L-\varepsilon)^k$ for every $\varepsilon > 0$.

Remark 2. We also note that in the base m positional notation for $m \ge 4$ the situation seems to be analogous: the number P_k^m denoting the number of k-digit superprimes in the base m positional notation has been calculated for $4 \le m \le 12$ and it turns out that it grows roughly as a^k, where a is slightly smaller than the number $\pi(m-1)$. (Here $\pi(x)$ is the prime-counting function, so $\pi(m-1)$ is the number of primes used in the base m positional notation.) Hence Conjecture 1 and Conjecture 2 formulated above with respect to the decimal representation can analogously be formulated in the base m positional notation for arbitrary $m \ge 4$.

2. Generalized Dirichlet Theorem

The well-known Dirichlet's Theorem on Primes in Arithmetic Progressions was first published in 1837. In his article [4], P.G.L. Dirichlet stated it as follows: *"each unlimited arithmetic progression, with the first member and the difference being coprime, will contain infinitely many primes."* We present it formally as a theorem below.

Dirichlet's Theorem. Assume that a, b are coprime positive numbers. There is an infinite number of primes in the arithmetical sequence

$$a, a+b, a+2b, a+3b, \dots.$$

Our aim here is to conjecture a Generalized Dirichlet Theorem. Let us call by *Dirichlet sequence* the sequence in Dirichlet's Theorem. We shall consider the sequence $(x_n)_{n=0}^{\infty}$ defined by the recursive formula

$$x_{n+1} = ax_n + b \tag{1}$$

where a, b and x_0 are integers such that b is coprime to $a \cdot x_0$. We shall call it *Generalized Dirichlet sequence.* We note that one obtains Dirichlet sequence from it in the special case $a = 1$.

For our Generalized Dirichlet (GD) sequence we have the explicit formula

$$x_n = \begin{cases} a^n\left(x_0 + \frac{b}{a-1}\right) - \frac{b}{a-1} = a^n x_0 + \frac{b(a^n-1)}{a-1} & \text{for } a \neq 1, \\ x_0 + n\,b & \text{for } a = 1. \end{cases}$$

We note that the cases $a \in \{-1, 0\}$ are trivial and in case $b = -(a-1)x_0$ our sequence is constant. So, we shall assume $a \notin \{-1, 0, 1\}$ and $b \neq -(a-1)x_0$. Throughout this section we shall also need to consider as primes all members x_n for which $|x_n|$ is prime - so even the negative integers. (This is not unusual, we note that also Mathematica treats prime numbers the same way and the commands PrimeQ[-3] or ProvablePrimeQ[-3] give answers 'True'). Such a consideration of prime numbers will help us to simplify the statements in this section and yet it will not negatively influence our main goal here which is to conjecture a Generalized Dirichlet Theorem.

Remark 3. If $a \neq -1$ a $b \neq -(a-1)x_0$, then the explicit formula above yields that all members of the GD sequence are different.

Also, the explicit formula above yields immediately the following statement.

Proposition 2.1. *Let $(x_n)_{n=0}^{\infty}$ be a GD sequence. If $a \cdot x_0$ and b have a divisor $d > 1$, then all x_n for $n \geq 1$ are divisible by d, and consequently, the GD sequence contains at most three primes.*

Example 2.2. Let us take in a GD sequence $a = 3$, $b = -12$ and $x_0 = 5$. Then the GD sequence $(5, 3, -3, -21, -75, \dots)$ contains only the primes 5, 3 and -3.

Remark 4. We note that also a partial converse to Proposition 2.1 is true: if two consecutive members x_n and x_{n+1} of the GD sequence have a common divisor $d > 1$, then $a \cdot x_0$ and b have the divisor d, too.

Example 2.3. (i) Let us take in the GD sequence $x_0 = 3$, $a = 5$ and $b = 1$. Hence $x_{n+1} = 5x_n + 1$. Then $x_{n+2} = 25x_n + 6$, which means that all members x_{2k} are divisible by 3. Since $x_1 = 16$, all members x_{2k+1} are even and the sequence $(x_n)_{n=0}^{\infty}$ contains only the prime $x_0 = 3$.

(ii) Let us consider $x_0 = 14$, $a = 16$ and $b = 1$. We have $x_{n+1} = 16x_n + 1$. Then $x_{n+3} = 4096x_n + 273$. We note that $x_0 = 14 = 7 \cdot 2$, $x_1 = 225 = 3 \cdot 75$, $x_2 = 3601 = 13 \cdot 277$ and $273 = 3 \cdot 7 \cdot 13$.

It can easily be seen from the formulas above that all members x_{3k} are divisible by 7, all x_{3k+1} are divisible by 3 and all x_{3k+2} are divisible by 13. The GD sequence $(x_n)_{n=0}^{\infty}$ does not contain any prime.

The previous examples show that even if b is coprime to $a \cdot x_0$, the GD sequence $(x_n)_{n=0}^{\infty}$ can be partitioned into k subsequences, of which each has its own nontrivial divisor, and so the sequence $(x_n)_{n=0}^{\infty}$ contains only a finite number of primes. Our aim is to study conditions forcing the GD sequence to contain only a finite number of primes.

Let us put

$$\begin{aligned} A_k &:= 1 + a + \dots a^{k-1} = \frac{a^k - 1}{a - 1} \text{ for } k \geq 0 \\ B_k &:= A_k b \\ y_n^{(k,j)} &:= x_{kn+j} \text{ for } k \geq 2 \text{ and } 0 \leq j \leq k-1 \end{aligned}$$

Obviously, $y_{n+1}^{(k,j)} = A_k y_n^{(k,j)} + B_k$ and $y_0^{(k,j)} = x_j$.

Proposition 2.4. *Assume that there exists $k \geq 2$ such that the following conditions hold:*

(a_k) *for all $j \in \{0, \dots, k-1\}, A_k$ has a common divisor $d_j > 1$ with x_j.*

Equivalently,

(b_k)

for all $j \in \{0, \dots, k-1\}, A_k$ has a common divisor $d_j > 1$ with $x_0 - A_j b$.

Then the GD sequence $(x_n)_{n=0}^{\infty}$ contains only a finite number of primes.

Proof. Assume that there exists $k \geq 2$ such that, for all $0 \leq j \leq k-1$, A_k has a common divisor $d_j > 1$ with $x_j = y_0^{(k,j)}$. Then d_j is a common divisor of $B_k = A_k b$ and x_j, which means that, for all $0 \leq j \leq k-1$, the sequence $(y_n^{(k,j)})_{n=0}^{\infty}$ contains a finite number of primes. Consequently, the GD sequence $(x_n)_{n=0}^{\infty}$ contains only a finite number of primes.

Now we show that (a_k) is equivalent to (b_k). If $j = 0$, then $x_j = x_0 - A_j b$, thus (a_k) immediately implies (b_k). For $1 \le j \le k-1$ we continue as follows. If, by (a_k), A_k has a common divisor $d_j > 1$ with x_j, it also has the common divisor d_j with $x_j - A_k b = a^j x_0 + (A_j - A_k)b = a^j(x_0 - A_{k-j}b)$. Since A_k is coprime to a, it has the common divisor d_j with $(x_0 - A_{k-j}b)$. Hence A_k has a common divisor with $x_0 - A_j b$ for all $1 \le j \le k-1$. Consequently, (b_k) holds.

Conversely, to show that (b_k) implies (a_k), let for all $0 \le j \le k-1$, A_k has a common divisor $d_j > 1$ with $x_0 - A_j b$. If $j = 0$, then A_k has the common divisor d_0 with x_0. If $1 \le j \le k-1$, then A_k has the common divisor d_j with $a^{k-j}(x_0 - A_j b) = a^{k-j}x_0 - A_{k-j}b + A_k b$. This means that A_k has the common divisor d_j with $a^{k-j}x_0 - A_{k-j}b = x_{k-j}$. Hence A_k has a common divisor with x_j for all $j \in \{0, \dots, k-1\}$. The proof is complete. $\square$

Remark 5. We note that in the conditions (a_k) and (b_k) we could write B_k instead of A_k. The equality $B_k = A_k b$ means that if B_k has a common divisor $d_j > 1$ with x_j, but A_k is coprime to x_j, then b has a common divisor with x_j. Then b has a common divisor with ax_{j-1}. Now if b has a common divisor with a, we may apply Proposition 2.1. If b is coprime to a, it has a common divisor with x_{j-1} and then, by induction, it has a common divisor with x_n for $n \ge 0$. So, these cases are in fact already covered by Proposition 2.1.

We also note that it follows from above that if we want the GD sequence to contain an infinite number of primes, then we have to guarantee that the conditions (a_k) and (b_k) are not satisfied for all $k \ge 2$, which is not a simple task. The most convenient way to guarantee it seems to be to show that A_k is coprime to x_0 or to x_1 as we do it later with respect to Mersenne and Fermat primes (see Remark 8 and Remark 9, respectively). An alternative way is to show that A_k is coprime to $x_0 - b = x_0 - A_1 b$, which is applied at the very end of Section 3.

Remark 6. (i) For $k = 2$ the condition (a_k) above means that $a+1$ has common divisors with x_0 and x_1, and the equivalent condition (b_k) means that $a + 1$ has common divisors with x_0 and $x_0 - b$.

Our following observations concerning the conditions (a_k) and (b_k) for $k > 2$ are based on our computations via Mathematica and C++.

(ii) For $k = 3$ the conditions (a_k) and (b_k) are not satisfied for all a, b, x_0 with the integer values in the interval from -15 to 15 provided $a \cdot x_0$ is coprime to b.

(iii) For $k = 5$ the conditions (a_k) and (b_k) are not satisfied for all a, b, x_0 with the integer values in the interval from -361 to 361 provided $a \cdot x_0$ is coprime to b.

If 361 above is replaced with 1000, then there is exactly one example of a, b, x_0 with the given integer values where the conditions (a_k) and (b_k) are satisfied, namely $a = -139$, $b = 67$, $x_0 = 362$.

(iv) Similarly, for $k = 7$ the conditions (a_k) and (b_k) are not satisfied for all a, b, x_0 with the integer values in the interval from -1500 to 1500 provided that $a \cdot x_0$ is coprime to b.

(v) An interesting situation occurs in case $k = 12$. The conditions (a_{12}) and (b_{12}) are satisfied for the values $a = -11$, $b = 7$, $x = -9$ (as well as for $a = 7$, $b = -23$, $x_0 = 25$), but for the same values the conditions (a_4) and (a_6) (as well as (b_4) and (b_6)) are not satisfied. Hence the validity of the conditions is not transferred from k's to their divisors.

The following statements are related to properties of (generalized Mersenne) numbers $\frac{a^n-1}{a-1}$ where $a \notin \{-1, 0, 1\}$, which can be primes only when n is a prime. However, as we shall see, they can be primes for only a finite number of values n.

Proposition 2.5. *Let $c \notin \{-1, 0, 1\}$ and $m > 1$ be integers. Then $u_n = \frac{c^{mn}-1}{c^m-1}$ is not prime for $n > m$.*

Proof. The number $c^{mn} - 1$ is divisible by numbers $c^m - 1$ and $c^n - 1$, and so is divisible by their least common multiple which we denote by M. We can consider $M > 0$. Obviously, $M \geq |c^n - 1|$.

First we shall show that $|c^n - 1| > |c^m - 1|$ for $n > m$. We have the inequalities $|c^n - 1| \geq |c|^n - 1 \geq |c|^{m+1} - 1 \geq 2|c|^m - 1 = |c|^m + |c|^m - 1 \geq |c|^m + 3 > |c|^m + 1 \geq |c^m - 1|$. Hence $M > |c^m - 1|$. Further, $|c^{mn} - 1| \geq |c^{mn}| - 1 \geq |c^{2n}| - 1 = (|c^n| - 1)(|c^n| + 1) > |c^m - 1||c^n - 1| \geq M$. So we obtain a non-trivial factorization $\frac{c^{mn}-1}{c^m-1} = \frac{c^{mn}-1}{M} \frac{M}{c^m-1}$. ☐

Proposition 2.6. *Let the GD sequence satisfy the condition*

$$(c)\ a = c^{km}, b = \pm\frac{c^{km}-1}{c^m-1}, x_0 = \pm\frac{c^{jm}-1}{c^m-1}, \textit{with } c \notin \{-1,0,1\}, j \geq 0, k \geq 1, m \geq 2$$

integer numbers and with a choice of the same signs $\pm$.

Then the GD sequence contains only a finite number of primes.

Proof. W.l.o.g., let us choose the sign $+$. We have that x_n is equal to

$$c^{kmn}\frac{c^{jm}-1}{c^m-1} + \frac{c^{kmn}-1}{c^{km}-1}\frac{c^{km}-1}{c^m-1} = \frac{c^{m(j+kn)}-c^{kmn}}{c^m-1} + \frac{c^{kmn}-1}{c^m-1} = \frac{c^{m(j+kn)}-1}{c^m-1}.$$

If $j + kn > m$, then x_n is not a prime by Proposition 2.5. ☐

Though $a = -4c^4$ is not a power of an integer, the next proposition shows that it behaves similarly to $a = c^m$, which is likely related to the fact that $-4c^4 = (1 + \mathrm{i})^4 c^4$.

Proposition 2.7. *Let $c \geq 1$ and $n \geq 1$. Then the integer $u_n = \frac{(-4c^4)^n-1}{-4c^4-1}$ is prime only for $c = 1$ and $n = 2$.*

Proof. We note that $u_0 = 0$, $u_1 = 1$ and $u_2 = 1 - 4c^4 = (1 - 2c^2)(1 + 2c^2)$. If $2 < n = 2k$, then

$$u_n = \frac{(-4c^4)^{2k} - 1}{-4c^4 - 1} = \frac{((-4c^4)^k - 1)((-4c^4)^k + 1)}{-4c^4 - 1} = \frac{(-4c^4)^k - 1}{-4c^4 - 1}((-4c^4)^k + 1)\,.$$

If $1 < n = 2k + 1$ then, with $x = 2^k c^{2k+1}$, we use the following identity due to Sophie Germain:

$$4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1).$$

We obtain

$$\begin{aligned} u_n &= \frac{\left(4c^4\right)^{2k+1} + 1}{4c^4 + 1} = \frac{4 \cdot (2^k c^{2k+1})^4 + 1}{4c^4 + 1} \\ &= \frac{(2 \cdot (2^k c^{2k+1})^2 + 2 \cdot 2^k c^{2k+1} + 1)(2 \cdot (2^k c^{2k+1})^2 - 2 \cdot 2^k c^{2k+1} + 1)}{4c^4 + 1}\,. \end{aligned}$$

□

Proposition 2.8. *Let the GD sequence satisfies the condition*

(*d*) $a = (-4c^4)^k, b = \pm\dfrac{(-4c^4)^k - 1}{-4c^4 - 1}, x_0 = \pm\dfrac{(-4c^4)^j - 1}{-4c^4 - 1}$ *with* $c \geq 1, j \geq 0, k \geq 1$ *integer numbers and with a choice of the same signs* $\pm$ *.*

Then the GD sequence contains at most one prime.

Proof. W.l.o.g., let us choose the sign +. We have that x_n is equal to

$$(-4c^4)^{kn}\frac{(-4c^4)^j - 1}{-4c^4 - 1} + \frac{(-4c^4)^{kn} - 1}{(-4c^4)^k - 1}\frac{(-4c^4)^k - 1}{-4c^4 - 1} = \frac{(-4c^4)^{(j+kn)} - 1}{-4c^4 - 1}.$$

If $j + kn \neq 2$, then x_n is not a prime by Proposition 2.7. □

The next statements are related to the factorizations of $a^n - b^n$ or to the identity of Sophie Germain that we already used in the proof of Proposition 2.7.

Proposition 2.9. *Let* $c \notin \{-1, 0, 1\}$, $d \neq 0$, $\alpha \neq 0$ *and* $m > 1$ *be integers. Then the number* $u_n = \alpha^m c^{mn} - d^m$ *is prime for only a finite number of values* n*.*

Proof. The number $\alpha c^n - d$ is obviously a divisor of u_n. The equalities $\alpha c^n - d = \pm 1$ and $\alpha c^n - d = \pm u_n$ can be satisfied for only a finite number of values n. □

Proposition 2.10. *Let the GD sequence satisfies the condition*

(*e*) $a = c^m,\ b = \pm d^m(c^m - 1),\ x_0 = \pm\left(\alpha^m - d^m\right)$ *with* $d \neq 0, c \notin \{-1, 0, 1\}, \alpha \neq 0$ *integer numbers and a choice of the same signs* $\pm$ *.*

Then the GD sequence contains only a finite number of primes.

Proof. W.l.o.g., let us choose the sign $+$. We have

$$x_n = c^{mn}(\alpha^m - d^m) + \frac{d^m(c^m-1)(c^{mn}-1)}{c^m-1} = \alpha^m c^{mn} - d^m.$$

$\square$

Remark 7. If c, d are not coprime, then a, b are not coprime, too, so in the previous proposition we could add the condition that c, d are not coprime.

Proposition 2.11. *Let $c \geq 2$, $d \geq 1$ and $\alpha \geq 1$ be integers. Then the numbers $u_n = \alpha^4 c^{4n} + 4d^4$ and $v_n = 4\alpha^4 c^{4n} + d^4$ are prime only for $d = 1$, $\alpha = 1$ and $n = 0$.*

Proof. Again, the identity due to Sophie Germain,

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2) = \left((x+y)^2 + y^2\right)\left((x-y)^2 + y^2\right),$$

leads to the factorizations

$$\begin{aligned} u_n &= \alpha^4 c^{4n} + 4d^4 = \left((\alpha c^n + d)^2 + d^2\right)\left((\alpha c^n - d)^2 + d^2\right) \\ v_n &= 4\alpha^4 c^{4n} + d^4 = \left((d + \alpha c^n)^2 + c^{2n}\right)\left((d - \alpha c^n)^2 + \alpha^2 c^{2n}\right). \end{aligned}$$

All factors are greater than 1 excepting the case $d = 1$, $\alpha = 1$ and $n = 0$. $\square$

Proposition 2.12. *Let the GD sequence satisfies one of the the conditions*

(f)

$a = c^4,\ b = \pm 4d^4(1 - c^4),\ x_0 = \pm\left(\alpha^4 + 4d^4\right),$

(g)

$a = c^4,\ b = \pm d^4(1 - c^4),\ x_0 = \pm\left(4\alpha^4 + d^4\right),$

with $c \geq 2$, $d \geq 1, \alpha \geq 1$ integer numbers and a choice of the same signs $\pm$.

Then the GD sequence contains at most one prime.

Proof. Assuming that the condition (f) (the condition (g)) is satisfied, one obtains $x_n = \pm u_n$ ($x_n = \pm v_n$) from Proposition 2.11. $\square$

Propositions 2.6, 2.8 2.10 and 2.12 give us another necessary conditions for the GD sequence to contain an infinite number of primes, namely that the integers a, b and x_0 cannot have the values given in the conditions (c), (d), (e), (f) and (g). We are now ready to conjecture Generalized Dirichlet Theorem.

Conjecture 3. (**Generalized Dirichlet Theorem**) Let a, b and x_0 be integers such that b is coprime to $a \cdot x_0$. Consider the GD sequence $(x_n)_{n=0}^{\infty}$ defined by the recursive formula

$$x_{n+1} = ax_n + b$$

in which none of the following conditions is satisfied:

(i) the conditions (a_k) (equivalently (b_k)) from 2.4, for $k \geq 2$;

(ii) the condition (c) from 2.6;
(iii) the condition(d) from 2.8;
(iv) the condition(e) from 2.10;
(v) the conditions (f) and (g) from 2.12.

Then the GD sequence $(x_n)_{n=0}^{\infty}$ contains an infinite number of primes.

Remark 8. In the special case $x_0 = 0, a = 2, b = 1$ one obtains in the GD sequence $x_n = 2^n - 1$ which is a prime only if n is a prime meaning x_n is a *Mersenne prime.* We note that, for all $k \geq 2$, the condition (a_k) is indeed not satisfied as $A_k = 2^k - 1$ and $x_1 = 1$ are coprime. None of the other conditions (c) - (g) is satisfied, too. Thus our Generalized Dirichlet Theorem implies a well-known conjecture saying that there is an infinite number of Mersenne primes.

Remark 9. In the special case $x_0 = 2, a = 2, b = -1$ we obtain the GD sequence $x_n = 2^n + 1$ which is a prime only if $n = 2^k$ meaning x_n is a *Fermat prime.* Now the numbers $A_k = 2^k - 1$ and $x_0 = 2$ are coprime. Hence, similarly, as above, our Generalized Dirichlet Theorem implies a famous conjecture saying that there is an infinite number of Fermat primes.

Remark 10. In the case $a = 10$, $b = 1$, $x_0 = 0$, we have that A_k is coprime to x_1, and we get the GD sequence with $x_n = (10^n - 1)/9 = 1 \cdots 1$, that is, with members x_n consisting only of the digits 1 for $n \geq 1$. Generalized Dirichlet Theorem implies that there is an infinite number of primes whose decimal representation has only digits 1. Here Mathematica found the primes for

$$n = 2, 19, 23, 317 \text{ and } 1031.$$

More primes in this sequence have not been found.

Example 2.13. Let $a = 18$, $b = 1$ and $x_0 = 0$. Then $x_n = \frac{18^n - 1}{17}$. Using Mathematica, we have found out that x_n is not prime for $2 < n \leq 25000$. However, we do not see any explanation for this fact.

Therefore also a weaker form of the conjecture seems to be interesting.

Conjecture 4. (**Weak Generalized Dirichlet Theorem**) There are integers $a \neq \pm 1$, x_0 and $b \neq (1 - a)x_0$ such that the GD sequence $(x_n)_{n=1}^{\infty}$ defined by

$$x_{n+1} = ax_n + b$$

contains an infinite number of primes.

3. Variations of the problem

There are several variations of the main problem regarding the infinite number of superprimes. For example, it looks as there is an infinite number of superprimes consisting only of arbitrary two fixed prime digits. Even a variation of Conjecture 2 saying that there is such specific k-digit superprime for any $k > 0$ seems to be true. We look more closely to superprimes consisting of the digits 2 and 3.

Example 3.1. To generate, via Mathematica, all superprimes having at most r digits from the set $\{2, 3\}$, one can easily modify the command from Example 1.3. Here is the output obtained for $r = 8$: 2, 3, 23, 223, 233, 2333, 3323, 23333, 32233, 32323, 33223, 222323, 232333, 233323, 323233, 323333, 333233, 333323, 2222333, 2223233, 2232323, 2233223, 2332333, 2333323, 3222223, 3223223, 3223333, 3233323, 3233333, 3332233, 3333233, 22222223, 22223323, 22232233, 22232323, 23222233, 23223223, 23223323, 23322223, 32322223, 32323223, 32333333, 33222223, 33323333.

In the table below, P_k denotes the number of k-digit primes of the considered type. One can ask what is the limit $L := \lim_{k\to\infty} \sqrt[k]{P_k}$. For L we again have the asymptotic inequality $P_k > (L - \varepsilon)^k$ for every $\varepsilon > 0$.

k	P_k	$\sqrt[k]{P_k}$	k	P_k	$\sqrt[k]{P_k}$	k	P_k	$\sqrt[k]{P_k}$
1	2	2.000000000	8	13	1.377980015	15	1337	1.615878716
2	1	1.000000000	9	39	1.502397860	16	1922	1.604111626
3	2	1.259921050	10	52	1.484568818	17	4549	1.641237856
4	2	1.189207115	11	104	1.525340028	18	7778	1.644975106
5	4	1.319507911	12	197	1.553121812	19	15926	1.664039040
6	7	1.383087554	13	382	1.579866021	20	25210	1.659887454
7	13	1.442562919	14	618	1.582545917	21	57882	1.685729112

Example 3.2. Here is the command and the output in Mathematica for all numbers $0 < n \le 11000$ such that there is a superprime with the first digit 2 followed by n digits 3:

$$\mathrm{Do}[p = 2 * 10^n + (10^n - 1)/3; \mathrm{If}[\mathrm{PrimeQ}[q], \mathrm{Print}[n]], \{n, 1, 11000\}];$$

$n = 1, 2, 3, 4, 10, 16, 22, 53, 91, 94, 106, 138, 210, 282, 522, 597, 1049, 2227, 6459, 10582.$

That is, the first five superprimes of this specific form are

$$23, 233, 2333, 23333, 23333333333.$$

Based on the above computation we state a stronger version of Problem 1 and the following two conjectures:

Problem 2. *Is there an infinite number of superprimes consisting of the digits* 2 *and* 3*?* More generally, *is there, for any pair of distinct prime digits, an infinite number of superprimes with only these two fixed digits?*

Conjecture 5. There is an infinite number of superprimes with the first digit 2 which is followed by n digits 3 ($n > 0$). This is also true if the pair of prime digits $(2, 3)$ is replaced with any pair (p, q) of distinct prime digits where $q \notin \{2, 5\}$.

Conjecture 6. For any integer $k > 0$ there is a k-digit superprime consisting only of the digits 2 and 3. This is also true if the prime digits $2, 3$ are replaced with any two distinct prime digits.

Remark 11. The sequence of numbers 2, 23, 233, 2333, ... from Example 3.2 can be obtained from our Generalized Dirichlet Theorem in the special case $a = 10$, $b = 3$ a x_0=2. In this case we have that $x_0 = 2$ is coprime to $A_k = 1 \cdots 1$.

Hence Generalized Dirichlet Theorem implies an affirmative answer to the the first part of Problem 2 and, of course, implies the first part of the Conjecture 5, too.

A stronger version of the main problem we consider here asks if there is an infinite number of superprimes with a stronger property that every subchain of the superprime's decimal representation consisting of the two subsequent digits is again a decimal representation of a prime number. For example, 373 is the first such superprime with 3 digits as both 37 and 73 are primes.

The following example indicates that there might be an infinite number of the superprimes having this stronger property. Let us call them *strong superprimes.* (We note that also strong superprimes can be considered in arbitrary base m positional notation for $m \geq 4$.)

Example 3.3. To generate, via Mathematica, all strong superprimes having at most r digits, one can again easily modify the command from Example 1.3. Here is the list of the first 7 strong superprimes with at least 3 digits:

$$373, 237373, 537373, 5373737, 53737373, 53737373737, 237373737373.$$

The output indicates that there are three types of the strong superprimes:

(i) Type A: 23 followed by n copies of 73, the first one is 237373 ($n = 2$);
(ii) Type B: 53 followed by n copies of 73, the first one is 537373 ($n = 2$);
(iii) Type C: 5 followed by n copies of 37, the first one is 5373737 ($n = 3$).

We have generated, via Mathematica, the strong superprimes of the given three types with at most 2000 digits by a modification of the command from Example 3.2:

(i) Type A strong superprimes: $n = 2, 5, 20, 441$;
(ii) Type B strong superprimes: $n = 2, 3, 12, 21, 23, 483$;
(iii) Type C strong superprimes: $n = 3, 5, 8, 11, 15, 24, 53, 369, 710$.

Conjecture 7. There is an infinite number of strong superprimes of each of the three types A, B, C described above.

We again note that our Generalized Dirichlet Theorem in section 2 implies the Conjecture 7. To see this, let us put in Generalized Dirichlet Theorem in all three cases $a = 100$. For $b = 73$ and $x_0 = 23$ one obtains the type A, for $b = 73$ and $x_0 = 53$ the type B and for $b = 53$ and $x_0 = 5$ type C. In all three cases we have $A_1 = 1$ and $A_k = 10\dots 101$ for $k \geq 2$. Since $23 - 73 = -50$, $53 - 73 = -20$ a $5 - 37 = -32$, the requirement that $x_0 - A_1 b$ is coprime to A_k is satisfied in all three cases. Here one can see that sometimes checking the condition (b_k) might be more convenient then checking (a_k). In all three cases the conditions (c)-(g) are obviously not satisfied.

We finally consider the question whether there is an infinite number of superprimes with even a stronger property that all subchains of the superprime's decimal representation consisting of the two and three subsequent digits are again decimal representations of prime numbers. Here 373 is the first such superprime as all of 37, 73 and 373 are primes. It is obviously the only such strong superprime among the types A, B, C, because the number 737 is not prime. Hence strengthening further the concept of a strong superprime introduced here does not seem to be fruitful anymore.

Remark 12. We note that in the base m positional notation for $m \geq 4$ the situation is quite different than in the above case $m = 10$. We have been searching (using simple modifications of the given commands in Mathematica) for strong superprimes in the base m positional notation for $4 \leq m \leq 16$.

Just to illustrate our findings, we note that for $m = 5$ there are two 2-digit superprimes $13 = 23_5$ and $17 = 32_5$, one 3-digit strong superprime $67 = 232_5$ and one 5-digit strong superprime $2213 = 32323_5$. We have found a 17-digit strong superprime $1540415445963 = 32323232323232323_5$. For the base $m = 6$ there are two 2-digit superprimes $17 = 25_6$ and $23 = 35_6$; from this it can be easily shown that for $k > 2$ there are no k-digit strong superprimes in the base 6 positional notation. For $m = 8$ there are eight 2-digit superprimes: $19 = 23_8$, $23 = 27_8$, $29 = 35_8$, $31 = 37_8$, $43 = 53_8$, $47 = 57_8$, $59 = 73_8$ and $61 = 75_8$. Then also the number of k-digit strong superprimes is of course greater in the base m positional notation for $m = 8$ than for $m = 10$.

4. Observations on sequences related to the problem

Our first observation in this section concerns the increasing sequence $(a_n)_{n=1}^{\infty}$ of all natural numbers (not necessarily superprimes) consisting only of the prime digits 2, 3, 5 and 7. This is the sequence

$$2, 3, 5, 7, 22, \cdots, 77, 222, \cdots, 777, 2222, \cdots, 7777, \cdots .$$

Assume that the n-th member a_n consists of k digits. Then

$$\frac{2}{9}\left(10^k - 1\right) \leq a_n \leq \frac{7}{9}\left(10^k - 1\right)$$

and

$$4+16+\cdots+4^{k-1} < n \le 4+16+\cdots+4^{k},$$

whence

$$\frac{4^k-4}{3} < n \le \frac{4^{(k+1)}-4}{3}.$$

Thus

$$\frac{4^k-1}{3} \le n < \frac{4^{(k+1)}-1}{3}$$

which yields

$$4^k \le 3n+1 < 4^{k+1}.$$

Therefore

$$k \le \frac{\log(3n+1)}{\log 4} < k+1$$

whence

$$k = \left[\frac{\log(3n+1)}{\log 4}\right].$$

Consequently, for k we have

$$\frac{\log(3n+1)}{\log 4} - 1 < k \le \frac{\log(3n+1)}{\log 4}.$$

Hence we obtain the inequalities

$$\frac{2}{9}\left(10^{\frac{\log(3n+1)}{\log 4}-1} - 1\right) < a_n \le \frac{7}{9}\left(10^{\frac{\log(3n+1)}{\log 4}} - 1\right),$$

which can be rewritten as

$$\frac{2}{90}(3n+1)^{\log_4 10} - \frac{2}{9} < a_n \le \frac{7}{9}(3n+1)^{\log_4 10} - \frac{7}{9}.$$

The last inequalities yield

$$\liminf_{n\to\infty} \frac{a_n}{n^{\log_4 10}} \ge \frac{2}{90}\, 3^{\log_4 10}$$

and

$$\limsup_{n\to\infty} \frac{a_n}{n^{\log_4 10}} \ge \frac{7}{9}\, 3^{\log_4 10}.$$

We conclude that the sequence $(a_n)_{n=1}^{\infty}$ behaves as $(n^a)_{n=1}^{\infty}$, where

$$a = \log_4 10 = 1.660964\ldots.$$

Our second observation concerns the sequence $(b_n)_{n=1}^{\infty}$ of all natural numbers, which are not primes. We first note the well-known Prime Number Theorem says that the number of primes among the first n natural numbers is asymptotically $\frac{n}{\log n}$. This yields that the sequence $(b_n)_{n=1}^{\infty}$ is growing ‘slowly’.

More precisely, let $\pi(n)$ denote the number of primes less than or equal to a natural number n and let p_n be the n-th prime. It is well-known that

$$\lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1\,.$$

This implies that

$$\lim_{n\to\infty} \frac{p_n}{n\,\ln n} = 1\,.$$

Moreover, by [5],

$$p_n > n\,\ln n \qquad \text{for all natural numbers } n\,.$$

We shall show that

$$n\left(1+\frac{1}{\ln n+2}\right) < b_n < n\left(1+\frac{1}{\ln n-5}\right) \ \text{ for } n > \mathrm{e}^6\,,\ \text{ i.e. } \ n \geq 404\,.$$

We note that the right inequality is an improvement of the asymptotic inequality

$$b_n < (1+\varepsilon)n \text{ for every } \varepsilon > 0.$$

On the other hand, the left inequality shows that the right inequality cannot be essentially improved.

For $n \geq 55$ we have (we refer to [5])

$$\frac{n}{\ln n+2} < \pi(n) < \frac{n}{\ln n-4}\,.$$

This is our starting point for the following observation. Let $n > \mathrm{e}^5 > 55$, hence $\ln n > 5$. Let us denote $m := b_n$. Obviously, $m > n$ and

$$n = m-\pi(m) > m\left(1-\frac{1}{\ln m-4}\right) > m\left(1-\frac{1}{\ln n-4}\right) = m\,\frac{\ln n-5}{\ln n-4}\,.$$

Therefore

$$m < n\,\frac{\ln n-4}{\ln n-5} = n\left(1+\frac{1}{\ln n-5}\right)\,.$$

From this it follows $\ln m < \ln n + \ln\left(1+\frac{1}{\ln n-5}\right) < \ln n + \frac{1}{\ln n-5}$ and

$$n = m-\pi(m) < m\left(1-\frac{1}{\ln m+2}\right) < m\left(1-\frac{1}{\ln n+2+\frac{1}{\ln n-5}}\right) = m\,\frac{\ln^2 n-4\ln n-4}{\ln^2 n-3\ln n-9}\,.$$

Hence

$$m > n\,\frac{\ln^2 n-3\ln n-9}{\ln^2 n-4\ln n-4} = n\left(1+\frac{\ln n-5}{\ln^2 n-4\ln n-4}\right)\,.$$

Under the condition that $\ln n \geq 6$ we have

$$\frac{\ln n-5}{\ln^2 n-4\ln n-4} \geq \frac{1}{\ln n+2}\,.$$

Thus

$$m > n\left(1 + \frac{1}{\ln n + 2}\right).$$

So we conclude that the growth of the sequence $(b_n)_{n=1}^{\infty}$ is comparable with the growth of the sequence of the natural numbers and yet it does not contain any prime. The sequence $(p_n)_{n=1}^{\infty}$ is growing a bit faster than the sequence of the natural numbers and yet it contains all (and only) primes. From this it follows that having a sequence of natural numbers, one cannot conclude anything about as whether it contains primes or not.

Remark 13. The inequalities above can even be slightly improved. We note that for $17 \le n < \mathrm{e}^{100}$ as well as for $n > \mathrm{e}^{200}$ we have (we refer again to [5])

$$\frac{n}{\ln n} < \pi(n) < \frac{n}{\ln n - 2}.$$

From this one can analogously as above derive

$$n\left(1 + \frac{1}{\ln n}\right) < a_n < n\left(1 + \frac{1}{\ln n - 3}\right) \text{ for } n > \mathrm{e}^{200}.$$

References

1. M. Aigner and G.M. Ziegler, *Proofs from THE BOOK*, Berlin-Heidelberg: Springer-Verlag, 1998.
2. F. Borneman, *PRIMES is in P: a breakthrough for "Everyman"*, Notices of AMS **50** (2003), pp. 545–552.
3. J. Derbyshire, *Prime Obsession*, Washington, DC: Joseph Henry Press, 2003.
4. P.G.L. Dirichlet, *There are infinitely many prime numbers in all arithmetic progressions with first term and difference coprime*, Abhandlungen der Königlich Preussischen Akademie der Wissenschaften von 1837, (1837), 45–81. (An English translation from German by R. Stephan available at http://arxiv.org/PS_cache/arxiv/pdf/0808/0808.1408v1.pdf.)
5. J.B. Rosser, *The n-th prime is greater than* $n \log n$, Proc. London Math. Soc. (2) **45** (1939), 21–44.
6. J.B. Rosser, *Explicit bounds for some functions of prime numbers*, Amer. J. Math. **63** (1941), 211–232.
7. J.B. Rosser and L. Schoenfeld, *Approximate Formulas for Some Functions of Prime Numbers*, Illinois Journal Math. **6** (1962), 64–94.
8. M. Sautoy, *The Music of the Primes*, Harper Collins, 2003.

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FACIAL NON-REPETITIVE EDGE COLOURING OF SEMIREGULAR POLYHEDRA

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Dedicated to the 70th birthday of Alfonz Haviar

Abstract. A sequence $r_1, r_2, \dots, r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \le i \le n$, is called a *repetition*. A sequence S is called *non-repetitive* if no subsequence of consecutive terms of S is a repetition. Let G be a graph whose edges are coloured. A trail in G is called *non-repetitive* if the sequence of colours of its edges is non-repetitive. If G is a plane graph, a *facial non-repetitive edge-colouring* of G is an edge-colouring such that any *facial trail* is non-repetitive. We denote $\pi'_f(G)$ the minimum number of colours needed. In this paper we prove that for graphs of Platonic, Archimedean and prismatic polyhedra $\pi'_f(G)$ is either 3 or 4.

1. Introduction

A *polyhedron* P in the three-dimensional Euclidean space is a finite collection of planar convex polygons, called the *faces*, such that every edge of every polygon is an edge of precisely one other polygon. The edge set of a polyhedron is the set of intersections of adjacent faces, and the vertex set is the set of intersections of adjacent edges. A polyhedron P is called *semiregular* if all of its faces are regular polygons and there exists a sequence $\sigma = (p_1, p_2, \dots, p_q)$ called the *cyclic sequence* of P, such that every vertex of P is surrounded by a p_1-gon, a p_2-gon, …, a p_q-gon, in this order within rotation and reflexion. A semiregular polyhedron P is called the $(p_1, p_2, \dots, p_q)$-polyhedron if it is determined by the cyclic sequence $(p_1, p_2, \dots, p_q) = \sigma$ (see [7], [10]). The five polyhedra with equal regular faces that can be inscribed in a sphere (the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron) are known as *Platonic solids*. Thirteen polyhedra, which were discovered by Archimedes and are contained by equilateral and equiangular but not similar polygons are known as *Archimedean solids* (see [3]). The pseudo-Archimedean solid that has congruent solid angles but they are not all equivalent, satisfy the above conditions too and is known as a *Miller*

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solid, a *Ashkinuze polyhedron*, a *pseudo rhomb-cub-octahedron* or a $(3,4,4,4)$-polyhedron. The family of semiregular polyhedra completes a set of *prismatic polyhedra* consists of two infinite families: the *prisms* i.e. $(4,4,n)$-polyhedra for every $n \geq 3$, $n \neq 4$, and the *antiprisms* i.e. $(3,3,3,n)$-polyhedra for every $n \geq 4$ (see [3]).

The study of the semiregular polyhedra began with the abstraction of regularity in Euclide's Book XIII of *Elements*. Since those times they continually treat a lot of attention. Thanks to Steinitz theorem [5] that asserts that the graph is a graph of a convex polyhedron if and only if it is planar and 3-connected, instead of a study of combinatorial properties of convex polyhedra it is enough to study their graphs. Hence we use the same name for a polyhedron and its graph. The family of graphs of semiregular polyhedra is very inspirating and many questions that deal with their graphs were asked.

Maehara asked for the the smallest integer n such that the graph of a semiregular polyhedra can be represented as the intersection graph of a family of unit-diameter spheres in Euclidean n-dimensional space. Such n is called the *sphericity* of the graph and in [12] is determined for graphs of semiregular polyhedra except for a few prisms. The generalized Archimedean solids were studied by Karabáš and Nedela (see [8], [9]). They gave a complete census of Archimedean solids of genera from two to five. But study of properties of semiregular polyhedra does not occur only in mathematics; man can find it also in chemistry (see [11]); architecture, art, cartography (see [3]); ... and so on. A lot of posed questions relate to colouring of semiregular polyhedra and determining some colouring characteristic of it: The rainbowness of semiregular polyhedra, the parameter $rb(P)$, had been studied by Jendroľ and Schrötter in [7]. They found the exact value of $rb(P)$ for all graphs of semiregular polyhedra except of three Archimedean solids for which the parameter is only estimated. A. Kemnitz and P. Wellmann in [10] determined the circular chromatic number $\chi_c(G)$ for Platonic solid graphs, Archimedean solid graphs and regular convex prism graphs. In this paper we determine a variant of non-repetitive edge-colouring for plane graphs of semiregular polyhedra introduced in [6].

A sequence $r_1, r_2, \ldots, r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$, is called a *repetition*. A sequence S is called *non-repetitive* if no subsequence of consecutive terms of S is a repetition. Thue [13] states that arbitrarily long non-repetitive sequences can be formed using only three symbols.

An *edge k-colouring* of G is a mapping $\varphi : E(G) \to \{1, 2, \ldots, k\}$. Alon et al. [1] introduced a natural generalization of Thue's sequences for edge-colouring of graphs. An edge-colouring φ of a graph G is *non-repetitive* if the sequence of colours on any path in G is non-repetitive. The minimum numbers of colours $\pi'(G)$ needed in any non-repetitive colouring of G is called the *Thue chromatic index* of G.

For a face f, the *size* (or degree) of f is defined to be the length of the shortest closed facial walk containing all edges from the boundary of f. The face of degree r is known as an r-gonal face.

Let G be a plane graph. A *facial trail* in G is a trail made of consecutive edges of the boundary walk of some face. A *facial non-repetitive edge colouring* of G is an edge colouring of G such that any facial trail is non-repetitive. The *facial Thue chromatic index* of G, denoted $\pi'_f(G)$, is the minimum number of colours of a facial non-repetitive edge colouring of G. Note that the facial Thue chromatic index depends on the embedding of the graph. In the following, all the graphs we will consider come along with an embedding in the plane.

We show the exact value of Thue chromatic index for graphs of all semiregular polyhedra, that is the first step towards the Conjecture 18 setted in [6].

The notation and terminology used but not defined in this paper can be found in [2].

2. Basic preliminaries

Thue's sequences (see [13]) show that the Thue chromatic index of a path is at most 3. Actually, $\pi'(P_n) = 3$, for all $n \geq 5$, as it is easy to see that every sequence of length 4 on two symbols contains a repetition. An immediate corollary is that the Thue chromatic index of a cycle is at most 4. In [4], Currie showed that $\pi(C_n) = 4$ only for $n \in \{5, 7, 9, 10, 14, 17\}$. For other values of $n \geq 3$, $\pi(C_n) = 3$.

From the above remarks it is easy to see that for our less constrained parameter $\pi'_f(G)$ the following holds (see [6]):

Theorem 1. *Let G be a cycle C_n.*

(i) if $n = 2$, then $\pi'_f(G) = 2$;
(ii) if $n \notin \{2, 5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) = 3$ and
(iii) if $n \in \{5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) = 4$.

Corollary 2. *Let G be a plane graph and let a facial trail of one of its faces be isomorphic to C_n.*

(i) If $n = 2$, then $\pi'_f(G) \geq 2$;
(ii) if $n \notin \{2, 5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) \geq 3$ and
(iii) if $n \in \{5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) \geq 4$.

3. Prismatic polyhedra

An *r-sided antiprism A_r* is defined as follows: The vertex set $V(A_r) = \{u_{r+1} = u_1, u_2, \ldots, u_r, v_{r+1} = v_1, v_2, \ldots, v_r\}$, $r \geq 3$. The edge set $E(A_r) = \{ \{u_i u_{i+1}\} \cup \{v_i v_{i+1}\} \cup \{u_i v_i\} \cup \{u_{i+1} v_i\},\ i = 1, \ldots, r \}$. The face set of A_r consists of two r-gonal faces f and h where $f = [u_1, \ldots, u_r]$, $h = [v_1, \ldots, v_r]$ and $2r$ faces $f_i = [u_i, u_{i+1}, v_i]$ and $h_i = [v_i, v_{i+1}, u_{i+1}]$, $i = 1, \ldots, r$, indices taken modulo r.

Theorem 3. *Let A_r be the graph of antiprism. If $r \in \{5, 7, 9, 10, 14, 17\}$ then $\pi'_f(A_r) = 4$; else $\pi'_f(A_r) = 3$.*

Proof. According to Theorem 2 the lower bound is clear.
Upper bound: Colour the edges of the cycle on vertices u_1, u_2, …, u_r nonrepetitively using 4 colours when $r = 5$, 7, 9, 10, 14 or 17; else use only 3 colours.
For $i = 1, \dots, r$, indices modulo r, use the colour of the edge $u_i u_{i+1}$ for colouring the edges $u_{i+1} v_{i+1}$ and $v_{i+1} v_{i+2}$.
Note that in such a case the cycle on vertices v_1, v_2, …, v_r is coloured non-repetitively too. Noncoloured edges are diagonals of the 4-gonal faces coloured with two colours. Thus there is still at least one colour more that can be used to obtain facial non-repetitive colouring of each 3-gonal face. □

An r*-sided prism* D_r, $r \geq 3$, is defined as follows: The vertex set $V = \{u_{r+1} = u_1,\ u_2, \dots,\ u_r,\ v_{r+1} = v_1,\ v_2, \dots,\ v_r\}$ and the edge set $E = \{\{u_i, u_{i+1}\} \cup \{v_i, v_{i+1}\} \cup \{u_i, v_i\}$, for $i = 1, \dots, r$ $\}$. The set of faces of D_r consists of two r-gonal faces: the outer face $f = [u_1, \dots, u_r]$ and the inner face $h = [v_1, \dots, v_r]$; and r quadrangles $[u_i, u_{i+1}, v_{i+1}, v_i]$ for any $i = 1, \dots, r$, indices taken modulo r.

Theorem 4. *Let D_r be a graph of prism. Then for $r \geq 4$ $\pi'_f(D_r) = 4$ and $\pi'_f(D_3) = 3$.*

Proof. It is easy to see that $\pi_f(D_r) \geq 3$ and that $\pi_f(D_3) = 3$.
Now we show the upper bound for D_r; $r > 3$: According to the Theorem of Thue [13] there exists a non-repetitive edge 3-colouring of the path $P = v_1, v_2, \dots, v_r$, see Figure 1, that uses the colours 1, 2 and 3. Let us colour the edges of the path $Q = u_2, u_3, \dots, u_r, u_1$ with the colours 1, 2 and 3 in such a way that an edge $u_{i+1} u_{i+2}$ has the same colour as an edge $v_i v_{i+1}$ for $i = 1, 2, \dots, r-1$. Colour the edges $v_r v_1$ and $u_1 u_2$ with the colour 4.

Then we have to distinguish four situations to show that our colouring fulfills the required conditions:

Without loss of generality we can assume, that edge $v_{r-1} v_r$ received colour 1 and a $v_{r-2} v_{r-1}$ colour 2.

Case 1: If colour $v_1 v_2$ is 1 and $v_2 v_3$ is 2, then we shall colour the edges $u_r v_r$, $u_1 v_1$ and $u_2 v_2$ with the colour 3 and the remaining edges with the colour 4.

Case 2: If colour $v_1 v_2$ is 1 and $v_2 v_3$ is 3, then we shall colour the edges $u_r v_r$, $u_1 v_1$ with the colour 3; the edge $u_2 v_2$ with the colour 2 and the remaining edges with the colour 4.

Case 3: If colour $v_1 v_2$ is 2, then we shall colour the edges $u_r v_r$ and $u_1 v_1$ with the colour 3.
If the colour of the edge $v_2 v_3$ is 1, then we shall colour the edge $u_2 v_2$ with the

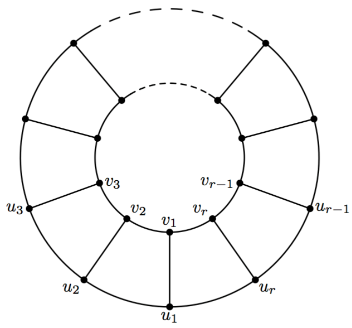

FIGURE 1. The prism

colour 3 too, otherwise we shall colour it 1. For colouring of the remaining edges we can use colour 4.

Case 4: If colour v_1v_2 is 3, then we shall colour the edge u_rv_r with the colour 3 and the edge u_1v_1 with the colour 2.
If the colour of the edge v_2v_3 is 2, then we shall colour the edge u_2v_2 with the colour 1; otherwise we shall colour it with the colour 2. For colouring of the remaining edges we shall use colour 4.

It is easy to see that in each case the obtained colouring is a facial non-repetitive 4-edge-colouring.
Now we are going to show that the lower bound of the facial Thue chromatic index of D_r is 4; $r \geq 4$: Suppose, that there exist facial non-repetitive 3-edge-colouring of D_r, $r \geq 4$. In this case on the r-gonal face of D_r there exist a sequence of edges v_iv_{i+1}, $v_{i+1}v_{i+2}$, $v_{i+2}v_{i+3}$, $v_{i+3}v_{i+4}$ coloured with colours a, b, a, c. Thus both of the edges $v_{i+1}u_{i+1}$ and $v_{i+2}u_{i+2}$ have to be coloured with the colour c and the edge $v_{i+3}u_{i+3}$ has to be coloured with b. Hence the edge $u_{i+2}u_{i+3}$ have to be coloured with the colour a. But then the colour a, as well as b and c, could not be used for colouring the edge $u_{i+1}u_{i+2}$ – a contradiction. $\square$

4. PLATONIC POLYHEDRA

The set of Platonic solids consists of five polyhedra:

(i) the tetrahedron or the $(3, 3, 3) - \textit{polyhedron}$,
(ii) the cube or the $(4, 4, 4) - \textit{polyhedron}$,
(iii) the octahedron or the $(3, 3, 3, 3) - \textit{polyhedron}$,
(iv) the dodecahedron or the $(5, 5, 5) - \textit{polyhedron}$ and
(v) the icosahedron or the $(3, 3, 3, 3, 3) - \textit{polyhedron}$.

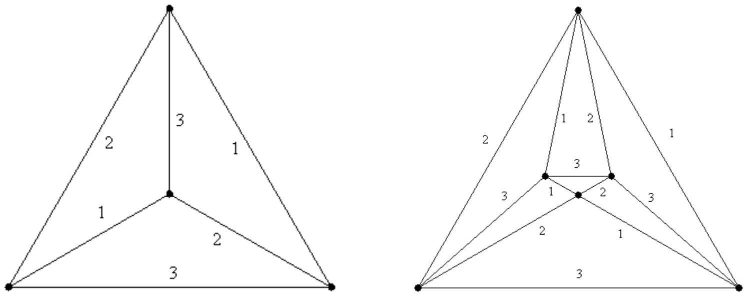

Figure 2. The tetrahedron and the octahedron

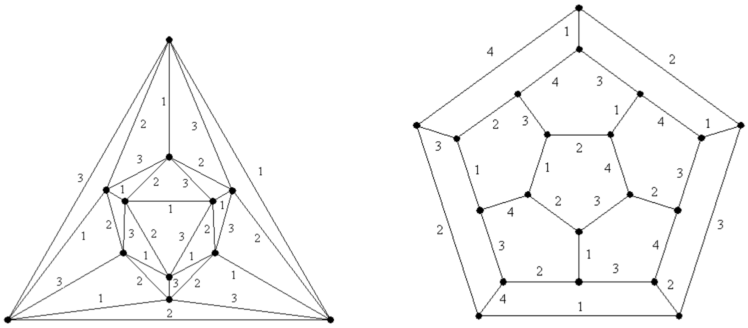

Figure 3. The isosahedron and the dodecahedron

Theorem 5. *If G is the tetrahedron, the octahedron or the icosahedron, then $\pi'_f(G) = 3$. If G is the dodecahedron or the cube, then $\pi'_f(G) = 4$.*

Proof. Theorem 2 implies that $\pi'_f(G) \geq 3$ for G being the tetrahedron, the octahedron, the cube or the icosahedron and $\pi'_f(G) \geq 4$ for G being the dodecahedron. From Figures 2 and 3 we can observe that except of the cube these bounds are achieved. The cube Q is in a family of prisms hence according to the Theorem 4 we have $\pi'_f(Q) = 4$. □

5. Archimedean polyhedra

The set of Archimedean solids consists of thirteen polyhedra:

(i) the cub-octahedron or the $(3,4,3,4) - polyhedron$,
(ii) the rhomb-cub-octahedron or the $(3,4,4,4) - polyhedron$,
(iii) the snub cube or the $(3,3,3,3,4) - polyhedron$,
(iv) the truncated dodecahedron or the $(3,10,10) - polyhedron$,
(v) the truncated icosi-dodecahedron or the $(4,6,10) - polyhedron$ or the great rhomb-icosi-dodecahedron,

(vi) the truncated icosahedron or the $(5,6,6)-polyhedron$,
(vii) the icosi-dodecahedron or the $(3,5,3,5)-polyhedron$,
(viii) the rhomb-icosi-dodecahedron or the $(3,4,5,4)-polyhedron$,
(ix) the snub dodecahedron or the $(3,3,3,3,5)-polyhedron$,
(x) the truncated tetrahedron or the $(3,6,6)-polyhedron$,
(xi) the truncated octahedron or the $(4,6,6)-polyhedron$,
(xii) the truncated cube or the $(3,8,8)-polyhedron$ and
(xiii) the truncated cub-octahedron or the $(4,6,8)-polyhedron$, or the great rhomb-cub-octahedron.

Theorem 6. *If G is a plane graph of the $(3,4,3,4)$-polyhedron, the $(3,4,4,4)$-polyhedron or the $(3,3,3,3,4)$-polyhedron, then $\pi'_f(G)=3$.*
If G is a plane graph of the $(3,10,10)$-polyhedron, the $(4,6,10)$-polyhedron, the $(5,6,6)$-polyhedron, the $(3,5,3,5)$-polyhedron, the $(3,4,5,4)$-polyhedron or the $(3,3,3,3,5)$-polyhedron, then $\pi'_f(G)=4$.
If G is a plane graph of the $(3,6,6)$-polyhedron, the $(4,6,6)$-polyhedron, the $(3,8,8)$-polyhedron or the $(4,6,8)$-polyhedron, then $\pi'_f(G)=4$.

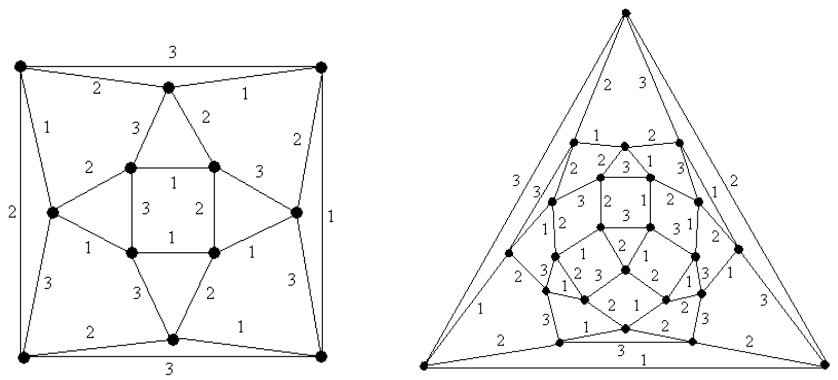

Figure 4. The $(3,4,3,4)$-polyhedron and the $(3,4,4,4)$-polyhedron

Proof. Theorem 2 gives the lower bound for the facial Thue chromatic index of Archimedean solids. From Figures $4-9$ we can observe that except of the $(4,6,6)$-polyhedron, the $(3,6,6)$-polyhedron, the $(3,8,8)$-polyhedron and the $(4,6,8)$-polyhedron these bounds are achieved.

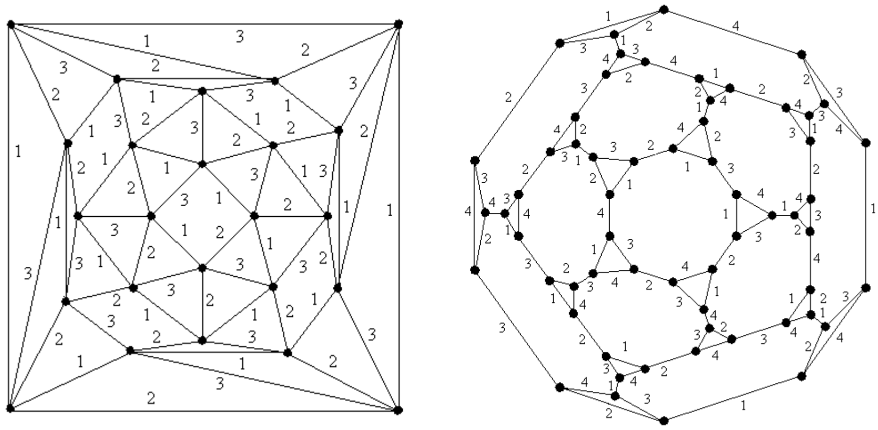

FIGURE 5. The $(3,3,3,3,4)$-polyhedron and the $(3,10,10)$-polyhedron

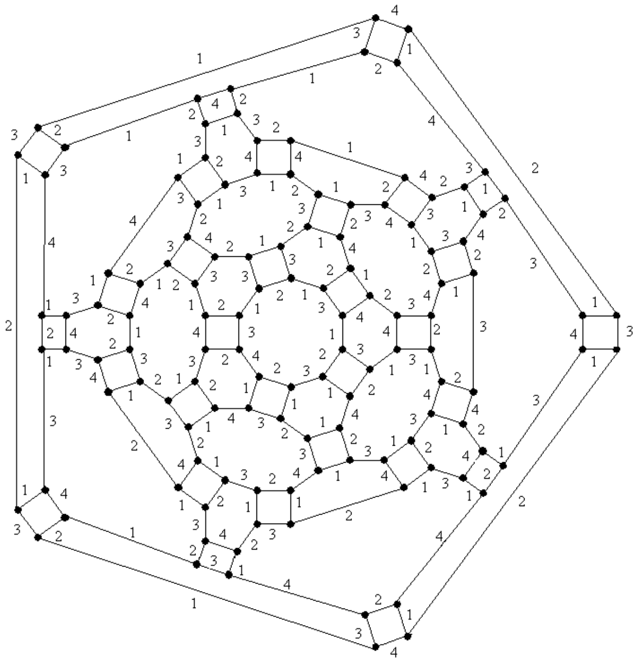

FIGURE 6. The $(4,6,10)$-polyhedron

For these four exceptions Theorem 2 gives $\pi'_f(G) \geq 3$. In what follows we show that 3 colours are not enough to colour their edges facially non-repetitively.

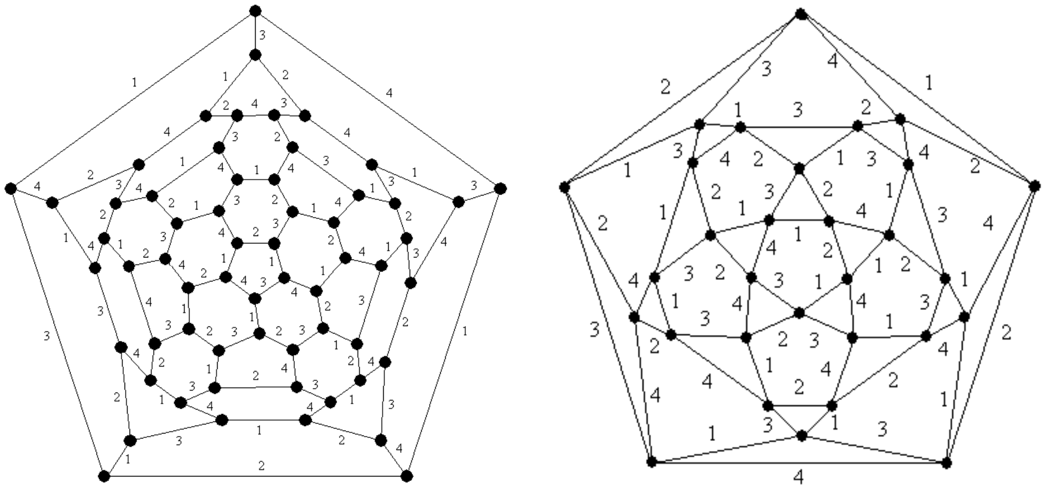

Figure 7. The $(5, 6, 6)$-polyhedron and the $(3, 5, 3, 5)$-polyhedron

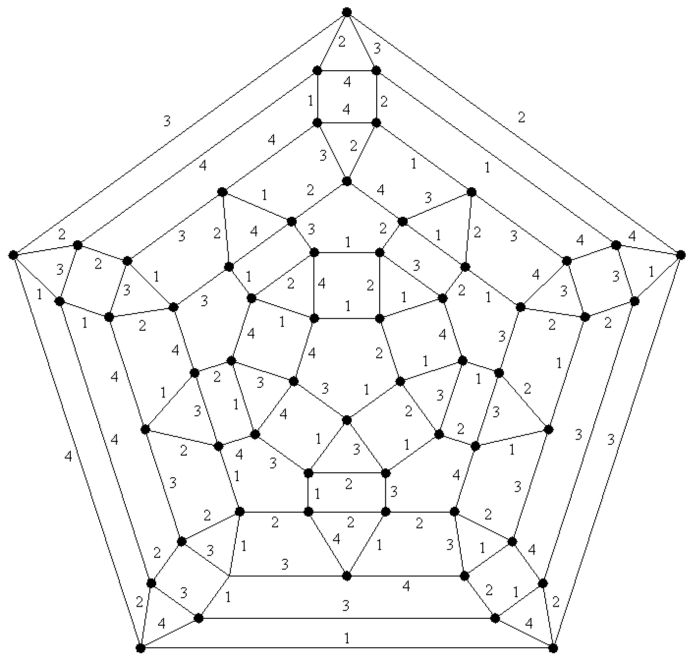

Figure 8. The $(3, 4, 5, 4)$-polyhedron

Case 1: The $(4, 6, 6)$-polyhedron
Consider a graph of the $(4, 6, 6)$-polyhedron depicted on the Figure 10 (do not

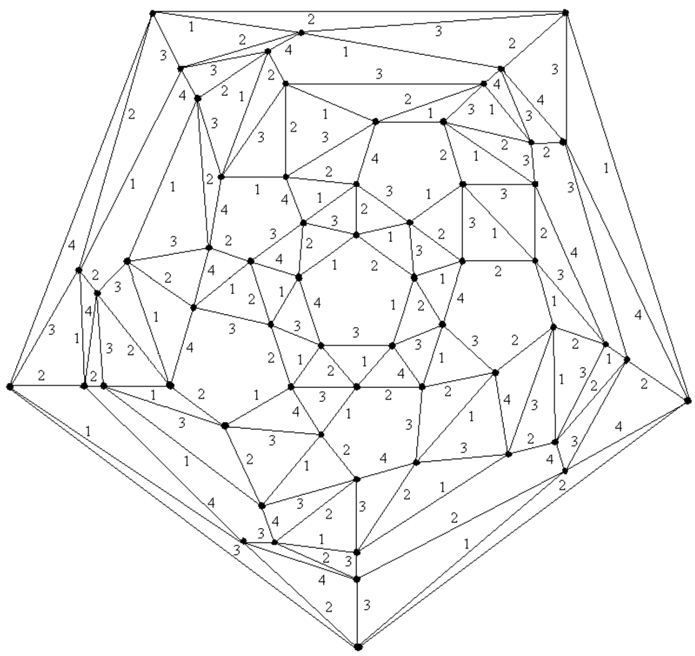

FIGURE 9. The $(3,3,3,3,5)$-polyhedron

consider the edge labelling there).
By a way of contradiction let us suppose that there exists a facial non-repetitive edge 3-colouring of the $(4,6,6)$-polyhedron. In such a case there exist a 4-gonal face that edges v_1v_2, v_2v_3, v_3v_4, v_4v_1 are coloured w.l.o.g. with colours 1, 2, 1, 3. Then the edges v_2v_6 and v_3v_7 have to have the colour 3 and the edges v_4v_8 and v_1v_5 have to have the colour 2. The edge v_5v_{10} has to have the colour 1, because in other case there would be either a repetition 2, 2 or a repetition 2, 3, 2, 3. Hence the edge v_5v_{11} has to have the colour 3. By the similar reasons the edge v_6v_{13} has to have the colour 1. Thus the edge v_6v_{12} has to have the colour 2. But in that case the edge $v_{11}v_{12}$ has to have the colour 1 and there is a repetitive sequence of colours 2, 3, 1, 2, 3, 1 on edges of one face of $(4,6,6)$-polyhedron – a contradiction.

For a facial non-repetitive 4-edge-colouring of $(4,6,6)$-polyhedron see Figure 10 where the numbers on edges are colours of these edges.

Case 2: The $(3,6,6)$-polyhedron
Consider a graph of the $(3,6,6)$-polyhedron depicted on the Figure 11 (do not consider the edge labelling).
By a way of contradiction let us suppose that there exists a facial non-repetitive

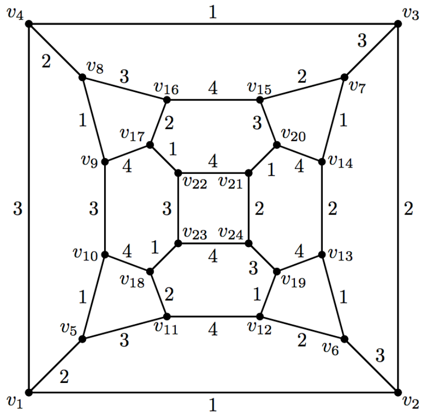

FIGURE 10. Case 1 - the $(4, 6, 6)$-polyhedron

edge 3-colouring of the $(3, 6, 6)$-polyhedron. Then there exist a 3-gonal face with edges v_1v_2, v_2v_3, v_3v_1 coloured w.l.o.g. 1, 2 and 3. Hence the edge v_2v_5 have to have the colour 3 and the edge v_3v_6 have to have the colour 1. Then one of the edges v_5v_{10}, v_5v_{11} is coloured with the colour 1 and the other one with the colour 2. Thus the edge $v_{10}v_{11}$ is coloured with the colour 3. Similarly one of the edges v_6v_8, v_6v_9 is coloured with the colour 2 and the other one with the colour 3. Hence the edge v_8v_9 is coloured with the colour 1. But then the edge v_9v_{10} has to have the colour 2 and the edge v_5v_{10} has to have the colour 1. But then the edge v_6v_9 has to have the colour 3 and thus there is a repetitive sequence of colours 1, 2, 3, 1, 2, 3 on edges of one face of the $(3, 6, 6)$-polyhedron – a contradiction. For a facial non-repetitive edge 4-colouring of $(3, 6, 6)$-polyhedron see Figure 11.

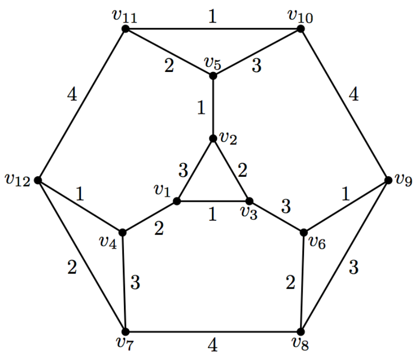

FIGURE 11. Case 2 - the $(3, 6, 6)$-polyhedron

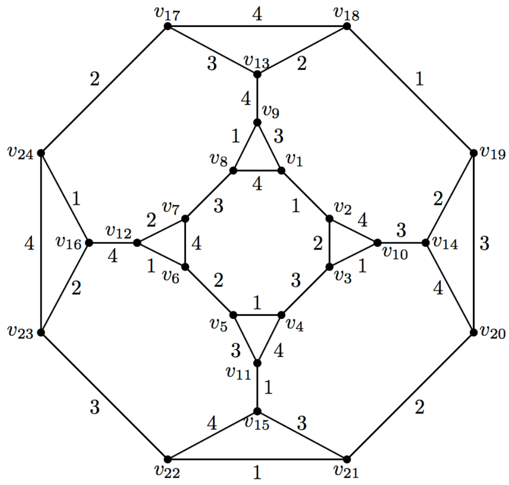

FIGURE 12. Case 3 - the $(3,8,8)$-polyhedron

Case 3: the $(3,8,8)$-polyhedron
Consider a graph of the $(3,8,8)$-polyhedron depicted on the Figure 12 (do not consider the edge labelling). By a way of contradiction let us suppose that there exists a facial non-repetitive edge 3-colouring of the $(3,8,8)$-polyhedron. Notice that there exist unique facial non-repetitive edge colouring of the cycle C_8 with three symbols. Thus there exists an 8-gonal face of $(3,8,8)$-polyhedron that edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_6v_7, v_7v_8, v_8v_1 are coloured either with the sequence of colours $S_1 = 1,2,1,3,2,1,2,3$, or with the sequence of colours $S_2 = 3,1,2,1,3,2,1,2$.

If the 8-gonal face is coloured with the sequence of colours S_1, the edges v_2v_{10} and v_3v_{10} have to have the colour 3 – a contradiction.

Now suppose that the 8-gonal face mentioned above is coloured with the sequence of colours S_2. In such a case the edges v_1v_9 and v_6v_{12} have to have the colour 1, the edges v_4v_{11}, v_7v_{12} and v_8v_9 have to have the colour 3 and the edge v_5v_{11} has to have the colour 2. Hence the edges v_9v_{13} and $v_{12}v_{16}$ have to have the colour 2 too and the edge $v_{11}v_{15}$ has to have the colour 1. Then one of the edges $v_{15}v_{21}$, $v_{15}v_{22}$ is coloured with the colour 2 and the other one with the colour 3 thus the edge $v_{21}v_{22}$ has to have the colour 1. Similarly one of the two edges $v_{16}v_{23}$, $v_{16}v_{24}$, likewise one of the two edges $v_{13}v_{17}$, $v_{13}v_{18}$, has to have the colour 1 and the other one the colour 3. Thus the edges $v_{23}v_{24}$, $v_{17}v_{18}$ have to have the colour 2. Hence the edge $v_{22}v_{23}$ has to have the colour 3; the edge $v_{15}v_{22}$ has to have the colour 2 and the edge $v_{16}v_{23}$ the colour 1. But then the edge $v_{17}v_{24}$ has to have the colour 1 and both of the edges $v_{13}v_{17}$, $v_{16}v_{24}$ have to have the colour

3. Hence there is a repetition 1, 3, 2, 3, 1, 3, 2, 3, of colours on edges of one face of the $(3, 8, 8)$-polyhedron – a contradiction.

For a facial non-repetitive 4-edge-colouring of $(3, 8, 8)$-polyhedron see Figure 12.

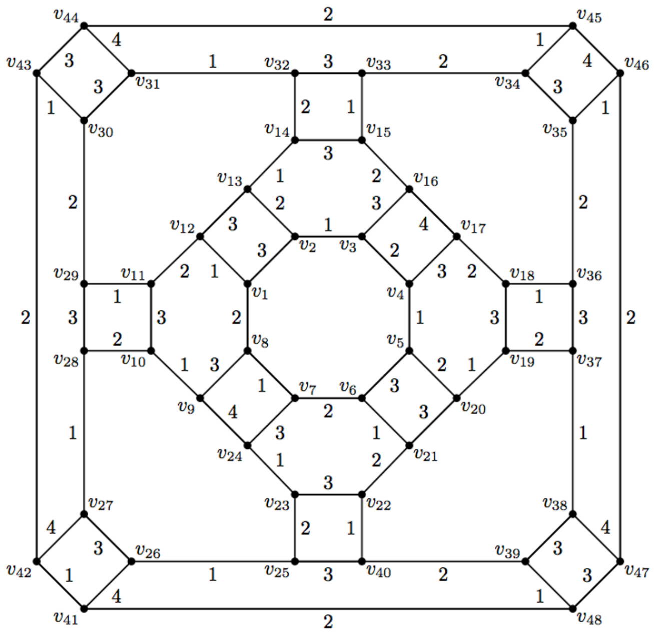

FIGURE 13. Case 4 - the $(4, 6, 8)$-polyhedron

Case 4: the $(4, 6, 8)$-polyhedron
Consider a graph of the $(4, 6, 8)$-polyhedron depicted on the Figure 13 without labellings of edges.
By a way of contradiction let us suppose that there exists a facial non-repetitive 3-edge-colouring of the $(4, 6, 8)$-polyhedron.
The unique non-repetitive colouring of C_8 gives two possibilities how the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_6v_7, v_7v_8 and v_8v_1 of an 8-gonal face of $(4, 6, 8)$-polyhedron are coloured.

First let us suppose that the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_6v_7, v_7v_8 and v_8v_1 are coloured with the sequence of colours 1, 2, 1, 3, 2, 1, 2, and 3, respectively. In such a case the edges v_2v_{13} and v_3v_{16} have to have the colour 3 and the edge v_4v_{17} has to have the colour 2. Thus the edge $v_{16}v_{17}$ has to have the colour 1. But then the edge $v_{15}v_{16}$ has to have the colour 2 and there is a repetition 2, 3, 2, 3 of colours on edges of one face of the $(4, 6, 8)$-polyhedron – a contradiction.

Now suppose that the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_6v_7, v_7v_8 and v_8v_1 of 8-gonal face are coloured consequently with colours 3, 1, 2, 1, 3, 2, 1, 2. Then the edges v_3v_{16} and v_4v_{17} have to have the colour 3, the edge v_1v_{12} has to have the colour 1 and the edge v_2v_{13} has to have the colour 2. Hence the edge $v_{16}v_{17}$ has to have the colour 1 and the edges $v_{15}v_{16}$, $v_{17}v_{18}$ have to have the colour 2. Then the edges $v_{12}v_{13}$ and $v_{14}v_{15}$ have to have the colour 3 and the edge $v_{13}v_{14}$ has to have the colour 1. But in such a case the edge $v_{15}v_{33}$ has to have the colour 1 and there is a repetition 1, 2, 1, 2 of colours on edges of one face of the $(4, 6, 8)$-polyhedron – a contradiction.

For a facial non-repetitive edge 4-colouring the $(4, 6, 8)$-polyhedron see Figure 13. □

6. Pseudo-Archimedean polyhedron

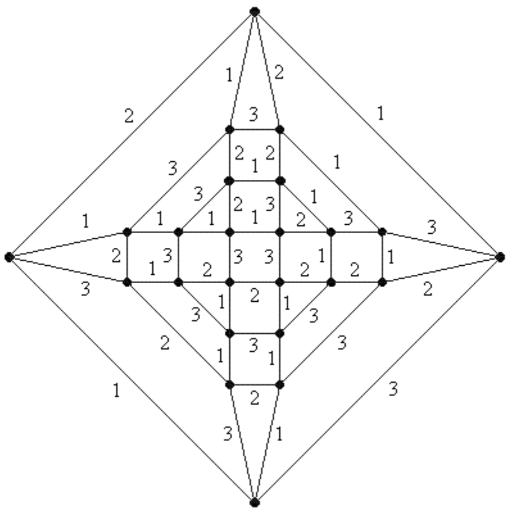

Figure 14. the Miller polyhedron

Theorem 7. *Let G be a graph of the Miller polyhedron. Then $\pi'_f(G) = 3$.*

Proof. Theorem 2 gives $\pi'_f(G) \geq 3$ for G being a graph of Miller polyhedron. A facial non-repetitive edge 3-colouring of G is at Figure 14, thus $\pi'_f(G) = 3$. □

7. Discussion

In [6] there was conjectured that for every 3-connected plane graph G the facial Thue chromatic index $\pi'_f(G) \leq 6$. In the present paper we have found the exact values of the facial Thue chromatic index for semiregular polyhedra. We showed that $\pi'_f(G)$ is either 3 or 4 for graphs of semiregular polyhedra, which is the first step towards the conjecture mentioned.

By Theorem 1 for every cycle C_n, where $n \in \{2, 5, 7, 9, 10, 14, 17\}$ holds $\pi'_f(C_n) = 4$. We showed that even if the 3-connected plane graph does not contain any face of degree $n \in \{2, 5, 7, 9, 10, 14, 17\}$ its facial Thue chromatic index could be 4 (see Figure 10 – 13) or greater in general case.

The existence of a plane graph G for which $\pi'_f(G) \geq 5$ is still an open question.

Acknowledgment

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References

1. N. Alon, J. Grytczuk, M. Hałuszczak, O. Riordan, *Non-repetitive colourings of graphs*, Random. Struct. Algor. 21 (2002), 336–346.
2. J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer Verlag, 2008.
3. P. R. Cromwell, *Polyhedra*, Cambridge University Press, Cambridge, 1997.
4. J. D. Currie, *There are ternary circular square-free words of length n for $n \geq 18$*, Electron. J. Combin. 9 (2002), #N10, 7pp.
5. B. Grünbaum, *Convex polytopes*, Springer Verlag, 2004 (2nd edition).
6. F. Havet, S. Jendroľ, R. Soták, E. Škrabuľáková, *Facial non-repetitive edge-colouring of plane graphs*, IM Preprint Series A No. 2/2009.
7. S. Jendroľ Š. Schrötter, *On rainbowness of semiregular polyhedra*, Czechoslovak Mathematical Journal, 58 (133) (2008), 359–380.
8. J. Karabáš, R. Nedela, *Archimedean solids of genus two*, Electronic Notes in Discrete Mathematics 28 (2007), 331–339.
9. J. Karabáš, R. Nedela, *Archimedean solids of higher genera*, Mathematics of Computation (submitted).
10. A. Kemnitz, P. Wellmann, *Circular chromatic numbers of certain planar graphs and small graphs*, Congr. Numeratium 169 (2004), 199–209.
11. I. Lukovits, S. Nikoli, N. Trinajsti, *Theoretical and Computational Developments: resistance distance in regular graphs*, International Journal of Quantum Chemistry 71 (1999), 217–225.
12. H. Maehara, *On the sphericity of the graphs of semiregular polyhedra*, Discrete Math. 58 (1986), 311–315.
13. A. Thue, *Über unendliche Zeichenreichen*, Norske Vid. Selsk. Skr., I Mat. Nat. Kl., Christiana 7 (1906), 1–22 (In German).

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HOMOMORPHIC EXTENSIONS OF PSEUDOCOMPLEMENTED SEMILATTICES

TIBOR KATRIŇÁK AND JAROSLAV GURIČAN

Dedicated to the 70th birthday of Alfonz Haviar

Abstract. Our aim is to study and characterize extensions to a homomorphism in the class of pseudocomplemented semilattices. We present here such a description.

1. Introduction

We shall deal with the question in which circumstances a mapping f from a generating set X of a pseudocomplemented semilattice S into a pseudocomplemented semilattice M can be extended to a homomorphism $g : S \to M$. Such an extension, if it exists, is uniquely determined.

It is a well-known fact (see [5]) that the class of all pseudocomplemented semilattices is equational with only one non-trivial subvariety, namely, the class of Boolean algebras. The preceding question found an answer for Boolean algebras (see [9] and especially Sikorski's extension criterion). We shall use these results as a motivation for our task.

2. Preliminaries

A pseudocomplemented semilattice (= PCS) is an algebra $(S; \wedge,^*, 0, 1)$ of type (2,1,0,0), where $(S; \wedge, 0, 1)$ is a bounded meet-semilattice and, for every $a \in S$, the element a^* is a *pseudocomplement* of a, i.e. $x \leq a^*$ if and only if $x \wedge a = 0$. A PCS S is said to be *non-trivial*, whenever $\mid S \mid \geq 2$. An element $a \in S$ is called *closed*, if $a = a^{**}$. Let $B(S)$ denote the set of all closed elements of S. It is known that

$$(B(S); +, \wedge,^*, 0, 1)$$

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forms a Boolean algebra with

$$a + b = (a^* \wedge b^*)^*$$

(see [1] and [3]). (Clearly, a PCS S is a Boolean algebra if and only if S satisfies the identity $x = x^{**}$.)

Here are some rules of computation with * and $\wedge$ (see [1] or [3]):

(1) $x \wedge x^* = 0$.

(2) $x \leq y$ implies that $x^* \geq y^*$.

(3) $x \leq x^{**}$.

(4) $x^* = x^{***}$.

(5) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$.

(6) $0^* = 1$ and $1^* = 0$.

The following result can be easily verified (see [7]).

Lemma 2.1. *Let S be a PCS and let $X \subseteq S$. Then S is generated by X, i.e. $S = [X]$ if and only if $[X^{**}]_{Bool} = B(S)$ and $S = [X \cup B(S)]_{sem}$, that means, $B(S)$ is generated by $X^{**} = \{x^{**} : x \in X\}$ as a Boolean algebra and S is generated by $X \cup B(S)$ as a semilattice.*

Let S and T be PCSs. A function $f : S \to T$ is called a *homomorphism* (of PCSs) if $f(x \wedge y) = f(x) \wedge f(y)$, $f(x)^* = f(x^*)$ for $x, y \in S$. We observe that $f(0) = 0$, and $f(1) = 1$.

The definitions of the concepts discussed in this paper may be found in [1] and [3].

3. Extensions

Let S and K be PCSs and let K be a subalgebra of S, that means, S is an *extension* of K. (Notation: $K \leq S$.) In addition, we set $K[X] = [K \cup X]$, whenever $X \subseteq S$. We say that S is a *finite* (*simple*) extension of its subalgebra K, if $S = K[X]$ for some finite (one-element) set $X \subseteq S$.

Proposition 3.1. *Let K and S be PCSs. Then S is a simple extension of K, that means, $S = K[x]$ for some $x \in S$, if and only if*

(i) $B(S) = [B(K) \cup \{x^{**}\}]_{Bool}$,

(ii) $S_1 = [B(S) \cup K]_{sem}$ *is a subalgebra of S and*

(iii) $S = [S_1 \cup \{x\}]_{sem}$.

Proof. Assume first $S = K[x]$. Then (i) is straightforward (see Lemma 2.1). (ii) We have only to show that $u \in S_1$ implies $u^* \in S_1$. But this follows from the fact that $u^* \in B(S) \subseteq S_1$. Thus S_1 is a subalgebra of S. (iii) Set $M = [S_1 \cup \{x\}]_{sem}$. We claim that M is a subalgebra of S. Similarly as above, we have only to show that $u \in M$ implies $u^* \in M$. Since $B(S) \subseteq S_1 \subseteq M$ and $u^* \in B(S)$, we see that $u^* \in M$. Finally, since $K \cup \{x\} \subseteq M$, we obtain $M = S$.

To prove the converse, assume that the conditions (i)-(iii) are satisfied. It is easy to see that $K \leq S$. Therefore, $K[x] = [K \cup \{x\}] \subseteq S$. On the other hand, $B(S) \subseteq K[x]$ by (i). Consequently, $S \subseteq K[x]$ by (ii) and (iii), and the proof is complete. $\square$

Proposition 3.1 generalizes immediately to arbitrary set X (instead of one-element set $\{x\}$).

Theorem 3.2. *Let K and S be PCSs. Then $S = K[X]$ for some $X \subseteq S$ if and only if*

(i) $B(S) = [B(K) \cup X^{**}]_{bool}$,
(ii) $S_1 = [B(S) \cup K]_{sem}$ *is a subalgebra of* S *and*
(iii) $S = [S_1 \cup X]_{sem}$.

Corollary 3.3. *Let $S = K[X]$ and let $u \in S$. Then there exist $s \in K$ and a finite $U \subseteq X$ such that*

$$u = u^{**} \wedge s \wedge \bigwedge(x : x \in U).$$

For our next result we need the following concept:

Definition 3.4. *Let K and S be bounded meet-semilattices* (PCSs) *such that $K \leq S$. Then K is said to be* relatively complete *in S, if for each $b \in S$ there exists a smallest $a \in K$ such that $b \leq a$. In notation:*

$$a = \Pr(b) = \Pr_K^S(b) = \min\{x \in K \mid b \leq x\}.$$

Write $K \leq_{rc} S$ if K is relatively complete in S. See also [6] or [9] for relatively complete lattices or Boolean algebras.

Using the notation from the preceding theorem, we can formulate the following result:

Corollary 3.5. *Let $K \leq S$ for PCSs. Then $K \leq_{rc} S$ if and only if*

$$K \leq_{rc} S_1 \leq_{rc} S,$$

where $S_1 = [B(S) \cup K]_{sem}$.

Proof. Let $K \leq_{rc} S$. (Clearly, $S = K[X]$ for some $X \subseteq S$.) It follows that $B(K) \leq_{rc} B(S)$ and $K \leq_{rc} S_1$. It remains to prove $S_1 \leq_{rc} S$. Let $u \in S$ and $u \leq v$ for some $v \in S_1$. It is easy to see that $v = a \wedge t$ for some $a \in B(S)$ and $t \in K$. Now, $u \leq v$ if and only if $u \leq a$ and $u \leq t$ in S. But $u \leq a$ if and only if $u^{**} \leq a$. The second relation $u \leq t$ is equivalent to $u \leq \Pr_K^S(u) \leq t$. Therefore,

$$u \leq u^{**} \wedge \Pr_K^S(u) \leq a \wedge t = v.$$

Since $u^{**} \wedge \Pr_K^S(u) \in S_1$, we have $S_1 \leq_{rc} S$. The converse implication is straightforward. $\square$

4. Extension to a homomorphism

In this section we shall examine the following situation: Let K, M and $S = K[X]$ be PCSs. Let $f_0 : K \to M$ be a homomorphism and $f : X \to M$ be a mapping. The question concerning f is whether or not there exists a homomorphism $g : S \to M$ such that $g\restriction_{K\cup X} = f_0 \cup f$ (= the restriction of g to $K \cup X$). It is easy to see that g, whenever it exists, is uniquely determined. In this case we say that g is an *extension of* $f_0 \cup f$ *to a homomorphism.*

Notice that a specialization of our question for Boolean algebras has been considered by R. Sikorski. He found a useful characterization of those mappings f, for which there exists an extension to a Boolean homomorphism (see Sikorski's extension criterion in [9]).

The next theorem is concerned with a more general situation and will frequently be useful:

Theorem 4.1. *Let K, M and S be PCSs and let S be an extension of K, that means, $S = K[X]$ for some $X \subseteq S$. Assume that $f_0 : K \to M$ is a homomorphism and let $f : X \to M$ be a mapping. Then there exists a homomorphism $g : S \to M$ extending $f_0 \cup f$ if and only if the following conditions are fulfilled:*

(i) *there is a Boolean homomorphism $h : B(S) \to B(M)$, which is an extension of $(f_0)_B : B(K) \to B(M)$ (i.e. $(f_0)_B$ is a restriction of f_0 to $B(K)$) such that*

$$h(x^{**}) = f(x)^{**}$$

for every $x \in X$;

(ii) *if $S_1 = [B(S)\cup K]_{sem}$, then there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$ such that f_1 is an extension of $f_0 \cup h$;*

(iii) *there exists a meet-semilattice homomorphism $g : S \to M$ which is an extension of $f_1 \cup f$.*

In addition, the homomorphism $g : S \to M$, if it exists, is uniquely determined. If $u \in S$, then

$$g(u) = h(u^{**}) \wedge f_0(s) \wedge \bigwedge(f(x) : x \in U) = f_1(u^{**} \wedge s) \wedge \bigwedge(f(x) : x \in U)$$

for some $s \in K$ and a finite $U \subseteq X$ (see Corollary 3.3).

Proof. The necessity of (i)-(iii) is straightforward (see Lemma 2.1 and Theorem 3.2). Conversely, assume conditions (i) - (iii). First we show that $f_1 : S_1 \to M$ is a PCS-homomorphism. Really, suppose $u \in S_1$. By Theorem 3.2, $u = a \wedge s$ for some $a \in B(S)$ and $s \in K$. Therefore,

$$f_1(u) = f_1(a \wedge s) = h(a) \wedge f_0(s),$$

by (ii). Now,

$$\begin{aligned} f_1(u)^{**} &= (h(a) \wedge f_0(s))^{**} = h(a)^{**} \wedge f_0(s)^{**} = h(a) \wedge f_0(s^{**}) \\ &= h(a) \wedge h(s^{**}) = h(a \wedge s^{**}) = h(u^{**}) = f_1(u^{**}), \end{aligned}$$

by (i) and (ii). Hence,

$$f_1(u)^* = f_1(u)^{***} = h(u^{**})^* = h(u^*) = f_1(u^*),$$

as h is a Boolean homomorphism. Clearly, f_1 is a PCSs homomorphism and an extension of $f_0 \cup h$.

Now, we can show that meet-semilattice homomorphism $g : S \to M$ satisfies $g(u)^* = g(u^*)$ for any $u \in S$ as well. Really, take $u \in S$. By Theorem 3.2, either $u \in S_1$ or $u = s \wedge (\bigwedge X_1)$ for some $s \in S_1$ and a finite non-empty $X_1 \subseteq X$. The first case is straightforward: $g(u) = f_1(u)$. Let us consider the second event. By hypothesis,

$$g(u) = g(s \wedge (\bigwedge X_1)) = g(s) \wedge \bigwedge(g(y) : y \in X_1) = f_1(s) \wedge \bigwedge(g(y) : y \in X_1).$$

Since $g(y)^{**} = f(y)^{**} = h(y^{**})$, for $y \in X_1$, we get

$$g(u)^{**} = f_1(s)^{**} \wedge \bigwedge(g(y)^{**} : y \in X_1) = h(s^{**}) \wedge h(\bigwedge X_1^{**}) = h(u^{**}).$$

It follows that

$$g(u)^* = g(u)^{***} = (g(u)^{**})^* = h(u^{**})^* = h(u^*) = f_1(u^*) = g(u^*),$$

by (i) - (iii). Now, it is easy to see that g is the required homomorphism extending $f_0 \cup f$. The last statement follows from Theorem 3.2 and Corollary 3.3. The proof is complete. □

Corollary 4.2. *Under the assumptions of Theorem 4.1 and the additional hypothesis that $B(K) = B(S)$, the following statements are equivalent:*

(i) *There exists a PCS-homomorphism $g : S \to M$, which is an extension of $f_0 \cup f$.*

(ii) *There exists a meet-semilattice homomorphism $g : S \to M$, which is an extension of $f_0 \cup f$.*

Proof. Clearly, $B(K) = B(S)$ yields that $h \subseteq f_0$. Hence $f_1 = f_0$ and the rest follows from Theorem 4.1. □

Theorem 4.1 shows that an extension of a PCS-homomorphism can be reduced to three special parts: one extension of a Boolean homomorphism and two extensions of bounded meet-semilattice homomorphisms. More precisely, let K, M and S be PCSs and let $S = K[X]$. Assume that there exist a PCS-homomorphism $f_0 : K \to M$ and a mapping $f : X \to M$. Then there exists

(I) a Boolean homomorphism $(f_0)_B : B(K) \to B(M)$, where $(f_0)_B$ is the restriction of f_0 to $B(K)$ (Lemma 2.1). In addition, there is a mapping $f^+ : X^{**} \to B(M)$ defined by the rule

$$f^+(x^{**}) = f(x)^{**}.$$

The first question concerning $(f_0)_B$ is whether or not there is an extension of $(f_0)_B \cup f^+$ to a Boolean homomorphism $h : B(S) \to B(M)$. (Notice that $[B(K) \cup X^{**}]_{Bool} = B(S)$ by Lemma 2.1.) The answer to this question comes from the following lemma, due to R. Sikorski (see [9], Theorem 5.5). First we need a new notation: Let B be a Boolean algebra. For $x \in B$ and $\varepsilon \in \{+1, -1\}$, define the element x^ε of B by

$$x^{+1} = x, \quad x^{-1} = x^*.$$

Lemma 4.3. *A Boolean homomorphism $h : B(S) \to B(M)$ is an extension of $(f_0)_B \cup f^+$ if and only if*

$$a^{\varepsilon_0} \wedge (x_1^{**})^{\varepsilon_1} \wedge \cdots \wedge (x_k^{**})^{\varepsilon_k} = 0$$

in $B(S)$ for $a \in B(K)$, $x_1^{**}, \cdots, x_k^{**} \in X^{**}$ and $\varepsilon_i \in \{+1, -1\}$ implies

$$f_0(a)^{\varepsilon_0} \wedge f(x_1^{**})^{\varepsilon_1} \wedge \cdots \wedge f(x_k^{**})^{\varepsilon_k} = 0$$

in $B(M)$.

(II) Suppose now that a Boolean homomorphism $h : B(S) \to B(M)$ exists and h is an extension of $(f_0)_B \cup f^+$. In addition, there exists $S_1 \leq S$ and we can ask again whether or not there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$. The answer can be formulated as follows:

Lemma 4.4. *Let $h : B(S) \to B(M)$ be a Boolean homomorphism and an extension of $(f_0)_B \cup f^+$. Then there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$ if and only if*

$$a \wedge s = b \wedge t$$

implies

$$h(a) \wedge f_0(s) = h(b) \wedge f_0(t)$$

for any $a, b \in B(S)$ and $s, t \in K$.

The result requires only routine verification, and the proof can be omitted.

(III) It remains to establish the third part. We thus have a semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$. Since $S = [S_1 \cup X]_{sem}$ (Theorem 3.2), it is reasonable to ask again whether or not there exists a meet-semilattice homomorphism $g : S \to M$, which is an extension of $f_1 \cup f$. The following lemma yields a solution:

Lemma 4.5. *Let $f_1 : S_1 \to M$ be a semilattice homomorphism extending $f_0 \cup h$. Then there exists a semilattice homomorphism $g : S \to M$, which is an extension of $f_1 \cup f$ if and only if*

$$s \wedge \bigwedge(y : y \in Y) = t \wedge \bigwedge(z : z \in Z)$$

implies

$$f_1(s) \wedge \bigwedge(f(y) : y \in Y) = f_1(t) \wedge \bigwedge(f(z) : z \in Z)$$

for any $s, t \in S_1$ and arbitrary finite $Y, Z \subseteq X$.

The proof is routine.

We conclude this section by observing that Lemmas 4.3-4.5 complete Theorem 4.1. The interested reader should have no serious difficulty in reconstructing the corresponding theorem.

5. Simple extensions

In the last section (Theorem 4.1) we saw how a PCS-homomorphism $f_0 : K \to M$ can be extended to a PCS-homomorphism $g : S \to M$, where $K \leq S$. Unfortunately, our characterization is of general nature, that means, the result is not useful enough. The purpose of this section is to find sufficient conditions under which we can easily say that an extension exists or not. For this reason we perform some specializations (simple extensions, retractions) and a generalization (meet-semilattices). (See the discussion in the preceding section.)

Proposition 5.1. *Let $f : T \to M$ be a homomorphism of non-trivial bounded meet-semilattices. Assume that the bounded meet-semilattice $S = T[x]$ is a simple extension of T and u is an element of M. Moreover, assume that the element $\mathrm{Pr}_T^S(x)$ exists and, that we have a retraction $\alpha : T[x] \to T$, that means, $\alpha(a) = a$ for any $a \in T$, such that $\alpha(x) = \mathrm{Pr}_T^S(x)$. Then there exists a meet-semilattice homomorphism*

$$g : S = T[x] \to M$$

extending f and mapping x to $u \in M$ if and only if

$$a \leq x \text{ in } S \text{ and } a \in T \text{ imply } f(a) \leq u \leq f(\mathrm{Pr}_T^S(x)) \text{ in } M.$$

Proof. Necessity of the condition is obvious. As to sufficiency, it is known that an arbitrary element $v \in S$ can be written in the form

$$v = a \wedge x^r,$$

where $r \in \{0, 1\}$ and $a \in T$. (Note that $x^0 = 1$ and $x^1 = x$.) Now, we can define

$$g : S \to M$$

by

$$g(v) = f(a) \wedge u^r.$$

First we have to show that g is well-defined, that is,

$$c \wedge x^r = d \wedge x^s \quad \text{implies} \quad f(c) \wedge u^r = f(d) \wedge u^s,$$

for $c, d \in T$. We have to verify two cases only:

$$c \wedge x = d \wedge x \quad \text{and} \quad c = d \wedge x.$$

Writing $\mathrm{Pr}(x)$ for $\mathrm{Pr}_T^S(x)$ we get in the first event

$$\alpha(c \wedge x) = c \wedge \mathrm{Pr}(x) = d \wedge \mathrm{Pr}(x) = \alpha(d \wedge x),$$

by the hypothesis on α. Therefore,

$$f(c) \wedge f(\mathrm{Pr}(x)) = f(c \wedge \mathrm{Pr}(x)) = f(d \wedge \mathrm{Pr}(x)) = f(d) \wedge f(\mathrm{Pr}(x)),$$

as f is a homomorphism. Since $u \leq f(\mathrm{Pr}(x))$, we obtain

$$f(c) \wedge u = f(d) \wedge u.$$

Considering the second case $c = d \wedge x$, we see that $c \leq x$. Hence $f(c) \leq u$, by the hypothesis on f. Using the same reasoning as above, we obtain

$$f(c) = f(d) \wedge u,$$

and g is well-defined. The element 0 in S can be expressed as $0 = 0 \wedge x$. Therefore,

$$g(0) = f(0) \wedge u = 0$$

in M. Similarly, $g(1) = 1$. Now, it can be readily shown that g is a meet-semilattice homomorphism extending f with the required properties. □

Lemma 5.2. *Let $S = K[x]$ be a simple extension of PCSs. Assume that there exists $\mathrm{Pr}_K^S(x)$. Then there exists $\mathrm{Pr}_{S_1}^S(x)$ (for S_1 see Section 3) and*

$$\mathrm{Pr}_{S_1}^S(x) = x^{**} \wedge \mathrm{Pr}_K^S(x).$$

Proof. Clearly, $x \leq x^{**} \wedge \mathrm{Pr}_K^S(x) \in S_1$. On the other hand, let $x \leq v$ for some $v \in S_1$. By Theorem 3.2, $v = a \wedge t$ for some $a \in B(S)$ and $t \in K$. Now, $x \leq a \wedge t$ implies $x^{**} \leq a$ in $B(S)$ and $x \leq t$ in K. Hence

$$x^{**} \wedge \mathrm{Pr}_K^S(x) \leq a \wedge t = v.$$

□

As a consequence of these results we have

Theorem 5.3. *Let K, M and S be PCSs, let $S = K[x]$ be a simple extension of K for some $x \in S$ and let $u \in M$. Let $f_0 : K \to M$ be a PCS-homomorphism. Assume that the element $\mathrm{Pr}_K^S(x)$ exists and that we have (in the notation of Section 3) a retraction $\alpha : S_1[x] \to S_1$ such that $\alpha(x) = x^{**} \wedge \mathrm{Pr}_K^S(x)$. Then there exists a PCS-homomorphism*

$$g : S \to M$$

extending f_0 and mapping x to $u \in M$ if and only if

(i) *there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$ which is an extension of $f_0 \cup h$ (see Theorem 4.1) and, we have*

$$f_1(x^{**}) = h(x^{**}) = u^{**},$$

(ii) *$t \le x$ in S and $t \in S_1$ imply $f_1(t) \le u \le f_1(\mathrm{Pr}^S_{S_1}(x))$ in M.*

Proof. Suppose that $g : S \to M$ is an extension of f_0 such that $g(x) = u$. Since g is a PCS-homomorphism, condition (ii) follows easily. Condition (i) is a consequence of Theorem 4.1.

To prove the remaining half, let us suppose (i) and (ii). We shall proceed by Theorem 4.1. We start by establishing a Boolean homomorphism $h : B(S) \to B(M)$ which is an extension of $(f_0)_B$ (see Theorem 4.1) such that $h(x^{**}) = u^{**}$. It is easy to check that $[B(K) \cup \{x^{**}\}]_{Bool} = B(S)$. Moreover, from (ii) and the hypothesis that $\mathrm{Pr}^S_K(x)$ exists, it follows that

$$a^{**} \le x^{**} \le b^{**} \text{ in } S \text{ implies } f_0(a^{**}) \le u^{**} \le f_0(\mathrm{Pr}^S_K(x)^{**}) \le f_0(b^{**}) \text{ in } M$$

for any $a, b \in K$. Now we can apply ([9], Corollary 5.8) of Sikorski's extension criterion for Boolean algebras. It does ensure that there is a Boolean homomorphism $h : B(S) \to B(M)$ extending $(f_0)_B : B(K) \to B(M)$ such that $h(x^{**}) = u^{**}$.

By (i) we see that $f_1 : S_1 \to M$ is a meet-semilattice homomorphism extending $f_0 \cup h$. It remains to show that there exists a meet-semilattice homomorphism $g : S \to M$ extending $f_1 \cup \{(x, u)\}$. Evidently, $S = S_1[x]$ is a simple meet-semilattice extension. Now, we can apply Proposition 5.1. By Lemma 5.2 and the hypothesis that f_1 ia a meet-semilattice homomorphism, we get

$$u^{**} \wedge f_0(\mathrm{Pr}^S_K(x)) = h(x^{**}) \wedge f_1(\mathrm{Pr}^S_K(x)) = f_1(\mathrm{Pr}^S_{S_1}(x)).$$

Now, setting T for S_1 in (ii), we get the main condition of Proposition 5.1. It follows that there exists a meet-semilattice homomorphism $g : S \to M$ extending $f_1 \cup \{(x, u)\}$. Ultimately Theorem 4.1 implies that g is a PCS-homomorphism, and the proof of the theorem is complete. □

References

1. Balbes, R. and Dwinger, Ph., *Distributive lattices*, Univ. Missouri Press, Missouri, 1974.
2. Frink, O., *Pseudo-complements in semi-lattices*, Duke Math. J. *29*(1962), 505-514.
3. Grätzer, G., *General lattice theory*, Birkhäuser Verlag, Basel, Boston, Berlin, 1998 (sec. ed.).
4. Howie, J. M., *An introduction to semigroup theory*, Acad. Press, London, N.Y., San Francisco, 1976.
5. Jones, G. T., *Projective pseudocomplemented semilattices*, Pacific J. Math. *32* (1974), 443-456.
6. Jónsson, B., *Relatively free products of lattices*, Algebra Univ. *1* (1972), 362-371.
7. Katriňák, T., *Free p-algebras*, Algebra Univ. *15* (1982), 176-182.

8. Katriňák, T. and Guričan, J., *Projective extensions of bounded semilattices*, Algebra Univ. *59* (2008), 97-110.
9. Koppelberg, S., *Hanbook of Boolean algebras*, vol. *1* of: Monk, J. D. with R. Bonnet (eds.) North-Holland, Amsterdam etc. 1989.

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ON FORMATIONS OF LATTICES

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Dedicated to the 70th birthday of Alfonz Haviar

Abstract. A class of lattices is said to be a formation if it is closed under homomorphic images and finite subdirect products. Let us denote by $\mathbb{F}$ the collection of all formations of lattices. Then $\mathbb{F}$ can be partially ordered by the class–theoretical inclusion. We study the properties of this partially ordered class; e.g., there are described all atoms of $\mathbb{F}$.

1. Introduction

A class of algebras is said to be a formation if it is closed under homomorphic images and finite subdirect products. This concept appeared first in the 1970's in the connection with finite groups. Formations of groups were studied by several authors. Let us mention at least the monograph [3] of Shemetkov, which deals with formations of finite groups. Nevertheless, Chapter I of [3] contains a detailed presentation of basic notions of the theory without assuming the finiteness of the groups under consideration. In fact, the above definition can be used for any class of similar algebras. Formations of lattice ordered groups and GMV–algebras were investigated by Jakubík [2].

Let $\mathbb{F}$ be the collection of all formations of lattices. For $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}$ we write $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_1$ is a subclass of $\mathcal{F}_2$. The collection $\mathbb{F}$ is large; namely, there exists an injective mapping of the class of all infinite cardinals into the collection $\mathbb{F}$. Nevertheless, with respect to the relation $\leq$ in $\mathbb{F}$, we will use the usual notions and the notation of the theory of partially ordered sets. We will prove that for any indexed system of elements of $\mathbb{F}$, both supremum and infimum exist.

For any class $\mathcal{K}$ of lattices, we will describe the least formation $\mathsf{form}(\mathcal{K})$ containing $\mathcal{K}$. Each formation of lattices, except the least one, contains subdirectly irreducible lattices. But, in contrast with varieties of lattices, different formations of lattices can have the same subclass of subdirectly irreducible lattices.

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In Section 5, we will describe all atoms in $\mathbb{F}$. They form a proper class, just like antiatoms of $\mathbb{F}$.

Finally, we will show that the class of formations of distributive lattices contains both large chains and large antichains.

2. Preliminaries

We will use the terminology and the notation as in Grätzer [1].

The direct product of an indexed system $(L_i)_{i\in I}$ of lattices is defined in the usual way; we apply the notation $\prod_{i\in I} L_i$ or $L_1 \times L_2 \times \ldots \times L_n$ if $I = \{1, 2, \ldots, n\}$. For $x = (x_i)_{i\in I}$ in $\prod_{i\in I} L_i$, x_i is the component of x in L_i; we write also $x_i = x(L_i)$. Let $K \subseteq \prod_{i\in I} L_i$ and $i_0 \in I$; we put $K(L_{i_0}) = \{x(L_{i_0}) : x \in K\}$. If K is a sublattice of $\prod_{i\in I} L_i$ and $K(L_i) = L_i$ for each $i \in I$, then K is said to be a subdirect product of the system $(L_i)_{i\in I}$. In such a case we write $K \leq \prod_{i\in I} L_i$. If the index set I is finite, K will be referred to as a finite subdirect product.

If L is a lattice, θ a congruence relation on L and $a \in L$, the symbol $[a]\theta$ will be used for the congruence class containing the element a.

3. The class $\mathsf{form}(\mathcal{K})$

Let $\mathcal{L}$ be the class of all lattices. For any class $\mathcal{K}$ of lattices we denote by

$\mathbf{H}(\mathcal{K})$– the class of all homomorphic images of elements of $\mathcal{K}$;

$\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$– the class of all finite subdirect products of elements of $\mathcal{K}$.

A class $\mathcal{F}$ of lattices is said to be a *formation* if is closed with respect to the operators $\mathbf{H}$ and $\mathbf{P}_{\mathrm{FS}}$.

It is easy to see that each variety of lattices is also a formation. The converse does not hold in general; e.g., the class of all finite lattices is a formation which fails to be a variety.

Let $\mathcal{K}$ be any class of lattices. We will describe the least formation containing $\mathcal{K}$. If $\mathcal{K} = \emptyset$, then it is evidently the class of all one–element lattices. Suppose that $\mathcal{K} \neq \emptyset$. It is $\mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K}) \subseteq \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$; this can be shown in the same way as the well–known inclusion $\mathbf{P}_{\mathrm{S}}\,\mathbf{H}(\mathcal{K}) \subseteq \mathbf{H}\,\mathbf{P}_{\mathrm{S}}(\mathcal{K})$, where $\mathbf{P}_{\mathrm{S}}$ stands for the operator of forming subdirect products. Using also the idempotency of the operators $\mathbf{H}$ and $\mathbf{P}_{\mathrm{FS}}$, we obtain:

Theorem 3.1. *Let $\mathcal{K}$ be any nonempty class of lattices. Then $\mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$ is a formation of lattices. Moreover, it is the least one containing $\mathcal{K}$.*

For any $\mathcal{K} \subseteq \mathcal{L}$, the least formation containing $\mathcal{K}$ will be denoted by $\mathsf{form}(\mathcal{K})$. So, if $\mathcal{K} \neq \emptyset$, then $\mathsf{form}(\mathcal{K}) = \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$. Let us remark that $\mathbf{H}\,\mathbf{P}_{\mathrm{S}}(\mathcal{K})$ is the least variety of lattices containing $\mathcal{K}$ (cf. [1],Corollary 5.1.5).

We will show that if $\mathcal{K}$ contains only distributive lattices, then $\mathsf{form}\,\mathcal{K} = \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K})$, i.e., the operators $\mathbf{H}$, $\mathbf{P}_{\mathrm{FS}}$ can be applied in arbitrary order. We will use the following assertions.

Proposition 3.2 ([1], Theorem 2.3.6)**.** *Let K be a sublattice of a distributive lattice L. Any congruence relation θ of K can be extended to L; that is, there exists a congruence relation ϕ on L such that $x\ \phi\ y$ iff $x\ \theta\ y$ for $x, y \in K$.*

Proposition 3.3 ([1], Theorem 1.3.13)**.** *Let L and K be lattices, let θ be a congruence relation of L, and let ϕ be a congruence relation of K. Define the relation $\theta \times \phi$ on $L \times K$ by*

$$(a, b)\ \theta \times \phi\ (c, d) \quad \text{iff} \quad a\ \theta\ c \quad \text{and} \quad b\ \phi\ d.$$

Then $\theta \times \phi$ is a congruence relation on $L \times K$. Conversely, every congruence relation of $L \times K$ is of this form.

Theorem 3.4. *Let $\mathcal{K}$ be a class containing only distributive lattices. Then* $\mathsf{form}(\mathcal{K}) = \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K})$*.*

Proof. In view of Theorem 3.1, it suffices to prove the inclusion $\mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K}) \subseteq \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K})$. Let $L \in \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\mathcal{K})$. Then there exist lattices $A_1, \ldots, A_n \in \mathcal{K}$, $B \leq A_1 \times \ldots \times A_n$ and a homomorphism φ of B onto L. Let $\theta = \mathrm{Ker}\ \varphi$, ϕ an extension of θ to a congruence relation of $A_1 \times \ldots \times A_n$. Further, let $\phi = \phi_1 \times \ldots \times \phi_n$ with ϕ_i being a congruence relation of A_i for $i = 1, \ldots, n$.

We are going to show that L is isomorphic to a subdirect product of $(A_i/\phi_i)_{i\in\{1,\ldots,n\}}$. Let us define $\psi : L \to A_1/\phi_1 \times \ldots \times A_n/\phi_n$ by

$$\psi(a) = ([b_1]\phi_1, \ldots, [b_n]\phi_n),$$

where $b = (b_1, \ldots, b_n)$ is any element of B with $\varphi(b) = a$.

It is easy to see that the definition of ψ is correct and that ψ is a one–to–one homomorphism. Moreover, if $a_i \in A_i$ and c is any element of B with $c(A_i) = a_i$, we have

$$(\psi\,(\varphi(c)))\,(A_i/\phi_i) = [a_i]\phi_i.$$

Since $A_i/\phi_i \in \mathbf{H}(\mathcal{K})$ for all $i \in \{1, \ldots, n\}$, we have proved $L \in \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\mathcal{K})$. $\square$

Let L be a nontrivial lattice, ω the least congruence relation of L. If ω is a completely meet–irreducible (a meet irreducible) element in the complete lattice *Con* L of all congruence relations on L, then L is said to be a subdirectly irreducible (a finitely subdirectly irreducible) lattice.

The following theorem is a slight modification of the well–known Jónsson's lemma ([1], Theorem 5.1.9).

Theorem 3.5. *Let $\mathcal{K}$ be any class of lattices. If A is a finitely subdirectly irreducible lattice, $A \in$* $\mathsf{form}(\mathcal{K})$*, then $A \in \mathbf{H}(\mathcal{K})$.*

Proof. Let A be a finitely subdirectly irreducible lattice, $A \in \mathsf{form}(\mathcal{K})$. By Theorem 3.1, there exist lattices $A_1, \ldots, A_n \in \mathcal{K}$, $B \leq A_1 \times \ldots \times A_n$, $\theta \in Con\ B$ such that $A \cong B/\theta$.

For $i \in I = \{1, \dots, n\}$, let π_i be the projection of B onto A_i. We are going to show that there exists $i_0 \in I$ such that Ker $\pi_{i_0} \subseteq \theta$. Evidently $\bigcap_{i \in I} \text{Ker } \pi_i = \omega \subseteq \theta$, so that

$$\theta = \theta \vee (\bigcap_{i \in I} \text{Ker } \pi_i) = \bigcap_{i \in I} (\theta \vee \text{Ker } \pi_i).$$

As the lattice B/θ is finitely subdirectly irreducible, we have $\theta = \theta \vee \text{Ker } \pi_{i_0} \supseteq \text{Ker } \pi_{i_0}$ for some $i_0 \in I$.

Now using the second isomorphism theorem we obtain that B/θ is a homomorphic image of $B/\text{Ker } \pi_{i_0}$. But $B/\text{Ker } \pi_{i_0}$ is isomorphic to A_{i_0} and thus $B/\theta \in \mathbf{H}(\{A_{i_0}\}) \subseteq \mathbf{H}(\mathcal{K})$. Consequently $A \in \mathbf{H}(\mathcal{K})$. □

Since evidently each subdirectly irreducible lattice is also finitely subdirectly irreducible, in the way described in the preceding theorem, all subdirectly irreducible lattices of $\mathsf{form}(\mathcal{K})$ are discovered. Each formation, except the least one, contains subdirectly irreducible lattices. Namely, if L is any lattice, $|L| > 1$, then L is a subdirect product of subdirectly irreducible lattices, $L \leq \prod_{i \in I} L_i$, in any variety containing L, where I need not be finite. Nevertheless, each L_i, as a homomorphic image of L, belongs to each formation containing L.

Let $\mathsf{Si}(\mathcal{F})$ denote the class of all subdirectly irreducible lattices belonging to the formation $\mathcal{F}$. Let us note that $\mathcal{F}$ is not uniquely determined by $\mathsf{Si}(\mathcal{F})$. For example, each formation of distributive lattices contains the only subdirectly irreducible lattice, the two–element chain.

4. The class of formations

Let $\mathbb{F}$ be the collection of all formations of lattices. For $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}$ we write $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_1$ is a subclass of $\mathcal{F}_2$. The collection $\mathbb{F}$ is large; it is easy to see that for any infinite cardinal $\varkappa$, the class of all lattices of cardinality not exceeding $\varkappa$, is a formation. Nevertheless, with respect to the relation $\leq$ in $\mathbb{F}$, we can apply for $\mathbb{F}$ the usual notions and notation of the theory of partially ordered sets. Thus, for $\{\mathcal{F}_i : i \in I\} \subseteq \mathbb{F}$, the symbols $\sup\{\mathcal{F}_i : i \in I\}$ or $\bigvee_{i \in I} \mathcal{F}_i$ denote the least upper bound of $\{\mathcal{F}_i : i \in I\}$ in $\mathbb{F}$; the symbols $\inf\{\mathcal{F}_i : i \in I\}$, $\bigwedge_{i \in I} \mathcal{F}_i$ have a dual meaning.

It is easy to see that the intersection of any non–empty collection of formations is a formation. Moreover, $\mathbb{F}$ contains a least element, the class of all one–element lattices and the greatest element, the class $\mathcal{L}$ of all lattices. So we have:

Theorem 4.1. *The collection $\mathbb{F}$ of all formations of lattices is a complete lattice in the sense, that $\bigwedge_{i \in I} \mathcal{F}_i$ and $\bigvee_{i \in I} \mathcal{F}_i$ exist for any nonempty collection of formations $\{\mathcal{F}_i : i \in I\}$. Moreover,*

$$\bigwedge_{i \in I} \mathcal{F}_i = \bigcap_{i \in I} \mathcal{F}_i \ , \quad \bigvee_{i \in I} \mathcal{F}_i = \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\bigcup_{i \in I} \mathcal{F}_i).$$

Theorem 4.2. *A formation $\mathcal{F}$ of lattices is a compact element in $\mathbb{F}$ if and only if it is generated by a single lattice.*

Proof. Let $\mathcal{F} = \mathsf{form}(\{L\})$, $L \in \mathcal{L}$. Assume that $\mathcal{F} \leq \bigvee_{i\in I} \mathcal{F}_i$, where $\{\mathcal{F}_i : i \in I\} \subseteq \mathbb{F}$. Then $L \in \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} \mathcal{F}_i)$, hence there exist lattices $L_1, \ldots, L_n \in \bigcup_{i\in I} \mathcal{F}_i$, $B \leq L_1 \times \ldots \times L_n$ and a homomorphism of B onto L. If $L_1 \in \mathcal{F}_{i_1}, \ldots, L_n \in \mathcal{F}_{i_n}$, we have $L \in \mathbf{H}\,\mathbf{P}_{\mathrm{FS}}(\bigcup_{j=0}^{n} \mathcal{F}_{i_j}) = \mathcal{F}_{i_1} \vee \ldots \vee \mathcal{F}_{i_n}$, which implies $\mathcal{F} \leq \mathcal{F}_{i_1} \vee \ldots \vee \mathcal{F}_{i_n}$.

Conversely, suppose that $\mathcal{F} \in \mathbb{F}$ is compact. Let $\mathcal{F} = \{L_i : i \in I\}$. As evidently $\mathcal{F} \leq \bigvee_{i\in I} \mathsf{form}(\{L_i\})$, we have $\mathcal{F} \leq \mathsf{form}(\{L_1\}) \vee \ldots \vee \mathsf{form}(\{L_n\})$ for some $L_1, \ldots, L_n \in \mathcal{F}$. But then $\mathcal{F} = \mathsf{form}(\{L_1\}) \vee \ldots \vee \mathsf{form}(\{L_n\}) = \mathsf{form}(\{L_1 \times \ldots \times L_n\})$. $\square$

Using the trivial fact that any formation $\mathcal{F}$ of lattices can be expressed as $\sup\{\mathsf{form}(\{L\}) : L \in \mathcal{F}\}$, we obtain:

Corollary 4.3. *The collection $\mathbb{F}$ of all formations of lattices is an algebraic lattice.*

Let $\mathbb{F}_d$ denote the collection of all formations of distributive lattices.

Theorem 4.4. *The collection $\mathbb{F}_d$ is a complete sublattice of $\mathbb{F}$; moreover, the relation*

$$\mathcal{F} \wedge \bigvee_{i\in I} \mathcal{F}_i = \bigvee_{i\in I} (\mathcal{F} \wedge \mathcal{F}_i)$$

is valid for any $\mathcal{F}, \mathcal{F}_i \in \mathbb{F}_d$.

Proof. It suffices to verify the relation

$$\mathcal{F} \wedge (\bigvee_{i\in I} \mathcal{F}_i) \subseteq \bigvee_{i\in I} (\mathcal{F} \wedge \mathcal{F}_i).$$

Using Theorem 3.4 and the fact that each $\mathcal{F}_i$ is closed under homomorphic images, we obtain $\mathcal{F} \wedge (\bigvee_{i\in I} \mathcal{F}_i) = \mathcal{F} \cap \mathsf{form}(\bigcup_{i\in I} \mathcal{F}_i) = \mathcal{F} \cap \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\bigcup_{i\in I} \mathcal{F}_i) = \mathcal{F} \cap \mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} \mathbf{H}(\mathcal{F}_i)) = \mathcal{F} \cap \mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} \mathcal{F}_i)$.

Now if $L \in \mathcal{F} \cap \mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} \mathcal{F}_i)$, then $L \in \mathcal{F}$ and $L \leq L_1 \times \ldots \times L_k$ for some $L_1, \ldots, L_k \in \bigcup_{i\in I} \mathcal{F}_i$. Each L_j, as a homomorphic image of L, belongs to $\mathcal{F}$, so each L_j belongs to $\mathcal{F} \cap (\bigcup_{i\in I} \mathcal{F}_i) = \bigcup_{i\in I} (\mathcal{F} \cap \mathcal{F}_i)$. Thus $L \in \mathbf{P}_{\mathrm{FS}}(\bigcup_{i\in I} (\mathcal{F} \cap \mathcal{F}_i)) \subseteq \mathsf{form}(\bigcup_{i\in I} (\mathcal{F} \cap \mathcal{F}_i)) = \bigvee_{i\in I} (\mathcal{F} \wedge \mathcal{F}_i)$. $\square$

The question, if this infinite distributive law or at least finite distributive law is valid in $\mathbb{F}$, is open.

Consider the following condition concerning a subclass $\mathcal{M}$ of $\mathcal{L}$:

$(*)$ $\qquad L \in \mathbf{H}(\mathcal{M})$, L is subdirectly irreducible $\Rightarrow$ $L \in \mathcal{M}$.

The following assertion is obvious.

Lemma 4.5. *Let $\mathcal{F}$ be any formation of lattices. Then* $\mathsf{Si}(\mathcal{F})$ *fulfils the condition* $(*)$.

To show that the condition $(*)$ is also sufficient for a class $\mathcal{M}$ of subdirectly irreducible lattices to be $\mathsf{Si}(\mathcal{F})$ for a formation $\mathcal{F}$, let us notice that the following holds:

Lemma 4.6. *Let $\{\mathcal{F}_i : i \in I\}$ be a nonempty class of formations of lattices. Then*

$$\mathsf{Si}(\bigwedge_{i\in I} \mathcal{F}_i) = \bigcap_{i\in I} \mathsf{Si}(\mathcal{F}_i) \ , \quad \mathsf{Si}(\bigvee_{i\in I} \mathcal{F}_i) = \bigcup_{i\in I} \mathsf{Si}(\mathcal{F}_i).$$

Proof. The first equality is evident, just like the inclusion $\bigcup_{i\in I} \mathsf{Si}(\mathcal{F}_i) \subseteq \mathsf{Si}(\bigvee_{i\in I} \mathcal{F}_i)$. Now let $L \in \mathsf{Si}(\bigvee_{i\in I} \mathcal{F}_i)$. Then the lattice L is subdirectly irreducible and $L \in \mathsf{form}(\bigcup_{i\in I} \mathcal{F}_i)$. By Theorem 3.5, $L \in \mathbf{H}(\bigcup_{i\in I} \mathcal{F}_i) = \bigcup_{i\in I}(\mathbf{H}(\mathcal{F}_i)) = \bigcup_{i\in I} \mathcal{F}_i$, so that $L \in \bigcup_{i\in I} \mathsf{Si}(\mathcal{F}_i)$. □

Lemma 4.7. *Let $\mathcal{M}$ be any class of subdirectly irreducible lattices satisfying the condition $(*)$. Then formations $\mathcal{F}$ with $\mathsf{Si}(\mathcal{F}) = \mathcal{M}$ form an interval in $\mathbb{F}$. The least element of this interval is* $\mathsf{form}(\mathcal{M})$.

Proof. First of all, let us notice that $\mathsf{Si}(\mathsf{form}(\mathcal{M})) = \mathcal{M}$. The implication $\mathsf{Si}(\mathsf{form}(\mathcal{M})) \subseteq \mathcal{M}$ follows from Theorem 3.5 and $(*)$, while the converse one is obvious. So $\mathcal{F}_0 = \mathsf{form}(\mathcal{M})$ is the least one of all formations $\mathcal{F}$ satisfying $\mathsf{Si}(\mathcal{F}) = \mathcal{M}$.

Further, let $\mathcal{F}_1$ be the least upper bound of the collection of all formations $\mathcal{F}$ with $\mathsf{Si}(\mathcal{F}) = \mathcal{M}$. By 4.6, $\mathsf{Si}(\mathcal{F}_1) = \mathcal{M}$. If $\mathcal{F}_0 \subseteq \mathcal{F} \subseteq \mathcal{F}_1$, then also $\mathsf{Si}(\mathcal{F}) = \mathcal{M}$. We have proved that $\{\mathcal{F} \in \mathbb{F} : \mathsf{Si}(\mathcal{F}) = \mathcal{M}\}$ is the interval $[\mathcal{F}_0, \mathcal{F}_1]$. □

Let C_2 be a two–element chain. Then $\mathcal{M} = \{C_2\}$ evidently satisfies $(*)$. It is easy to see that $\mathcal{F} \in \mathbb{F}$ with $\mathsf{Si}(\mathcal{F}) = \{C_2\}$ are just formations belonging to the interval $[\mathcal{F}_0, \mathcal{F}_1]$, where $\mathcal{F}_0$ is the formation containing all finite distributive lattices and $\mathcal{F}_1$ that of all distributive lattices.

Let $\mathbb{M}$ be the collection of all classes $\mathcal{M}$ of subdirectly irreducible lattices satisfying the condition $(*)$. It is easy to see that $\mathbb{M}$ is closed under arbitrary (not only finite) intersections and unions so that $(\mathbb{M}, \subseteq)$ can be considered as a complete lattice.

The following assertion is evident.

Theorem 4.8. *Let $\equiv$ be a binary relation on $\mathbb{F}$ defined by*

$$\mathcal{F} \equiv \mathcal{F}' \quad \Leftrightarrow \quad \mathsf{Si}(\mathcal{F}) = \mathsf{Si}(\mathcal{F}').$$

Then $\equiv$ is a congruence relation and the mapping $f : \mathbb{F}/\equiv\, \to \mathbb{M}$ defined by $f([\mathcal{F}]\equiv) = \mathsf{Si}(\mathcal{F})$ is an isomorphism.

5. Atoms and antiatoms

Let L be a lattice with a least element 0. An element $a \in L$ is said to be an atom of L if a covers 0. If $b \in L \setminus \{0\}$ and there is no atom a with $a \leq b$, then b is referred to as an antiatom. We are able to describe all atoms of $\mathbb{F}$.

Consider the following condition concerning a lattice L:

$$(**) \qquad L' \in \mathbf{H}(\{L\}),\ L' \text{ is subdirectly irreducible} \quad \Rightarrow \quad L \in \mathbf{H}(\{L'\}).$$

Theorem 5.1. *A formation $\mathcal{F}$ of lattices is an atom of $\mathbb{F}$ if and only if $\mathcal{F} = \mathsf{form}(\{L\})$ for a subdirectly irreducible lattice L satisfying the condition $(**)$.*

Proof. Let $\mathcal{F}$ be an atom. As we have remarked, $\mathcal{F}$ contains a subdirectly irreducible lattice L. Then $\mathsf{form}(\{L\}) \leq \mathcal{F}$, so that $\mathcal{F} = \mathsf{form}(\{L\})$, as $\mathcal{F}$ is an atom. We are going to show that L satisfies $(**)$. Let L' be a subdirectly irreducible lattice with $L' \in \mathbf{H}(\{L\})$. Thus it is also $\mathcal{F} = \mathsf{form}(\{L'\})$. But then $L \in \mathsf{form}(\{L'\})$ implies $L \in \mathbf{H}(\{L'\})$ by Theorem 3.5.

Conversely, let $\mathcal{F} = \mathsf{form}(\{L\})$, where L is a subdirectly irreducible lattice fulfilling $(**)$. Let $\mathcal{F}'$ be a formation of lattices different from the least one satisfying $\mathcal{F}' \leq \mathcal{F}$. Take any subdirectly irreducible lattice $L' \in \mathcal{F}'$. Then $L' \in \mathcal{F} = \mathsf{form}(\{L\})$, so that $L' \in \mathbf{H}(\{L\})$ by Theorem 3.5. Using $(**)$ we obtain $L \in \mathbf{H}(\{L'\})$, which implies $\mathcal{F} = \mathsf{form}(\{L\}) \leq \mathsf{form}(\{L'\}) \leq \mathcal{F}'$. Thus $\mathcal{F}' = \mathcal{F}$ and $\mathcal{F}$ is an atom. $\square$

Evidently each simple lattice is a subdirectly irreducible lattice satisfying $(**)$. So each simple lattice generates an atom in $\mathbb{F}$, non–isomorphic lattices generate different atoms. Let κ be any cardinal, $\kappa \geq 3$, I any set of cardinality κ. Set $M_\kappa = \{0, 1\} \cup \{a_i : i \in I\}$ and define $\leq$ on M_κ by $0 < a_i < 1$ for all $i \in I$, a_i mutually non–comparable.

Evidently M_κ are simple lattices, mutually non-isomorphic. So we obtain:

Corollary 5.2. *Atoms of $\mathbb{F}$ form a proper class.*

If a formation $\mathcal{F}$ contains finite lattices with more than one element, then $\mathcal{F}$ contains also simple finite lattices, so that there exist atoms which lie in $\mathbb{F}$ under $\mathcal{F}$. Thus in the case that we are interested in antiatoms, we must concentrate upon formations containing, besides the one–element lattices, only infinite ones. The aim is to prove that antiatoms form a proper class, too.

Let us have an infinite ascending chain of cardinals $(\kappa_i)_{i\in\mathbb{N}}$, $\kappa_1 < \kappa_2 < \dots$, $\kappa_1 \geq 3$. Let M_{κ_i} be as above, with κ_i instead of κ. Define lattices L_i for $i \in \mathbb{N}$ by induction as follows:

$$L_1 = M_{\kappa_1}$$
$$L_{n+1} = (L_n \to M_{\kappa_{n+1}}) \quad \text{for } n \in \mathbb{N},$$

where $(L_n \to M_{\kappa_{n+1}})$ means a lattice obtained from $M_{\kappa_{n+1}}$ by interchanging one of its "middle" elements by L_n.

If we take, e.g., the sequence $3 < 4 < 5 < \dots$, we obtain a sequence of lattices, whose first three members are depicted in Fig. 1.

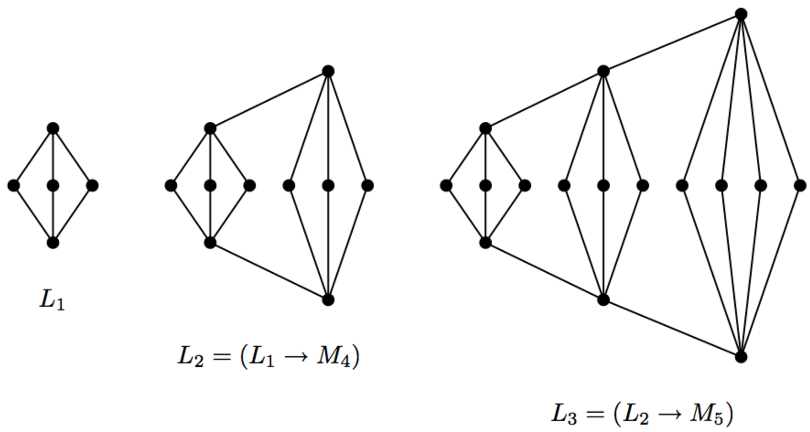

Fig. 1

It is easy to see that $(L_i)_{i\in\mathbb{N}}$, with natural embeddings $f_i : L_i \to L_{i+1}$, form a direct family of lattices. Let $L((\kappa_i)_{i\in\mathbb{N}})$ be the direct limit of this direct family. We remark, that the direct limit in this case is nothing else than a directed (set–theoretical) union. (When we consider the natural embeddings as set inclusions.)

The following assertion is easy to verify.

Lemma 5.3. *The congruence lattice of L_n is an $(n+1)$–element chain, that of $L((\kappa_i)_{i\in\mathbb{N}})$ is isomorphic to the ordinal $\omega_0 + 1$. Hence both L_n $(n \in \mathbb{N})$ and $L((\kappa_i)_{i\in\mathbb{N}})$ are subdirectly irreducible lattices.*

Lemma 5.4. *Homomorphic images of the lattice $L((\kappa_i)_{i\in\mathbb{N}})$ are just those isomorphic to $L((\kappa_{n+i})_{i\in\mathbb{N}})$ for $n \in \mathbb{N}_0$, and one–element lattices.*

Proof. Let $\theta_0 \subset \theta_1 \subset \dots$ be the sequence of all congruence relations of $L = L((\kappa_i)_{i\in\mathbb{N}})$ different from the greatest one. Then L/θ_0 is isomorphic to L; L/θ_1 is isomorphic to $L((\kappa_{1+i})_{i\in\mathbb{N}})$, and so on. In particular, for all $n \in \mathbb{N}_0$, L/θ_n is isomorphic to $L((\kappa_{n+i})_{i\in\mathbb{N}})$. □

Theorem 5.5. *Let $L = L((\kappa_i)_{i\in\mathbb{N}})$, Then the formation $\mathsf{form}(\{L\})$ is an antiatom in $\mathbb{F}$.*

Proof. By way of contradiction, let $\mathcal{F}$ be an atom in $\mathbb{F}$ with $\mathcal{F} \leq \mathsf{form}(\{L\})$. Then $\mathcal{F} = \mathsf{form}(\{M\})$ for a subdirectly irreducible lattice M satisfying the condition

($**$), $M \in \mathsf{form}(\{L\})$. By Theorem 3.5, $M \in \mathbf{H}(\{L\})$, so that M is isomorphic to $L((\kappa_{n+i})_{i\in\mathbb{N}})$ for some $n \in \mathbb{N}_0$. As $L((\kappa_{n+1+i})_{i\in\mathbb{N}}) \in \mathbf{H}(L((\kappa_{n+i})_{i\in\mathbb{N}})) = \mathbf{H}(\{M\})$, using ($**$) we obtain $M \in \mathbf{H}(\{L((\kappa_{n+1+i})_{i\in\mathbb{N}})\})$. This contradicts Lemma 5.4. $\square$

Theorem 5.6. *There exists a proper class of mutually non–comparable antiatoms in* $\mathbb{F}$.

Proof. Let $(\kappa_i)_{i\in\mathbb{N}}$ and $(\varkappa_i)_{i\in\mathbb{N}}$ be infinite ascending sequences of cardinals with $\kappa_i < \varkappa_j$ for all $i, j \in \mathbb{N}$. Denote $\mathcal{F}_\kappa$, $\mathcal{F}_\varkappa$ the formation generated by the lattice $L((\kappa_i)_{i\in\mathbb{N}})$, $L((\varkappa_i)_{i\in\mathbb{N}})$, respectively.

Suppose that $\mathcal{F}_\kappa \leq \mathcal{F}_\varkappa$. Then $L((\kappa_i)_{i\in\mathbb{N}}) \in \mathsf{form}(L((\varkappa_i)_{i\in\mathbb{N}}))$ and using Theorem 3.5 we obtain $L((\kappa_i)_{i\in\mathbb{N}}) \in \mathbf{H}(L((\varkappa_i)_{i\in\mathbb{N}}))$. By Lemma 5.4, $L((\kappa_i)_{i\in\mathbb{N}})$ is isomorphic to $L((\varkappa_{n+i})_{i\in\mathbb{N}})$ for some $n \in \mathbb{N}_0$, which implies $\kappa_1 = \varkappa_{n+1}$, a contradiction. Similarly, $\mathcal{F}_\varkappa \leq \mathcal{F}_\kappa$ implies $\varkappa_1 = \kappa_{m+1}$, for some $m \in \mathbb{N}_0$, again a contradiction.

In order to complete the proof, it is sufficient to find a proper class of such sequences of cardinals. Obviously, $\{(\aleph_{\alpha+i})_{i\in\mathbb{N}} : \alpha \text{ limit ordinal }\}$ forms a proper class and for limit ordinals α, β with $\alpha < \beta$ and $i, j \in \mathbb{N}$, we have $\aleph_{\alpha+i} < \aleph_{\beta+j}$. $\square$

6. Formations of distributive lattices

In Section 4, we have introduced the denotation $\mathbb{F}_d$ for the collection of all formations of distributive lattices. This collection is a proper class. For any infinite cardinal κ, let $\mathcal{F}_d(\kappa)$ be the class of all distributive lattices with cardinalities not exceeding κ. Then $\mathcal{F}_d(\kappa)$, for various infinite cardinals κ, form a large chain. We are going to show that $\mathbb{F}_d$ contains also large antichains.

Lemma 6.1. *Let α, β be any limit ordinals. Then $\beta \in \mathbf{H}(\{\alpha\})$ if and only if α contains a cofinal subset of the type β.*

Proof. Let f be a homomorphism of α onto β. For $y \in \beta$, let $x(y)$ be the least element of $f^{-1}(y)$. It is easy to see that $\{x(y) : y \in \beta\}$ is a cofinal subset of α isomorphic to β.

Conversely, let $X \subseteq \alpha$ be a cofinal subset of α, g an isomorphism of X onto β. For any $a \in \alpha$, let x_a be the least element of the set $\{x \in X : x \geq a\}$. Set $f(a) = g(x_a)$. Then f is a homomorphism of α onto β. $\square$

For any limit ordinal $\alpha > 0$, let $\mathrm{cf}(\alpha)$ denote the cofinality of α. If $\mathrm{cf}(\alpha) = \alpha$, then α is said to be a regular ordinal. In fact, each regular ordinal is an initial ordinal, i.e., cardinal. In the sequel we will denote the initial ordinals as usual by ω_α, $\alpha \in \mathit{Ord}$. The Axiom of Choice guarantees the existence of a proper class of regular ordinals, in particular for each $\alpha \in \mathit{Ord}$, $\omega_{\alpha+1}$ is a regular ordinal.

If ω_α is regular and $L \in \mathbf{H}(\{\omega_\alpha\})$, then L is isomorphic to ω_α or to a successor ordinal less then ω_α, by Lemma 6.1.

Theorem 6.2. *Let $\omega_\alpha, \omega_\beta$ be any different regular ordinals. Then formations generated by ω_α and ω_β are non–comparable.*

Proof. Let us suppose that $\omega_\alpha < \omega_\beta$. Since $\mathsf{form}(\{\omega_\alpha\})$ contains only lattices L with $|L| \leq \aleph_\alpha$, ω_β does not belong to $\mathsf{form}(\{\omega_\alpha\})$.

Further, we will show that $\omega_\alpha \notin \mathsf{form}(\{\omega_\beta\})$. By way of contradiction, let $\omega_\alpha \in \mathsf{form}(\{\omega_\beta\}) = \mathbf{P}_{\mathrm{FS}}\,\mathbf{H}(\{\omega_\beta\})$, due to Theorem 3.4. Then ω_α is a subdirect product of some L_i, $(i = 1, \ldots, n)$, $L_i \in \mathbf{H}(\{\omega_\beta\})$. Each L_i is a homomorphic image of ω_α, too. A homomorphic image of ω_α must be a well ordered chain, which (for the cardinality reason) cannot be isomorphic to a cofinal subset of ω_β. By 6.1, L_i cannot be a limit ordinal, which means that L_i has a greatest element. Hence, the same holds for ω_α, a contradiction. $\square$

Corollary 6.3. *Formations generated by regular ordinals form an antichain which is a proper class.*

References

1. G. Grätzer, *General Lattice Theory, Second Edition*, Birkhäuser, Basel 1998.
2. J. Jakubík, *Formations of lattice ordered groups and GMV–algebras*, Math. Slovaca 58, No. 5, 521–534 (2008).
3. L. A. Shemetkov, *Formations of Finite Groups*, Nauka, Moskva, 1978. (Russian).

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STABILITY OF HOMOMORPHISMS BETWEEN COMPACT ALGEBRAS

PAVOL ZLATOŠ

Dedicated to the 70th birthday of Alfonz Haviar

Abstract. We generalize a result on stability of continuous homomorphisms between compact groups to continuous homomorphisms between compact topological algebras. In general, a continuous function between such algebras which is almost a homomorphism need not be uniformly close to a homomorphism. Positive results can be obtained introducing some control over the continuity of the functions resembling homomorphisms by means of a "continuity scale."

The question we are dealing with in this paper can be roughly speaking formulated as follows: if a continuous function $g\colon A \to B$ between two topological universal algebras A and B of the same similarity type behaves almost like a homomorphism, is it then necessarily uniformly close to a genuine continuous homomorphism $h\colon A \to B$? Question of this type, made precise for various types of algebras and functions, use to be called *stability problems.*

The problem of ε-stability of additive functions $\mathbb{R} \to \mathbb{R}$, as well as its generalization to mappings between arbitrary metrizable groups, was raised by S. M. Ulam — cf. [9], [10]. Since then the topic was thoroughly examined and generalized in various respects — see [2], [4], [5] and [7] for surveys of further development.

There are many known examples showing that "arbitrarily good almost homomorphisms" need not be close to homomorphisms. In the paper [8] by J. Špakula and the present author a stability result for continuous homomorphisms between compact topological groups was established. This was enabled by controlling the continuity of the almost homomorphisms by means of a "continuity scale." We also remarked there that this result could readily be generalized to homomorphisms between any compact universal algebras of a *finite* type. In this note we

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prove such a generalization for compact universal algebras of an *arbitrary* similarity type (signature). Finally, we state separately the particular version of this result for metrizable algebras and quote a family of counterexamples from [8] showing that one cannot resign on the continuity control.

1. The stability theorem

Our standard references for universal algebra and general topology are the books [3] by G. Grätzer and [1] by R. Engelking, respectively.

Under the term *universal algebra*, or just *algebra* we always mean an algebra of the same fixed but otherwise arbitrary similarity type, with the set of (finitary) operation symbols denoted by F. However, unlike in [3], we denote the algebra $\left(A, f^A\right)_{f\in F}$ by the same character as its underlying set A. A *topological (universal) algebra* A is usually defined as an algebra endowed with a topology making all the operations $f^A\colon A^n \to A$, $f \in F$, continuous. We additionally include the Hausdorff or T_2 separation property into this definition. A topological algebra A is called *compact* or *completely regular* if the topological space A is compact or completely regular, respectively.

If A is a completely regular topological algebra then the topology of A can be induced by a uniformity $\mathcal{U}$ on A. However, the operations $f^A\colon A^n \to A$ are neither explicitly required nor need to be (unless A is compact) uniformly continuous with respect to $\mathcal{U}$.

Definition 1. Let A, B be topological algebras of the same type, such that the topology of B is induced by a uniformity $\mathcal{U}$ on B. Given an entourage $U \in \mathcal{U}$, a function $g\colon A \to B$ is called a *U-homomorphism* if for each n-ary operation symbol $f \in F$ and all $a_1, \dots, a_n \in A$ we have

$$\big(gf^A(a_1, \dots, a_n),\ f^B(ga_1, \dots, ga_n)\big) \in U.$$

Two functions $g, h\colon A \to B$ are U-close if $\big(g(a), h(a)\big) \in U$ for each $a \in A$. The pair (A, B) is said *to have stable homomorphisms with respect to $\mathcal{U}$* if for each $V \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that for every continuous U-homomorphisms $g\colon A \to B$ one can find a continuous homomorphism $h\colon A \to B$ such that g is V-close to h.

The answer to the stability problem is negative in general. A counterexample, comprising an infinite family of pairs of compact metrizable abelian groups (A, B) none of which has stable homomorphisms, will be presented in the final part of the next section, devoted to the metrizable case.

Thus in order to get some positive results, some additional assumptions are unavoidable. One possibility consists in introducing a kind of control over the continuity of functions $A \to B$.

Definition 2. Let X and Y be two topological spaces such that the topology of Y is induced by a uniformity $\mathcal{U}$ on Y with a basis $\mathcal{U}_0 \subseteq \mathcal{U}$. An $(X, Y, \mathcal{U})$-*continuity scale* is any mapping Γ assigning to each pair $(x, U) \in X \times \mathcal{U}_0$ a neighborhood $\Gamma(x, U)$ of the point $x \in X$. Then a function $g\colon X \to Y$ is said to be Γ-*continuous* if for all $U \in \mathcal{U}_0$ and $x, y \in X$ the condition $y \in \Gamma(x, U)$ implies $\big(g(x), g(y)\big) \in U$.

Obviously, a Γ-continuous function $g\colon X \to Y$ is continuous. The point is that any family of Γ-continuous function $g\colon X \to Y$ already is *equicontinuous*. The other way round, using the axiom choice one can easily show that any equicontinuous family of functions $X \to Y$ is Γ-continuous with respect to some $(X, Y, \mathcal{U})$-continuity scale Γ.

Definition 3. Let A, B be topological algebras such that the topology of B is induced by a uniformity $\mathcal{U}$ and Γ be an $(A, B, \mathcal{U})$-continuity scale. The pair (A, B) is said *to have stable homomorphisms with respect to the continuity scale* Γ if for each $V \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that for every Γ-continuous U-homomorphism $g\colon A \to B$ one can find a continuous homomorphism $h\colon A \to B$ such that g is V-close to h.

Now, everything is ready to state and prove the announced stability theorem.

Theorem 1. *Let A, B be compact topological algebras of the same similarity type. Then the pair (A, B) has stable homomorphisms with respect to every $(A, B, \mathcal{U})$-continuity scale Γ, where $\mathcal{U}$ is the (unique) uniformity inducing the topology of B.*

Proof. Assume the contrary and fix some compact topological algebras A, B, a uniformity $\mathcal{U}$ on B with a basis $\mathcal{U}_0$, an $(A, B, \mathcal{U})$-continuity scale Γ with domain $A \times \mathcal{U}_0$ and an entourage $V \in \mathcal{U}$ witnessing it.

Let D denote the set of all functions $g\colon A \to B$ such that g is not V-close to any continuous homomorphism $h\colon A \to B$. For $U \in \mathcal{U}$ denote by E_U the set of all Γ-continuous U-homomorphisms $g\colon A \to B$. Obviously, $E_U \subseteq E_{U'}$ for any $U \subseteq U'$ in $\mathcal{U}$. By the assumption, $D \cap E_U \neq \emptyset$ for each $U \in \mathcal{U}$. Then $D \cap E_U$ is an equicontinuous family of functions $A \to B$, and — as B is compact — the set

$$\{g(x) \mid g \in D \cap E_U\} \subseteq B$$

is relatively compact in B for each $x \in A$. Hence, by the Arzelà-Ascoli theorem, the closure

$$H_U = \mathrm{cl}(D \cap E_U)$$

is a compact subset of the space $\mathcal{C}(A, B)$ of all continuous functions $A \to B$ with the compact-open topology (which, by the compactness of A, coincides with the topology of uniform convergence on A). As $\emptyset \neq H_U \subseteq H_{U'}$ for $U \subseteq U'$ in $\mathcal{U}$, the intersection

$$H = \bigcap_{U \in \mathcal{U}} H_U$$

is nonempty, as well. Take any $h \in H$; it obviously is a continuous function $A \to B$. (Moreover, if all the entourages in the basis $\mathcal{U}_0$ are closed as subsets of the topological space $B \times B$, which can be assumed without loss of generality, then h is even Γ-continuous.)

We show that h is a homomorphisms. Take any n-ary operation symbol $f \in F$, $a_1, \ldots, a_n \in A$, and a symmetric entourage $U \in \mathcal{U}$. By the continuity of f^B one can find a symmetric $U' \in \mathcal{U}$ such that $U' \subseteq U$ and

$$\big(f^B(ha_1, \ldots, ha_n),\ f^B(b_1, \ldots, b_n)\big) \in U$$

for all $b_1, \ldots, b_n \in B$ satisfying $\big(h(a_i), b_i\big) \in U'$. As $h \in H_U$, there is a $g \in D \cap E_U$ such that $\big(h(x), g(x)\big) \in U'$ for each $x \in A$. Hence, in particular,

$$\big(hf^A(a_1, \ldots, a_n),\ gf^A(a_1, \ldots, a_n)\big) \in U'.$$

Since g is an U-homomorphism,

$$\big(gf^A(a_1, \ldots, a_n),\ f^B(ga_1, \ldots, ga_n)\big) \in U.$$

Finally, as $\big(h(a_i), g(a_i)\big) \in U'$ for $i \leq n$, we have

$$\big(f^B(ga_1, \ldots, ga_n),\ f^B(ha_1, \ldots, ha_n)\big) \in U.$$

Consequently,

$$\big(hf^A(a_1, \ldots, a_n),\ f^B(ha_1, \ldots, ha_n)\big) \in U' \circ U \circ U \subseteq U^3.$$

As the entourages of the form U^3, with $U \in \mathcal{U}$ symmetric, form a basis of the Hausdorff uniformity $\mathcal{U}$, we get the homomorphy condition

$$f^B(ha_1, \ldots, ha_n) = hf^A(a_1, \ldots, a_n).$$

Choose a symmetric $W \in \mathcal{U}$ such that $W^4 \subseteq V$. We will show that for any continuous homomorphism $\varphi\colon A \to B$ there is an $x \in A$ such that $\big(h(x), \varphi(x)\big) \notin W$. Assume this is not the case, i.e., there exists a continuous homomorphism $\varphi\colon A \to B$ which is W-close to h. Then for any $U \in \mathcal{U}$, $U \subseteq V$, there is a $g_U \in D \cap E_U$ such that $\big(g_U(x), h(x)\big) \in U$ for all $x \in A$, and an $x_U \in A$ such that $\big(g_U(x_U), \varphi(x_U)\big) \notin V$. As A is compact, there is a basis $\mathcal{U}_1 \subseteq \mathcal{U}$ such that $U \subseteq V$ for each $U \in \mathcal{U}_1$ and the net $(x_U)_{U \in \mathcal{U}_1}$ converges to a point $x \in A$. By the continuity of the functions h and φ there is a neighborhood $N \subseteq A$ of x such that for any $y \in N$ we have $\big(h(x), h(y)\big) \in W$, as well as $\big(\varphi(x), \varphi(y)\big) \in W$. Then there is a $U_1 \in \mathcal{U}_1$ such that for each $U \in \mathcal{U}_1$ the condition $U \subseteq U_1$ implies $x_U \in N$. Choose any $U \in \mathcal{U}_1$ such that $U \subseteq U_1 \cap W$. Then

$$\begin{aligned}
\big(g_U(x_U), h(x_U)\big) &\in U, \\
\big(h(x_U), h(x)\big) &\in W, \\
\big(h(x), \varphi(x)\big) &\in W, \\
\big(\varphi(x), \varphi(x_U)\big) &\in W,
\end{aligned}$$

hence

$$\big(g_U(x_U), \varphi(x_U)\big) \in U \circ W \circ W \circ W \subseteq W^4 \subseteq V,$$

and we have a contradiction.

As h itself is a continuous homomorphisms $A \to B$, it follows that in particular h is not W-close to h. This contradiction concludes the proof of the theorem. □

2. The metrizable case

Assume that the topologies of A and B stem from metrics ρ and σ, respectively. Then for any strictly decreasing sequence $(\eta_k)_{k\in\mathbb{N}}$ of positive reals converging to 0, the relations $\big\{(u,v) \in B \times B \mid \sigma(u,v) \le \eta_k\big\}$ form a basis of the uniformity $\mathcal{U}_\sigma$ on B. Due to compactness of A the notions of continuity and uniform continuity coincide for functions $A \to B$. Hence a closed and symmetric $(A, B, \mathcal{U}_\sigma)$-continuity scale can be represented as a sequence $\varGamma = \big((\gamma_k, \eta_k)\big)_{k\in\mathbb{N}}$, where $(\gamma_k)_{k\in\mathbb{N}}$ is a decreasing sequence of positive reals. It is more natural to call the sequence $\varGamma$ a (ρ,σ)-*continuity scale* in this case. Then a function $g\colon A \to B$ is called $\varGamma$-*continuous* if for each k and all $x, y \in A$ the condition $\rho(x,y) \le \gamma_k$ implies $\sigma(g(x), g(y)) \le \eta_k$.

For completeness' sake we still quote the obvious translations of the intuitive concepts of closeness and almost homomorphy into the metric terms. Let $\varepsilon > 0$. Two functions $g, h\colon A \to B$ are said to be ε-*close* if $\sigma(g(a), h(a)) \le \varepsilon$ for each $a \in A$; a function $g\colon A \to B$ is called an ε-*homomorphism* if for each n-ary operation symbol $f \in F$ and all $a_1, \ldots, a_n \in A$ we have

$$\sigma\big(gf^A(a_1, \ldots, a_n),\ f^B(ga_1, \ldots, ga_n)\big) \le \varepsilon.$$

For metrizable algebras the stability theorem 1 can be stated in the following more usual form.

Theorem 2. *Let A, B be compact topological algebras of the same similarity type, ρ and σ be two metrics inducing the topology of A and B, respectively, and $\varGamma = \big((\gamma_k, \eta_k)\big)_{k\in\mathbb{N}}$ be a (ρ,σ)-continuity scale. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that every $\varGamma$-continuous δ-homomorphism $g\colon A \to B$ is ε-close to a continuous (even $\varGamma$-continuous) homomorphism $h\colon A \to B$.*

Let us close with the announced counterexample, showing that one cannot even prove the stability of continuous homomorphisms between compact metrizable abelian groups, unless some additional assumptions are fulfilled. In particular, one cannot get rid of mentioning the continuity scale $\varGamma$ in theorems 1 and 2. The construction is taken from [8]; it is based on an example from [6], forming its initial part.

Example. Let p be an arbitrary prime and $\mathbb{Z}_p$ denote the compact metric abelian group of p-adic integers, i.e., the completion of the ring $\mathbb{Z}$ with respect to the norm

$$|a|_p = p^{-o_p(a)},$$

where $p^{o_p(a)}$ is the highest power of p dividing the integer $a \neq 0$, and $|0|_p = 0$. Mapping the remainder $x \in \{0, 1, \ldots, p^n - 1\}$ mod p^n onto the corresponding integer $g_n(x) = x \in \mathbb{Z} \subseteq \mathbb{Z}_p$ defines a p^{-n}-homomorphisms of the finite cyclic group $\mathbb{Z}/(p^n)$ into $\mathbb{Z}_p$ for every $n \in \mathbb{N}$. Indeed, the difference $g_n(x) + g_n(y) - g_n(x+y)$ is either 0 or p^n for any $x, y \in \mathbb{Z}/(p^n)$, hence

$$\big|g_n(x) + g_n(y) - g_n(x+y)\big|_p \leq |p^n|_p = p^{-n}.$$

However, as $\mathbb{Z}_p$ is torsionfree, there is no homomorphism $\mathbb{Z}/(p^n) \to \mathbb{Z}_p$ except for the trivial one.

The direct product $A_p = \prod_{n \in \mathbb{N}} \mathbb{Z}/(p^n)$ with the product topology is a compact metrizable abelian group; denote by $\pi_n \colon A_p \to \mathbb{Z}/(p^n)$ the projection onto the nth factor and $\iota_n \colon \mathbb{Z}/(p^n) \to A_p$ its embedding into the product. Then $g_n \circ \pi_n \colon A_p \to \mathbb{Z}_p$ is a continuous p^{-n}-homomorphisms, however, for each homomorphism $h \colon A_p \to \mathbb{Z}_p$ we have

$$\begin{aligned} \sup_{u \in A_p} \big|(g_n \circ \pi_n)(u) - h(u)\big|_p \quad &\geq \max\nolimits_{x \in \mathbb{Z}/(p^n)} \big|g_n(x) - (h \circ \iota_n)(x)\big|_p \\ &= \max\nolimits_{0 \leq x < p^n} |x|_p = 1, \end{aligned}$$

since the homomorphism $h \circ \iota_n \colon \mathbb{Z}/(p^n) \to \mathbb{Z}_p$ necessarily is constantly 0. We can conclude that, for any prime p, the pair $(A_p, \mathbb{Z}_p)$ of compact metrizable abelian groups does not have stable homomorphisms.

References

1. R. Engelking, General Topology, PWN — Polish Scientific Publishers, Warszawa, 1977.
2. G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50**(1995), 143–190.
3. G. Grätzer, Universal Algebra (2nd ed.), Springer-Verlag, New York-Berlin-Heidelberg, 1979.
4. D. H. Hyers, G. Isac, Th. M. Rassias, Stability of functional equations in several variables, Birkhäuser Verlag, Basel-Boston, 1998.
5. D. H. Hyers, Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44**(1992), 125–153.
6. D. Kazhdan, *On ε-representations*, Israel J. Math **43**(1982), 315–323.
7. Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Applicanda Math. **62**(2000), 23–130.
8. J. Špakula, P. Zlatoš, *Almost homomorphisms of compact groups*, Illinois J. Math. **48**(2004), 1183–1189.
9. S. M. Ulam, A collection of mathematical problems, Interscience Publishers, New York, 1960.
10. S. M. Ulam, *An anecdotal history of the Scottish Book*, in: R. D. Mauldin (ed.), The Scottish Book, Birkhäuser Verlag, Basel-Boston, 1981.

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