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ASYMPTOTIC MOTIONS OF THREE-PARAMETRIC ROBOT MANIPULATORS

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Abstract. The Lie algebra of screws in matrix form is used for introducing asymptotic and geodesic motions of robot manipulators. These motions are characterized as motions the covariant acceleration elements of which are tangential. In this paper the asymptotic motions of all three-parametric robot manipulators are described.

1. Introduction

We deal with robot manipulator motions which have a strong geometrical background. We use the screw method of the investigation, see [1, 7, 8, 9]. Motions of the robot effectors are determined by curves on the Lie group $E(3)$ of all Euclidean motions in the Euclidean space E_3 the Lie algebra $e(3)$ of which is the algebra of velocity twists. All motions of a n-parametric robot manipulator with n joints which are controlled by n parameters $(u) = (u_1, \dots, u_n)$ form at any position (u) the subspace $VT(u) \subset e(3)$ of the velocity twists, the subspace $AC(u) \subset e(3)$ spanned on the set of all Lie bracket values $[VT(u), VT(u)]$, the subspace $Cov(u) = span(VT(u) \cup AC(u))$ of the so-called covariant acceleration twists, the Klein subspace $K(u) \subset VT(u)$ of all twists orthogonal to $VT(u)$ according to the Klein bilinear scalar form KL on $e(3)$. We introduce the so-called asymptotic motion as a motion the covariant acceleration twists of which lie in the spaces $VT(u)$. The simple examples of the asymptotic motions are the motions when only one joint works. We describe all asymptotic motions for all three-parametric robot manipulators with revolute and prismatic joints only.

2. Basic notions of robot manipulators

The geometrical background of a kinematic and dynamic problems of the robots are both the Lie group $E(3)$ of the Euclidean motions in the Euclidean space E_3

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and its Lie algebra $e(3)$, see for example [4, 5, 7, 8]. We use the homogeneous matrix presentation of $E(3)$, the matrix and the screw (twist) presentations of $e(3)$. We introduce a brief explanation of main ideas. Throughout our paper the notion of the robot will mean a robot manipulator with n links.

Let $\mathcal{S}_0 = (\bar{i}_0, \bar{j}_0, \bar{k}_0, O)$ or $\mathcal{S}_n = (\bar{i}_n, \bar{j}_n, \bar{k}_n, P^T)$ be the cartesian coordinate system fixed on the robot base or on the robot effector, respectively. There is a unique Euclidean transformation $\mathbf{H}$ in which the system $\mathcal{S}_0$ goes into $\mathcal{S}_n$. We get the matrix relation $\mathcal{S}_n = \mathcal{S}_0 H$ where $H = \left(\begin{smallmatrix} A & P \\ 0 & 1 \end{smallmatrix}\right)$ is the homogeneous transformation matrix; i.e., A is a $(3,3)$-matrix the columns of which are the successively coordinates of the vectors $\bar{i}_n, \bar{j}_n, \bar{k}_n$ in the base $(\bar{i}_0, \bar{j}_0, \bar{k}_0)$ and $P = (p_x, p_y, p_z)^T$ are the coordinates of the origin P of $\mathcal{S}_n$ in $\mathcal{S}_0$. Therefore $AA^T = I$ where I is the identity matrix and A^T denotes the transposed matrix to A.

Let us recall two relations according to the matrix H which we will use.

a) If $L_n = (x_n, y_n, z_n, 1)^T$ denotes the homogeneous coordinates of a point L of the effector in $\mathcal{S}_n$ then $L_0 = HL_n$ are the homogeneous coordinates of L in $\mathcal{S}_0$.
b) If $L_0 = (x_0, y_0, z_0, 1)^T$ are the coordinates of L in $\mathcal{S}_0$ and $L_0' = (x_0', y_0', z_0', 1)^T$ are the coordinates of the image of L in the Euclidean motion h in which the system $\mathcal{S}_0$ goes into $\mathcal{S}_n$ then $L_0' = HL_0$. Evidently the Cartesian system $\mathcal{S}_0$ states the map $h \mapsto H$ which is the matrix representation of $E(3)$. We identify $E(3)$ with the group of matrices H.

A motion of the effector is expressed by the equation $\mathcal{S}_n(t) = \mathcal{S}_0 H(t)$. Then the equation $L_0(t) = H(t)L_0$ expresses the corresponding motion of an effector point L in the coordinate system $\mathcal{S}_0$. Differentiating this equation with respect to t we get $\dot{L}_0(t) = \dot{H}(t)L_0 = \dot{H}H^{-1}L_0(t)$, where $HH^{-1} = I$, $H^{-1} = \left(\begin{smallmatrix} A^T & -A^T P \\ 0 & 1 \end{smallmatrix}\right)$, $\dot{H}H^{-1} = \left(\begin{smallmatrix} \dot{A}A^T & -\dot{A}A^T P + \dot{P} \\ 0 & 0 \end{smallmatrix}\right)$ and the dot over letters denotes differentiation with respect to t. The relation $AA^T = I$ inherits the equation $\dot{A}A^T + A\dot{A}^T = 0$ and thus $\dot{A}A^T$ is skew symmetric $(\dot{A}A^T)^T = -\dot{A}A^T$. There is a unique vector $\overline{\omega} = (\omega_x, \omega_y, \omega_z)$ such that $\dot{A}A^T = \left(\begin{smallmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{smallmatrix}\right)$. A skew symmetric matrix determined in such way by the vector $\overline{\omega}$ will by denoted by C^ω. Let us recall the well known relation $C^\omega \overline{v}^T = (\overline{\omega} \times \overline{v})^T$, where $\overline{\omega} \times \overline{v}$ denotes the vector product of the vectors $\overline{\omega}$ and $\overline{v}$. If we denote $\overline{b} = B^T$, $B := -\dot{A}A^T P + \dot{P}$ we see that the matrix $\mathcal{H} := \dot{H}H^{-1}$ determines the vector couple $(\overline{\omega}, \overline{b})$ which is called briefly a twist, see [7]. The matrix $\left(\begin{smallmatrix} C^\omega & \overline{b}^T \\ 0 & 0 \end{smallmatrix}\right)$ is said to be the matrix presentation of the twist $(\overline{\omega}, \overline{b})$.

From the geometrical point of view $H(t)$ is a curve on the manifold $E(3)$ and $\dot{H}(t)$ is its tangent vector at $H(t)$. Then the right group translation by the element $H^{-1}(t) \in E(3)$ transforms $\dot{H}(t)$ into the tangent vector $\mathcal{H}(t) =$

$\dot{H}(t)H^{-1}(t)$ at the unit $I \in E(3)$. The vector space of all tangent vectors on a Lie group at its unit has a Lie algebra structure which is $e(3)$ in our case of $E(3)$. So $e(3)$ is the vector space of all twists. The Lie bracket (product) has the standard matrix representation $[\mathcal{H}_1, \mathcal{H}_2] = \mathcal{H}_1\mathcal{H}_2 - \mathcal{H}_2\mathcal{H}_1$ which corresponds with its following twist representation $[(\overline{\omega}_1, \overline{b}_1), (\overline{\omega}_2, \overline{b}_2)] = (\overline{\omega}_1 \times \overline{\omega}_2, \overline{\omega}_1 \times \overline{b}_2 - \overline{\omega}_2 \times \overline{b}_1)$.

Any twist $Y = (\overline{\omega}, \overline{b})$ determines the motion m_Y (called canonical) in the following way. If $\overline{\omega} \neq \overline{0}$ then m_Y is the uniform helical motion around the axis p of Y with the angular velocity $\overline{\omega}$ and with the translation velocity $\upsilon\overline{\omega}$ where $\upsilon = \overline{\omega} \cdot \overline{b}/\overline{\omega}^2$ is the so-called the pitch of this motion. So it is a rotation if the scalar product $\overline{\omega} \cdot \overline{b}$ is zero. Let us recall that the velocity of a point L at this motion is $\overline{v} = \overline{\omega} \times \overline{CL} + \upsilon\overline{\omega}$ and thus $\overline{v} = \overline{\omega} \times \overline{CO} + \upsilon\overline{\omega} = \overline{b}$ is the velocity of the origin O of $\mathcal{S}_0$. Let us emphasize that in the rotation case $\overline{b}$ is orthogonal to the plane (O, p). If $\overline{\omega} = \overline{0}$ then the canonical motion m_Y of $Y = (\overline{0}, \overline{b})$ means the translation with the velocity $\overline{b}$. Conversely, the uniform helical motion m around the axis $p = (C, \overline{\omega})$ where $\overline{OC}$ is perpendicular to p or $C \equiv O$, $\overline{\omega}$ is its angular velocity, $\upsilon\overline{\omega}$ is its translation velocity and $\overline{b}$ is the velocity of the origin O at this motion, determines the twist $(\overline{\omega}, \overline{b})$ the canonical motion of which is just the motion m. Evidently the translation τ with the velocity $\overline{b}$ determines the twist $(\overline{0}, \overline{b})$ the canonical motion of which is just τ. So a twist $Y = (\overline{\omega}, \overline{b})$ is called rotational or helical or translating if $\overline{\omega} \cdot \overline{b} = 0$, $\overline{\omega} \neq \overline{0}$ or $\overline{\omega} \cdot \overline{b} \neq 0$ or $\overline{\omega} = \overline{0}$ respectively.

Remark 1 (On the exponential map $\exp : e(3) \mapsto E(3)$)**.** Let $(\overline{\omega}, \overline{b})$ be a twist and $\mathcal{H} = \begin{pmatrix} C^{\omega} & \overline{b}^T \\ 0 & 0 \end{pmatrix}$ be its matrix presentation. Then

$$\exp t(\overline{\omega}, \overline{b}) = \exp t\mathcal{H} = I + t\mathcal{H} + t^2\mathcal{H}^2/2! + t\mathcal{H} + t^3\mathcal{H}^3/3! + \cdots \equiv \gamma(t)$$

and $\dot{\gamma}(t) = \mathcal{H} \exp t\mathcal{H}$. Let us consider the motion $L_0(t) = \gamma(t)L_0$ of a point L in the base coordinate system $\mathcal{S}_0$, $L_0 = L_0(0)$. Then for the velocity of the point $L_0(t)$ we get $\dot{L}_0(t) = \dot{\gamma}(t)L_0 = \dot{\gamma}\gamma^{-1}L_0(t) = \mathcal{H}L_0(t)$. Then in the case $\overline{\omega} \neq \overline{0}$: $\dot{L}_0(t) = \begin{pmatrix} C^{\omega} & \overline{b}^T \\ 0 & 0 \end{pmatrix} L_0(t) = (\overline{\omega} \times \overline{OL}_0(t) + \overline{b})^T = (\overline{\omega} \times (\overline{OC} + \overline{CL}_0(t)) + \overline{\omega} \times \overline{CO} + \upsilon\overline{\omega})^T = \overline{\omega} \times \overline{CL}_0(t) + \upsilon\overline{\omega}$. If $\overline{\omega} = \overline{0}$ then $\dot{L}_0(t) = \overline{b}^T$. We conclude: $\exp t(\overline{\omega}, \overline{b})$ is the canonical motion of the twist $(\overline{\omega}, \overline{b})$.

Remark 2. Let us recall that if $\exp t(\overline{\omega}, \overline{b})$ expresses a motion in the coordinate system $\mathcal{S}_n$, $\mathcal{S}_n = \mathcal{S}_0 H$, then $H \exp t(\overline{\omega}, \overline{b})H^{-1}$ is its expression in $\mathcal{S}_0$ and thus $Ad_H(\overline{\omega}, \overline{b}) := H\mathcal{H}H^{-1}$, $\mathcal{H} = \begin{pmatrix} C^{\omega} & \overline{b}^T \\ 0 & 0 \end{pmatrix}$ is the matrix presentation of $(\overline{\omega}, \overline{b})$ in $\mathcal{S}_0$.

Remark 3. The map $H \mapsto Ad_H$ is the representation of $E(3)$ in $GL(e(3))$. The axis p' of $Ad_H(\overline{\omega}, \overline{b})$ is the H-image of the axis p of $(\overline{\omega}, \overline{b})$, $p' = H(p)$.

Remark 4. The notion of the *twist* is not unified. Some authors use notions the *infinitesimal motion* or the *velocity operator*. We will refer to the *velocity twist* or the *velocity element*.

A n-parametric robot $\mathcal{R}$ consists of a sequence of n rigid links connected by joints such that every joint J_i enables the successive links to rotate around the axis o_i of J_i or to translate in the direction of o_i. Such a joint is called rotational or prismatic. So every joint determines a twist $(\overline{\omega}, \overline{b})$ in $\mathcal{S}_0$ where $\overline{\omega}^2 = 1$, $\overline{\omega} \cdot \overline{b} = 0$ or $\overline{\omega} = \overline{0}$, $\overline{b}^2 = 1$ and $\exp t(\overline{\omega}, \overline{b})$ is the unit uniform motion enabled by this joint. We will refer to the "canonical twist" of the corresponding joint. Let $X_i = (\overline{\omega}_i, \overline{b}_i)$ be the canonical twist of the ith joint J_i at the initial position $(u) = (0)$ of the robot in $\mathcal{S}_0$. Let $\exp u^i X_i$ be any motion enabled by J_i. Then the robot activity states a map $\rho : U_n \to E(3)$, $(u) = (u_1, \dots, u_n) \mapsto \exp u^1 X_1 \dots \exp u^n X_n$, where $U_n \subset \mathbb{R}^n$ is an open neighbourhood of $(0) = (0, \dots, 0) \in \mathbb{R}^n$ of the joint parameters (of the control joint variables) the admissible values of which are determined by the robot construction. Let us consider the motion $L_0(t) = H(t)L_0$, $H(t) = \exp u^1(t)X_1 \dots \exp u^n(t)X_n$, $\dot{L}_0(t) = \dot{H}(t)H^{-1}(t)L_0(t)$. We get

$$Y(t) := \dot{H}(t)H^{-1}(t) = \dot{u}^1 Y_1 + \dots + \dot{u}^n Y_n, \tag{1}$$

$Y_1 = X_1, Y_i = H_i X_i H_i^{-1} = Ad_{H_i} X_i, H_i = \exp u^1(t)X_1 \dots \exp u^{i-1}(t)X_{i-1}$, $\dot{L}_0(t) = Y L_0(t)$. The last equation inspire us to call Y the velocity twist or shortly a v-twist. As $Y_i = Ad_{H_i} X_i$ the axis of Y_i is the actual position of the joint axis o_i at time t. We will say that the joint J_i works or does not work at t_0 if $\dot{u}^i(t_0) \neq 0$ or $\dot{u}^i(t_0) = 0$, respectively. By the notion a "position" $(u) = (u^1, \dots, u^n)$ of the robot we mean an element $\rho(u) \in E(3)$. We say about a motion cross a position (u_0) if there is t_0 such that $(u_0) = (u(t_0))$.

Let $(u_0) = (u_0^1, \dots, u_0^n)$ be a fixed position. The curve $\gamma_i(t) = H_i(t_0)\exp(u_0^i + t)$ $X_i \exp u_0^{i+1} X_{i+1} \dots \exp u_0^n X_n$ is called an u^i-curve cross (u_0). Its tangent vector $\dot{\gamma}_i(t_0)$ will be denoted by $\partial u_i(u_0)$. By (1) it is clear that $Y_i(u_0) := Y_i(t_0) = \partial u_i(u_0)H^{-1}(u_0)$, i.e. $Y_i(u_0)$ is the image of $\partial u_i(u_0)$ by the right group translation stated by the element $H^{-1}(t_0)$. In general, a u^i-motion cross (u_0) will mean a motion $H_i(t_0)\exp u^i(t)X_i \exp u_0^{i+1} X_{i+1} \dots \exp u_0^n X_n$ when only the ith joint works.

Differentiating the equation $\dot{L}_0(t) = Y(t)L_0(t)$ we obtain $\ddot{L}_0(t) = (\dot{Y} + YY)L_0(t)$. As $Y(t) \in e(3)$ is a curve in the vector space $e(3)$ then $\dot{Y} \in e(3)$, i.e. $\dot{Y}$ is a twist. In contrary YY is not a twist. We have $\dot{Y} = \sum_{i=1}^n \ddot{u}^i Y_i + \sum_{i=1}^n \dot{u}^i \sum_{i=1}^n \frac{\partial Y_i}{\partial u^k} \dot{u}^k$. As Y_i depends on the parameters $u^1, \dots, u^{i-1}$ only then $\frac{\partial Y_i}{\partial u^k} = 0$ for $k > i$ and the direct computation leads to the expression $\frac{\partial Y_i}{\partial u^k} = [Y_k, Y_i]$, $k = 1, \dots, i-1$, see [5, 8]. We obtain

$$\dot{Y} = \dot{Y}_J + \dot{Y}_C, \qquad \dot{Y}_J = \sum_{i=1}^{n} \ddot{u}^i Y_i, \qquad \dot{Y}_C = \sum_{k<i} [Y_k, Y_i] \dot{u}^k \dot{u}^i \tag{2}$$

$$\ddot{L}_0 = (\dot{Y}_J + \dot{Y}_C + \dot{Y}_S) L_0, \qquad \dot{Y}_S = YY \tag{3}$$

Definition 1. An element $\dot{Y}_J$ or $\dot{Y}_C$ will be called a joint acceleration twist (shortly J-twist) or a skew acceleration twist (shortly C-twist) of the motion respectively.

3. Kinematic twist subspaces, asymptotic motions

Let us recall that the Jacobian $J(u)$ of a n-parametric robot at u is the differential $T(\rho_R^{-1}(u)\rho)$ of the composition of two maps: $\rho : U \to E(3)$ and the right side group translation $\rho_R^{-1}(u)$ by the group element $[\rho(u)]^{-1}$. Then the rank of the robot at (u) is the number $rank J(u)$.

Let us denote $VT(u) := span(Y_1(u), \ldots, Y_n(u)) \subset e(3)$ the vector space of the twists at (u). Evidently $rank J(u) = \dim VT(u)$. We write shortly $VT := VT(0) = span(X_1, \ldots, X_n)$.

Definition 2. A position (u) of a n-parametric robot is called regular or singular if $\dim VT(u) = n$ or $\dim VT(u) < n$ respectively. A motion is singular if its all positions are singular. A n-parametric robot is said to be the nr-robot if $\dim VT = n$.

We confine ourself on nr-robots, $\dim VT = n$. Then there is a neighourhood $U_0 \subset U$ that $\dim VT(u) = n$, $u \in U_0$ and thus the map ρ is an immersion on U_0. Then there is a neighborhood $V \subset U_0$ such that $\rho(V)$ is an immersed submanifold, see [6].

Definition 3. The subspace $AC(u) := span\{[A, B]; A, B \in VT(u)\}$ where $[A, B]$ denotes the Lie bracket of elements A, B in $e(3)$ is called the Coriolis subspace. Its elements are C-twists.

Remark 5. Let us recall that a subspace $VT(u)$ is a subalgebra of $e(3)$ iff $AC(u) \subset VT(u)$. If VT is a subalgebra then $\rho(V)$ is a submanifold of the subgroup stated by VT and $VT(u) = VT$, $u \in V$. We give the survey of all subalgebras of $e(3)$, see [4, 8]. They are:

- (α) All 1-dimensional subspaces of $e(3)$.
- (β) (a) All 2-dimensional subspaces of the translating twists.
  - (b) All 2-dimensional subspaces of the twists $(k\overline{\omega}, k\overline{b} + c\overline{\omega})$, $\overline{\omega} \neq \overline{0}$, $k, c \in \mathbb{R}$ with the same axis.
- (γ) (a) All 3-dimensional subspaces of the rotational twists the axes of which have a common point or all translating twists.

(b) All 3-dimensional subspaces spanned on a rotational twist and on two translating twists where the axis of the rotational twist is orthogonal to the directions of the translating twists.

(δ) All 4-dimensional subspaces spanned on a rotational twist and on 3 independent translating twists.

Definition 4. The subspace $Cov(u) = span(VT(u) + AC(u))$ is called the covariant subspace. Its elements $\dot{Y}$ introduced by the relation (2) will be briefly called Cov-twists. A motion $u(t)$ is called geodesic if $\dot{Y}(u(t)) = 0$.

Definition 5. A motion $u(t)$ will be called asymptotic at $u_0 = u(t_0)$ or asymptotic if $\dot{Y}(t_0) \in VT(u_0)$ or $\dot{Y}(t) \in VT(u)$ for any admissible $u(t)$ respectively. A position (u) is called flat if every motion cross (u) is asymptotic at (u). The robot is flat if there is an open set $W \subset U$ such that the robot is flat at any $(u) \in W$. The subspace $AC(u) \cap VT(u)$ will be called asymptotic, its elements will be called the asymptotic twists.

The following assertions are evident.

Lemma 1.

a$_1$) If a robot motion is asymptotic at (u) then its C-twist $\dot{Y}_C(u)$ is asymptotic. If $\dot{Y}_C(u) = 0$ then this motion is asymptotic and thus every u^i-motion and every motions due to work of prismatic joints only are asymptotic.

a$_2$) Every geodesic motion is asymptotic. An asymptotic motion is geodesic if its C-twist is opposite to its J-twist, $\dot{Y}_C = -\dot{Y}_J$.

a$_3$) A position (u) or a robot is flat iff $VT(u)$ or VT is a subalgebra.

a$_4$) All motions given by work of such joints $J_{i_1}, \dots, J_{i_k}$ that the subspaces $span(Y_{i_1}, \dots, Y_{i_k})$ are subalgebras, are asymptotic.

Definition 6. A motion given by work of such joints $J_{i_1}, \dots, J_{i_k}$ that the subspaces $span(Y_{i_1}, \dots, Y_{i_k})$ are subalgebras, is called trivially asymptotic (shortly t-asymptotic). The others motions will be shortly called nt-asymptotic.

Let us recall that all u^i-motions, all motions given by work of prismatic joints only, all motions given by work of a rotational joint and a prismatic joint with parallel axes are t-asymptotic. We concentrate on nt-asymptotic motions especially on the ones with non zero C-twists.

Let us remind that in $e(3)$ the Klein bilinear form KL is defined as follows. If $X_i = (\overline{\omega}_i, \overline{b}_i) \in e(3)$, $i = 1, 2$, then $KL(X_1, X_2) := \overline{\omega}_1 \cdot \overline{b}_2 + \overline{\omega}_2 \cdot \overline{b}_1$. It means that KL is regular, symmetric and Ad-invariant.

Definition 7. Twists $Z_1, Z_2 \in e(3)$ will be called KL-orthogonal if $KL(Z_1, Z_2) = 0$. Let $V \subset e(3)$ be a vector subspace. The subspace of all twists in $e(3)$ which are KL-orthogonal to every twist $Z \in V$ will be denoted by V^{or}. The subspace $K(u) = VT(u) \cap [VT(u)]^{or}$ will be called the Klein's subspace and its elements are called the Klein's twists. We say that $VT(u)$ is isotropic if $K(u) = VT(u)$.

It is valid that any two translating twists are KL-orthogonal.

Let $Y = t_1Y_1 + \cdots + t_nY_n$ be a velocity twist in $VT(u)$, $Y_i = (\overline{\omega}_i, \overline{b}_i)$. Then $Y \in K(u)$ iff

$$Kl(Y_i, Y) = t_1Kl(Y_i, Y_1) + \cdots + t_nKl(Y_i, Y_n) = 0, \qquad i = 1, \dots, n \tag{4}$$

The symmetric matrix of the system (4) is the matrix of the restriction $KL|_{VT(u)}$ of the Klein form KL to the subspace $VT(u)$ and thus Y is rotational or translating iff $Kl|_{VT(u)}(Y, Y) = 0$: i.e., iff

$$0 = \left(\sum_{i=1}^{n} t_i\overline{\omega}_i\right) \cdot \left(\sum_{j=1}^{n} t_j\overline{b}_j\right) = \sum_{i,j=1}^{n} (\overline{\omega}_i \cdot \overline{b}_j)t_it_j. \tag{5_1}$$

Y i.e. translating iff

$$\overline{0} = \sum_{i=1}^{n} t_i\overline{\omega}_i. \tag{5_2}$$

Remark 6 (On screws, see [1])**.** Let $\dim VT(u)$ be $k+1$. Then all one-dimensional vector subspaces in $VT(u)$, the so-called screws, form a projective space $\mathcal{P}_k$, $\dim \mathcal{P}_k = k$. The screws in $VT(u)$ are helical, rotational, translating if their non-zero twists are helical, rotational, translating respectively. Evidently all non-zero twists which are belonging to the same screw have the same axis. All rotational and translating screws in $\mathcal{P}_k$ form a quadric subspace (q) in $\mathcal{P}_k$. The subspace (q) is given by the equation (5_1) and the projective subspace of translating screws given by (5_2) lies on the subspace (q). Their axes lie on a quadratic straight line space (Q) in E_3 extended by the infinity points. The system (4) is the system for singular points of (q). Every Klein twist is either rotational or translating and it exists only if (q) degenerates.

Let us remind the following well known properties, see [3, 8].

Lemma 2.

(a) Two rotational twists are KL-orthogonal iff their axes are intersecting or parallel. A rotational or helical twist is KL-orthogonal to a translating twist iff its axis is orthogonal to the direction of the translating twist.

(b) Let X, Y be two not translating twists with the axes x, y. Then the axis of $[X, Y]$ orthogonally intersects the axes x, y. If Y is translating with the direction $\overline{y}$ then $[X,Y]$ is translating and its direction is orthogonal to x and $\overline{y}$. If X,Y are translating then $[X,Y]=0$.

Corollary 1.
(a) The twist $[X,Y]$ is KL-orthogonal to X and to Y.
(b) If a Klein twist $\hat{Y}$ is rotational then its axis intersects or is parallel with the axis of any rotational twist in $VT(u)$ and it is orthogonal to all directions of all translating twists in $VT(u)$. If $\hat{Y}$ is translating then its direction is orthogonal to the axes of all rotational twists in $VT(u)$.

Remark 7 (The Background of the notions "asymptotic", "geodesic")**.** In our consideration an essential role has the right group translations on $E(3)$. The parallel transport stated by these translations introduces an affine connection ${}^{R}\Gamma$ on $E(3)$, see [6, 8]. In general, let Γ be an affine connection on an arbitrary manifold M and let ${}^{\Gamma}\nabla$ be the covariant derivative stated by the parallel transport of Γ. Let $N\subset M$ be a submanifold. A curve γ on N is Γ-asymptotic or geodesic if ${}^{\Gamma}\nabla_{\dot{\gamma}}\dot{\gamma}$ is tangent on N or ${}^{\Gamma}\nabla_{\dot{\gamma}}\dot{\gamma}=0$ respectively. Our concept of asymptotic and geodesic motions coincides with the ones above with respect to the connection ${}^{R}\Gamma$ on $E(3)$.

4. Three-parametric robot manipulators

Definition 8. A $(3,k)$-robot means a three-parametric robot the maximal number of the independent directions of their rotational joint axes is just k. By $\tau(u)$ or $\tau_D(u)$ we will denote the subspace of all translating twists in $VT(u)$ or the subspace of their directions in E_3 respectively.

4.1. **(3,0)-robots.** All joints of a $(3,0)$-robot are translating and thus $AC(u)=0$ and $VT(u)$ is a subalgebra, $\dot{Y}_C(u)=0$. A motion of $(3,0)$-robot is geodesic iff $\ddot{u}^1(t)=0$, $\ddot{u}^2(t)=0$, $\ddot{u}^3(t)=0$; i.e., iff this motion is uniform. As $rank(KL|_{VT})=0$ therefore for the Klein space it holds $K(u)=VT(u)$ and $VT(u)$ is isotropic.

4.2. **(3,1)-robots.** These robots are described in detail in the paper [2]. Therefore we give here a short survey of properties of these robots only.

Let us remind that in the case of the $(3,1)$-robots $\rho(V)$ lies on a 4-dimensional subgroup generated by a 4-dimensional subalgebra spanned on VT and all translating twists in $e(3)$. This fact has no influence on our considerations.

All rotational joint axes of the $(3,1)$-robot are parallel at any position (u). Therefore every velocity twist Y_i, $i=1,2,3$, is of the form $Y_i=(\delta_i\overline{\omega},\overline{m}_i)$, $\delta_i\in\mathbb{R}$. Neglecting ordering there are three cases: $\{Y_1=(\overline{\omega},\overline{m}_1),Y_2=(\overline{0},\overline{m}_2),Y_3=(\overline{0},\overline{m}_3)\}$, $\{Y_1=(\overline{\omega},\overline{m}_1),Y_2=(\overline{\omega},\overline{m}_2),Y_3=(\overline{0},\overline{m}_3)\}$, $\{Y_1=(\overline{\omega},\overline{m}_1),Y_2=$

$(\overline{\omega}, \overline{m}_2), Y_3 = (\overline{\omega}, \overline{m}_3)\}$, $\overline{\omega}\cdot\overline{m}_i = 0$. It means that in $VT(u)$ there are always velocity elements $B_1 = (\overline{s}, \overline{b}_1), B_2 = (\overline{0}, \overline{b}_2), B_3 = (\overline{0}, \overline{b}_3)$, $\overline{s}\cdot\overline{b}_i = 0$, $i = 1, 2, 3$, such that $VT(u) = span(B_1, B_2, B_3)$. Now $AC(u) = span([B_1, B_2] = (\overline{0}, \overline{s}\times\overline{b}_2), [B_1, B_3] = (\overline{0}, \overline{s}\times\overline{b}_3), [B_2, B_3] = (\overline{0}, \overline{0}))$, $\tau = span(B_2, B_3)$, $\tau_D = span(\overline{b}_2, \overline{b}_3)$ and the equations (5_1) and (5_2) are of the forms $\left[(\overline{s}\cdot\overline{b}_2)t_2 + (\overline{s}\cdot\overline{b}_3)t_3\right] t_1 = 0$ and $t_1 = 0$. From this we infer the following properties.

Proposition 1. Let ρ be a $(3, 1)$-robot. Then

(1) A position (u) is singular iff $\dim\tau_D = 1$. At a singular position $VT(u)$ can not be a subalgebra.
(2) At a regular position $VT(u)$ is a subalgebra iff the direction $\overline{s}$ of the rotational joint axes is orthogonal to τ_D.
(3) The rank of $KL|_{VT(u)}$ is 0 or 2. The relation $rank\left(KL|_{VT(u)}\right) = 0$, i.e. $K(u) = VT(u)$, is true iff $\overline{s}$ is orthogonal to τ_D. There is not a helical twist in $VT(u)$ in this case. If $\overline{s}$ is not orthogonal to τ_D then rank of $KL|_{VT(u)}$ is 2, $\dim K(u) = 1$, $K(u) = span(\hat{Y})$, $\hat{Y} = (\overline{0}, \overline{m}) \in \tau$ where $\overline{m}$ is orthogonal to $\overline{s}$. In this case the axes of all rotational twists in $VT(u)$ form a bundle of the parallel straight lines the plane of which is orthogonal to the direction $\overline{m}$ of the Klein twist $\hat{Y}$.
(4) $\dim AC(u) = 1$ if and only if the direction of the rotational joint axes is parallel with τ_D or (u) is singular. In this case the asymptotic space $AC(u)\cap VT(u)$ is zero.
(5) If the direction of the rotational joint axes is not parallel with τ_D and (u) is regular then $\dim AC(u) = 2$ and $AC(u)$ is a subspace of all translating twists the directions of which are orthogonal to the rotational joint axes. If $AC(u) = \tau$ then $VT(u)$ is a subalgebra. If $AC(u) \neq \tau$ then $AC(u)\cap VT(u) = K(u) = span(\hat{Y})$ and thus a motion is asymptotic iff $\dot{Y}_C = \lambda\hat{Y}$.

We will specify these assertions for concrete cases of robots in the corollary form.

Corollary 2. For $(3, 1)$-robots with one rotational joint, (RTT, TRT, TTR), we get:

(a) A singular position there is only in the case TRT when $\angle(o_1, o_2) = \angle(o_3, o_2)$.
(b) The subspace $VT(u)$ is a subalgebra iff (u) is regular and the rotational joint axis is orthogonal to the prismatic joint axes.
(c) Nontrivial asymptotic motions there are in two cases:
  c_1) In the cases RTT, TTR when all joint axes are complanar, the rotational joint works and the ratio of the prismatic joint velocities is equal to the ratio of the corresponding coordinates of the unit vector of the angular velocity (of the rotational joint) in the base formed by the unit direction vectors of the prismatic joint axes.

c$_2$) If the joint axes are not complanar and $VT(u)$ is not a subalgebra then $AC(u) \cap VT(u) = K(u) = span(\hat{Y})$ and just the motions satisfying the equation $\dot{Y}_C = \lambda \hat{Y}$, $\lambda \neq 0$ are nt-asymptotic.

Corollary 3. In the case of (3,1)-robots with two rotational joints, RRT, RTR, TRR, we have:

(a) The subspace τ_D is spanned on a direction vector of the prismatic joint axis and a normal vector of the plane ξ of the rotational joint axes. Thus a position (u) is singular iff the prismatic joint axis is orthogonal to the plane ξ.

(b) $VT(u)$ is a subalgebra iff the prismatic joint axis is orthogonal to the rotational joint axes but it is not orthogonal to the plane ξ.

(c) The rotational joint axes are parallel with τ_D at any position (u) iff all joint axes are parallel. There are only t-asymptotic motions in this case.

(d) A nt-asymptotic motion there is only in the case when the rotational joint axes are not parallel with τ_D and $VT(u)$ is not a subalgebra. It is determined by the equation $\dot{Y}_C = \lambda \hat{Y}$, $\lambda \neq 0$, $AC(u) \cap VT(u) = K(u) = span(\hat{Y})$.

Corollary 4. At a (3,1)-robot with three rotational joints, RRR, we have. The subspace τ_D is spanned on the normal vectors of the planes $\xi_1 = (o_1, o_2)$, $\xi_2 = (o_1, o_3)$ and thus $VT(u)$ is always subalgebra excepting singular positions when $\xi_1 = \xi_2$. There are not nt-asymptotic motions.

4.3. **(3,2)-robots.** Excluding helical joints there are the cases: RRT, RTR, TRR, RRR. A robot RRR is a $(3,2)$-robot at any position only in two cases: $o_1 \parallel o_2$ or $o_3 \parallel o_2$. At any position (u) there are at least two rotational joints with non parallel axes and thus in every space $VT(u)$ there are such velocity twists $B_1 = (\overline{s}_1, \overline{b}_1)$, $\overline{s}_1 \cdot \overline{b}_1 = 0$, $B_2 = (\overline{s}_2, \overline{b}_2)$, $\overline{s}_2 \cdot \overline{b}_2 = 0$, $\overline{s}_1 \times \overline{s}_2 \neq \overline{0}$, $B_3 = (\overline{0}, \overline{b}_3 \neq \overline{0})$ that $VT(u) = span(B_1, B_2, B_3)$, $\tau = span(B_3)$, $\tau_D = span(\overline{b}_3)$, $AC(u) = span([B_1, B_2] = (\overline{s}_1 \times \overline{s}_2, \overline{s}_1 \times \overline{b}_2 + \overline{s}_2 \times \overline{b}_1, [B_1, B_3] = (\overline{0}, \overline{s}_1 \times \overline{b}_3), [B_2, B_3] = (\overline{0}, \overline{s}_2 \times \overline{b}_3))$.

Proposition 2. In the case of a $(3,2)$-robot there is not a singular position and $VT(u)$ can not be a subalgebra.

Proof. As $\overline{s}_1 \times \overline{s}_2 \neq \overline{0}$ then $\overline{s}_1 \times \overline{s}_2 \notin span(\overline{s}_1, \overline{s}_2)$ and thus $VT(u)$ can not be a subalgebra. A position (u) is singular iff $\overline{b}_3 = \overline{0}$, i.e. $\tau_D = \overline{0}$. It is impossible at RRT, RTR, TRR and also at RRR as $o_1 \neq o_2$, $o_2 \neq o_3$. $\square$

Lemma 3. At any three-parametric robot if $\dim AC(u) = 3$ and $AC(u) \cap VT(u) = \overline{0}$ then a motion is asymptotic iff it is an u^i-motion, $i = 1, 2, 3$.

Proof. If $AC(u) \cap VT(u) = \overline{0}$ then a motion is asymptotic iff $\overline{0} = \dot{Y}_C(u) = \sum_{i<j}^{3} \dot{u}^i \dot{u}^j [Y_i, Y_j]$. As $\dim AC(u) = 3$ then $\dot{Y}_C = \overline{0}$ iff $\dot{u}^1 \dot{u}^2 = 0$, $\dot{u}^1 \dot{u}^3 = 0$, $\dot{u}^2 \dot{u}^3 = 0$. It completes our proof. $\square$

Let $\Omega(u)$ denote the vector space spanned on the direction vectors of all rotational joint axes at a position (u).

Lemma 4. At any $(3,2)$-robot we have: $\dim AC(u) < 3$ iff $\tau_D(u) \subset \Omega(u)$.

Proof. $\dim AC(u) < 3$ iff $\overline{0} = (\overline{s}_1 \times \overline{b}_3) \times (\overline{s}_2 \times \overline{b}_3)$; i.e., iff $(\overline{s}_1 \times \overline{s}_2) \cdot \overline{b}_3 = 0$. The proof is finished. $\square$

Proposition 3. At any $(3,2)$-robot the equation $AC(u) \cap VT(u) = \overline{0}$ is valid.

Proof. he relation $k_1[B_1, B_2] + k_2[B_1, B_3] + k_3[B_2, B_3] = c_1 B_1 + c_2 B_2 + C_3 B_3$ is true iff $k_1 \overline{s}_1 \times \overline{s}_2 = c_1 \overline{s}_1 + c_2 \overline{s}_2$, $k_1(\overline{s}_1 \times \overline{b}_2 - \overline{s}_2 \times \overline{b}_1) + k_2 \overline{s}_1 \times \overline{b}_3 + k_3 \overline{s}_2 \times \overline{b}_3 = c_1 \overline{b}_1 + c_2 \overline{b}_2 + c_3 \overline{b}_3$; i.e., iff $k_1 = 0$, $c_1 = 0$, $c_2 = 0$ and $(k_2 \overline{s}_1 + k_3 \overline{s}_2) \times \overline{b}_3 = c_3 \overline{b}_3$. Evidently $\dim AC(u)$ is 2 or 3. If $\dim AC(u) = 2$ then we can suppose that $AC(u) = span([B_1, B_2], [B_1, B_3])$; i.e.; $k_3 = 0$. Then $k_2 \overline{s}_1 \times \overline{b}_3 = c_3 \overline{b}_3$ is true iff $k_2 = 0$, $c_3 = 0$. In the case $\dim AC(u) = 3$ the equation $(k_2 \overline{s}_1 + k_3 \overline{s}_2) \times \overline{b}_3 = c_3 \overline{b}_3$ is equivalent to $k_2 = k_3 = c_3 = 0$. It completes our proof. $\square$

Corollary 5. At any $(3,2)$-robot we have if $\dim AC(u) = 3$ then by *Proposition 3* and *Lemma 4* only u^i-motions are asymptotic.

In the case $\tau_D(u) \subset \Omega(u)$, $\overline{b}_3 = c_1 \overline{s}_1 + c_2 \overline{s}_2$, we compute successively:

$$(6_1) \qquad \begin{gathered} RRT : Y_1 = B_1, Y_2 = B_2, Y_3 = B_3, \\ \dot{Y}_C = \dot{u}^1 \dot{u}^2 [B_1, B_2] + \dot{u}^3 (c_2 \dot{u}^1 - c_1 \dot{u}^2)(\overline{0}, \overline{s}_1 \times \overline{s}_2), \end{gathered}$$

$$(6_2) \qquad \begin{gathered} RTR : Y_1 = B_1, Y_2 = B_3, Y_3 = B_2, \\ \dot{Y}_C = \dot{u}^1 \dot{u}^3 [B_1, B_2] + \dot{u}^2 (c_2 \dot{u}^1 + c_1 \dot{u}^3)(\overline{0}, \overline{s}_1 \times \overline{s}_2), \end{gathered}$$

$$(6_3) \qquad \begin{gathered} TRR : Y_1 = B_3, Y_2 = B_1, Y_3 = B_2, \\ \dot{Y}_C = \dot{u}^2 \dot{u}^3 [B_1, B_2] + \dot{u}^1 (-c_2 \dot{u}^2 + c_1 \dot{u}^3)(\overline{0}, \overline{s}_1 \times \overline{s}_2), \end{gathered}$$

$$(6_4) \qquad \begin{gathered} RRR, o_1 \parallel o_2 : Y_1 = B_1, Y_2 = B_3 + B_1, Y_3 = B_2, \\ \dot{Y}_C = \dot{u}^3 (\dot{u}^1 + \dot{u}^2)[B_1, B_2] + \dot{u}^2 (c_2 \dot{u}^1 + c_1 \dot{u}^3)(\overline{0}, \overline{s}_1 \times \overline{s}_2), \end{gathered}$$

$$(6_5) \qquad \begin{gathered} RRR, o_2 \parallel o_3 : Y_1 = B_1, Y_2 = B_2, Y_3 = B_3 + B_2, \\ \dot{Y}_C = \dot{u}^1 (\dot{u}^2 + \dot{u}^3)[B_1, B_2] + \dot{u}^3 (c_2 \dot{u}^1 - c_1 \dot{u}^2)(\overline{0}, \overline{s}_1 \times \overline{s}_2). \end{gathered}$$

In the case of RRR, $\tau_D = span(\overline{b}_3)$ is orthogonal to the plane ξ of the parallel joint axes. Evidently the relation $\tau \subset \Omega(u)$ at RRT, RTR, TRR means complanarity of all joint axes. This property is preserved at all positions in a RTR-robot but in the case of robots RRT, TRR only iff the prismatic joint axis is parallel with the axis of the neighboring rotational joint. In the case of robots RRT, TRR there always is a position $u_2 = \tilde{u}_2$ so that $\tau_D(\tilde{u}) \subset \Omega(\tilde{u})$ at $(\tilde{u}) = (u_1, \tilde{u}_2, u_3)$. At RRR-robots $\overline{b}_3 = \overline{n}_\xi$ is a normal vector of the plane ξ. Therefore at RRR-robots the relation $\tau_D \subset \Omega(u)$ is true at a position $u_2 = \overset{\circ}{\rightarrow} u_2$ when the joint axis which is not parallel with o_2 is complanar with the plane ξ. This parameter $u_2 \overset{\circ}{=\!\!\rightarrow} u_2$ always exists but this property is not preserved. By *Proposition 3* a motion is asymptotic iff $\dot{Y}_C = 0$. We conclude from the relations (6).

Proposition 4. The motions, when $\tau_D \subset \Omega(u)$ for all positions $(u(t))$, can not be nt-asymptotic. They can be only t-asymptotic: u^i-motions or motions when only the prismatic and the rotational joint with parallel axes work.

Corollary 6. At any (3,2)-robots there are not nontrivial asymptotic motions.

Let us turn to rotational velocity twists in $VT(u)$ and to the Klein form $KL|_{VT(u)}$.

The equation (5_1) for $Y = t_1 B_1 + t_2 B_2 + t_3 B_3$ to be rotational or translating have the form

$$(5_1') \qquad \begin{gathered} t_1 t_2(\overline{s}_1 \cdot \overline{b}_2 + \overline{s}_2 \cdot \overline{b}_1) + t_1 t_3(\overline{s}_1 \cdot \overline{b}_3) + t_2 t_3(\overline{s}_2 \cdot \overline{b}_3) = 0, \\ \det KL|_{VT(u)} = 2(\overline{s}_1 \cdot \overline{b}_2 + \overline{s}_2 \cdot \overline{b}_1)(\overline{s}_1 \cdot \overline{b}_3)(\overline{s}_2 \cdot \overline{b}_3). \end{gathered}$$

A twist Y is translating iff $Y \in \tau$; i.e., iff $t_1 = 0, t_2 = 0$. Y is rotational iff $t_1^2 + t_2^2 \neq 0$ and $(5_1')$ is satisfied. By *Lemma 2* the form $KL|_{VT(u)}$ is singular; i.e., $\det KL|_{VT(u)} = 0$, in two cases: a) two rotational joint axes are intersecting, b) at least one rotational joint axis is perpendicular to τ_D. Then $KL|_{VT(u)}$ is always singular at a (3,2)-robot RRR. Evidently the point of intersection of the neighboring rotational joint axes is preserved. At RTR the point of intersection of the axes o_1, o_3 is preserved iff the axes o_1, o_2, o_3 are complanar.

Let us recall that the equation $(5_1')$ determines in the projective plane $\mathcal{P}_2$ of all screws in $VT(u)$ a quadratic curve (q) on which the translating screw τ lies. Therefore the infinity axis of τ belongs to the set Q of all axes of the velocity screws from (q) and thus if $KL|_{VT(u)}$ is regular then Q is a system of straight lines in a hyperbolic paraboloid. If (q) is singular then the Klein space $K(u)$ is not zero. We describe the set Q using *Corollary 1*. Evidently the rank of $KL|_{VT(u)}$ is 0 or 2. The rank of $KL|_{VT(u)} = 0$ iff two rotational joint axes are intersecting and they are orthogonal to τ_D. In this case $K(u) = VT(u)$. There are no helical twists in $VT(u)$ and Q is the space of all straight lines in the plane orthogonal to τ_D. If the rank of $KL|_{VT(u)}$ is 2 then $\dim K(u) = 1$, $K(u) = span(\hat{Y})$. There are two cases:

(a) $\hat{Y}$ is not translating. Then its axis $\hat{o}$ intersects or it is parallel with the rotational joint axes and it is orthogonal to τ_D. The set Q is decomposed on the bundle Q_1 and Q_2. The bundle Q_1 forms intersecting lines involving $\hat{o}$ and the rotational joint axis which is not orthogonal to τ_D. The bundle Q_2 forms parallel lines (including $\hat{o}$ and the rotational joint axis which is orthogonal to τ) the plane of which is orthogonal to τ_D.

(b) $\hat{Y}$ is translating, $\hat{Y} = (\overline{0}, \overline{m})$, $\tau_D = span(\overline{m})$. Then the axis of any rotational twist in $VT(u)$ is orthogonal to $\overline{m}$. The set Q is decomposed on two bundles Q_1, Q_2 of parallel straight lines the planes of which are orthogonal to τ_D.

4.4. **(3,3)-robots.** Let us analyze a singular position (u) of (3,3)-robots. The velocity joint twists Y_1, Y_2, Y_3 at a position (u) are of the forms: $Y_1 = (\overline{\omega}_1, \overline{0})$, $Y_2 = (\overline{\omega}_2, \overline{m}_2)$, $Y_3 = (\overline{\omega}_3, \overline{m}_3)$, $\overline{\omega}_2 \cdot \overline{m}_2 = 0$, $\overline{\omega}_3 \cdot \overline{m}_3 = 0$, $\overline{\omega}_1 \times \overline{\omega}_2 \neq \overline{0}$. Then $\dim VT(u) < 3$ iff $k_1\overline{\omega}_1 + k_2\overline{\omega}_2 + k_3\overline{\omega}_3 = \overline{0}$, $k_2\overline{m}_2 + k_3\overline{m}_3 = \overline{0}$, $k_1, k_2, k_3 \in \mathbb{R}$, $k_1^2 + k_2^2 + k_3^2 \neq 0$. As $\overline{\omega}_1 \times \overline{\omega}_2 \neq \overline{0}$ therefore $k_3 \neq 0$. The following cases are possible: a) $k_2 = 0$. Then $\overline{m}_3 = \overline{0}$, $\overline{\omega}_3 = k\overline{\omega}_1$ and thus $o_1 \equiv o_3$; i.e., at the rotation around o_2 the joint axis o_3 is identified with o_1. It is possible iff the axes o_1, o_3 lie on the same rotational cone or belong to the same system of straight lines of a one sheet rotational hyperboloid with axis o_2. b) $k_2 \neq 0$. Then $\overline{m}_3 = k\overline{m}_2$ and thus the plane (O, o_2) is identified with the plane (O, o_3). As $\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3$ are complanar then $o_1 \in (O, o_2)$ and o_1 intersect o_2. If we choose the origin $O = o_1 \cap o_2$ then $\overline{m}_2 = \overline{0} = \overline{m}_3$. It means that o_1, o_2, o_3 are intersecting at a common point. We conclude: if a (3,3)-robot all joint axes of which are intersecting at a common point or the axes o_1, o_3 belong to the same system of straight lines of a one sheet rotational hyperboloid the axis of which is o_2 then the (3,3)-robot has singular positions for some value $u^2 = \overline{u}^2$.

The equation (5_1) for a velocity twist $Y = t_1Y_1 + t_2Y_2 + t_3Y_3$ to be rotational or translating have the form

$$(5_1") \qquad \begin{aligned} t_1t_2(\overline{\omega}_1 \cdot \overline{m}_2) + t_1t_3(\overline{\omega}_1 \cdot \overline{m}_3) + t_2t_3(\overline{\omega}_2 \cdot \overline{m}_3 + \overline{\omega}_3 \cdot \overline{m}_2) = 0, \\ \det KL|_{VT(u)} = 2(\overline{\omega}_1 \cdot \overline{m}_2)(\overline{\omega}_1 \cdot \overline{m}_3)(\overline{\omega}_2 \cdot \overline{m}_3 + \overline{\omega}_3 \cdot \overline{m}_2). \end{aligned}$$

As the axes o_1, o_2 and o_2, o_3 can not be parallel then due to the equation (5_2) at a position (u) there is a translating twist in $VT(u)$ iff $\overline{\omega}_3 \in span(\overline{\omega}_1, \overline{\omega}_2)$; i.e., iff the axes o_1, o_2, o_3 are complanar. Then if $KL|_{VT(u)}$ is regular and $\overline{\omega}_3 \notin span(\overline{\omega}_1, \overline{\omega}_2)$ or $\overline{\omega}_3 \in span(\overline{\omega}_1, \overline{\omega}_2)$ then the set Q of all axes of all rotational twists in $VT(u)$ is a system of the straight lines on an one sheet hyperboloid or in a hyperbolic paraboloid respectively.

The form $KL|_{VT(u)}$ is singular; i.e., $\det KL|_{VT(u)} = 0$, iff at least two joint axes are intersecting. Its rank can be 0 or 2. It is zero in two cases: a) All joint axes have a common point (a spherical robot) b) The axis o_2 intersects o_1 and o_3 in two different points and the axis o_3 rotating around o_2 gets at a value $u^2 = \overset{\circ}{\rightarrow} u^2$ into plane (o_1, o_2) and intersects o_1. The rank of $KL|_{VT(u)}$ is two iff joint axes are intersecting excluding two of them. In this case the Klein space $K(u)$ is spanned on a twist $\hat{Y}$ and the axes of all rotational twists in $VT(u)$ form two bundles of the straight lines with the common axis $\hat{o}$ of $\hat{Y}$. If the joint axes are complanar at (u) then there is a twist $(\overline{0}, \overline{m}) \in VT(u)$ and thus one bundle is a bundle of the parallel straight lines of the plane which is orthogonal to $\overline{m}$.

Let us turn to the investigation of asymptotic motions. In the case when all joint axes have a common point (a spherical robot) all motions at the regular position are asymptotic. A singular motions can be asymptotic only if it is u^i-motion.

Let use analyze cases when o_1, o_2 or o_2, o_3 are intersecting and all axes have no a common point. We have the following cases:

a) $o_1 \cap o_2 = M$. We choose $M = O$ as the origin of $\mathcal{S}_0$. Then $Y_1 = (\overline{\omega}_1, \overline{0})$, $Y_2 = (\overline{\omega}_2, \overline{0})$, $Y_3 = (\overline{\omega}_3, \overline{m}_3)$, $\overline{m}_3 \neq \overline{0}$, $\overline{\omega}_3 \cdot \overline{m}_3 = 0$, $[Y_1, Y_2] = (\overline{\omega}_1 \times \overline{\omega}_2, \overline{0})$, $[Y_1, Y_3] = (\overline{\omega}_1 \times \overline{\omega}_3, \overline{\omega}_1 \times \overline{m}_3)$, $[Y_2, Y_3] = (\overline{\omega}_2 \times \overline{\omega}_3, \overline{\omega}_2 \times \overline{m}_3)$. Then $Y = t_1[Y_1, Y_2] + t_2[Y_1, Y_3] + t_3[Y_2, Y_3]$ lies in $VT(u)$ iff $Y = k_1Y_1 + k_2Y_2 + k_3Y_3$; i.e., iff the following equalities are valid:

$$k_1\overline{\omega}_1 + k_2\overline{\omega}_2 + k_3\overline{\omega}_3 = t_1\overline{\omega}_1 \times \overline{\omega}_2 + t_2\overline{\omega}_1 \times \overline{\omega}_3 + t_3\overline{\omega}_2 \times \overline{\omega}_3 \tag{7}$$

$$k_3\overline{m}_3 = t_2\overline{\omega}_1 \times \overline{m}_3 + t_3\overline{\omega}_2 \times \overline{m}_3 = (t_2\overline{\omega}_1 + t_3\overline{\omega}_2) \times \overline{m}_3. \tag{8}$$

We are interested in the case $Y \neq 0$ (we find asymptotic motions that are not u^i-motions). The vector scalar multiplication of (8) by $\overline{m}_3$ gives $k_3 = 0$. Then (8) is satisfied iff $\overline{m}_3 = k(t_2\overline{\omega}_1 + t_3\overline{\omega}_2)$, $0 \neq k \in \mathbb{R}$; i.e., iff the plane $\beta = (O, o_3)$ is orthogonal to the plane $\alpha = (o_1, o_2)$; i.e., iff the plane β includes the straight line p orthogonal to α and going cross O. Then o_3 intersects p or it is parallel with p. This property is preserved at a rotation around o_2 iff β is orthogonal to o_2. When β is not orthogonal to o_2 then the straight line o_3 creates at a rotation around o_2 a surface $\mathcal{F}$. If p intersects or does not intersect $\mathcal{F}$ then there is a value $u^2 = \tilde{u}^2$ such that o_3 intersects p or o_3 is parallel with p and thus β is orthogonal to α at $\tilde{u} = (u^1, \tilde{u}^2, u^3)$. At a motion $u^2 = \tilde{u}^2$, $\dot{u}^2 = 0$, it holds $\dot{Y}_C = \dot{u}^1\dot{u}^3[Y_1, Y_3]$. For $Y = [Y_1, Y_3]$; i.e., for $t_1 = 0$, $t_2 = 1$, $t_3 = 0$, the equations (7), (8) are satisfied; i.e., $[Y_1, Y_3] \in VT(\tilde{u})$, iff $k_1\overline{\omega}_1 + k_2\overline{\omega}_2 = \overline{\omega}_1 \times \overline{\omega}_3$, $\overline{\omega}_1 \times \overline{m}_3 = \overline{0}$. In this case $\overline{m}_3 = c\overline{\omega}_1$, (i.e., $\overline{\omega}_1 \cdot \overline{\omega}_3 = 0$), $k_1\overline{\omega}_2 \cdot \overline{\omega}_3 = 0$, $\overline{\omega}_2 \cdot \overline{\omega}_3 = 0$ and so o_1 is orthogonal to β, o_3 is orthogonal to α, $\overline{\omega}_1 \times \overline{\omega}_3$ is a direction vector of the axis $\hat{o} = \alpha \cap \beta$ of the Klein twist $\hat{Y}$. Let us analyze the case when β is orthogonal to o_2. Then β is always orthogonal to α, $\overline{m}_3 = k\overline{\omega}_2$ and the relation (8) is satisfied iff $k_3 = 0$, $t_2 = 0$. As $\hat{o} = \alpha \cap \beta$ then $\hat{Y} = (\hat{\overline{\omega}}, \overline{0})$, $\hat{\overline{\omega}} \cdot \overline{\omega}_2 = 0$, $\hat{\overline{\omega}} \in span(\overline{\omega}_1, \overline{\omega}_2)$, $\hat{\overline{\omega}} \in span((\overline{\omega}_1 \times \overline{\omega}_2), (\overline{\omega}_2 \times \overline{\omega}_3))$ as $\hat{o} \in \beta$. Then (7) is satisfied iff $k_1\overline{\omega}_1 + k_2\overline{\omega}_2 = \hat{\overline{\omega}} = t_1\overline{\omega}_1 \times \overline{\omega}_2 + t_3\overline{\omega}_2 \times \overline{\omega}_3$. It means that $AC(u) \cap VT(u) = span(\hat{Y}(u))$ for any (u). Then a motion is asymptotic iff $\dot{Y}_C = \lambda\hat{Y}$; i.e., iff $\dot{u}^1\dot{u}^2[Y_1, Y_2] + \dot{u}^1\dot{u}^3[Y_1, Y_3] + \dot{u}^2\dot{u}^3[Y_2, Y_3] = \lambda(c_1[Y_1, Y_2] + c_2[Y_2, Y_3])$, $c_1^2 + c_2^2 \neq 0$, i.e. iff $\dot{u}^1\dot{u}^2 = \lambda c_1$, $\dot{u}^1\dot{u}^3 = 0$, $\dot{u}^2\dot{u}^3 = \lambda c_2$. If $\lambda = 0$ then only u^i-motions can be asymptotic. When $\lambda \neq 0$ then we have the following cases: i) If $\dot{u}^1 = 0$ then $c_1 = 0$. Then $\hat{Y} = c_2[Y_2, Y_3]$; i.e., $\hat{\overline{\omega}} = c_2\overline{\omega}_2 \times \overline{\omega}_3$. Then o_3 is orthogonal to α. This position is possible only for a special value $u^2 = \tilde{u}^2$. Then $\dot{u}^2 = 0$ and $c_2 = 0$. Then $\hat{Y} = \overline{0}$. It is impossible. ii) If $\dot{u}^3 = 0$ then $c_2 = 0$. We get $\hat{Y} = c_1[Y_1, Y_2]$, i.e. $\hat{\overline{\omega}} = c_1\overline{\omega}_1 \times \overline{\omega}_2$. It is impossible. We summarize the previous results.

Proposition 5. Let o_1, o_2 be intersecting in $O = o_1 \cap o_2$ and $O \notin o_3$. Let o_3 does not lie in the plane orthogonal to o_2 cross O. Let $\hat{o}$ be the straight line in the plane $\alpha = (o_1, o_2)$ orthogonal to o_1 cross O. Then only if o_3 lies in the plane κ orthogonal to o_2 and its distance from o_2 is equal to $|MN|$, $M = \kappa \cap o_2$, $N = \kappa \cap \hat{o}$, there are asymptotic motions with non zero C-twists. They are given by the equation $u^2 = \tilde{u}^2$, $\dot{u}^2 = 0$ where $\tilde{u} = (u^1, \tilde{u}^2, u^3)$ is a position when $o_3(\tilde{u})$ is orthogonal to α. Their C-twists are of the form $\dot{Y}_C(\tilde{u}) = \dot{u}^1\dot{u}^3 k\tilde{Y}(\tilde{u})$ where $\tilde{Y}(\tilde{u})$ is the Klein twist in $VT(\tilde{u})$. If o_3 lies in the plane orthogonal to o_2 cross O then $VT(u) \cap AC(u) = K(u) = span(\hat{Y})$ at any (u) but there is not an asymptotic motion the C-twist of which is $\lambda\hat{Y}$, $\lambda \neq 0$.

b) In the case when o_2, o_3 are intersecting and o_2 does not intersect o_1 we have:

Proposition 6. Let o_2, o_3 be intersecting in $M = o_2 \cap o_3$ and o_1 does not intersect o_2. Let p be the straight line orthogonal to the plane $\alpha = (o_1, M)$ going cross M. Let $o_2 \neq p$. Then only in the case when o_1 is orthogonal to the plane $\beta = (o_2, o_3)$ and $\angle(o_2, o_3) = \angle(o_2, p)$ there are asymptotic motions with non zero C-twists. They are given by the equations $u^2 = \tilde{u}^2$, $\dot{u}^2 = 0$, $\dot{Y}_C = \dot{u}^1\dot{u}^3 k\hat{Y}$, $k \neq 0$ where $\tilde{u} = (u^1, \tilde{u}^2, u^3)$ is a position at which $o_3 = p$ and $\hat{Y}$ is the Klein twist. If $p = o_2$ then $AC(u) \cap VT(u) = span(\hat{Y}(u))$ at any (u) but an asymptotic motion the C-twist of which is $\lambda\hat{Y}$, $\lambda \neq 0$, there is iff o_1 is orthogonal to the plane (o_2, o_3). Its equation is $\dot{u}^3 = 0$.

Proof. Evidently $\alpha \cap \beta = \hat{o}$ is the axis of the Klein twist $\hat{Y} \neq \overline{0}$ and intersects o_1 at $N = o_1 \cap \hat{o}$. Now $\beta = (o_2, \hat{o})$. When the only second joint works the plane α is stable and the point N is moving along o_1. Evidently $VT(u) = span(Y_1, \hat{Y}, Y_2)$. We choose N as the origin of $\mathcal{S}_0$ at $u(t)$. Then $Y_1 = (\overline{\omega}_1, \overline{0})$, $\hat{Y} = (\hat{\overline{\omega}}, \overline{0})$, $Y_2 = (\overline{\omega}_2, \overline{m}_2)$, $\overline{\omega}_2 \cdot \overline{m}_2 = 0$, $\overline{m}_2 \neq \overline{0}$, $\hat{\overline{\omega}} \cdot \overline{m}_2 = 0$, $VT(u) = span(Y_1, \hat{Y}, Y_2)$, $[Y_1, \hat{Y}] = (\overline{\omega}_1 \times \hat{\overline{\omega}}, \overline{0})$, $[Y_1, Y_2] = (\overline{\omega}_1 \times \overline{\omega}_2, \overline{\omega}_1 \times \overline{m}_2)$, $[\hat{Y}, Y_2] = (\hat{\overline{\omega}} \times \overline{\omega}_2, \hat{\overline{\omega}} \times \overline{m}_2)$, $Y_3 = (\overline{\omega}_3, \overline{m}_3) = z_1\hat{Y} + z_2Y_2$, $\overline{m}_3 = z_2\overline{m}_2$, $z_1 \neq 0$. Solving the question of existence of non zero asymptotic twists $Y \in AC(u) \cap VT(u)$ we can repeat our considerations on the relation (7) and (8) where instead Y_2, Y_3 we put $\hat{Y}, Y_2$ respectively. If there are non zero asymptotic twists then the plane β must be orthogonal to α; i.e., the plane β includes the line p. There are the following cases: a) If $o_2 \neq p$ then there are only special values $u_2 = \tilde{u}_2$ when $\beta \equiv \xi = (o_2, p)$. At the position $\tilde{u} = (u^1, \tilde{u}^2, u^3)$ β is orthogonal to α and $\overline{m}_2 = y_1\overline{\omega}_1 + y_2\hat{\overline{\omega}}$. For the motions $u^2 = \tilde{u}^2$, $\dot{u}^2 = 0$ we have $\dot{Y}_C = \dot{u}^1\dot{u}^3[Y_1, Y_3]$. Then $\dot{Y}_C$ is asymptotic iff $[Y_1, Y_3] = k_1Y_1 + k_2\hat{Y} + k_3Y_2$; i.e., iff $z_1(\overline{\omega}_1 \times \hat{\overline{\omega}}) + z_2(\overline{\omega}_1 \times \overline{\omega}_2) = k_1\overline{\omega}_1 + k_2\hat{\overline{\omega}} + k_3\overline{\omega}_2$, $z_2\overline{\omega}_1 \times \overline{m}_2 = k_3\overline{m}_2$. The last relation is true iff $k_3 = 0$, $z_2 = 0$ or $k_3 = 0$, $\overline{\omega}_1 = k\overline{m}_2$, $0 \neq k \in \mathbb{R}$. In the former, $(o_3 \equiv \hat{o})$, the first relation $z_1(\overline{\omega}_1 \times \hat{\overline{\omega}}) = k_1\overline{\omega}_1 + k_2\hat{\overline{\omega}}$ can not be satisfied. In the further case, scalar multiplication of the firs relation by $\overline{\omega}_1 = k\overline{m}_2$ gives $k_1 = 0$. Then $\overline{\omega}_1 \times (z_1\hat{\overline{\omega}} + z_2\overline{\omega}_2) = \overline{\omega}_1 \times \overline{\omega}_3 = k_2\hat{\overline{\omega}}$ is satisfied iff $\overline{\omega}_3$ is orthogonal to α. Now $\dot{Y}_C = \dot{u}^1\dot{u}^3 k_2\hat{Y} \in VT(\tilde{u})$. It means that if $p \neq o_2$ then only in the case when $\overline{\omega}_1$ is orthogonal to β and $\angle(o_2, o_3) = \angle(o_2, p)$ there are positions $\tilde{u} = (u^1, \tilde{u}^2, u^3)$ at which $\overline{0} \neq \dot{Y}_C(\tilde{u}) \in VT(\tilde{u})$. Let us analyse the case b) $o_2 = p$. Now β is always orthogonal to α; i.e., $\overline{m}_2 \in span(\overline{\omega}_1, \hat{\overline{\omega}})$. The relations (7), (8) in the base $Y_1, \hat{Y}, Y_2$ are of the form:

$$(7') \qquad k_1\overline{\omega}_1 + k_2\hat{\overline{\omega}} + k_3\overline{\omega}_2 = t_1\overline{\omega}_1 \times \hat{\overline{\omega}} + t_2\overline{\omega}_1 \times \overline{\omega}_2 + t_3\hat{\overline{\omega}} \times \overline{\omega}_2$$

$$(8') \qquad k_3\overline{m}_2 = t_2\overline{\omega}_1 \times \overline{m}_2 + t_3\hat{\overline{\omega}} \times \overline{m}_2 = (t_2\overline{\omega}_1 + t_3\hat{\overline{\omega}}) \times \overline{m}_2$$

The relation $(8')$ is true iff $k_3 = 0$, $t_2\overline{\omega}_1 + t_3\hat{\overline{\omega}} = k\overline{m}_2$. Multiplying the equation $(7')$ by $\overline{\omega}_2$ we get $t_1 = 0$. Then $(7')$ is of the form $k_1\overline{\omega}_1 + k_2\hat{\overline{\omega}} = k\overline{m}_2 \times \overline{\omega}_2$. Let us multiply the last equation by $\overline{m}_2$. As $\overline{\omega}_1 \cdot \overline{m}_2 \neq 0$ we have $k_1 = 0$ and then $k_2\hat{\overline{\omega}} = k\overline{m}_2 \times \overline{\omega}_2$. It means that $VT(u) \cap AC(u) = span(\hat{Y}(u))$ at any (u). A motion is asymptotic iff $\dot{Y}_C = \lambda\hat{Y}$; i.e., iff $(\dot{u}^1\dot{u}^2 + z_2\dot{u}^1\dot{u}^3)[Y_1, Y_2] + \dot{u}^1\dot{u}^3 z_1[Y_1, \hat{Y}] - \dot{u}^2\dot{u}^3 z_1[\hat{Y}, Y_2] = \lambda(t_2[Y_1, Y_2] + t_3[\hat{Y}, Y_2]$; i.e., iff $\dot{u}^1\dot{u}^2 + z_2\dot{u}^1\dot{u}^3 = \lambda t_2$, $\dot{u}^1\dot{u}^3 z_1 = 0$, $-\dot{u}^2\dot{u}^3 z_1 = \lambda t_3$. If $\lambda = 0$ then only u^i-motions can be asymptotic. For $\lambda \neq 0$ we have following cases: i) $\dot{u}^1 = 0$, $t_2 = 0$. Then $t_3\hat{\overline{\omega}} = k\overline{m}_2$. It is impossible. ii) $\dot{u}^3 = 0$, $t_3 = 0$. Then $t_2\overline{\omega}_1 = k\overline{m}_2$; i.e., o_1 is orthogonal to β. It completes our proof. □

Finally we will deal with the case when any joint axes are not intersecting at a position (u). If (u) is a regular position then in $VT(u)$ there are velocity twists $B_1 = (\overline{s}_1, \overline{0}), B_2 = (\overline{s}_2, \overline{b}_2), B_3 = (\overline{s}_3, \overline{b}_3)$ such that $(\overline{s}_1, \overline{s}_2, \overline{s}_3)$ is an orthonormal base and $\overline{s}_1 \times \overline{s}_2 = \overline{s}_3$, $\overline{s}_1 \times \overline{s}_3 = -\overline{s}_2$, $\overline{s}_2 \times \overline{s}_3 = \overline{s}_1$. Let us remark that B_2, B_3 can be helical. Let $\overline{b}_i = b_i^1\overline{s}_1 + b_i^2\overline{s}_2 + b_i^3\overline{s}_3$, $i = 1, 2, 3$, $b_1^k = 0$, $k = 1, 2, 3$. We get $[B_1, B_2] = (\overline{s}_1 \times \overline{s}_2 = \overline{s}_3, \overline{s}_1 \times \overline{b}_2 = b_2^2\overline{s}_3 + b_2^3\overline{s}_2)$, $[B_1, B_3] = (\overline{s}_1 \times \overline{s}_3 = -\overline{s}_2, \overline{s}_1 \times \overline{b}_3 = b_3^2\overline{s}_3 - b_3^3\overline{s}_2)$, $[B_2, B_3] = (\overline{s}_2 \times \overline{s}_3 = \overline{s}_1, \overline{s}_2 \times \overline{b}_3 - \overline{s}_3 \times \overline{b}_2 = \overline{s}_1(b_3^3 + b_2^2) - b_2^1\overline{s}_2 - b_3^1\overline{s}_3)$. A velocity twist $Y = k_1B_1 + k_2B_2 + k_3B_3$ lies in the asymptotic space $A_3(u) \cap AC(u)$ iff there are real numbers t_1, t_2, t_3 such that $t_1[B_1, B_2] + t_2[B_1, B_3] + t_3[B_2, B_3] = k_1B_1 + k_2B_2 + k_3B_3$; i.e., iff $t_1 = k_3, t_2 = -k_2, t_3 = k_1$ and

$$(9) \qquad \begin{aligned} (b_2^2 + b_3^3)k_1 - b_2^1k_2 - b_3^1k_3 &= 0 \\ -b_2^1k_1 + (b_2^2 - b_3^3)k_2 + (-b_2^3 - b_3^2)k_3 &= 0 \\ -b_3^1k_1 + (-b_2^3 - b_3^2)k_2 + (b_2^2 - b_3^3)k_3 &= 0. \end{aligned}$$

Let D be the determinant of the system (9). If $D = 0$ then there is a non zero C-twist $\dot{Y}_C \in AC(u) \cap VT(u)$ in $VT(u)$. If $D \neq 0$ then $AC(u) \cap VT(u) = \overline{0}$. As $\dim AC(u) = 3$ then by *Lemma 3* only u^i-motions are asymptotic. The technical and geometrical interpretation of the condition $D = 0$ or $D \neq 0$ rest open.

Remark 8. As $\det KL|_{VT(u)} \neq 0$ then $K(u) = 0$; i.e., there is not a non zero Klein twist in $VT(u)$. If the axes o_1, o_2, o_3 are not complanar then there is not a translating twist in $VT(u)$ and the axes of all rotational velocity twists in $VT(u)$ form a system of the straight lines in a one sheet hyperboloid.

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REGULAR ORIENTED HYPERMAPS UP TO FIVE HYPERFACES

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Abstract. The chiral hypermaps with at most four hyperfaces were classified in [A. Breda d'Azevedo and R. Nedela, *Chiral hypermaps with few hyperfaces*, Math. Slovaca, 53 (2003), n.2, 107–128]. It arises from this classification that all chiral hypermaps are "canonical metacyclic", that is, the one-step rotation about a hypervertex, or about a hyperedge, or about a hyperface, generates a normal subgroup in the orientation-preserving automorphism group. In this paper we complete the above classification by classifying the reflexible regular oriented hypermaps with three and four hyperfaces, and extend the classification to five hyperfaces. The chiral hypermaps arising in this work will be either canonical metacyclic or coverings of canonical metacyclic hypermaps. All have metacyclic monodromy groups and cyclic chirality groups.

1. Introduction

Regular oriented hypermaps algebraically correspond to two-generated groups G with a prescribed couples of generators a and b. Geometrically they determine cellular embeddings of hypergraphs (bipartite graphs) in orientable compact and connected surfaces. Endowing the compact surface with an orbifold-induced metric, the edges of the bipartite map can be seen as geodesics. The genus of the compact surface is the *genus* of the hypermap. If $\mathcal{H} = (G; a, b)$ is a hypermap, the Euler characteristic of $\mathcal{H}$ (that is, the Euler characteristic of its underlying surface) is calculated according to the formula

$$\chi(\mathcal{H}) = V + E + F - |G|,$$

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where V is the number of orbits of $\langle b \rangle$ (the *hypervertices* of $\mathcal{H}$), E is the number of orbits of $\langle ab \rangle$ (the *hyperedges* of $\mathcal{H}$) and F is the number of orbits of $\langle a \rangle$ (the *hyperfaces* of $\mathcal{H}$), under the right action of $G = \langle a, b \rangle$ on itself. If ab is an involution, $\mathcal{H}$ is a *map*. A common central problem in the theory of maps/hypermaps has been the classification of regular oriented hypermaps either by size (order of G) [22, 25], by number of hyperfaces [26, 6], by underlying graph [18, 24, 12], by automorphism group [3], or by genus [14, 15, 16, 21]. As a consequence of the well known Hurwitz bound, the number of regular oriented hypermaps of genus $g > 1$ is finite and bounded by $84(g-1)$. Regular oriented hypermaps on the sphere are easily deduced from the Euler formula, viz. the five Platonic solids, two infinite families of types $(1, n, n)$ and $(2, 2, n)$, plus their duals. A classification of the regular oriented maps on the torus can be seen in Coxeter and Moser [11]. The generalisation to hypermaps was done by Corn and Singerman [10]. The classification problem for double torus was settled in [4] and for higher genera only partial results are known. Conder and Dobcsanyi [9], with computational support, classified all regular oriented maps[1] from genus 8 to 15, raising previous classifications of Sherk [21], Grek [14, 15, 17] and Garb [13]. On the other hand, Breda and Nedela [7] classified the chiral hypermaps of genus up to 4. According to this classification any chiral hypermap must have at least 3 hyperfaces. Chiral hypermaps are regular oriented hypermaps that are not isomorphic with their mirror images (see for instance [1, 6, 5, 11, 19, 23]). We say that a regular oriented hypermap $\mathcal{H} = (G; a, b)$ is *canonical metacyclic* if the rotation one-step about a hyperface, or about a hypervertex or about a hyperedge, generates a normal cyclic subgroup of G (the automorphism group of $\mathcal{Q}$). This is equivalent to say that a rotation one-step about a hyperface (or a hyperedge or a hypervertex) fixes all the hyperfaces (resp. all the hyperedges or all the hypervertices). The monodromy (or automorphism) group of a canonical metacyclic hypermap is a metacylcic group, but the converse is not true. One feature standing out from the classification [7] is that the chiral hypermaps with 3 and 4 hyperfaces are all canonical metacyclic.

The classification by number of faces appears most frequently inside other classifications. In [8] it was classified the *reflexible* hypermaps (the regular oriented hypermaps that are isomorphic to their mirror images) with one and two hyperfaces. In [6] we find a classification of chiral hypermaps up to 4 hyperfaces and in [26] a classification of the non-orientable reflexible maps with a prime number of faces and the non-orientable reflexible hypermaps with 1, 2, 3 and 5 hyperfaces. With 4 hyperfaces only a partial result has been established. Non-orientable reflexible hypermaps $\mathcal{H}$ are regular hypermaps on non-orientable compact surfaces - these correspond to groups G with prescribed involutory triples of generators

[1] By the time we have finished writing this paper Conder released in his web-homepage a computer-classification of all regular and chiral hypermaps up to genus 101.

r_0, r_1, r_2 such that the even words r_0r_1 and r_1r_2 still generate G. In this settlement, hypervertices, hyperedges and hyperfaces correspond to orbits of $\langle r_1, r_2\rangle$, $\langle r_2, r_0\rangle$ and $\langle r_0, r_1\rangle$. If r_0r_1 and r_1r_2 generate instead a proper subgroup G^+ (necessarily a normal subgroup of index 2 in G), the reflexible hypermap is orientable and therefore accomplished by the reflexible regular oriented hypermap $\mathcal{H}^+ = (G^+; r_0r_1, r_1r_2)$; in other words, both $\mathcal{H}$ and $\mathcal{H}^+$ determine the same hypergraph cellular embedding on the same compact orientable surface.

Contrary to hypermap theory, in map theory classifying by size is the same as classifying by the number of edges since the size of a regular oriented map with E number of edges is $2E$. This has been done extensively by Wilson [22] with a quasi complete classification of regular oriented maps up to 100 edges, now collected and better completed in the census [25]. More recently Orbanic′ [20] gave a classification of reflexible maps up to 100 edges which are not parallel-product decomposable. On the other hand, we can find a classification of reflexible hypermaps of size $2p$ (p prime) in [2] and a classification of non-orientable reflexible hypermaps of size a power of 2 in [26].

In this paper we complete the classification [6] by computing the reflexible hypermaps with 3 and 4 hyperfaces, and extend this classification to 5 hyperfaces. A complete list of the regular oriented hypermaps (up to duality) with at most 5 hyperfaces can be seen in the Table 1.1.

Most of the definitions and notations are borrowed from [6], where we can also find a more deep introduction to maps, hypermaps and chirality. For short, by a *reflexible hypermap* we mean a reflexible regular oriented hypermap (usually refered as *regular hypermap*).

# faces	extra relations $(\langle a, b \mid a^n = 1, \text{xtra relations}\rangle)$	κ	X
1	$b = a^s$ (cyclic group C_n)	1	1
2	$[a, b] = 1$, $b^2 = a^u$	1	1
2	$b^2 = (ab)^2 = 1$ (dihedral group D_n)	1	1
2	$(ab)^2 = 1$, $b^2 = a^{\frac{n}{2}}$ n even	1	1
3	$[a, b] = 1$, $b^3 = a^u$	1	1
3	$b^3 = a^u$, $bab^{-1} = a^t$ $n \geq 7$, $(t-1)u = 0 \bmod n$, $t^3 = 1 \bmod n$, $t \neq 1 \bmod n$	$\frac{n}{(n,t^2-1)}$	$\langle a^{t^2-1}\rangle$
3	$[a^2, b] = 1$, $b^3 = a^u$, $(ab)^2 = a^v$ n, u, v even and $3v - 2u = 6 \bmod n$	1	1
4	$b^4 = a^u$, $[a, b] = 1$	1	1
4	$b^4 = 1$, $bab^{-1} = a^{-1}$	1	1

4	$b^4 = a^{\frac{n}{2}},\ bab^{-1} = a^{-1}$ n even	1	1
4	$b^4 = a^u,\ bab^{-1} = a^t$ $n \geq 5,\ t^4 = 1 \bmod n,\ t^2 \neq 1 \bmod n,\ u(t-1) = 0 \bmod n$	$\frac{n}{(n,t^2-1)}$	$\langle a^{t^2-1} \rangle$
4	$[a^2, b^2] = 1, b^4 = a^u, (ab)^2 = a^v, b^{-2}ab^2 = a^t$ $n,\ u,\ v$ even, t odd, $2(t-1) = 0 \bmod n$ and $(2v-u-t-3)\frac{u}{2} = (2v-u-t-3)\frac{v}{2} = 0 \bmod n$ $(2v-u-t-3)\frac{t-1}{2} = 0 \bmod n$	1	1
4	$[a^3, b] = 1, b^3 = a^u, (ab)^2 = a^v$ $n,\ u,\ v = 0 \bmod 3$ and $-4u + 6v = 12 \bmod n$	1	1
4	$[a^3, b] = 1, b^3 = a^u, (ab)^3 = a^{3u+3v-3}, (ab^{-1})^2 = a^v$ $n,\ u,\ v = 0 \bmod 3$ and $4u + 6v = 12 \bmod n$	1	1
5	$[a, b] = 1,\ b^5 = a^u$	1	1
5	$b^5 = a^u,\ bab^{-1} = a^t$ $n \geq 5,\ t^5 = 1 \bmod n,\ t \neq 1 \bmod n,\ u(t-1) = 0 \bmod n$	$\frac{n}{(n,t^2-1)}$	$\langle a^{t^2-1} \rangle$
5	$b^5 = a^u, (ab)^2 = a^v, b^{-1}ab^{-1} = a^{3-v}$ $n,\ u,\ v$ even and $-2u + 5v = 10 \bmod n$	1	1
5	$[a^4, b] = 1, b^4 = a^{4t}, (ab)^2 = a^{2t+2}, b^2ab^{-1} = a^{t+1}$ $n = 0 \bmod 4$ and $t = 1 \bmod 4$	5	$\langle [a, b] \rangle$
5	$[a^4, b] = 1, b^5 = a^{5(t-1)}, b^2ab^{-1} = a^t$ $n = 0 \bmod 4$ and $t = 1 \bmod 4$	5	$\langle [a, b] \rangle$

Table 1.1: The regular oriented hypermaps (up to duality and a chiral pair) with 1, 2, 3, 4 and 5 hyperfaces ("faces" in the table) and their chirality indices κ and chirality groups X. Chirality index 1 means reflexible. All the information were collected from [6, 8] and this paper.

Apart from the brief introduction to the classification made in section 2 where the necessary tools and notation are given, the rest of the paper is organised in two sections, one dealing with reflexible hypermaps with 3 and 4 hyperfaces and the other with reflexible and chiral hypermaps with 5 hyperfaces. The sensation of repetition cannot be avoided at all since the relations that appear are different in each case.

2. Preamble to the classification

In what follows, let $\mathcal{Q} = (D; a, b)$ be a regular oriented hypermap of type (l, m, n) with n hyperfaces. We use the bipartite map representation of a hypermap: black and white vertices are the hypervertices and the hyperedges respectively, while faces are the hyperfaces. The set D is the set of *darts* (of the bipartite map), that is, a pair of hyperedge-hypervertex incident flags (see Fig. 1), where a *flag* is a (local) mutually incident triple hypervertex-hyperedge-hyperface (usually represented by a little triangle). The permutation a permutes the darts around

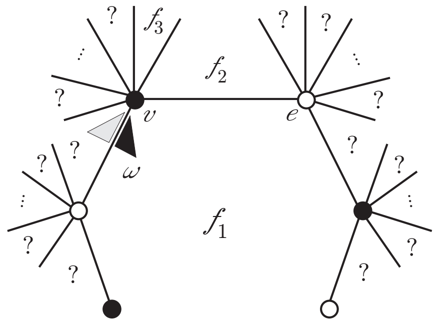

FIGURE 1

hyperfaces as local one-step counter-clockwise face-rotations, while permutation b permutes the darts around hypervertices as local one-step counter-clockwise vertex-rotations. The group Q generated by a and b is called the *monodromy* group of $\mathcal{Q}$ and often denoted by $Mon(\mathcal{Q})$. We have $\mathcal{Q} \cong (Q; a, b)$, where a and b "act" on Q by right multiplication (an *isomorphism* $(D; a, b) \longrightarrow (D'; a', b')$ is a bijective function $\phi : D \to D'$ such that $a\phi = \phi a'$ and $b\phi = \phi b'$). $\mathcal{Q}$ is *reflexible* if $\mathcal{Q}$ is isomorphic to its *mirror pair* $(D; a^{-1}, b^{-1})$. All actions in this paper are right actions.

Let r denote the number of hyperfaces about a hypervertex and s the number of hyperfaces about a hyperedge. As observed in [6], if $\mathcal{Q}$ has 3 or more hyperfaces then $r, s \geq 2$ and one of the following possibilities must occur, either $r \geq 3$ or $s \geq 3$. Up to a $(0, 1)$-duality we assume that $r \geq s$. By a σ-duality we mean the duality operation that the permutation $\sigma \in S_3$ induces by interchanging the role of the hypervertices (0-cells), hyperedges (1-cells) and hyperfaces (2-cells); so $(0, 1)$-duality just changes hypervertices with hyperedges resulting most of the times a new hypermap.

Let f_1 be a hyperface of $\mathcal{Q}$, v be a hypervertex incident to f_1, e be a hyperedge incident to both v and f_1, and f_2 be the hyperface incident to f_1, v and e. Label the other hyperfaces of $\mathcal{Q}$ as f_3, f_4, ... , f_n. The monodromy group $Mon(\mathcal{Q})$ of a regular oriented hypermap acts regularly on the darts as well as on half of the flags. Fix a dart's root-flag ω (the black flag pictured below) which we identify with the identity of $Mon(\mathcal{Q})$. Acting $Mon(\mathcal{Q})$ on ω, each dart will be marked with a root-flag.

Each element $\gamma \in Mon(\mathcal{Q})$ induces an automorphism φ_γ of $\mathcal{Q}$ by sending each dart $g \in G$ to the dart $\gamma^{-1}g$. The regularity of $\mathcal{Q}$ implies that each automorphism ϕ of $\mathcal{Q}$ is of the form φ_γ for some $\gamma \in G$. Moreover, $Aut(\mathcal{Q}) \cong Q$ and $\mathcal{Q} \cong (Aut(\mathcal{Q}); \varphi_a, \varphi_b)$. Since we have assigned the identity of $Mon(\mathcal{Q})$ to a fixed root-flag of f_1, the automorphism φ_a is a one-step clockwise rotation about the hyperface f_1 while φ_b is a one-step clockwise rotation about the hypervertex v.

The action of $Aut(\mathcal{Q})$ on the hyperfaces induces a permutation of hyperfaces π_g by assigning each hyperface f of $\mathcal{Q}$ to $f\varphi_\gamma$ (functions are written on the right). The automorphisms φ_a and φ_b induce the permutations π_a and π_b. Of course the composition $\varphi_a\varphi_b = \varphi_{ab}$ induces the permutation π_{ab}. For convenience let $A = \pi_a^{-1}$ and $B = \pi_b^{-1}$. By the way f_1, v and e were chosen, the first cycles of B and AB (that is, the cycles containing $1 = f_1$) are (123...) and (12...) respectively. The length of the first cycles of B and AB are r and s respectively. These two permutations must generate a transitive group on $\{1, 2, 3, .., n\}$. Hence the Schreier like diagram[2] induced by B and AB, the $B - AB$ *diagram*, must be connected. A relabelling of f_3, f_4 , ... , f_n produces a new permutation pair $(B', A'B')$ which we call a *relabelling pair.* Relabelling pairs correspond to conjugation pairs $(B^g, (AB)^g)$ by a permutation $g \in S_n$ centralising B and such that $(AB)^g$ sends 1 to 2. Let P be the group generated by A and B, and let $\mathcal{P}$ be the regular oriented hypermap $(P; A^{-1}, B^{-1})$. The permutations A^{-1} and B^{-1} when acting on the right (by right multiplication) give the generators of the monodromy group of $\mathcal{P}$ while when acting on the left (by left multiplication) give the generators of the automorphism group of $\mathcal{P}$. Since $(Aut(\mathcal{Q}); \varphi_a, \varphi_b)$ covers $(P; \pi_a, \pi_b) = (P; A^{-1}, B^{-1})$ then the function $a \mapsto A^{-1}$ and $b \mapsto B^{-1}$ gives rise to a covering from $\mathcal{Q}$ to $\mathcal{P}$. If $(P; A^{-1}, B^{-1})$ is not isomorphic to $(P; A, B)$ then these two hypermaps form a chiral pair. The classification is done up to a chiral pair.

3. Regular oriented hypermaps with 3 and 4 hyperfaces

In this section we complete the classification [6] by analysing the cases that lead to reflexible hypermaps. In Table 3.1 we reprint the list of all possible enumerations of the hyperfaces (through the permutations B, AB and A) of a generic regular hypermap with 3 and 4 hyperfaces, up to a duality and a relabelling of hyperfaces. These permutations determine a regular oriented hypermap $\mathcal{P} = (P; A, B)$ and in this table we also display the associated *H-sequence* of $\mathcal{P}$, that is, a sequence $[l, m, n, V, E, F, |P|]$ formed by the type $(l; m; n)$, the number of hypervertices V, the number of hyperedges E, the number of hyperfaces F and the order $|P|$ of the group generated by A and B.

[2]The diagram whose vertices are 1,...,n, and whose edges reflects the action of A and AB on 1,...,n.

#hyperfaces	Case	B	AB	A	[	l	m	n	V	E	F	$\|P\|$	]
3	**I**	$(1,2,3)$	$(1,2)$	$(2,3)$	[	3	2	2	2	3	3	6	]
3	**II** ✓	$(1,2,3)$	$(1,2,3)$	$()$	[	3	3	1	1	1	3	3	]
4	**III**	$(1,2,3)$	$(1,2)(3,4)$	$(2,3,4)$	[	3	2	3	4	6	4	12	]
4	**IV**	$(1,2,3)$	$(1,2,4)$	$(2,4,3)$	[	3	3	3	4	4	4	12	]
4	**V**	$(1,2,3,4)$	$(1,2)(3,4)$	$(2,4)$	[	4	2	2	2	4	4	8	]
4	**VI** ✓	$(1,2,3,4)$	$(1,2,3,4)$	$()$	[	4	4	1	1	1	4	4	]

Table 3.1: List of all possible enumerations of the 3 and 4 hyperfaces.

Cases II and VI give rise mostly to chiral hypermaps. The chiral ones where classified in [6] so here we classify only the reflexible hypermaps.

Case I. In this case we have $a^n = 1$, $b^3 = a^u$, $(ab)^2 = a^v$ and $b^{-1}ab^{-1} = a^t$, for some n even and $u, v, t \in \{0, \ldots, n-1\}$. Let G be the group generated by a, b subject to these relations and let K be the subgroup generated by a. This has index three in G and so G is partitioned into 3 cosets K, Kb and Kb^2. From the relations we deduce that $Kba^i = Kb$ or Kb^2, according as i is even or odd. Since $Kba^u = Kba^v = Kb$ and $Kba^t = Kb^2$ we find that u and v are even and t is odd. From the second and third relations we derive $b^{-1}a^2b = ba^2b^{-1} = a^{v+t-1}$ which shows that a^2 commutes with b^2, and thus a^2 also commutes with b. Hence $v + t - 1 = 2 \mod n \Leftrightarrow t = 3 - v \mod n$. Then $a^{3-v} = b^{-1}ab^{-1} = b^2ab^2a^{-2u} = ba^{v-1}ba^{-2u} = a^{v-2}baba^{-2u} = a^{2v-3-2u}$ which gives $3v-2u-6 = 0 \mod n$. The last relation can then be replaced by $[a^2, b] = 1$ and we have

$$G = \langle a, b | a^n = 1,\, b^3 = a^u,\, (ab)^2 = a^v,\, [a^2, b] = 1\rangle\,,$$

where n, u and v are even and $3v - 2u - 6 = 0 \mod n$. We now show that these congruencies are enough to describe a group of order $3n$. As G is indexed by 3 even numbers, n, u and v, we rewrite G as $G^{\mathrm{I}}_{(n,u,v)}$. Consider the particular case $(n, u, v) = (n, 0, 2)$, which satisfies the congruency $3v - 2u - 6 = 0 \mod n$. Let now G be $G^{\mathrm{I}}_{(n,0,2)} = \langle a, b | a^n = 1,\, b^3 = 1,\, (ab)^2 = a^2,\, [a^2, b] = 1\rangle$. Changing generators $\alpha = a$, $\beta = ba^{-v+u+2}$, where u, v are even integers satisfying $3v - 2u - 6 = 0 \mod n$, we get $G^{\mathrm{I}}_{(n,u,v)}$. Hence $G^{\mathrm{I}}_{(n,u,v)}$ has $3n$ elements if and only if G has $3n$ elements. Consider the normal subgroup H of index 2 of G generated by b and a^2. The set $T = \{1, a\}$ is a transversal for H in G. By the Reidmaster-Schreier Rewriting Process, H is freely generated by $A = a^2$ and $B = b$ subject to the relations $A^{\frac{n}{2}} = 1$, $B^3 = A^{\frac{u}{2}}$ and $[A, B] = 1 \Leftrightarrow B^A = A$. Hence H is an abelian metacyclic group of order $3\frac{n}{2}$. Thus $|G| = 2|H| = 3n$ and so $|G^{\mathrm{I}}_{(n,u,v)}| = 3n$.

Case II gives a metacyclic group $G^{\mathrm{II}}_{n,u,t} = \langle a, b \mid a^n = 1, b^3 = a^u, a^b = a^t\rangle$ with $u(t-1) = 0 \mod n$ and $t^3 = 1 \mod n$, and in [6] it was shown that this induces a reflexible hypermap $\mathcal{H}^{\mathrm{II}}_{(n,u)} = (G^{\mathrm{II}}_{n,u,1}; a, b)$ when $t = 1 \mod n$. These two cases show,

Theorem 1. *Any reflexible hypermap with 3 hyperfaces is, up to a duality, isomorphic to* $\mathcal{H}^{\mathrm{I}}_{(n,u,v)} = (G^{\mathrm{I}}_n; a, ba^{u-v+2})$ *for some n, u and v even such that* $3v - 2u = 6 \mod n$*, where G^{I}_n is the group with presentation* $\langle a, b | a^n = 1,\, b^3 = 1,\, (ab)^2 = a^2,\, [a^2, b] = 1\rangle$*, or to* $\mathcal{H}^{\mathrm{II}}_{(n,u)} = (G^{\mathrm{II}}_{n,u,1}; a, b)$ *for some n and u, where $G^{\mathrm{II}}_{n,u,1}$ is the abelian metacyclic group (cyclic C_{3n} or a direct product $C_n \times C_3$) with presentation* $\langle a, b \mid a^n = 1, b^3 = a^u, a^b = a\rangle$.

The H-sequences of $\mathcal{H}^{\mathrm{I}}_{(n,u,v)}$ and $\mathcal{H}^{\mathrm{II}}_{(n,u)}$ are respectively $[\frac{3n}{(n,u)}, \frac{2n}{(n,v)}, n; (n,u), \frac{3}{2}(n,v), 3; 3n]$ and $[\frac{3n}{(n,u)}, \frac{3n}{(n,u+3)}, n; (n,u), (n,u+3), 3; 3n]$.

Case III. In this case $a^n = 1$, $b^3 = a^u$, $(ab)^2 = a^v$ and $b^{-1}a^2b^{-1} = a^t$, for some $n = 0 \mod 3$ and $u, v, t \in \{0, \ldots, n-1\}$. As bab^{-2} and $b^2a^2b^{-1}$ are elements of $K = \langle a\rangle$, any coset-word Kw can be reduced to one of 4 cosets, K, Kb, Kb^2 and Kb^2a. This means that the oriented monodromy group G of the hypermap corresponding to this case has presentation

$$\langle a, b \mid a^n = 1, b^3 = a^u, (ab)^2 = a^v, b^{-1}a^2b^{-1} = a^t\rangle$$

for some $n = 0 \mod 3$, $u, v, t \in \{0, \ldots, n-1\}$. Since $b^{-1}a^3b = ba^3b^{-1} = a^{v+t-1}$ then b^2 commutes with a^3 and since $b^3 \in Z(G)$, b commutes with a^3. From $Kba = Kb^2$ and $Kba^3 = Kb$ one has $Kba^i = Kb$, Kb^2 or Kba^2 according as $i = 0, 1$ or $2 \mod 3$. Since $Kba^u = Kba^v = Kb$ and $Kba^t = kb^2$ we get $u = 0 \mod 3$, $v = 0 \mod 3$ and $t = 1 \mod 3$. Using the relation $[a^3, b] = 1$ we deduce $b^{-1}a^2b^{-1} = b^{-1}a^{-1}b^{-1}a^3 = a^{4-v}$, that is, the relation 4 can be replaced by $[a^3, b] = 1$. From $b^{a^{-1}b} = b^{-1}aba^{-1}b = b^{-2}a^{v-2}b = a^{v-3}b^{-2}ab = a^{v-u-3}bab = a^{2v-u-4}$ we deduce that $a^u = a^{6v-3u-12}$. Thus

$$G = G^{\mathrm{III}}_{(n,u,v)} = \langle a, b \mid a^n = 1, b^3 = a^u, (ab)^2 = a^v, [a^3, b] = 1\rangle$$

with n, u, $v = 0 \mod 3$ and $4u - 6v + 12 = 0 \mod n$. To show that under these conditions $G_{(n,u,v)}$ has exactly $4n$ elements take the normal subgroup $H = \langle a^3\rangle$ which factors G onto A_4. Using the Schreier transversal $T = \{1, a, a^{-1}, b, b^{-1}, ab, ba, a^{-1}b, ba^{-1}, b^{-1}a, ab^{-1}, ab^{-1}a\}$ of H in G in the Reidmaster-Schreier Rewriting Process we get

$$H = \langle x \mid x^{\frac{n}{3}} = 1, x^{\frac{4u-6v+12}{3}} = 1\rangle\,.$$

Since $\frac{4u-6v+12}{3} = 0 \mod \frac{n}{3}$, H is a cyclic group of order $\frac{n}{3}$ and thus G has order $4n$. Denote by $\mathcal{H}^{\mathrm{III}}_{(n,u,v)}$ the family of reflexible regular oriented hypermaps $(G^{\mathrm{III}}_{(n,u,v)}; a, b)$, where n, u, $v = 0 \mod 3$, $6v - 4u = 12 \mod n$ and a, b generate the group $G^{\mathrm{III}}_{(n,u,v)}$ subject to the relations of the above presentation.

Case IV. In this case we have $a^n = 1$ and $b^{-3}, (ab^{-1})^2, ba^2b \in K = \langle a\rangle$. Thus this is just the ψ-dual of Case III where ψ is the automorphism of the free group $\Delta^+ = F(a,b)$ determined by $a \mapsto a$, $b \mapsto b^{-1}$. Denote by $\mathcal{H}^{\mathrm{IV}}_{(n,u,v)} = D_\psi(\mathcal{H}^{\mathrm{III}}_{(n,u,v)}) = (G^{\mathrm{III}}_{(n,u,v)}; a, b^{-1})$, where a, b generates $G^{\mathrm{III}}_{(n,u,v)}$ subject to the relations that defines this group.

Case V. Here $a^n = 1$, $b^4 = a^u$, $(ab)^2 = a^v$, $b^{-2}ab^2 = a^t$ and $b^{-1}ab^{-1} = a^w$ for some even n and $u, v, t, w \in \{0, \dots, n-1\}$. As before we let $K = \langle a\rangle$. As $Kba = Kb^{-1}$, $Kb^2a = Kb^2$ and $Kb^{-1}a = Kb$, the index of K in G is at most four, namely $G/_r H = \{K, Kb, Kb^2, Kb^{-1}\}$. Moreover $Kba^i = Kb$ or Kb^{-1}, according as i is even or odd. Thus $Kba^u = Kba^v = Kb$ and $Kba^w = Kb^{-1}$ implies that w is odd and u, v are even. From the third and the last equations we get $b^{-1}a^2b = a^{v+w-1} = ba^2b^{-1}$, thus $a^2 \rightleftharpoons b^2$. Then $b^{-1}ab^{-1} = b^{-2}a^{v-1}b^{-2} = a^vb^{-2}a^{-1}b^{-2} = a^{v-u}b^{-2}a^{-1}b^2 = a^{v-u-t}$ and so the last equation is equivalent to $[a^2, b^2] = 1$. As $w = v - u - t \mod n$, t is odd. Moreover $a^2 = b^{-2}a^2b^2 = a^{2t}$, thus $2(t-1) = 0 \mod n$. Now a^u and a^v are in the centre of G and as $b^{-1}a^{t-1}b = b^{-3}ab^2a^{-1}b = a^{-u}bab^2a^{-1}b = a^{-u+v-1}ba^{-1}b = a^{t-1}$, a^{t-1} is also in the centre of G. From the equality $b^{-1}a^2b = a^{2v-u-t-1}$ we deduce that $a^u = b^{-1}a^ub = a^{(2v-u-t-1)\frac{u}{2}}$, $a^v = b^{-1}a^vb = a^{(2v-u-t-1)\frac{v}{2}}$ and $a^{t-1} = b^{-1}a^{t-1}b = a^{(2v-u-t-1)\frac{t-1}{2}}$. The hypermaps in this case have monodromy group

$$G^{\mathrm{V}}_{(n,u,v,t)} \equiv G = \langle a, b | a^n = 1, b^4 = a^u, (ab)^2 = a^v, b^{-2}ab^2 = a^t, [a^2, b^2] = 1\rangle\,,$$

where n, u, v are even, t is odd, $2(t-1) = 0 \mod n$ and $(2v-u-t-3)\frac{u}{2} = (2v-u-t-3)\frac{v}{2} = (2v-u-t-3)\frac{t-1}{2} = 0 \mod n$. For any value of these parameters we always get a group of order $4n$. In fact, the normal closure $H = \overline{\langle a^2, b^2\rangle}$ in G factors G onto $C_2 \times C_2$. Considering the Schreier transversal $T = \{1, a, b, ab\}$ for H in G and applying the Reidmaster-Schreier Rewriting Process we get $H = \langle A, B \mid A^{\frac{n}{2}} = 1, B^2 = A^{\frac{u}{2}}, [A, B] = 1\rangle$, which is an abelian metacyclic group of order $\frac{n}{2}2 = n$. Consequently G has order $|G| = 4n$.

Let $\mathcal{H}^{\mathrm{V}}_{(n,u,v,t)}$ denote the reflexible hypermap $(G^{\mathrm{V}}_{(n,u,v,t)}; a, b)$ where a, b generate $G^{\mathrm{V}}_{(n,u,v,t)}$ subject to the above relations.

Recalling [6], case VI gives rise to a metacyclic group $G^{\mathrm{VI}}_{n,u,t} = \langle a, b \mid a^n = 1, b^4 = a^u, bab^{-1} = a^t\rangle$ with $t^4 = 1 \mod n$, and $u(t-1) = 0 \mod n$. This induces a reflexible hypermap $\mathcal{H}^{\mathrm{VI}}_{(n,u,t)} = (G^{\mathrm{VI}}_{n,u,t}; a, b)$ only when $t^2 = 1 \mod n$. This leads to two families of reflexible hypermaps, $\mathcal{H}^{\mathrm{VIa}}_{(n,u)} = \mathcal{H}^{\mathrm{VI}}_{(n,u,1)}$ and $\mathcal{H}^{\mathrm{VIb}}_{(n,u)} = \mathcal{H}^{\mathrm{VI}}_{(n,u,-1)}$, the first of which all its members are abelian while in the second (where $2u = 0 \mod n$) its members are abelian only when $n \leq 2$. This proves,

Theorem 2. *Any reflexible hypermap with 4 hyperfaces is, up to a duality, isomorphic to $\mathcal{H}^{\mathrm{III}}_{(n,u,v)}$, or $\mathcal{H}^{\mathrm{IV}}_{(n,u,v)}$, for some n, u, $v = 0 \mod 3$ and $6v - 4u = 12 \mod n$, or $\mathcal{H}^{\mathrm{V}}_{(n,u,v,t)}$ for some n, u, v even, t odd, $t = 1 \mod \frac{n}{2}$ and $(2v - u -$*

$t-3)\frac{u}{2} = (2v-u-t-3)\frac{v}{2} = (2v-u-t-3)\frac{t-1}{2} = 0 \mod n$, *or the abelian* $\mathcal{H}^{\mathrm{VIa}}_{(n,u)}$ *for some* n *and* u, *or* $\mathcal{H}^{\mathrm{VIb}}_{(n,u)}$ *for some* n *and* u *such that* $2u = 0 \mod n$.

The H-sequences of these hypermaps are displayed in the following table

$$\begin{array}{lcl}
\mathcal{H}^{\mathrm{III}}_{(n,u,v)} & : & [\frac{3n}{(n,u)}, \frac{2n}{(n,v)}, n\,;\, \frac{4}{3}(n,u), 2(n,v), 4\,;\, 4n] \\
\mathcal{H}^{\mathrm{IV}}_{(n,u,v)} & : & [\frac{3n}{(n,u)}, \frac{n}{(n,v-u-1)}, n\,;\, \frac{4}{3}(n,u), 4(n,v-u-1), 4\,;\, 4n] \\
\mathcal{H}^{\mathrm{V}}_{(n,u,v,t)} & : & [\frac{4n}{(n,u)}, \frac{2n}{(n,v)}, n\,;\, (n,u), 2(n,v), 4\,;\, 4n] \\
\mathcal{H}^{\mathrm{VIa}}_{(n,u)} & : & [\frac{4n}{(n,u)}, \frac{4n}{(n,u+4)}, n\,;\, (n,u), (n,u+4), 4\,;\, 4n] \\
\mathcal{H}^{\mathrm{VIb}}_{(n,0)} & : & [4, 4, n\,;\, n, n, 4\,;\, 4n] \\
\mathcal{H}^{\mathrm{VIb}}_{(n,\frac{n}{2})} & : & [8, 8, n\,;\, \frac{n}{2}, \frac{n}{2}, 4\,;\, 4n] \quad (n \text{ even})
\end{array}$$

4. Regular oriented hypermaps with five hyperfaces

The number of hyperfaces fixed by a rotation (as an automorphism) about a hyperface must divide 5 ([26], Corollary 13) and so it must be 1 or 4. This means that the permutation A has support $\{2,3,4,5\}$ or is the identity. Not counting relabelling pairs, and having into account the form of A, we easily find 21 possible pairs (B, AB) determining a connected $B - AB$ diagram and such that $r \geq s$ (Table 4.1), each pair defining a hypermap $\mathcal{P}$.

#	B	AB	A	[	l	m	n	V	E	F	$\|P\|$	]
1	$(1,2,3)$	$(1,2)(3,4,5)$	$(2,3,4,5)$	[	3	6	4	40	20	30	120	]
2	$(1,2,3)$	$(1,2,4)(3,5)$	$(2,4,3,5)$	[	3	6	4	40	20	30	120	]
3	$(1,2,3)(4,5)$	$(1,2)(3,4)$	$(2,3,5,4)$	[	6	2	4	20	60	30	120	]
4	$(1,2,3)(4,5)$	$(1,2,4)$	$(2,5,4,3)$	[	6	3	4	20	40	30	120	]
5	$(1,2,3)(4,5)$	$(1,2,4)(3,5)$	$(2,5)(3,4)$	[	6	6	2	20	20	60	120	]
6	$(1,2,3,4)$	$(1,2)(3,5)$	$(2,4,3,5)$	[	4	2	4	5	10	5	20	]
7	$(1,2,3,4)$	$(1,2)(4,5)$	$(2,4,5,3)$	[	4	2	4	5	10	5	20	]
8	$(1,2,3,4)$	$(1,2)(3,5,4)$	$(2,4)(3,5)$	[	4	6	2	30	20	60	120	]
9	$(1,2,3,4)$	$(1,2,5)$	$(2,5,4,3)$	[	4	3	4	30	40	30	120	]
10	$(1,2,3,4)$	$(1,2,4,5)$	$(2,3)(4,5)$	[	4	4	2	5	5	10	20	]
11	$(1,2,3,4)$	$(1,2,5,3)$	$(2,5)(3,4)$	[	4	4	2	5	5	10	20	]
12	$(1,2,3,4,5)$	$(1,2)$	$(2,5,4,3)$	[	5	2	4	24	60	30	120	]
13	$(1,2,3,4,5)$	$(1,2)(3,5)$	$(2,5)(3,4)$	[	5	2	2	2	5	5	10	]
14	$(1,2,3,4,5)$	$(1,2)(3,5,4)$	$(2,5,3,4)$	[	5	6	4	24	20	30	120	]
15	$(1,2,3,4,5)$	$(1,2,4)$	$(2,3)(4,5)$	[	5	3	2	12	20	30	60	]
16	$(1,2,3,4,5)$	$(1,2,4)(3,5)$	$(2,3,4,5)$	[	5	6	4	24	20	30	120	]
17	$(1,2,3,4,5)$	$(1,2,4,3)$	$(2,3,5,4)$	[	5	4	4	4	5	5	20	]
18	$(1,2,3,4,5)$	$(1,2,5,3)$	$(2,4,3,5)$	[	5	4	4	24	30	30	120	]
19	$(1,2,3,4,5)$	$(1,2,5,4)$	$(2,4,5,3)$	[	5	4	4	4	5	5	20	]
20	$(1,2,3,4,5)$	$(1,2,3,4,5)$	$()$	[	5	5	1	1	1	5	5	]
21	$(1,2,3,4,5)$	$(1,2,5,4,3)$	$(2,4)(3,5)$	[	5	5	2	12	12	30	60	]

Table 4.1: The 21 cases and the corresponding H-sequences of $\mathcal{P}$.

One observes from this table that most of the hypermaps $\mathcal{P}$ have more than 5 hyperfaces, a situation that cannot occur. This eliminate most of the items in the above table, leaving only 6 possible cases behind, namely the cases 6, 7, 13, 17, 19 and 20.

Looking at the action of $\langle B, AB\rangle$ on the hyperfaces $\{1,2,3,4,5\}$ and taking into account the regularity of $\mathcal{Q}$, the following word relations are common to all cases and are therefore omitted in Table 4.2:

$$a^n = 1\,, \quad \text{where } n = 0 \mod |A|\,;$$
$$b^r = a^u\,, \quad \text{for some } u \in \{0,\dots,n-1\}\,;$$
$$(ab)^s = a^v\,, \quad \text{for some } v \in \{0,\dots,n-1\}\,.$$

For each case we derive extra word relations (shown in Table 2). The letters u, v, x, y, z, w, t appearing in these relations are integers in $\{0,...,n-1\}$.

#	B	AB	A	Extra relations for $Mon(\mathcal{Q})$
6	(1,2,3,4)	(1,2)(3,5)	(2,4,3,5)	$b^2ab^{-1} = a^w, b^{-1}aba^{-1}b = a^t, b^{-1}a^2b^2 = a^z$
7	(1,2,3,4)	(1,2)(4,5)	(2,4,5,3)	$b^{-1}ab^2 = a^w, ba^{-1}bab^{-1} = a^t, b^2a^2b^{-1} = a^z$
13	(1,2,3,4,5)	(1,2)(3,5)	(2,5)(3,4)	$b^{-1}ab^{-1} = a^z, b^{-2}ab^{-2} = a^x, b^2ab^2 = a^y$
17	(1,2,3,4,5)	(1,2,4,3)	(2,3,5,4)	$b^{-1}ab^{-2} = a^x, b^{-2}ab = a^y, bab^2 = a^z, b^2ab^{-1} = a^t$
19	(1,2,3,4,5)	(1,2,5,4)	(2,4,5,3)	$b^{-2}ab^{-1} = a^x, bab^{-2} = a^y, b^2ab = a^z, b^{-1}ab^2 = a^t$
20	(1,2,3,4,5)	(1,2,3,4,5)	1	$bab^{-1} = a^t, b^2ab^{-2} = a^x, b^{-1}ab = a^y, b^{-2}ab^2 = a^z$

Table 4.2: The 6 cases with their extra relations.

Before we start with the classification we remark that four of these six regular oriented hypermaps form two chiral pairs. Denote by $\mathcal{P}^i$ the regular oriented hypermap corresponding to item i in table 4.1. In item 6 we have $\mathcal{P}^6 = (P^6; A^{-1}, B^{-1})$ where $B^{-1} = (1,4,3,2)$ and $A^{-1} = (2,5,3,4)$. Relabelling the hyperfaces according to the permutation $(2,4)$, we get $B^{-1} = (1,2,3,4)$, $A^{-1} = (2,4,5,3)$, which are the permutations B and A in line 7, and $\mathcal{P}^6 = (P^6; A, B)$ where $A = (2,4,5,3)$ and $B = (1,2,3,4)$ (line 7). This shows that $P^6 = P^7$. Its chiral pair is

$$chiral(\mathcal{P}^6) = (P^6; A^{-1}, B^{-1}) = (P^7; A^{-1}, B^{-1}) = \mathcal{P}^7\,.$$

It is not difficult to see that $\mathcal{P}^6$ is the toroidal chiral map $\{4,4\}_{2,1}$. Similarly $\mathcal{P}^{17}$ and $\mathcal{P}^{19}$ form a chiral pair. The hypermaps $\mathcal{P}^{13}$ and $\mathcal{P}^{20}$ are easily seen to be reflexible.

4.1. **The Reflexible hypermaps with five hyperfaces.** In this section we analyse the reflexible oriented hypermaps with 5 hyperfaces that appear in the cases 13 and 20. Only two families of reflexible hypermaps will arise and they are exhibited in Theorem 3. In the next section we will see that the remaining cases will give rise to chiral hypermaps.

Theorem 3. *If $\mathcal{H}$ is a reflexible hypermap with 5 hyperfaces (of valency $n > 0$) then, up to a $(0,1)$-duality and an isomorphism, $\mathcal{H}$ is either $\mathcal{H}^{13}_{n,u,v} = (G^{13}_n; a, ba^{u-2v+4})$ for some non-negative even numbers n, u, v (with $u, v < n$) such that $2u - 5v +$*

$10 = 0 \mod n$, *or* $\mathcal{H}^{20}_{n,u} = (G^{20}_{n,u}; a, b)$ *for some non-negative numbers* n *and* u. *Here* G^{13}_n *is the metacyclic group* $\langle a, b \mid a^n = b^5 = 1, a^{-1}ba = b^{-1}\rangle$ *and* $G^{20}_{n,u}$ *is the abelian metacyclic group* $\langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a\rangle$, *either a cyclic group* C_{5n} *or a direct product* $C_n \times C_5$.

The H-sequences of $\mathcal{H}^{13}_{(n,u,v)}$ and $\mathcal{H}^{20}_{(n,u)}$ are, respectively,

$$[\tfrac{5n}{(n,u)}, \tfrac{2n}{(n,v)}, n\,;\, (n,u), \tfrac{5}{2}(n,v), 5\,;\, 5n] \quad \text{and} \quad [\tfrac{5n}{(n,u)}, \tfrac{2n}{(n,u+5)}, n\,;\, (n,u), (n,u+5), 5\,;\, 5n]\,.$$

Proof. The hypermap $\mathcal{H}^{13}_{n,u,v}$ arises from case 13 while $\mathcal{H}^{20}_{n,u}$ arises from case 20. Actually this case gives rise to two families of regular oriented hypermaps with 5 hyperfaces (of valency n), one reflexible $\mathcal{H}^{20}_{n,u}$ and the other chiral $\mathcal{Q}^{20}_{n,u,t}$. We will skip the chiral part here and deal with it later in Theorem 4.

Case 13: Let G be the group with presentation

$$\langle a, b \mid a^n = 1, b^5 = a^u, (ab)^2 = a^v, b^{-1}ab^{-1} = a^z, b^{-2}ab^{-2} = a^x, b^2ab^2 = a^y\rangle\,,$$

where $u, v, z, x, y \in \{0, ..., n-1\}$ and $n = 0 \mod 2$. The monodromy group of $\mathcal{Q}$ is then a factor of G. Let K be the subgroup generated by a. One can see that K divides G into (no more than five) cosets K, Kb, Kb^2, Kb^3 and Kb^4, not necessarily distinct.

Clearly the elements b^5 and $(ab)^2 = (ba)^2$ are central in G. From the 3rd and 4th relations we get $a^{v-1}a^z = a^z a^{v-1} \Leftrightarrow ba^2b^{-1} = b^{-1}a^2b \Leftrightarrow b^2a^2 = a^2b^2$, that is, $a^2 \leftrightharpoons b^2$. Moreover, from the 2nd relation we get $b = a^x b^{-4}$, and so, $ba^2 = a^x b^{-4} a^2 = a^x a^2 b^{-4} = a^2 a^x b^{-4} = a^2 b$, which says that $b \leftrightharpoons a^2$. On the other hand, $Kba = Kb^{-1} = Kb^4$ (3th relation) and $Kba^2 = Kb$. Thus $Kba^i = Kb$ or Kb^4, according as $i = 0$ or $1 \mod 2$. Now, 4th and 2nd relations imply that $Kba^z = Kb^{-1} = Kb^4$, hence $z = 1 \mod 2$. The 5th relation is redundant, in fact, $b^{-2}ab^{-2} = b^{-1}(b^{-1}ab^{-1})b^{-1} = b^{-1}a^z b^{-1} = b^{-1}a^{z-1}ab^{-1} = a^{z-1}b^{-1}ab^{-1} = a^{2z-1}$. Also, $Kb^4a = Kb^{-1}a = Kb$ (from 4th and 2nd relations), hence $Kb^4a^i = Kb^4$ or Kb, according as $i = 0 \mod 2$ or $i = 1 \mod 2$, respectively. Then the 3rd and 2nd relations imply that $Kb^4a^{v-1} = Kb^{-1}a^{v-1} = Kb$, and so $v = 0 \mod 2$. But then the 6th relation is also redundant, $b^2ab^2 = b(bab)b = a^{v-2}bab = a^{2v-3}$.

Looking back at cosets, we also have $Kb^2a = Kb^{-2} = Kb^3$ (6th relation), therefore $Kb^2a^i = Kb^2$ or Kb^3, according as $i = 0$ or $1 \mod 2$, and $Kb^3a = Kb^{-2}a = Kb^2$ which gives $Kb^3a^i = Kb^3$ or Kb^2, according as $i = 0$ or $1 \mod 2$. Hence, the above relations are enough to reduce any word $Kw, w \in F(a, b)$, to one of the cosets, K, Kb, Kb^2, Kb^3 or Kb^4. Thus $|G| \le 5|K| = 5n$.

Now $u = 0 \mod 2$ since $Kb^4a^u = Kb^{-1}a^u = Kb^4$ (2nd relation). From $a^{v-1+z} = ba^2b^{-1} = a^2$ we get $a^{v+z} = a^3 \Leftrightarrow a^z = a^{3-v}$. For any integer $i \ge 0$, $b^iab^i = a^{i(v-2)+1}$. Then $a^{3-v} = b^{-1}ab^{-1} = a^{-u}b^4ab^4a^{-u} = a^{-2u}a^{4v-7} \Leftrightarrow$

$a^{2u-5v+10} = 1$, hence $2u - 5v + 10 = 0 \mod n$. Then

$$G = \overset{13}{G}_{n,u,v} = \langle a, b \mid a^n = 1, b^5 = a^u, (ab)^2 = a^v, b^{-1}ab^{-1} = a^{3-v} \rangle$$

with n, u, v even and $2u - 5v + 10 = 0 \mod n$.

The subgroup $F = \langle a^2 \rangle$ is normal in G and $G/F = P = \langle A = Fa, B = Fb \rangle = \langle A, B \mid A^2 = B^5 = (AB)^2 = 1 \rangle = D_5$; this corresponds to the insertion of the relation $a^2 = 1$ in the above presentation. The epimorphism $G \longrightarrow D_5$, mapping a to A and b to B, induces a covering from $\mathcal{H} = \overset{13}{\mathcal{H}}_{n,u,v} = (G; a, b)$ to the spherical reflexible map $\mathcal{D}_5$ with two vertices, 5 faces and automorphism group D_5. This shows that $\mathcal{H}$ has at least 5 faces.

The group $\overset{13}{G}_{n,0,2} = G_n$ with presentation

$$\langle a, b \mid a^n = b^5 = 1, (ab)^2 = a^2, b^{-1}ab^{-1} = a \rangle = \langle a, b \mid a^n = b^5 = 1, a^{-1}ba = b^{-1} \rangle$$

is clearly a metacyclic group. By changing generators $A = a$, $B = ba^{-2v+u+4}$ for u and v even such that $2u - 5v + 10 = 0 \mod n$, and having into account that a^2 belongs to the center of $\overset{13}{G}_{n,0,2}$, we get the presentation

$$\langle A, B \mid A^n = 1, B^5 = A^u, (AB)^2 = A^v, B^{-1}AB^{-1} = A^{3-v} \rangle$$

of $G_{n,u,v}$. Hence for all u, v even such that $2u - 5v + 10 = 0 \mod n$, $\overset{13}{G}_{n,u,v} = G_n$ is metacyclic of order $5n$. $\diamond$

Case 20: Let G be the group generated by a and b given by the case 20. As in the above case, the group Q (the monodromy group of $\mathcal{Q}$) is a factor group of G. In this case $K = \langle a \rangle \lhd G$ since $A = 1$, and since $G/K = \langle Kb \rangle$ is also cyclic, G is metacyclic. Hence $G = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a^t \rangle$, for some u, t such that $(t-1)u = 0 \mod n$ and $t^5 = 1 \mod n$. Theorem 8 of [6] says that $\mathcal{Q}' = (G; a, b)$ is reflexible if and only if $t^2 = 1 \mod n$. As $t^5 = 1 \mod n$ then $t^2 = 1 \mod n \Leftrightarrow t = 1 \mod n \Leftrightarrow bab^{-1} = a$, that is, G is abelian. Let $\overset{20}{G}_{n,u} := G = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a \rangle$ (the reflexible case) and $\overset{20}{Q}_{n,u,t} = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a^t \rangle$, where $(t-1)u = 0 \mod n$, $t^5 = 1 \mod n$ and $t \neq 1 \mod n$ (the chiral case). Both $\overset{20}{\mathcal{H}}_{n,u} = (\overset{20}{G}_{n,u}; a, b)$ and $\overset{20}{\mathcal{Q}}_{n,u,t} = (\overset{20}{Q}_{n,u,t}; a, b)$ have 5 hyperfaces of valency n (since a has order n and $\langle a \rangle$ has index 5 in G). Hence $\mathcal{Q} = \mathcal{Q}'$, that is, $\mathcal{Q} \cong \overset{20}{\mathcal{H}}_{n,u}$ or $\overset{20}{\mathcal{Q}}_{n,u,t}$. $\diamond$

The other cases are easily seen to give chiral hypermaps. The proof of Theorem 3 is finished. $\square$

4.2. **The Chiral hypermaps with five hyperfaces.** Each one of the remaining cases 6, 7, 17 and 19 will give rise to families of chiral hypermaps.

Theorem 4. *If $\mathcal{Q}$ is a chiral hypermap with 5 hyperfaces (of valency n) then, up to a (0,1)-duality, mirror-symmetry and an isomorphism, $\mathcal{Q}$ is either the canonical metacyclic hypermap $\overset{20}{\mathcal{Q}}_{n,u,t} = (\overset{20}{G}_{n,u,t}; a, b)$ for some n, u, t such that $(t-1)u = 0 \mod n$, $t^5 = 1 \mod n$ and $t \neq 1 \mod n$, or $\mathcal{Q}$ is $\overset{6}{\mathcal{Q}}_{n,t} = (G_n; a, ba^{t-1})$, or $\overset{17}{\mathcal{Q}}_{n,t} = (G_n; a^{-1}, ba^{-t})$, for some $n = 0 \mod 4$ and $t = 1 \mod 4$. Here $\overset{20}{Q}_{n,u,t} = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a^t\rangle$ and $G_n = \langle a, b \mid a^n = 1, b^4 = a^4, (ab)^2 = a^4, a^2b = b^2a\rangle$.*

The chirality groups and indices of these hypermaps, shown in the table below, are the last two entries of the H-sequences. In what follows $\gamma = t^4 + t^3 + t^2 + t + 1 + u$.

$\mathcal{Q}$	:	[type ; V, E, F ; $\lvert Mon(\mathcal{Q})\rvert$; $X(\mathcal{Q})$; κ]
$\overset{20}{\mathcal{Q}}_{n,u,t}$	:	$\left[\frac{5n}{(n,u)}, \frac{5n}{(n,\gamma)}, n\,;\, (n,u), (n,\gamma), 5\,;\, 5n\,;\, \langle a^{t^2-1}\rangle\,;\, \frac{n}{(n,t^2-1)}\right]$
$\overset{6}{\mathcal{Q}}_{n,t}$	:	$\left[\frac{n}{(\frac{n}{4},t)}, \frac{n}{\left(\frac{n}{2},t+1\right)}, n\,;\, 5(\frac{n}{4},t), 5\left(\frac{n}{2},t+1\right), 5\,;\, 5n\,;\, \langle [a,b]\rangle\,;\, 5\right]$
$\overset{17}{\mathcal{Q}}_{n,t}$	:	$\left[\frac{5n}{(n,5(t-1))}, \frac{n}{\left(\frac{n}{4},t\right)}, n\,;\, (n,5(t-1)), 5\left(\frac{n}{4},t\right), 5\,;\, 5n\,;\, \langle [b,a^{-1}]\rangle\,;\, 5\right]$

Proof. Before we go any further, let us go back to the case 20 where, as we have observed earlier, there is a family of canonical metacyclic chiral hypermaps.

Case 20: As seen in the proof of Theorem 3 (case 20), the group generated by a and b corresponding to this case is the metacyclic group $Q = \langle a, b | a^n = 1, b^5 = a^u, bab^{-1} = a^t\rangle$, where $(t-1)u = 0 \mod n$ and $t^5 = 1 \mod n$. The oriented hypermap $\mathcal{Q} = (Q; a, b)$ is chiral if and only if $t \neq 1 \mod n$, that is, Q is not abelian. By Corollary 9 of [6] the chirality group of $\overset{20}{\mathcal{Q}}_{n,u,t} = (\overset{20}{Q}_{n,u,t}; a, b)$, where $\overset{20}{Q}_{n,u,t} = Q$ with $t \neq 1 \mod n$, is cyclic, given by $X(\overset{20}{\mathcal{Q}}_{n,u,t}) = \langle a^{t^2-1}\rangle$, while its chirality index is given by $\frac{n}{(n,t^2-1)}$. $\diamond$

For the remaining cases 6, 7, 17 and 19 we observe that the cases 7 and 19 correspond to chiral pairs of 6 and 17 respectively. To avoid too much repetition let us fix G to be the group with presentation $\langle a, b | a^n = 1, b^r = a^u, (ab)^s = a^v, \mathcal{R} = 1\rangle$, where $n = 0 \mod |A|$ and $\mathcal{R}$ is the corresponding set of extra relators given in Table 4.2 (the monodromy group of $\mathcal{Q}$ will be a factor group of G) and K the subgroup of G generated by a.

Case 6: $a^n = 1$, $b^4 = a^u$, $(ab)^2 = a^v$, $b^2ab^{-1} = a^w$, $b^{-1}aba^{-1}b = a^t$, $b^{-1}a^2b^2 = a^z$, for some $u, v, w, t, z \in \{0, ..., n-1\}$, with $n = 0 \mod 4$. One may check that K divides G into at most 5 cosets, namely K, Kb, Kb^2, Kb^3 and Kba^2, hence $|G| \le 5n$. As $(ab)^2 \in Z(G)$ then $(ba)^2 = (ab)^2 \in Z(G)$. Now, relation 3 implies that $Kba = Kb^{-1} = Kb^3$; relation 4 implies that $Kb^2a = Kb$; and relation 5 implies that $Kb^{-1}ab = Kb^{-1}a$. Then $Kba^3 = Kb^{-1}a^2 = Kb^{-1}aba = Kb^{-2}(ba)^2 = K(ba)^2b^{-2} = Kb^{-2} = Kb^2$ and $Kba^4 = Kb^2a = Kb$, thus $Kba^i = Kb, Kb^3, Kba^2$ or Kb^2, according as $i = 0, 1, 2, 3 \mod 4$, respectively. The 6th relation was not used, it must be redundant; in fact, $Kb^{-1}a^2b^2 = K$ is equivalent to $Kb^4 = K$.

Since $Kb^2a = Kb$ then $Kb^2a^i = Kba^{i-1}$. As $Kb^3 = Kb^{-1} = Kb^{-2}a^w = Kb^2a^w = Kba^{w-1}$ and $Kba^t = Kba^{-1}b = Kb^2b = Kb^{-1}$ we have $w = 2 \mod 4$ and $t = 1 \mod 4$. Since $b^4, (ab)^2 \in Z(G)$, powering both sides of the fifth relation by 4 we get $a^u = a^{4t}$, squaring both sides of the fourth relation we get $a^v = a^{2w}$ and combining the fifth and sixth relations we get $a^z = a^{2w-t}$. We have just reduced the six parameters n, w, t, u, v, z to three parameters n, w and t,

$$G = \langle a, b | a^n = 1, b^4 = a^{4t}, (ab)^2 = a^{2w}, b^2ab^{-1} = a^w, b^{-1}aba^{-1}b = a^t \rangle .$$

A further reduction can be done. The equalities $ba^4b^{-1} = (bab)(b^{-1}a^2b^2)b^{-4}(b^2ab^{-1}) = a^{5w-5t-1}$ and $b^{-1}a^4b = b^{-1}a^2b^2b^{-4}b^2ab^{-1}bab = a^{5w-5t-1}$ implies that a^4 commutes with b^2. Then $b^{-1}a^{t-1}b = b^{-1}a^{-1}b^{-1}aba^{-1}bb = a^{-2w+2}ba^{-1}b^{-2}a^{4t} = a^{-3w+4t+2}$ and $ba^{w-2}b^{-1} = bb^2ab^{-1}a^{-2}b^{-1} = b^3abb^{-2}a^{-2}bb^{-2} = b^3aba^{t-2w}b^{-2} = b^2a^{2w-1}a^{t-2w}b^{-2} = a^{t-1}$. Thus $ba^{w-2} = a^{t-1}b = ba^{-3w+4t+2}$, that is, $a^{-4w+4t+4} = 1$. On the other hand, $b^{-1}a^2b^2 = a^{2w-t} \Leftrightarrow b^{-1}a^{-2}b^2 = a^{2w-t-4} \Rightarrow (bab)b^{-1}a^{-2}b^2 = a^{4w-t-5} \Leftrightarrow ba^{-1}b^{-2} = a^{4w-5t-5} \Leftrightarrow a^{-w} = a^{4w-5t-5} \Leftrightarrow a^{5w-5t-5} = 1$. Combining these two relations we get $a^w = a^{t+1}$. Now replacing a^w above we get $ba^4b^{-1} = b^{-1}a^4b = a^4$. Hence a^4 is central in G and

$$G = G^{6}_{n,t} = \langle a, b | a^n = 1, b^4 = a^{4t}, (ab)^2 = a^{2t+2}, b^2ab^{-1} = a^{t+1}, b^{-1}aba^{-1}b = a^t \rangle ,$$

where $n = 0 \mod 4$ and $t = 1 \mod 4$.

Consider the particular case of $n = 4$ and $t = 1$:

$$G_{4,1} = \langle \alpha, \beta | \alpha^4 = 1, \beta^4 = 1, (\alpha\beta)^2 = 1, \beta^2\alpha\beta^{-1} = \alpha^2, \beta^{-1}\alpha\beta\alpha^{-1}\beta = \alpha \rangle.$$

Last equation of this particular case is redundant,

$$\alpha^{-1}\beta^{-1}\alpha\beta\alpha^{-1}\beta = \beta\alpha^2\beta\alpha^{-1}\beta = \beta(\beta^2\alpha\beta^{-1})\beta\alpha^{-1}\beta = \beta^4 = 1 .$$

So the above presentation simplifies to

$$G_{4,1} = \langle \alpha, \beta | \alpha^4 = 1, \beta^4 = 1, (\alpha\beta)^2 = 1, \beta^2\alpha\beta^{-1} = \alpha^2 \rangle$$

and reveals the monodromy group of the toroidal hypermap $\{4, 4\}_{2,1} = \mathcal{P}^{6}$, with 5 hyperfaces, 5 hypervertices, 10 hyperedges, 20 darts and chirality index 5, see

[1]. As $n = 0 \mod 4$ and $t = 1 \mod 4$ the function $a \to \alpha, b \to \beta$ extends to an epimorphism $G_{n,t} \to G_{4,1}$. Consequently, all oriented hypermaps $\mathcal{Q}_{n,t} = (G_{n,t}, a, b)$ are coverings of $\{4,4\}_{2,1}$ and thus they all have five hyperfaces. On the other hand, adjoining the relation $b = a^t$ to the relations of $G_{n,t}$ we get $C_n = \langle a \mid a^n = 1\rangle$ and an epimorphism $G_{n,t} \longrightarrow C_n$, $a \mapsto a$ and $b \mapsto a^t$. This shows that a in $G_{n,t}$ has order n and hence $|G_{n,t}| = 5n$. Thus $Mon(\mathcal{Q}) = G_{n,t}$.

The chirality group of $\mathcal{Q}$ is the normal closure $X(Q) = \langle b^{-2}a^{-1}ba^{t+1},\ ba^{-1}b^{-1}ab^{-1}a^t\rangle^G$ where $G = G_{n,t}$. Now $b^{-2}a^{-1}ba^{t+1} = b^{-1}a^{-1}ba = [b,a]$ and $[b,a] = (b^{-1}a^{-1})ba = aba^{-2t-2}ba = ab^2aa^{-2t-2} = a(b^2ab^{-1})ba^{-2t-2} = aa^{t+1}ba^{-2t-2} = a^{-t}b$. On the other hand, $ba^{-1}b^{-1}ab^{-1}a^t = (b^{-3}a^{4t})a^{-1}b^{-1}ab^{-1}(b^{-1}aba^{-1}b) =$ $b^{-2}(b^{-1}a^{-1}b^{-1})a(b^{-2}a^{4t})aba^{-1}b = b^{-2}a^{-2t}(b^2ab^{-1})b^2a^{-1}b = b^{-2}a^{-2t}a^{t+1}b^2a^{-1}b =$ $b^{-2}a^{-t+1}b^2a^{-1}b = a^{-t}b$. Hence $X(Q) = \langle [b,a]\rangle^G$. Let $X = [b,a]$. Table below shows the conjugates X^d for $d \in \{a, a^2, a^3, a^4, b, b^2, b^3, b^4\}$,

d	a	a^2	a^3	a^4	b	b^2	b^3	b^4
X^d	Y	X^{-1}	Y^{-1}	X	Y	X^{-1}	Y^{-1}	X

where $Y = [b,a^{-1}]$. From the 4th relation we derive $[b^{-1},a^{-1}] = b^{-1}a^t$ (thus $[a^{-1},b^{-1}] = a^{-t}b = [b,a] = X$) and from the 5th relation we deduce $Y = [b,a^{-1}] = a^tb^{-1}$. Then $X^2 = [a^{-1},b^{-1}][b,a] = aba^{-1}b^{-2}a^{-1}ba = ba^{-1}b^{-2}ba = [b^{-1},a]$. But $X^2 = [b,a][a^{-1},b^{-1}] = a^{-t}(bab)a^{-1}b^{-1} = a^vb^{-1} = [b,a^{-1}]$, hence $[b^{-1},a] = [b,a^{-1}]$, or equivalently, $[a,b^{-1}] = [a^{-1},b]$. Thus $Y = [b,a^{-1}] = [b,a]^2 = X^2$ and hence $\mathcal{Q}$ has cyclic chirality group generated by $[a,b]$. Since $[b,a]^3 = [a,b]^2$, $\mathcal{Q}$ has chirality index $k = 5$. $\diamond$

Case 17: $G = \langle a,b | a^n = 1, b^5 = a^u, (ab)^4 = a^v, b^{-1}ab^{-2} = a^x, b^{-2}ab = a^y, bab^2 = a^z, b^2ab^{-1} = a^t\rangle$ for some $u,v,x,y,z,t \in \{0,...,n-1\}$, with $n = 0 \mod 4$. The cosets K, Kb, Kb^2, $Kb^3 = Kb^{-2}$ and $Kb^4 = Kb^{-1}$ is a complete set of right cosets. As $Kba = Kb^{-2}$, $Kba^2 = Kb^{-2}a = Kb^{-1}$, $Kba^3 = Kb^{-1}a = Kb^2$ and $Kba^4 = Kb^2a = Kb$ then $Kba^i = Kb, Kb^{-2}, Kb^{-1}$ or Kb^2, according as $i = 0,1,2,3 \mod 4$. Since $Kb^{-2} = Kba^x$, $Kb = Kb^2a^y = Kba^{3+y}$, $Kb^2 = Kb^{-1}a^z = Kba^{z+2}$ and $Kb^{-1} = Kb^{-2}a^t = Kba^{t+1}$ we conclude that $x,y,z,t = 1 \mod 4$.

Now b^5 and $(ab)^4 \in Z(G)$. Since $A^4 = 1$ the subgroup generated by a^4 is normal in G. The two equalities $ba^4b^{-1} = bab^2b^{-2}abb^{-1}ab^{-2}b^2ab^{-1} = a^{x+y+z+t}$ and $b^{-1}a^4b = b^{-1}ab^{-2}b^2ab^{-1}bab^2b^{-2}ab = a^{x+y+z+t}$ shows that a^4 commutes with b^2. Since a^4 commutes with b^5 then also a^4 commutes with $b = b^5b^{-4}$ and so $a^4 \in Z(G)$.

Equation $b^2ab^{-1} = a^t$ is equivalent to $b(ba)b^{-1} = a^t$. Powering by 4 we get $a^v = a^{4t}$. Because $a^{4t-1} = (bab)abab = a^z(b^{-1}ab)ab = a^va^zb^5b^{-2}ab = a^{z+x+u+y}$ we have $a^u = a^{4t-x-z-y-1}$. Since $a^{2x} = b^{-1}ab^{-2}b^{-1}ab^{-2} = b^{-1}aa^{-u}b^2ab^{-2} =$

$b^{-1}aa^{-u}a^tbb^{-2} \Leftrightarrow a^{2x+y+z} = b^{-1}a^{-u+t+1}b^{-1}(bab^2)(b^{-2}ab) = b^{-1}a^{-u+t+3}b = a^{-u+t+3}$ we have $a^x = a^{4-3t}$. As $a^{x+1} = b^{-1}a(b^{-2}a) = b^{-1}aa^yb^{-1} \Leftrightarrow a^{x+y+z+1} = b^{-1}a^{y+1}b^{-1}(bab^2)(b^{-2}ab) = b^{-1}a^{y+3}b = a^{y+3}$ we have $a^z = a^{2-x} = a^{3t-2}$. Finally, from $a^{x+t+z+y} = a^4$ we get $a^y = a^{2-t}$ and so $a^u = a^{5(t-1)}$. Therefore G is determined by two parameters n and t satisfying $n = 0 \mod 4$ and $t = 1 \mod 4$:

$$G = \langle a, b \mid a^n = 1, b^5 = a^{5(t-1)}, (ab)^4 = a^{4t}, b^{-1}ab^{-2} = a^{4-3t}, b^{-2}ab = a^{2-t}, bab^2 = a^{3t-2}, b^2ab^{-1} = a^t\rangle .$$

Adding $[a^4, b] = 1$ we get $G = \langle a, b \mid a^n = 1, b^5 = a^{5(t-1)}, [a^4, b] = 1, b^2ab^{-1} = a^t\rangle = G_{n,t}$.

The $(0, 2)$-dual of the particular case $n = 4$ and $t = 1$ yields a metacyclic group of order 20

$$\langle \alpha, \beta | \alpha^5 = 1, \beta^4 = 1, \beta\alpha\beta^{-1} = \alpha^2\rangle.$$

This is the monodromy group of a chiral hypermap with 4 hyperfaces, 5 hypervertices, 5 hyperedges and chirality index $k = 5$ see [6]. Thus $\mathcal{M} = (G_{4,1}; \alpha, \beta)$ is a chiral hypermap of order 20 with five hyperfaces. As $n = 0 \mod 4$ and $t = 1 \mod 4$ the function $a \to \alpha, b \to \beta$ extends to an isomorphism $G_{n,t} \to G_{4,1}$. Consequently each $\mathcal{Q}_{n,t} = (G_{n,t}; a, b)$ is a covering of $\mathcal{M}$ and therefore has five hyperfaces. Adjoining the relation $b = a^{t-1}$ to the above presentation of $G_{n,t}$ we get a cyclic group C_n and an epimorphism $G_{n,t} \longrightarrow C_n$ given by $a \mapsto a$, $b \mapsto a^{t-1}$. Thus $|a| = n$ and $|G_{n,t}| = 5n$, and therefore $Mon(\mathcal{Q}) = G_{n,t}$.

The chirality group of $\mathcal{Q} = \mathcal{Q}_{n,t}$ is the normal closure $X(\mathcal{Q}) = \langle X\rangle^G$, where

$$X = b^{-2}a^{-1}ba^t = b^{-2}a^{-1}b^3ab^{-1} = b^{-2}a^{-1}a^{2-t}b^3 = b^{-2}a^{1-t}b^3 = a^{1-t}b = [b, a^{-1}]$$

and $G = G_{n,t}$. Notice that $a^{1-t} \in Z(G)$. Looking at the conjugates X^Θ for $\theta \in \{a, a^2, a^3, a^4, b\}$ (table below)

Θ	a	a^2	a^3	a^4	b
X^Θ	X^{-2}	X^{-1}	X^2	X	X

one sees that $X(\mathcal{Q})$ is cyclic and generated by $X = a^{1-t}b$. Since X has order 5 ($X^5 = a^{5-5t}b^5 = 1$ and $X^i \neq 1$ for $0 < i < 5$), $\mathcal{Q}$ has chirality index $k = 5$. $\diamond$

Let G_n be the group $\overset{6}{G}_{n,1} = \langle a, b \mid a^n = 1, b^4 = a^4, (ab)^2 = a^4, a^2b = b^2a\rangle$. One easily computes that $(ba^{-1})^2 = a^{-1}b$. This shows that $(ba^{-1})^a = (ba^{-1})^b = (ba^{-1})^2$ and so ba^{-1} generates a normal subgroup. Hence G_n is metacyclic. From the covering $\overset{6}{\mathcal{Q}}_{n,1} \longrightarrow \overset{6}{\mathcal{P}}$ one sees that ba^{-1} has order at least 5. On the other hand, $(ba^{-1})^4 = (a^{-1}b)^2 = ab^{-1}$ and so ba^{-1} has order 5. Changing generators $a' = a$ and $b' = ba^{t-1}$ we get $\overset{6}{G}_{n,t}$ and changing generators $a' = a^{-1}$ and $b' = ba^{-t}$ we get $\overset{17}{G}_{n,t}$.

Finally, if $\mathcal{Q}^{6}_{n,t}$ and $\mathcal{Q}^{17}_{n,t}$ were canonical metacyclic then also $\mathcal{P}^{6}$ and $\mathcal{P}^{17}$ would be canonical metacyclic. But a quick checking shows that these are not canonical metacyclic.

The proof of Theorem 4 is finished. $\square$

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LATTICES WITH RELATIVE STONE CONGRUENCE LATTICES II

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Abstract. In this companion paper to [4] we present an alternative characterization of lattices with relative Stone congruence lattices. For the first time, the (RS)-modularity is given by a condition that, unlike its versions known so far, does not include the quantification via congruences of a lattice. It is also closer to the already known characterization in the semi-discrete case [5], [7]. We similarly present an alternative characterization of lattices whose congruence lattices satisfy the identities (E_n) of T. Hecht and T. Katriňák [8] which describe the subvarieties of relative Stone Heyting algebras. This second result generalizes an old characterization of G. Grätzer and E.T. Schmidt of lattices with Boolean congruence lattices [3].

1. Introduction

G. Grätzer and E. T. Schmidt in [3] characterized lattices whose congruence lattices are Boolean thereby answering G. Birkhoff's problem [1] (cf. [1, Problem 39]). Their result (Theorem 4.3) has been presented in terms of weak projectivity of quotients of the lattice. In this paper we generalize Grätzer-Schmidt's result within the subvarieties of relative Stone Heyting algebras which are characterized by the identity

$$(E_n) \quad (x_0 * x_1) \vee (x_1 * x_2) \vee \ldots \vee (x_{n-1} * x_n) = 1$$

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introduced by T. Hecht and T. Katriňák [8]. Our characterization is given in Theorem 4.9. It is alternative to the one given in [4] and the version of the (E_n)-modularity we use here does not include the unpleasant quantification via congruences of a lattice as the version in [4].

We also give a new characterization, alternative to those in [4] and [5], of lattices with relative Stone congruence lattices (Theorem 3.9). As in [4], we use here the identity (RS) $(x*y)\vee(y*x)=1$ characterizing the relative Stone lattices within the variety of Heyting algebras but our version of the (RS)-modularity is given by a nicer symmetric condition that, for the first time, does not include the quantification via congruences of a lattice. Comparing to [4] and [5], it is also the most natural of the generalizations of already known characterization in the semi-discrete case [5], [7].

This is very much a companion paper to [4], and as such, a continuation of the papers [9], [5], [7] and [6] by T. Katriňák and the second author. The results are presented in terms of weak projectivity of quotients of a lattice.

2. Preliminaries

We denote by $\operatorname{Con} L$ the lattice of all congruence relations on a lattice L. The smallest and the largest congruence relation, respectively, are denoted by Δ and ∇. The lattice $\operatorname{Con} L$ satisfies the infinite distributivity law

$$\theta \wedge \bigvee(\alpha_i : i \in I) = \bigvee(\theta \wedge \alpha_i : i \in I)$$

for any $\theta, \alpha_i \in \operatorname{Con} L$ (cf. [2]). Consequently, for any $\alpha, \beta \in \operatorname{Con} L$ there exists a largest congruence $\delta \in \operatorname{Con} L$ such that $\alpha \wedge \delta \leq \beta$. It is clear that

$$\delta = \bigvee(\sigma : \alpha \wedge \sigma \leq \beta).$$

The congruence δ is known as the *relative pseudocomplement of* α *with respect to* β and is denoted by $\alpha * \beta$. It follows that $\langle \operatorname{Con} L, \vee, \wedge, *, \Delta, \nabla \rangle$ is a complete relatively pseudocomplemented lattice, thus a complete Heyting algebra.

The *bounded relative Stone lattices*, that is those bounded distributive lattices in which all intervals are Stone lattices, are characterized as Heyting algebras satisfying the identity

$$(RS) \quad (x * y) \vee (y * x) = 1$$

(cf. [10, 2.9. and 2.10]). T. Hecht and T. Katriňák [8] were the first to discover that the lattice of all subvarieties of the variety of bounded relative Stone lattices is isomorphic to the chain of type $\omega + 1$ and that a Heyting algebra L belongs to its n-th ($n \geq 2$) subvariety if and only if it satisfies the identity

$$(E_n) \quad (x_0 * x_1) \vee (x_1 * x_2) \vee \ldots \vee (x_{n-1} * x_n) = 1.$$

We note that the subvariety satisfying (E_2) is exactly the variety of all Boolean algebras.

Traditionally, we use the notation $a/b \to c/d$ for the weak projectivity of quotients of the lattice L (for the detailed definition see [2, Chapter 3]). By a *non-trivial* quotient a/b we mean that $b < a$ in L and by a *proper* subquotient $a'/b' \subset a/b$ we mean that $b \leq b' \leq a' \leq a$ but $a'/b' \neq a/b$. By a *prime* quotient a/b we mean that a covers b in L. We recall that if $a/b \to c/d$ and $(a, b) \in \theta$ for a congruence $\theta \in \operatorname{Con} L$ then also $(c, d) \in \theta$. The importance of the *weak projectivity of quotients of a lattice* in the description of lattice congruences comes from the following results.

Lemma 2.1. ([2, Theorem III.1.2] or [6, Lemma 1]) *For any principal congruence $\theta_{a,b} \in \operatorname{Con} L$,*

$$(c, d) \in \theta_{a,b},$$

($d \leq c, b \leq a$) if and only if there is a finite chain $d = y_0 \leq \ldots \leq y_m = c$ such that $a/b \to y_{i+1}/y_i$ for all $i \in \{0, \ldots, m-1\}$.

As in [4], we use the following abbreviation: for a lattice L and quotients $a/b, c/d$ of L, we shall say that *there are projections from the quotient a/b into a chain in c/d* or that *a/b has projections into a chain in c/d* if there is a finite chain $d = y_0 \leq \ldots \leq y_m = c$ such that $a/b \to y_{i+1}/y_i$ for all $i \in \{0, \ldots, m-1\}$. Then Lemma 2.1 can be rephrased such that $(c, d) \in \theta_{a,b}$ if and only if a/b has projections into a chain in c/d.

Lemma 2.2. ([11, 1.4], [6, Lemma 2]) *Let L be a lattice and $\theta, \varphi \in \operatorname{Con} L$. Then the relative pseudocomplement of θ with respect to φ is*

$$\theta * \varphi = \bigvee (\theta_{u,v}, (u, v) \in S),$$

where S is the set of all pairs of elements (u, v) ($u, v \in L$) such that $u/v \to z/t$ and $(z, t) \in \theta$ implies $(z, t) \in \varphi$ for all $z, t \in L$.

Lemma 2.3. ([4, Corollary 2.3]) *Let L be a lattice, $\theta, \varphi \in \operatorname{Con} L$ and let $a, b \in L$, $b < a$.*

(i) *$(a, b) \in \theta * \varphi$ if and only if for every projection $a/b \to c/d$ ($c, d \in L$), $(c, d) \in \theta$ yields $(c, d) \in \varphi$;*

(ii) $(a,b) \notin \theta * \varphi$ *if and only if there is a projection* $a/b \to c/d$ $(c, d \in L)$ *such that* $(c,d) \in \theta$ *and* $(c,d) \notin \varphi$.

3. Lattices whose congruence lattices are relative Stone

To compare our new characterizations of lattices with relative Stone congruence lattices with those already known from [5] and [4], we start with recalling the main concepts and the results of [5] and [4].

Definition 3.1. ([5], Definition 1) *Let* L *be a lattice,* $\pi \in \operatorname{Con} L$ *and* $a/b, u/v$ *be quotients of* L*. Then* L *is said to be* π-**almost weakly modular** *whenever* $a/b \to u/v$ *and* $(u,v) \notin \pi$ *imply the existence of a subquotient* $a_1/b_1 \subseteq a/b$ *with* $(a_1, b_1) \notin \pi$ *such that for every quotient* r/s *with* $a_1/b_1 \to r/s$ *and* $(r,s) \notin \pi$ *there exists a quotient* z/t *with* $r/s \to z/t$ *,* $u/v \to z/t$ *and* $(z,t) \notin \pi$.

Definition 3.2. ([5], Definition 2) *Let* L *be a lattice and* $\theta, \pi \in \operatorname{Con} L$*. Then* θ *is said to be* π-**weakly separable** *if* $\pi \leq \theta$ *and for any* $a < b$ *in* L *there is a chain* $a = z_0 \leq z_1 \leq \ldots \leq z_m = b$ *such that for each* $i \in \{0, \ldots, m-1\}$ *either*

(i) $z_{i+1}/z_i \to u/v$ *and* $(u,v) \in \theta$ *imply* $(u,v) \in \pi$ *or*

(ii) *for every subquotient* $r/s \subseteq z_{i+1}/z_i$ *with* $(r,s) \notin \pi$*, there exists a quotient* u/v *with* $r/s \to u/v$ *and* $(u,v) \in \theta$*,* $(u,v) \notin \pi$.

Proposition 3.3. ([5], Theorem 2) Let L be a lattice. The lattice $\operatorname{Con} L$ is relatively Stone if and only if for every $\pi \in \operatorname{Con} L$ the following hold:

(1) L is π-almost weakly modular and

(2) every congruence of L is π-weakly separable.

The following concepts introduced in [4], alternative to those above, were already motivated by the symmetric identity (RS). The disadvantage of the version of the (RS)-weakly modularity used in [4] is the quantification of the given condition via arbitrary congruences $\theta, \pi \in \operatorname{Con} L$.

Definition 3.4. ([4, Definition 3.4]) Let L be a lattice. Let $a/b, u/v, z/t$ be quotients of L and let $\theta, \pi \in \operatorname{Con} L$. Then L is said to be **(RS)-weakly modular** whenever $a/b \to u/v$ and $a/b \to z/t$ with $(u,v) \in \theta$ and $(z,t) \in \pi$ imply that either one of (I) $(u,v) \in \pi$, (II) $(z,t) \in \theta$ holds or the following condition is satisfied:

(III) there are proper subquotients $a_1/b_1 \subset a/b$, $a_2/b_2 \subset a/b$ and quotients $u'/v', z'/t'$ such that $a_1/b_1 \to u'/v'$, $(u',v') \in \theta$, $(u',v') \notin \pi$ and $a_2/b_2 \to z'/t'$, $(z',t') \in \pi$, $(z',t') \notin \theta$.

Definition 3.5. ([4, Definition 3.5]) *Let L be a lattice and let θ, π be congruences of L. Then a pair θ, π is said to be* **(RS)-separable** *if for every $b < a$ in L there is a finite chain $b = x_0 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ one of the following conditions holds:*

(i) *for every proper subquotient $r_1/s_1 \subset x_{i+1}/x_i$ and every projection $r_1/s_1 \to u_1/v_1$, $(u_1, v_1) \in \theta$ implies $(u_1, v_1) \in \pi$;*

(ii) *for every proper subquotient $r_2/s_2 \subset x_{i+1}/x_i$ and every projection $r_2/s_2 \to u_2/v_2$, $(u_2, v_2) \in \pi$ implies $(u_2, v_2) \in \theta$.*

The characterization of lattices with relative Stone congruence lattices given in [4] was presented as follows:

Proposition 3.6. ([4, Theorem 3.6]) Let L be a lattice. Then $\operatorname{Con} L$ satisfies the identity

$$(RS) \qquad (\theta * \pi) \vee (\pi * \theta) = \nabla$$

if and only if

(1) L is (RS)-weakly modular and
(2) every pair θ, π of congruences of L is (RS)-separable.

The characterizations of lattices with relative Stone congruence lattices given in Propositions 3.3 and 3.6 can both be essentially simplified if the lattice L is semi-discrete. Let us recall that a lattice L is called *semi-discrete* if between every two comparable elements of L there exists a finite maximal chain. (Every finite lattice is of course semi-discrete.) The following characterization in the semi-discrete case has already been given in [5].

Proposition 3.7. ([5], [4, Theorem 3.6]) *Let L be a semi-discrete lattice. Then $\operatorname{Con} L$ is a relative Stone lattice if and only if for any prime quotients p, q, r of L, the projections $p \to q$ and $p \to r$ imply that $q \to r$ or $r \to q$.*

$$\begin{tikzcd}
p \arrow[r] \arrow[d] & q \arrow[dl, phantom, sloped, "\not\leftrightarrow"] \\
r &
\end{tikzcd}$$

We now introduce a new alternative version of the (RS)-modularity given by a nice symmetric condition that, for the first time, does not include the unpleasant quantification via arbitrary congruences of a lattice.

Comparing to [4] and [5], it is also the most natural of the generalizations of Proposition 3.7.

$$
\begin{tikzcd}
 & & & u'/v' & \\
 & & & a_1/b_1 \arrow[u, dashed] & \\
 & & & \cap & \\
z'/t' & a_2/b_2 \arrow[l, dashed] & \subset & a/b \arrow[r] \arrow[d, dashed] & u/v \arrow[uuul, dashed] \arrow[dl, Rightarrow, dashed, shift left] \\
 & & & z/t \arrow[ulll, dashed] \arrow[ur, Rightarrow, dashed, shift left] &
\end{tikzcd}
$$

Definition 3.8. *A lattice L is* **(RS)-modular** *if for all non-trivial quotients $a/b, u/v, z/t$ of L, $a/b \to u/v$ and $a/b \to z/t$ yield that either one of the following conditions holds,*

(I) *there are projections from the quotient z/t into a chain in u/v,*

(II) *there are projections from the quotient u/v into a chain in z/t,*

or the following condition is satisfied:

(III) *there are proper subquotients $a_1/b_1 \subset a/b$, $a_2/b_2 \subset a/b$ and quotients $u'/v', z'/t'$ with $a_1/b_1 \to u'/v'$, $u/v \to u'/v'$, $(u', v') \notin \theta_{z,t}$ and $a_2/b_2 \to z'/t'$, $z/t \to z'/t'$, $(z', t') \notin \theta_{u,v}$.*

In our main result of this section we use the above new condition of (RS)-modularity together with the already used (in [4]) condition of (RS)-separability.

Theorem 3.9. *Let L be a lattice. Then* $\operatorname{Con} L$ *satisfies the identity*

$$(RS) \qquad (\theta * \pi) \vee (\pi * \theta) = \nabla$$

and so is a relative Stone lattice if and only if

(1) *L is (RS)-modular and*

(2) *every pair θ, π of congruences of L is (RS)-separable.*

Proof. For the necessity, let $\operatorname{Con} L$ satisfy the identity (RS). We note that all the quotients considered below are non-trivial. To show that L is (RS)-modular, let $a/b \to u/v$ and $a/b \to z/t$. Set

$$\theta := \theta_{u,v}, \ \pi := \theta_{z,t}.$$

Since $(a, b) \in (\theta * \pi) \vee (\pi * \theta)$, there exists a chain $b = x_0 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. We shall distinguish the following three cases.

(I) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$. So $(a, b) \in \theta * \pi$ and since $a/b \to u/v$ and $(u, v) \in \theta$, by Lemma 2.3(i) we get $(u, v) \in \pi$. As $\pi = \theta_{z,t}$, by Lemma 2.1 there are projections from the quotient z/t into a chain in u/v. So the condition (I) of Definition 3.8 holds.

(II) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \pi * \theta$. Then $(a, b) \in \pi * \theta$ and similarly as above one can obtain the condition (II) of Definition 3.8.

(III) Now assume that (I) and (II) do not hold so $(x_{i+1}, x_i) \notin \theta * \pi$ and $(x_{j+1}, x_j) \notin \pi * \theta$ for some $i, j \in \{0, \ldots, m-1\}$ and proper quotients $x_{i+1}/x_i, x_{j+1}/x_j \subset a/b$. We set $a_1/b_1 := x_{i+1}/x_i$ and $a_2/b_2 := x_{j+1}/x_j$. Then $(a_1, b_1) \notin \theta * \pi$ and $(a_2, b_2) \notin \pi * \theta$ yield by Lemma 2.3(ii) that there exist projections $a_1/b_1 \to c_1/d_1$ with $(c_1, d_1) \in \theta, (c_1, d_1) \notin \pi$ and $a_2/b_2 \to c_2/d_2$ with $(c_2, d_2) \in \pi, (c_2, d_2) \notin \theta$. Since $(c_1, d_1) \in \theta_{u,v}$, there are projections from u/v into a chain in c_1/d_1. As $(c_1, d_1) \notin \pi$, there exists a subquotient $u'/v' \subseteq c_1/d_1$ such that $(u', v') \notin \pi$ and $u/v \to u'/v'$, $a_1/b_1 \to u'/v'$. Similarly, as $(c_2, d_2) \in \theta_{z,t}$, z/t has projections into a chain in c_2/d_2 and there is a subquotient $z'/t' \subseteq c_2/d_2$ such that $(z', t') \notin \theta$ and $z/t \to z'/t'$, $a_2/b_2 \to z'/t'$. We finally note that $(u', v') \notin \pi$ means $(u', v') \notin \theta_{z,t}$ and $(z', t') \notin \theta$ means $(z', t') \notin \theta_{u,v}$. We have proved (1).

The necessity of (2) was already shown in [4], but as the argument is short and we want this proof to be self-contained, we briefly repeat the argument. Let $\theta, \pi \in \operatorname{Con} L$ and $a, b \in L$, $b < a$. As $(a, b) \in (\theta * \pi) \vee (\pi * \theta)$, there exists a chain $b = x_0 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. If for $i \in \{0, \ldots m-1\}$ we have $(x_{i+1}, x_i) \in \theta * \pi$, then for every proper subquotient $r_1/s_1 \subset x_{i+1}/x_i$ also $(r_1, s_1) \in \theta * \pi$. Hence by Lemma 2.3(i), for every projection $r_1/s_1 \to u_1/v_1$, $(u_1, v_1) \in \theta$ implies $(u_1, v_1) \in \pi$. So the condition (i) of Definition 3.5 is satisfied. Analogously, if for $i \in \{0, \ldots m-1\}$ we have $(x_{i+1}, x_i) \in \pi * \theta$, then (ii) of Definition 3.5 is satisfied.

We will prove the sufficiency. Assume that (1) and (2) are satisfied and $\theta, \pi \in \operatorname{Con} L$, $a, b \in L$, $b < a$. By the (RS)-separability of θ, π, there is a chain $b = x_0 \leq \ldots \leq x_m = a$ such that (i) or (ii) of Definition 3.5 holds for each $i \in \{0, \ldots, m-1\}$. We shall distinguish the following two cases.

(a) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. Then $(a, b) \in (\theta * \pi) \vee (\pi * \theta)$. We show that the remaining case is impossible.

(b) Let (a) above does not hold. So there exists $i \in \{0, \dots, m-1\}$ such that $(x_{i+1}, x_i) \notin \theta * \pi$ and $(x_{i+1}, x_i) \notin \pi * \theta$. By Lemma 2.3(ii), there exist quotients $u/v, z/t$ and projections $x_{i+1}/x_i \to u/v$ with $(u, v) \in \theta$, $(u, v) \notin \pi$ and $x_{i+1}/x_i \to z/t$ with $(z, t) \in \pi$, $(z, t) \notin \theta$. Now we use that L is (RS)-modular. If (I) of Definition 3.8 holds with $a/b = x_{i+1}/x_i$, then z/t has projections into a chain in u/v. Since $(z, t) \in \pi$, this yields $(u, v) \in \pi$, a contradiction. In the same way, if (II) of Definition 3.8 holds, then $(z, t) \in \theta$, a contradiction.

We finally note that we do not need to consider (III) of Definition 3.8 in our case with $a/b = x_{i+1}/x_i$. The reason is that if there exists an element $x \in L$ such that $x_i < x < x_{i+1}$ (i.e. the quotient x_{i+1}/x_i is not prime) then for both the proper subquotients x/x_i and x_{i+1}/x of the quotient x_{i+1}/x_i, one of the conditions (i) respectively (ii) of Definition 3.5 applies, which gives us that $(x_{i+1}, x) \in \theta * \pi$ and $(x, x_i) \in \theta * \pi$ respectively $(x_{i+1}, x) \in \pi * \theta$ and $(x, x_i) \in \pi * \theta$; thus $(x_{i+1}, x_i) \in \theta * \pi$ respectively $(x_{i+1}, x_i) \in \pi * \theta$, which contradicts our assumption above that $(x_{i+1}, x_i) \notin \theta * \pi$ and $(x_{i+1}, x_i) \notin \pi * \theta$. The remaining case is that the quotient x_{i+1}/x_i is prime in which case (III) of Definition 3.8 cannot occur for our quotient $a/b = x_{i+1}/x_i$ and its projections into the quotients u/v and z/t. The proof is complete. □

We finally note that in semi-discrete lattices the chains in our conditions can be considered maximal and so the quotients can be considered prime; hence there are no proper subquotients of considered quotients. This means that for semi-discrete lattices, the conditions (I) and (II) in Definition 3.8 can be immediately simplified and the condition (III) requiring the existence of certain proper subquotients of the considered quotient a/b cannot be satisfied and so can be omitted. Also, we note that for semi-discrete lattices, the condition of (RS)-separability is satisfied vacuously and can be omitted from the characterization. So in the semi-discrete case, Theorem 3.9 gives us immediately Proposition 3.7.

4. Lattices whose congruence lattices satisfy the identity (E_n)

We start with repeating the characterization of lattices L with Boolean congruence lattices due to G. Grätzer and E.T. Schmidt [3].

Definition 4.1. ([3], cf. also [2, Definition III.1.8]) *Let L be a lattice and let $a, b, c, d \in L$, $b < a, d < c$. Then L is called* **weakly modular**

if $a/b \to c/d$ implies the existence of a proper subquotient $a'/b' \subset a/b$ satisfying $c/d \to a'/b'$.

Definition 4.2. ([3], cf. also [2, p. 155]) *Let L be a lattice. A congruence θ of the lattice L is called* **separable** *if, for all $a, b \in L$, $b < a$ there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta$ or $(u, v) \in \theta$ for no proper subquotient $u/v \subset x_{i+1}/x_i$.*

Proposition 4.3. ([3], cf. [2, Theorem III.4.9]) *Let L be a lattice. Then* $\operatorname{Con} L$ *is Boolean if and only if L is weakly modular and all congruences of L are separable.*

Our aim in this section is to give a generalization of the Grätzer-Schmidt result within the subvarieties of all relative Stone Heyting algebras defined by the identities (E_n) $(n \geq 2)$ which is alternative to the one presented in [4]. We start by recalling our companion characterization in [4]. To sort out the terminology settings, we shall call the version of (E_n)-modularity presented in [4] **($\mathbf{E_n}$)-weakly modularity**.

Definition 4.4. ([4, Definition 4.4]) Let L be a lattice and $n \geq 2$. Let $a/b, u_1/v_1, \ldots, u_n/v_n$ be quotients of L and $\theta_1, \ldots, \theta_{n+1} \in \operatorname{Con} L$. Then L is said to be **($\mathbf{E_n}$)-weakly modular** whenever

$$a/b \to u_j/v_j \quad \text{and} \quad (u_j, v_j) \in \theta_j,\ j = 1, \ldots, n$$

imply that either

- (I$_j$) there is $j \in \{1, \ldots, n\}$ with $(u_j, v_j) \in \theta_{j+1}$ or
- (II) for all $j \in \{1, \ldots, n\}$ there are proper subquotients $a_j/b_j \subset a/b$ and projections $a_j/b_j \to u'_j/v'_j$ with $(u'_j, v'_j) \in \theta_j$ and $(u'_j, v'_j) \notin \theta_{j+1}$.

Definition 4.5. ([4, Definition 4.5]) Let L be a lattice, $n \geq 2$ and let $\theta_1, \ldots, \theta_{n+1} \in \operatorname{Con} L$. Then the (unordered)(n+1)-tuple $\theta_1, \ldots, \theta_{n+1}$ is said to be **($\mathbf{E_n}$)-separable** if for any $b < a$ there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there exists $j \in \{1, \ldots, n\}$ with the following property:

- (j) for every proper subquotient $a'/b' \subset x_{i+1}/x_i$ and every quotient u'_j/v'_j, $a'/b' \to u'_j/v'_j$ and $(u'_j, v'_j) \in \theta_j$ imply $(u'_j, v'_j) \in \theta_{j+1}$.

Proposition 4.6. ([4, Theorem 4.6]) Let L be a lattice and $n \geq 2$. The congruence lattice $\operatorname{Con} L$ satisfies the identity (E_n) if and only if all of the following conditions hold:

1. $\operatorname{Con} L$ is relative Stone;

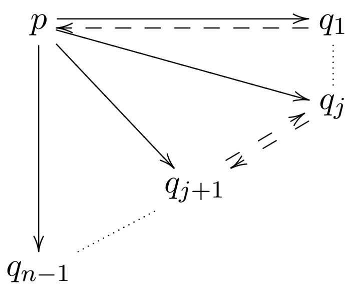
(2) L is (E_n)-weakly modular;
(3) every (n+1)-tuple of congruences on L is (E_n)-separable.

The presented characterization could again be essentially simplified in case the lattice L is semi-discrete. The result below was first proven in [5].

Proposition 4.7. ([5], [4, Corollary 5.1]) *Let L be a semi-discrete lattice. Then* $\operatorname{Con} L$ *is a relative Stone lattice satisfying the identity* (E_n) $(n \geq 2)$ *if and only if for any prime quotients $p, q_1, \dots, q_{n-1}$ of L the projections $p \to q_k$ for $k \in \{1, \dots, n-1\}$ imply that one of the following conditions holds:*

(i) *There exists $j \in \{1, \dots, n-1\}$ such that there is a projection $q_j \to p$.*
(ii) *There exist $i, j \in \{1, \dots, n-1\}$, $i \neq j$ such that there are projections $q_i \to q_j$ and $q_j \to q_i$.*

We note that by reordering the quotients above one can assume that $j = 1$ in the condition (i) and that $i = j + 1$ in the condition (ii) as the following figure indicates.



Our aim in this section is to introduce a new alternative version of the (E_n)-modularity given by a nicer condition that does not include the unpleasant quantification via arbitrary congruences of the considered lattice. Our result leads to a more natural generalization of Proposition 4.7 than our characterization given in Proposition 4.6.

Definition 4.8. *Let L be a lattice and $n \geq 2$. Let $a/b, u_1/v_1, \dots, u_{n-1}/v_{n-1}$ be quotients of L. Then L is said to be* $(\mathbf{E_n})$**-modular** *whenever*

$$a/b \to u_i/v_i, \ i = 1, \dots, n-1$$

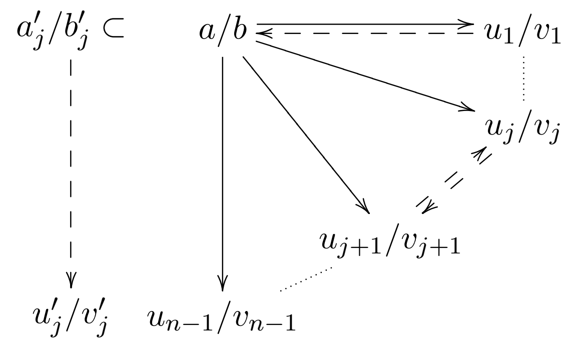
imply that either one of the following conditions holds,

(I) *u_1/v_1 has projections into a chain in a/b,*

(II) *there is* $j \in \{1, \ldots, n-2\}$ *with* u_j/v_j *having projections into a chain in* u_{j+1}/v_{j+1} *and* u_{j+1}/v_{j+1} *having projections into a chain in* u_j/v_j,

or the following condition is satisfied:

(III) *there exist* $i, j \in \{1, \ldots, n-1\}$ *with* $i \neq j$, *proper subquotients* $a_i'/b_i' \subset a/b$, $a_j'/b_j' \subset a/b$ *and quotients* $u_i'/v_i', u_j'/v_j'$ *such that* $(u_i', v_i') \notin \theta_{u_j,v_j}$, $(u_j', v_j') \notin \theta_{u_i,v_i}$ *and* $a_i'/b_i' \to u_i'/v_i'$, $u_i/v_i \to u_i'/v_i'$ *and* $a_j'/b_j' \to u_j'/v_j'$, $u_j/v_j \to u_j'/v_j'$.



In our main result of this section we employ the above new condition of (E_n)-modularity together with the condition of (E_n)-separability used in [4].

Theorem 4.9. *Let* L *be a lattice and* $n \geq 2$. *The congruence lattice* $\operatorname{Con} L$ *satisfies the identity* (E_n) *if and only if all of the following conditions hold:*

(1) $\operatorname{Con} L$ *is relative Stone;*
(2) L *is* (E_n)*-modular;*
(3) *every (n+1)-tuple of congruences on* L *is* (E_n)*-separable.*

Proof. Let L be a lattice and $n \geq 2$. We assume that $\operatorname{Con} L$ satisfies the identity

$$(E_n) \quad (\theta_0 * \theta_1) \vee (\theta_1 * \theta_2) \vee \ldots \vee (\theta_{n-1} * \theta_n) = \nabla.$$

We proved the necessity of (1) and (3) in [4], but as the arguments are short and we want this proof to be self-contained, we repeat the arguments briefly here. First we set $\theta = \theta_0 = \theta_2 = \ldots$ and $\pi = \theta_1 = \theta_3 = \ldots$ in the above congruence identity. Then we obtain that $\operatorname{Con} L$ satisfies the identity

$$(RS) \qquad (\theta * \pi) \vee (\pi * \theta) = \nabla.$$

Hence $\operatorname{Con} L$ is relative Stone and (1) holds.

To show the necessity of (3), let $\theta_0, \ldots, \theta_n$ be congruences of the lattice L. Since $(a, b) \in (\theta_0 * \theta_1) \vee (\theta_1 * \theta_2) \vee \ldots \vee (\theta_{n-1} * \theta_n)$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there exists $j \in \{0, \ldots, n-1\}$ with $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$. Let a'/b' be a proper subquotient of x_{i+1}/x_i and u'_j/v'_j be a quotient of L such that $a'/b' \to u'_j/v'_j$ and $(u'_j, v'_j) \in \theta_j$. As $(a', b') \in \theta_j * \theta_{j+1}$, Lemma 2.3(ii) gives us $(u'_j, v'_j) \in \theta_{j+1}$ as required.

Most of the work is with proving the necessity of (2). Let $a/b \to u_i/v_i$ for $i \in \{1, \ldots, n-1\}$. We shall distinguish two cases.

(a) By the (RS)-modularity of L, there are $i, j \in \{1, \ldots, n-1\}$, $i \neq j$ such that for the projections $a/b \to u_i/v_i$, $a/b \to u_j/v_j$ the condition (III) of Definition 3.8 holds. It is easy to see that in this case the condition (III) of Definition 4.8 is immediately satisfied.

(b) The condition (a) above is not satisfied so that the (RS)-modularity of L yields that for every pair of projections $a/b \to u_i/v_i$, $a/b \to u_j/v_j$ where $i, j \in \{1, \ldots, n-1\}$, $i \neq j$, one of the conditions (I), (II) of Definition 3.8 holds. Then there exists an ordering $i_1 \leq i_2 \leq \ldots \leq i_{n-1}$ of the set $\{1, \ldots, n-1\}$ such that there are projections from the quotient u_{i_j}/v_{i_j} into a chain in $u_{i_{j+1}}/v_{i_{j+1}}$ for $j = 1, \ldots, n-2$. Without loss of generality we can assume that $i_j = j$ for $j = 1, \ldots, n-1$. Now we set

$$\theta_0 := \theta_{a,b},\ \theta_1 := \theta_{u_1,v_1}, \ldots,\ \theta_{n-1} := \theta_{u_{n-1},v_{n-1}},\ \theta_n := \Delta.$$

Since $(a, b) \in (\theta_0 * \theta_1) \vee (\theta_1 * \theta_2) \vee \ldots \vee (\theta_{n-1} * \theta_n)$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there is $j \in \{0, \ldots, n-1\}$ with $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$.

We shall distinguish the following possible subcases of the case (b).

(b_1) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta_0 * \theta_1$. Then we have $(a, b) \in \theta_0 * \theta_1$. Since $(a, b) \in \theta_0$, by Lemma 2.3(i) we obtain $(a, b) \in \theta_1$. As $\theta_1 := \theta_{u_1,v_1}$, by Lemma 2.1 there are projections from u_1/v_1 into a chain in a/b.

(b_2) There exists $j \in \{1, \ldots, n-2\}$ such that for every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$. Then $(a, b) \in \theta_j * \theta_{j+1}$ whence $(u_j, v_j) \in \theta_j * \theta_{j+1}$ as $a/b \to u_j, v_j$. Since $(u_j, v_j) \in \theta_j$, it follows $(u_j, v_j) \in \theta_{j+1}$. Since we have $\theta_{j+1} := \theta_{u_{j+1},v_{j+1}}$, by Lemma 2.1 there are projections from u_{j+1}/v_{j+1} into a chain in u_j/v_j.

(b_3) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta_{n-1} * \theta_n$. Then we get $(a, b) \in \theta_{n-1} * \theta_n$, whence $(u_{n-1}, v_{n-1}) \in \theta_{n-1} * \theta_n$ as $a/b \to u_{n-1}/v_{n-1}$.

Now since $(u_{n-1}, v_{n-1}) \in \theta_{n-1}$, it follows $(u_{n-1}, v_{n-1}) \in \theta_n$, whence finally $u_{n-1} = v_{n-1}$, a contradiction.

($\neg b_{1-3}$) Now we assume that none of the cases (b_1)-(b_3) is satisfied. So we assume that for every $j \in \{0, \ldots, n-1\}$ there is $k_j \in \{0, \ldots, m-1\}$ such that $(x_{k_j+1}, x_{k_j}) \notin \theta_j * \theta_{j+1}$. Let $a_j/b_j := x_{k_j+1}/x_{k_j} \subset a/b$ be the proper subquotient with $(a_j, b_j) \notin \theta_j * \theta_{j+1}$. By Lemma 2.3(ii) there exists a quotient c_j/d_j such that $a_j/b_j \to c_j/d_j$ and $(c_j, d_j) \in \theta_j$, $(c_j, d_j) \notin \theta_{j+1}$. As $\theta_j := \theta_{u_j,v_j}$ for all $j \in \{1, \ldots, n-1\}$, by Lemma 2.1 there are projections from the quotient u_j/v_j into a chain in c_j/d_j. Let $j \in \{1, \ldots, n-1\}$. Since $(c_j, d_j) \notin \theta_{j+1}$, there is a subquotient $u'_j/v'_j \subseteq c_j/d_j$ such that $(u'_j, v'_j) \notin \theta_{j+1}$ and $u_j/v_j \to u'_j/v'_j$. We note that $(u'_j, v'_j) \notin \theta_{j+1}$ means $(u'_j, v'_j) \notin \theta_{u_{j+1},v_{j+1}}$.

So in case (b) one of the following conditions is always satisfied:

(I) There are projections from the quotient u_1/v_1 into a chain in a/b.
(II) There exists $j \in \{1, \ldots, n-2\}$ such that there are projections from the quotient u_j/v_j into a chain in u_{j+1}/v_{j+1} and there are projections from u_{j+1}/v_{j+1} into a chain in u_j/v_j.
(III) There exist $i, j \in \{1, \ldots, n-1\}$ with $i \neq j$, proper subquotients $a'_i/b'_i \subset a/b$, $a'_j/b'_j \subset a/b$ and quotients $u'_i/v'_i, u'_j/v'_j$ such that $(u'_i, v'_i) \notin \theta_{u_j,v_j}$, $(u'_j, v'_j) \notin \theta_{u_i,v_i}$ and $a'_i/b'_i \to u'_i/v'_i$, $u_i/v_i \to u'_i/v'_i$ and $a'_j/b'_j \to u'_j/v'_j$, $u_j/v_j \to u'_j/v'_j$.

We have proved (2).

Conversely, let the conditions (1)–(3) hold. Let $\theta_1, \ldots, \theta_{n+1} \in \operatorname{Con} L$ and let $b < a$ in L. By the (E_n)-separability of $\theta_1, \ldots, \theta_{n+1}$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for all $i \in \{0, \ldots, m-1\}$ there exists $j \in \{1, \ldots, n\}$ such that the condition (j) from Definition 4.5 is satisfied. We distinguish two cases.

(a) For every $i \in \{0, \ldots, m-1\}$ there exists some $j \in \{1, \ldots, n\}$ such that $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$. Then $(a, b) \in (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \ldots \vee (\theta_n * \theta_{n+1})$ as required. We show that the remaining case is impossible.

(b) If (a) above does not hold, then there exists $i \in \{0, \ldots, m-1\}$ such that for all $j \in \{1, \ldots, n\}$, $(x_{i+1}, x_i) \notin \theta_j * \theta_{j+1}$. By Lemma 2.3(ii), for all $j \in \{1, \ldots, n\}$ there exist quotients u_j/v_j and projections $x_{i+1}/x_i \to u_j/v_j$ with $(u_j, v_j) \in \theta_j$, $(u_j, v_j) \notin \theta_{j+1}$. Now we use that L is (E_n)-modular for the quotient $a/b = x_{i+1}/x_i$ and its projections $a/b \to u_2/v_2, \ldots, a/b \to u_n/v_n$. But we shall also use that $a/b \to u_1/v_1$ and $(u_1, v_1) \notin \theta_2$.

If the condition (I) of Definition 4.8 holds, then there are projections from u_2/v_2 into a chain in a/b. As $(u_2, v_2) \in \theta_2$, we get $(a, b) \in \theta_2$. Since $a/b \to u_1/v_1$, we obtain $(u_1, v_1) \in \theta_2$, a contradiction.

If the condition (II) holds, then there is $j \in \{2, \ldots, n-1\}$ such that there are projections from the quotient u_j/v_j into a chain in u_{j+1}/v_{j+1} and there are projections from u_{j+1}/v_{j+1} into a chain in u_j/v_j. As we have $(u_{j+1}, v_{j+1}) \in \theta_{j+1}$, we obtain $(u_j, v_j) \in \theta_{j+1}$, a contradiction.

We again note that we do not need to consider (III) of Definition 4.8 in our case. We recall that $a/b = x_{i+1}/x_i$. If there exists an element $x \in L$ such that $x_i < x < x_{i+1}$ (i.e. the quotient x_{i+1}/x_i is not prime) then for both the proper subquotients x/x_i and x_{i+1}/x of the quotient x_{i+1}/x_i, the condition (j) of Definition 4.5 applies for some $j \in \{1, \ldots, n\}$. This gives us that $(x_{i+1}, x) \in \theta_j * \theta_{j+1}$ and $(x, x_i) \in \theta_j * \theta_{j+1}$ for some $j \in \{1, \ldots, n\}$, which contradicts our assumption of case (b) that $(x_{i+1}, x_i) \notin \theta_j * \theta_{j+1}$ for all $j \in \{1, \ldots, n\}$. The remaining case is that the quotient x_{i+1}/x_i is prime in which case (III) of Definition 4.8 cannot occur. The proof is complete. $\square$

We again finally note that in semi-discrete lattices, the conditions (I) and (II) in Definition 4.8 can be immediately simplified and the condition (III) can be omitted. The condition of (E_n)-separability is satisfied vacuously and can be omitted from the characterization. So in the semi-discrete case, Theorem 4.9 gives us immediately Proposition 4.7. We emphasize this as an another important advantage of our new characterization in Theorem 4.9, comparing to the one in Proposition 4.6, because in [4] we needed a rather long proof to derive Proposition 4.7 from Proposition 4.6.

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CATEGORICALLY-ALGEBRAIC DUALITIES

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Abstract. The paper introduces a new technique for producing topological representations of algebraic structures called *categorically-algebraic* (*catalg*) *dualities*. Based on our recent results, generalizing the famous duality for bounded distributive lattices of H. Priestley and developed in the framework of catalg topology (subsuming both the crisp and the fuzzy approaches), the theory incorporates not only the representation theories of H. Priestley and M. Stone, but also *natural dualities* of D. Clark and B. Davey, bringing into light their catalg properties and serving as a tool for generating new topological representation theorems for algebraic structures. We apply the emerging theory to investigate the relations between topological representations of a given variety and its reduct, already considered by several researchers under the name of *piggyback dualities*. The results obtained are illustrated by the examples of *J-distributive lattices* of A. Petrovich and *¬-lattices* of S. Celani, providing a better insight into their properties.

1. Introduction

The important question of relation between algebra and topology has always occupied mathematician's mind. One of the most important steps in the developments was done in the mid-30's of the last century by the famous representation theorems of M. Stone for *Boolean algebras* [75] and *distributive lattices* [76], and L. Pontrjagin for *abelian groups* [51], which opened a truly novel topological outlook on the well-known algebraic concepts. Transforming algebraic problems, stated in an abstract symbolic language, into their dual topological ones, where geometric intuition comes to our help, the new machinery induced many researchers to consider topological counterparts of different algebraic structures.

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The well-known representation theorem for *monadic Boolean algebras* of P. Halmos [34], which is a useful tool in the realm of algebraic logic coined *monadic* [35], serves as a good example.

While the representation of Boolean algebras has been appreciated right from the start, the respective machinery for distributive lattices appeared to be less satisfactory. In 1970 H. Priestley [53] presented another approach in her celebrated duality theorem, which combined the results of M. Stone for Boolean algebras and G. Birkhoff [5] for finite distributive lattices. The crucial point of her setting was the enrichment of the representing topological space with a partial order (getting the so-called *Priestley space*) that simplified the framework dramatically, making it more application-friendly. Equipped with the new machinery (called *Priestley duality*), working algebraists constructed a plethora of topological representations of structures based on distributive lattices [8, 9, 10, 12, 14, 26, 30, 49]. Soon it became clear that the above-mentioned results share a common background which, expressed in the language of category theory, brought into light a powerful theory of *natural dualities* [15, 16, 18, 20, 52]. Its main interest lies in establishing a dual isomorphism between a quasi-variety generated by a finite algebra (a finite set of finite algebras) and a particular category of structured (actually, enriched) topological spaces called *topological quasi-variety.* The technique is based on the concept of the so-called *schizophrenic object*, which has two personalities: algebraic and topological, the former (resp. latter) generating the algebraic (resp. topological) quasi-variety in question. The approach provides a common framework for many existing representation theories (e.g., the above-mentioned results of H. Priestley and M. Stone fit perfectly into the new setting), on one hand, and serves as a useful tool for generating new ones, on the other.

The notion of (*L*-)*fuzzy set* of L. A. Zadeh [78] and J. A. Goguen [28] brought a fresh challenge into the theory of topological representations. The new area of mathematics called *fuzzy* necessitated a new setting for well-known notions, based on the procedure of *fuzzification* [28]. In particular, numerous attempts were made to provide a fuzzy version of the above-mentioned representation theorems of M. Stone and H. Priestley. The most successful ones are due to S. E. Rodabaugh [57, 59], who considered the representation theories of M. Stone, applying his new *point-set lattice-theoretic* (*poslat*) technique. The results obtained not only generalized the classical theory, but in some cases uniquely streamlined it via two explicit evaluation maps. Motivated by the achievements, we provided their generalizations [73, 74], relying on our own *categorically-algebraic* (*catalg*) approach (the word "categorically" stems from "category theory"). The crucial difference from the ideas of S. E. Rodabaugh was the use of an arbitrary variety of algebras instead of a fixed category of lattices of particular kind (mostly, semi-quantales, currently popular in the fuzzy community), in which the fuzzification in question was being made. This brought our theory more inline with the above-mentioned natural dualities of D. Clark and B. Davey [15].

The case of Priestley duality appeared to be more difficult to attack. Up to now, there has been no fruitful fuzzification of the machinery, suitable for applications. In [70] we tried to fill in the gap, introducing a catalg framework for the duality in question. The theory presented was based on two important steps. On the first one, we showed sufficient conditions for an adjunction to exist between the dual category of a variety of algebras and the category of (catalg) topological spaces enriched in a variety of relational structures [17]. On the second step, we singled out particular subcategories (the so-called *spatial algebras* and *sober topological spaces*), the restriction to which of the obtained adjunction provided an equivalence. The setting was based on the well-known sobriety-spatiality approach to the representation theorems of M. Stone promoted by P. T. Johnstone, A. Pultr, S. Vickers, *etc.* [39, 54, 77], and also involved certain aspects of the theory of natural dualities (the notion of schizophrenic object was generalized to construct the adjunction in question; the structured topological spaces, however, were truncated to relational ones). The underlying machinery was borrowed from our previous research on the Stone dualities [73, 74], relying on the concept of *powerset operator*, coming from a generalization of the classical *image* and *preimage* operators induced by a map.

Soon afterwards, it appeared that the obtained theory provides a common framework not only for both the Priestley and the Stone representation theorems, but also incorporates the above-mentioned natural dualities (the category of structured topological spaces is, in fact, a particular instance of the category of relational ones; moreover, our schizophrenic object needs to be neither finite nor have the discrete topology), bringing into light their catalg properties, the study of which is indispensable for their development. This paper presents the emerging approach, calling it the theory of *catalg dualities.* Its most crucial property is applicability to both crisp and fuzzy topologies (providing a fuzzification of natural dualities), that extends considerably the field of potential applications and makes another step towards our ultimate goal of erasing the border between traditional and fuzzy mathematics. The theory also underlines the advantage of our catalg approach over the poslat one of S. E. Rodabaugh [56], the latter one, tied to lattices, being unable to switch to arbitrary algebras.

Looking closely at Priestley duality (which served as a motivation for our theory), a working algebraist will easily notice that it is not the result itself, but the amount of representation theorems based on it, that constitutes its importance in mathematics. As an example, recall topological representations of *Q-distributive lattices* [12], *J-distributive lattices* [49], *$\neg$-lattices* [8, 9], (*weak-*) *quasi-Stone algebras* [10, 26], *complex algebras* [30], *etc.* All of them rely on the fact that the algebraic structure in question has a bounded distributive lattice as a reduct and therefore, Priestley duality is at hand. Additional operations on the lattices are then compensated by certain relations on their respective Priestley spaces, in such a way that the underlying duality for distributive lattices can be lifted to the new setting providing the desired topological representation. As was already mentioned, the gateway to topology makes the solution of some problems much easier. For example, it is possible to characterize congruences and thus, to get an insight into simple and subdirectly irreducible algebras [8, 27, 30, 49]. Moreover, R. Cignoli [13] uses the duality to construct free Q-distributive lattices from bounded distributive lattices, whereas H. Gaitán [26] does the same job for quasi-Stone algebras. In this paper, we show a catalg framework for the above-mentioned procedures. More particularly, given two varieties of algebras $\mathbf{C}$ and $\mathbf{C}'$ such that $\mathbf{C}'$ is a reduct of $\mathbf{C}$, we investigate thoroughly the possibilities of obtaining a topological representation for $\mathbf{C}$ (resp. $\mathbf{C}'$) from that for $\mathbf{C}'$ (resp. $\mathbf{C}$). The results obtained are illustrated by the examples of J-distributive lattices of A. Petrovich [49] and $\neg$-lattices of S. Celani [8], underlying almost all of the above-mentioned derived representation theorems. It is important to notice that the problem of reducts has already been considered in the theory of natural dualities under the name of *piggyback dualities* [15, 52]. The crucial difference of our approach from the already considered in the literature is the lack of an explicit construction of the duality in question in terms of particular algebras and topological spaces. Motivated by the essence of category theory itself, we provide instead a common catalg framework for the machinery employed, leaving it for the researcher to find its concrete realization in each particular case. The situation is similar to that of *adjoint situation* [36], which is defined in the abstract language of categories, without providing an explicit way of its obtainment (apart from general existence conditions) in every specific case.

The paper uses both category theory and algebra, relying more on the former. The necessary categorical background can be found in [1, 36, 45, 46]. For the notions of universal algebra we recommend [7, 17, 31, 46]. Although we tried to make the paper as much self-contained as possible, some details are still omitted and left to the reader.

2. Categorically-algebraic topology

In this section we recall from [70] basic concepts of *categorically-algebraic* (*catalg*) *topology* (see also [67, 68, 69, 66]). The approach is motivated (but is destined to replace) by the currently so popular *point-set lattice-theoretic* (*poslat*) one, introduced by S. E. Rodabaugh [56] and developed by P. Eklund, C. Guido, U. Höhle, T. Kubiak, A. Šostak and the initiator himself [24, 32, 33, 37, 42, 43, 58]. The main advantage of the new setting is the fact that the catalg framework ultimately erases the border between the traditional and the fuzzy developments, producing a theory which underlines the algebraic essence of the whole (not only fuzzy) mathematics and thus, propagating algebra as the main driving force of modern exact sciences.

The cornerstone of our approach is the notion of *algebra*. The structure is to be thought of as a set with a family of operations defined on it, satisfying certain identities, e.g., semigroup, monoid, group and also complete lattice, frame, quantale. In case of *finitary algebras*, i.e., those induced by a set of finitary operations, there are (at least) two ways of describing the resulting entities [7, 17, 31]. The categorical one uses the concept of *variety*, i.e., a class of algebras closed under homomorphic images, subalgebras and direct products. The algebraic one is based on the notion of *equational class*, namely, providing a set of identities and taking precisely those algebras which satisfy all of them. The well-known HSP-theorem of G. Birkhoff [4] says that varieties and equational classes coincide. Motivated by the algebraic structures used in fuzzy topology (where unions are usually represented as joins), this paper includes infinitary cases as well, extending the categorical approach of varieties to cover its needs, and leaving aside the infinitary algebraic machineries of *equationally-definable class* [46] and *equational category* [44, 64].

Definition 1.

- Let $\Omega = (n_\lambda)_{\lambda\in\Lambda}$ be a (possibly proper) class of cardinal numbers. An Ω-*algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$, which consists of a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$, called n_λ-*ary operations* on A. An Ω-*homomorphism* $(A, (\omega_\lambda^A)_{\lambda\in\Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda\in\Lambda})$ is a map $A \xrightarrow{\varphi} B$ making the diagram

$$\begin{tikzcd} A^{n_\lambda} \arrow[r, "\varphi^{n_\lambda}"] \arrow[d, "\omega_\lambda^A"'] & B^{n_\lambda} \arrow[d, "\omega_\lambda^B"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

  commute for every $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms, with the underlying functor denoted by $|-|$.
- Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of* Ω-*algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called *algebras* (resp. *homomorphisms*).
- Let $\mathbf{A}$ be a variety of Ω-algebras and let Ω' be a subclass of Ω. An Ω'-*reduct* of $\mathbf{A}$ is a pair $(\|-\|, \mathbf{B})$, where $\mathbf{B}$ is a variety of Ω'-algebras and $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ is a concrete functor.

The concept can be illustrated by several examples, all of which (except the last one) are currently rather popular in fuzzy topology [61, 63], due to the fact that their induced categories of fuzzified structures are topological over their ground categories. The last item in the list was motivated by our interest in *closure spaces* and their interrelationships with *state property systems* [2, 3, 72], introduced as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics and modeling an arbitrary physical system by means of its set of states, its set of properties, and a relation of "actuality of a certain property for a certain state". A catalg modification of the notion has been developed by us in [72].

Definition 2.

- Given $\Xi \in \{\bigvee, \bigwedge\}$, a Ξ-*semilattice* is a partially ordered set having arbitrary Ξ. $\mathbf{CSLat}(\Xi)$ is the variety of Ξ-semilattices.
- A *semi-quantale* (*s-quantale*) is a $\bigvee$-semilattice equipped with a binary operation $\otimes$ (*multiplication*). **SQuant** is the variety of s-quantales.
- An s-quantale is called *unital* (*us-quantale*) provided that its multiplication has the unit $\imath$. **USQuant** is the variety of us-quantales.
- An s-quantale is called *distributive* (*ds-quantale*) provided that its multiplication distributes across finite $\vee$ from both sides. **DSQuant** is the variety of ds-quantales.
- An s-quantale is called *DeMorgan* provided that it is equipped with an order-reversing involution $(-)'$. **DmSQuant** is the variety of DeMorgan s-quantales.
- A *quantale* is an s-quantale whose multiplication is associative and distributes across $\bigvee$ from both sides. **Quant** is the variety of quantales.
- A *semi-frame* (*s-frame*) is a us-quantale whose multiplication and unit are $\wedge$ and $\top$ respectively. **SFrm** is the variety of s-frames.
- A *frame* is an s-frame which is a quantale. **Frm** is the variety of frames.
- A *closure semilattice* (*c-semilattice*) is a $\bigwedge$-semilattice, with the singled out bottom element $\perp$. **CSL** is the variety of c-semilattices.

The reader should be aware that our concept of ds-quantale is a stronger version of *ordered s-quantale* of [61, 63], where monotonicity instead of $\vee$-distributivity is postulated. The new notion was motivated by the definition of variety, i.e., its closure under homomorphic images that fails in the weaker case. Also notice the simple facts: $\mathbf{CSLat}(\bigvee)$ is a reduct of **SQuant**; **SQuant** is a reduct of **USQuant**, **DSQuant** and **DmSQuant**; **DSQuant** is a reduct of **Quant**; **USQuant** is a reduct of **SFrm**; **UQuant** is a reduct of **Frm**; $\mathbf{CSLat}(\bigwedge)$ is a reduct of **CSL**.

For the sake of convenience, from now on we use the following notations (see, e.g., [23, 58, 61] for the motivation). An arbitrary variety is denoted by **A**, **B**, **C**, *etc.* (sometimes with indices). The categorical dual of a variety **A** is denoted by **LoA** (the "**Lo**" comes from "localic"), whose objects (resp. morphisms) are called *localic algebras* (resp. *homomorphisms*). Following the already accepted notations of [39], the dual of **Frm** is denoted by **Loc**, whose objects are called *locales*. Given a localic algebra A, $\mathbf{S}_A$ stands for the subcategory of **LoA** with the only morphism 1_A. To distinguish maps (or, more generally, morphisms) and homomorphisms, the former are denoted by f, g, h (α, β, γ in case of fuzzy sets), reserving φ, ψ, ϕ for the latter. Given a homomorphism φ, the respective localic one is denoted by φ^{op} and vice versa.

The second crucial notion of our approach is a mixture of *powerset theories* of [61, Definition 3.5] (see also [60, 63]) and *topological theories* of [1, Exercise 22B].

Definition 3. A *variety-based backward powerset theory* (*vbp-theory*) in a category $\mathbf{X}$ (the *ground category* of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$.

The intuition for the new concept comes from the so-called *image* (resp. *preimage*) operators [61], well-known for every working mathematician. Recall that given a set map $X \xrightarrow{f} Y$, there exist the maps $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ (resp. $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$) with $f^{\rightarrow}(S) = \{f(x) \,|\, x \in S\}$ (resp. $f^{\leftarrow}(T) = \{x \,|\, f(x) \in T\}$). The latter operator can be extended to a more general setting.

Example 4. Given a variety $\mathbf{A}$, every subcategory $\mathbf{C}$ of **LoA** induces a functor $\mathbf{Set} \times \mathbf{C} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$ defined by $((X, A) \xrightarrow{(f,\varphi)} (Y, B))^{\leftarrow} = A^X \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} B^Y$ with $(f, \varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$. Considered as a vbp-theory $\mathcal{S}_{\mathbf{A}}^{\mathbf{C}}$, the functor was used in our former approach to catalg topology of, e.g., [73, 74], incorporating at the same time a multitude of important subcases, some of which are listed below.

(1) $\mathbf{Set} \times \mathbf{S_2} \xrightarrow{\mathcal{P}=(-)^{\leftarrow}} \mathbf{LoCBool}$ with **CBool** the variety of *complete Boolean algebras* (complete, complemented, distributive lattices) and $\mathbf{2} = \{\bot, \top\}$, provides the above-mentioned preimage operator.
(2) $\mathbf{Set} \times \mathbf{S}_I \xrightarrow{\mathcal{Z}=(-)^{\leftarrow}_I} \mathbf{DmLoc}$ with $I = [0, 1]$ the unit interval, provides the fixed-basis fuzzy approach of L. A. Zadeh [78].
(3) $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}_1=(-)^{\leftarrow}_L} \mathbf{Loc}$ provides the fixed-basis L-fuzzy approach of J. A. Goguen [28]. The setting was changed to $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}_2=(-)^{\leftarrow}_L} \mathbf{LoUQuant}$ in [29]. The machinery can be generalized to an arbitrary variety $\mathbf{A}$ and the theory $\mathcal{S}_{\mathbf{A}}^{\mathbf{S}_A}$, uniting the previous items in one common fixed-basis framework.
(4) $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}_1^{\mathbf{C}}=(-)^{\leftarrow}} \mathbf{DmLoc}$ with $\mathbf{C}$ a subcategory of **DmLoc**, gives the variable-basis poslat approach of S. E. Rodabaugh [55]. The setting was generalized to $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}_2^{\mathbf{C}}=(-)^{\leftarrow}} \mathbf{LoUSQuant}$ in [61].

(5) $\mathbf{Set} \times \mathbf{FuzLat} \xrightarrow{\mathcal{E}=(-)^{\leftarrow}} \mathbf{FuzLat}$ provides the variable-basis approach of P. Eklund [24], motivated by those of S. E. Rodabaugh [55] and B. Hutton [38]. Notice that **FuzLat** is the dual of the variety **HUT** of completely distributive DeMorgan frames called *Hutton algebras* [58].

Two important points should be mentioned at once. Firstly, the topic of the paper restricts us to the ground categories of the form $\mathbf{Set} \times \mathbf{LoA}$. In [67, 71] more general categories come into account, motivated by *generalized topology* of M. Demirci [21, 22] and *non-commutative topology* of C. J. Mulvey and J. W. Pelletier [47, 48]. Secondly, Example 4 deals with the preimage operator, leaving the image one aside. The reason is that the current fuzzifications of the map are $\bigvee$-dependant (e.g., having the form of $(f_A^{\rightarrow}(\alpha))(y) = \bigvee_{f(x)=y} \alpha(x)$ in the fixed-basis case), whereas a general variety may lack even a partial order. An additional restriction on the variety in question (the existence of categorical *biproducts* [36]) allows one to restore the full framework [68].

The next concept is a modification of *composite topological theories* of [67]. The crucial change is that they are no longer the powerset theories in question (before going forward, the reader is advised to recall the construction of *product of categories* [36]). To avoid unnecessary complications (touched in [67]), from now on, "set-indexed" means "indexed by a *non-empty* set".

Definition 5. Let $\mathbf{X}$ be a category and let $\mathcal{T}_I = ((P_i, (\| - \|_i, \mathbf{B}_i)))_{i \in I}$ be a set-indexed family, where for every $i \in I$, $\mathbf{X} \xrightarrow{P_i} \mathbf{LoA}_i$ is a vbp-theory in the category $\mathbf{X}$ and $(\| - \|_i, \mathbf{B}_i)$ is a reduct of $\mathbf{A}_i$. A *composite variety-based topological theory* (*cvt-theory*) *in* $\mathbf{X}$ *induced by* $\mathcal{T}_I$ is the functor $\mathbf{X} \xrightarrow{T_I = \langle \| - \|_i^{op} \circ P_i \rangle_I} \prod_{i \in I} \mathbf{LoB}_i$ defined by commutativity of the diagram

$$\begin{tikzcd} \mathbf{X} \arrow[r, "P_j"] \arrow[d, dotted, "T_I"'] & \mathbf{LoA}_j \arrow[d, "\| - \|_j^{op}"] \\ \prod_{i \in I} \mathbf{LoB}_i \arrow[r, "\pi_j"'] & \mathbf{LoB}_j \end{tikzcd}$$

for every $j \in I$ (π_j is the respective projection).

Since a cvt-theory is completely determined by the respective family $\mathcal{T}_I$, we use occasionally the notation $((P_i, \mathbf{B}_i))_{i \in I}$ instead of T_I. A cvt-theory induced by a singleton family is denoted by T. We also employ the shorter T_i for $\| - \|_i^{op} \circ P_i$. All preliminaries on their places, we are ready to introduce catalg topology (notice that the use of the standard image operator in the definition of continuity was motivated by purely esthetic reasons).

Definition 6. Let T_I be a cvt-theory in the category $\mathbf{X}$. $\mathbf{CTop}(T_I)$ is the concrete category over $\mathbf{X}$, whose objects (called *composite variety-based topological spaces*) are pairs $(X, (\tau_i)_{i\in I})$ with X an $\mathbf{X}$-object and τ_i a subalgebra of $T_i(X)$ for every $i \in I$ ($(\tau_i)_{i\in I}$ is called *composite variety-based topology* on X), and whose morphisms $(X, (\tau_i)_{i\in I}) \xrightarrow{f} (Y, (\sigma_i)_{i\in I})$ are those $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ that satisfy $((T_i f)^{op})^{\rightarrow}(\sigma_i) \subseteq \tau_i$ for every $i \in I$ (called *composite continuity*). The underlying functor to the ground category $\mathbf{X}$ is defined by $|(X, (\tau_i)_{i\in I}) \xrightarrow{f} (Y, (\sigma_i)_{i\in I})| = X \xrightarrow{f} Y$.

For the sake of simplicity, $\mathbf{CTop}(T)$ is denoted by $\mathbf{Top}(T)$. The new concept was motivated by the multitude of approaches to topology in the fuzzy community. Our main purpose was to provide a common unifying framework suitable for exploring interrelations between different topological settings. The machinery employed was inspired by *(bi)topological theories* of S. E. Rodabaugh [61, 62] and T. Kubiak [42].

Example 7. The following provides a short list of examples illustrating the notion of catalg topology, to give the feeling of their abundance and the fruitfulness of the new unifying framework.

(1) $\mathbf{Top}((\mathcal{P}, \mathbf{Frm}))$ is isomorphic to the category $\mathbf{Top}$ of topological spaces and continuous maps.
(2) $\mathbf{Top}((\mathcal{P}, \mathbf{CSL}))$ is isomorphic to the category $\mathbf{Cls}$ of closure spaces and continuous maps studied by D. Aerts *et al.* [2, 3].
(3) $\mathbf{CTop}(((\mathcal{P}, \mathbf{Frm}))_{i\in\{1,2\}})$ is isomorphic to the category $\mathbf{BiTop}$ of bitopological spaces and bicontinous maps [41].
(4) $\mathbf{Top}((\mathcal{Z}, \mathbf{Frm}))$ is isomorphic to the category I-$\mathbf{Top}$ of fixed-basis fuzzy topological spaces introduced by C. L. Chang [11].
(5) $\mathbf{Top}((\mathcal{G}_2, \mathbf{UQuant}))$ is isomorphic to the category L-$\mathbf{Top}$ of fixed-basis L-fuzzy topological spaces of J. A. Goguen [29].
(6) $\mathbf{Top}((\mathcal{R}_i^{\mathbf{C}}, \mathbf{USQuant}))$ is isomorphic to the category $\mathbf{C}$-$\mathbf{Top}_i$ ($i \in \{1, 2\}$) for variable-basis poslat topology of S. E. Rodabaugh [55, 61].
(7) $\mathbf{CTop}(((\mathcal{R}_1^{\mathbf{S}_L}, \mathbf{Frm}))_{i\in\{1,2\}})$ is isomorphic to the category L-$\mathbf{BiTop}$ of fixed-basis L-bitopological spaces of T. Kubiak [42].
(8) $\mathbf{CTop}(((\mathcal{R}_2^{\mathbf{S}_L}, \mathbf{USQuant}))_{i\in\{1,2\}})$ is isomorphic to the category L-$\mathbf{BiTop}$ of fixed-basis L-bitopological spaces of S. E. Rodabaugh [62].
(9) $\mathbf{Top}((\mathcal{E}, \mathbf{Frm}))$ is isomorphic to the category $\mathbf{FUZZ}$ for variable-basis poslat topology of P. Eklund [24], motivated by those of S. E. Rodabaugh [55] and B. Hutton [38].
(10) $\mathbf{Top}((\mathcal{S}_{\mathbf{A}}^{\mathbf{S}_Q}, \mathbf{A}))$ (resp. $\mathbf{Top}((\mathcal{S}_{\mathbf{A}}^{\mathbf{LoA}}, \mathbf{A}))$) is isomorphic to the fixed- (resp. variable-) basis category Q-$\mathbf{Top}$ (resp. $\mathbf{LoA}$-$\mathbf{Top}$) used in our former approach to catalg topologies [74] (resp. [73]).

Notice the frequent use of the variety **USQuant**, whose objects are proposed by S. E. Rodabaugh [61] as the basic mathematical structure for doing poslat topology upon, since the axioms of s-quantales constitute the minimum allowing the obtained categories for topology (and these include many well-known categories) to be topological over their ground categories. The claim was justified through our catalg approach in [69].

The reader may be well aware of the important result of classical topology stating that continuity of a map can be checked on the elements of a subbase (for a full discussion of the (categorical) role of subbase in topology see [58]). It appears that the result can be readily extended to our current setting.

Definition 8.

- Let $\mathbf{A}$ be a variety of Ω-algebras and let $\Omega' \subseteq \Omega$ be a (possibly) subclass. Given an algebra A and a subset $S \subseteq A$, $\langle S\rangle_{\Omega'}$ denotes the smallest Ω'-subreduct of A containing S ($\langle S\rangle_{\Omega}$ is shortened to $\langle S\rangle$).
- Given a cvt-theory $\mathbf{X} \xrightarrow{T} \mathbf{LoB}$ with $\mathbf{B}$ an Ω'-reduct of $\mathbf{A}$, a subclass $\Omega'' \subseteq \Omega'$ and a $\mathbf{Top}(T)$-space (X,τ), a subset $S \subseteq T(X)$ is an Ω''-*base* for τ provided that $\tau = \langle S\rangle_{\Omega''}$. Ω'-bases are called *subbases*.

The next example provides the intuition for the new concept, justifying its fruitfulness.

Example 9. In the category $\mathbf{C}\text{-}\mathbf{Top}_2$, $\{\bigvee\}$-bases (resp. $\{\bigvee, \otimes, 1\}$-bases) are well-known bases (resp. subbases) of poslat topology as defined in, e.g., [62]. **Top** gives the classical definition of base (resp. subbase), where elements of the topology are unions of (resp. unions of finite intersections of) elements of the base (resp. subbase).

Lemma 10. *Let $A_1 \xrightarrow{\varphi} A_2$ be a homomorphism of a variety $\mathbf{A}$ of Ω-algebras and let $\Omega' \subseteq \Omega$.*

(1) For every Ω'-subreduct B of A_2, $\varphi^{\leftarrow}(B)$ is an Ω'-subreduct of A_1.
(2) For every subset $S \subseteq A_1$, $\varphi^{\rightarrow}(\langle S\rangle_{\Omega'}) = \langle \varphi^{\rightarrow}(S)\rangle_{\Omega'}$.

Proof. Since Item (1) is easy, we show Item (2). $S \subseteq (\varphi^{\leftarrow}\circ\varphi^{\rightarrow})(S) \subseteq \varphi^{\leftarrow}(\langle\varphi^{\rightarrow}(S)\rangle_{\Omega'})$ gives $\langle S\rangle_{\Omega'} \subseteq \varphi^{\leftarrow}(\langle\varphi^{\rightarrow}(S)\rangle_{\Omega'})$ by Item (1) and therefore $\varphi^{\rightarrow}(\langle S\rangle_{\Omega'}) \subseteq (\varphi^{\rightarrow}\circ \varphi^{\leftarrow})(\langle\varphi^{\rightarrow}(S)\rangle_{\Omega'}) \subseteq \langle\varphi^{\rightarrow}(S)\rangle_{\Omega'}$. Conversely, $S \subseteq \langle S\rangle_{\Omega'}$ gives $\varphi^{\rightarrow}(S) \subseteq \varphi^{\rightarrow}(\langle S\rangle_{\Omega'})$ and thus $\langle\varphi^{\rightarrow}(S)\rangle_{\Omega'} \subseteq \langle\varphi^{\rightarrow}(\langle S\rangle_{\Omega'})\rangle_{\Omega'} = \varphi^{\rightarrow}(\langle S\rangle_{\Omega'})$. $\square$

Corollary 11. *Let T_I be a cvt-theory in a category $\mathbf{X}$ and let $(X,(\tau_i)_{i\in I})$, $(Y,(\sigma_i)_{i\in I})$ be $\mathbf{CTop}(T_I)$-spaces with $\sigma_i = \langle S_i\rangle_{\Omega''_i}$ for every $i \in I$. An $\mathbf{X}$-morphism $X \xrightarrow{f} Y$ is continuous iff $((T_i f)^{op})^{\rightarrow}(S_i) \subseteq \tau_i$ for every $i \in I$.*

Proof. To show the sufficiency, use Lemma 10(2) in the following: $((T_i f)^{op})^{\rightarrow}(\sigma_i) = ((T_i f)^{op})^{\rightarrow}(\langle S_i\rangle_{\Omega''_i}) = \langle((T_i f)^{op})^{\rightarrow}(S_i)\rangle_{\Omega''_i} \subseteq \langle\tau_i\rangle_{\Omega''_i} = \tau_i$. $\square$

Corollary 11 provides a generalization of the above-mentioned achievement for subbases in various settings (cf., e.g., Theorem 3.2.6 of [58]). In particular, the result incorporates directly the cases of classical subbase and base (no additional calculation is required as in, e.g., [25, Proposition 1.4.1]).

It may be well-known to the reader that **Top** has products of objects (Cartesian products of the underlying sets with the initial topology induced by projections). In conclusion of the section, we show that the result holds in a more general setting of catalg topologies.

Lemma 12. *Let T_I be a cvt-theory in a category* $\mathbf{X}$ *and let* $\mathbf{X}$ *have products. Then the category* $\mathbf{CTop}(T_I)$ *has concrete products.*

Proof. Given a set-indexed family $((X_j, (\tau_{j_i})_{i\in I}))_{j\in J}$ of $\mathbf{CTop}(T_I)$-spaces, the desired product is given by $((\prod_{k\in J} X_k, (\prod_{k\in J} \tau_{k_i})_{i\in I}) \xrightarrow{\pi_j} (X_j, (\tau_{j_i})_{i\in I}))_{j\in J}$, where $(\prod_{k\in J} X_k \xrightarrow{\pi_j} X_j)_{j\in J}$ is the product of $(X_j)_{j\in J}$ in the category $\mathbf{X}$ (implying concreteness) and $\prod_{k\in J} \tau_{k_i} = \langle \bigcup_{j\in J}((T_i\pi_j)^{op})^{\rightarrow}(\tau_{j_i})\rangle$ for every $i \in I$. $\square$

A simple application of Lemma 12 runs as follows (recall the category Q-**Top** of Example 7(10)).

Corollary 13. *The category* Q-**Top** *has concrete products.*

Notice that the respective result for the category **LoA**-**Top** does not necessarily hold, since in general $\mathbf{A}$ may lack coproducts (cf., e.g., the variety **CLat** of complete lattices [1, Exercise 10S]).

3. Categorically-algebraic dualities

In this section we provide a catalg approach to the theory of *natural dualities* developed by D. Clark and B. Davey in [15]. The cited theory is based on the concept of the so-called *schizophrenic object*, i.e., a finite set M equipped with two structures: algebraic (providing an algebra M_A) and topological (assumed to be discrete) with some additional enrichment consisting of finitary total and partial operations as well as finitary relations (providing a *structured topological space* M_T). Under suitable conditions, the theory developed produces a dual equivalence (the so-called *natural duality*) between the algebraic quasi-variety generated by M_A and the appropriately defined topological quasi-variety obtained from M_T.

The approach of this paper differs from the above-mentioned one in several respects. The most important points are underlined below.

- Every requirement of finiteness on the structures in question is dropped.
- Topological enrichment is reduced to a family of relations, incorporating both total and partial operations as their particular kinds.

- Arbitrary topologies on the set M are allowed, extending the framework considerably.
- Algebraic (resp. topological) quasi-varieties are replaced with the notion of *spatiality* (resp. *sobriety*), producing an equivalence between the categories of spatial algebras and sober spaces in the sense of P. T. Johnstone [39] (see also [54, 77]).
- The category **Top** is replaced with the category Q-**Top** of Example 7(10), providing catalg fuzzification.

Having an informal description of what follows in hand, we turn to the explicit construction of the promised machinery. The results obtained come from [70], where we restricted our attention to the case of H. Priestley duality for bounded distributive lattices [53]. It is the purpose of this section to present the achievements in a more general light of natural dualities.

3.1. **Underlying adjunction.** For the sake of simplicity, we reduce the topological setting to the fixed-basis category $\mathbf{Top}((\mathcal{S}_{\mathbf{A}}^{\mathbf{S}_Q}, \mathbf{B}))$, denoted by $Q_{\mathbf{B}}$-**Top**, with the prefix "$Q_{\mathbf{B}}$" added to the respective topological stuff, e.g., "$Q_{\mathbf{B}}$-space", "$Q_{\mathbf{B}}$-topology", "$Q_{\mathbf{B}}$-continuity" (we also use the notation τ_X, to underline the set, a given $Q_{\mathbf{B}}$-topology τ is referring to), leaving the (composite) variable-basis generalization to the subsequent developments of the topic. On the next step, we show an analogue of the aforesaid structured topological space, suitable for our framework and motivated by *relational structures* of [17, Chapter V] (the reader should recall Definition 1).

Definition 14.

- Let $\Sigma = (m_\upsilon)_{\upsilon \in \Upsilon}$ be a (possibly proper) class of cardinal numbers. A Σ-*structure* is a pair $(R,(\varpi_\upsilon^R)_{\upsilon\in\Upsilon})$, which consists of a set R and a family of subsets $\varpi_\upsilon^R \subseteq R^{m_\upsilon}$, called m_υ-*ary relations* on R. A Σ-*homomorphism* $(R,(\varpi_\upsilon^R)_{\upsilon\in\Upsilon}) \xrightarrow{f} (S,(\varpi_\upsilon^S)_{\upsilon\in\Upsilon})$ is a map $R \xrightarrow{f} S$ such that $(f^{m_\upsilon})^{\rightarrow}(\varpi_\upsilon^R) \subseteq \varpi_\upsilon^S$ for every $\upsilon \in \Upsilon$. $\mathbf{Rel}(\Sigma)$ is the construct of Σ-structures and Σ-homomorphisms, with the underlying functor denoted by $|-|$.
- Let $\mathcal{R}$ be the class of all Σ-homomorphisms $R \xrightarrow{f} S$ such that for every $\upsilon \in \Upsilon$ and every $\langle r_i \rangle_{m_\upsilon} \in R^{m_\upsilon}$, $\langle f(r_i)\rangle_{m_\upsilon} \in \varpi_\upsilon^S$ implies $\langle r_i\rangle_{m_\upsilon} \in \varpi_\upsilon^R$, and let $\mathcal{M}$ (resp. $\mathcal{E}$) be the subclass of $\mathcal{R}$ with injective (resp. surjective) underlying maps. A *variety of* Σ-*structures* is a full subcategory of $\mathbf{Rel}(\Sigma)$ closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients. The objects (resp. morphisms) of a variety are called *structures* (resp. *homomorphisms*).

- Let $\mathbf{R}$ be a variety of Σ-structures and let Σ' be a subclass of Σ. A Σ'-*reduct* of $\mathbf{R}$ is a pair $(\|-\|, \mathbf{S})$, where $\mathbf{S}$ is a variety of Σ'-structures and $\mathbf{R} \xrightarrow{\|-\|} \mathbf{S}$ is a concrete functor.

From now on, varieties of Σ-structures are denoted by $\mathbf{R}$, $\mathbf{S}$, $\mathbf{T}$, *etc.* To avoid ambiguity with varieties of Ω-algebras, we add the letter "r" (stemming from "relational") to their name, i.e., *r-variety*, referring to their objects (resp. morphisms) as *r-structures* (resp. *r-homomorphisms*).

Example 15. The construct **Pos** of partially ordered sets and order-preserving maps is an r-variety induced by the category $\mathbf{Rel}(2)$, whose signature consists of a single binary relation.

Example 16. Given a category $\mathbf{Alg}(\Omega)$, define $\Sigma = (n_\lambda + 1)_{\lambda\in\Lambda}$ and get the category $\mathbf{Rel}(\Sigma)$. There exists a concrete functor $\mathbf{Alg}(\Omega) \xrightarrow{V} \mathbf{Rel}(\Sigma)$ defined by $V((A_1, (\omega_\lambda^{A_1})_{\lambda\in\Lambda}) \xrightarrow{\varphi} (A_2, (\omega_\lambda^{A_2})_{\lambda\in\Lambda})) = (A_1, (\varpi_\lambda^{A_1})_{\lambda\in\Lambda}) \xrightarrow{\varphi} (A_2, (\varpi_\lambda^{A_2})_{\lambda\in\Lambda})$ where $\varpi_\lambda^{A_j} = \mathrm{Grph}\,\omega_\lambda^{A_j} = \{\langle\langle a_i\rangle_{n_\lambda}, a\rangle \,|\, \omega_\lambda^{A_j}(\langle a_i\rangle_{n_\lambda}) = a\} \subseteq A_j^{n_\lambda} \times A_j$ for every $\lambda \in \Lambda$ and every $j \in \{1, 2\}$. The image under V of a variety $\mathbf{A}$ (denoted by $V^{\rightarrow}(\mathbf{A})$) is closed under the formation of products, but may miss the closure under $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients (every subset (resp. quotient set) of a Σ-structure gives rise to its $\mathcal{M}$-subobject (resp. $\mathcal{E}$-quotient), that almost never holds for algebras). The smallest r-variety containing $V^{\rightarrow}(\mathbf{A})$ (which exists since the family of r-varieties of the same signature is closed under (possibly class-indexed) intersections) gives the one corresponding to $\mathbf{A}$.

It is important to keep in mind that unlike the case of algebras, a bijective r-homomorphism is *not* necessarily an r-isomorphism. Due to the lack of space, we will not dwell upon other properties of r-varieties, modifying the category $Q_{\mathbf{B}}$-**Top** with their help instead.

Definition 17. Given an r-variety $\mathbf{R}$, $Q_{\mathbf{B}}$-**RTop** is the category, whose objects (called *r-$Q_{\mathbf{B}}$-spaces*) are pairs (R, τ) with R an r-structure and $(|R|, \tau)$ a $Q_{\mathbf{B}}$-space, and whose morphisms $(R, \tau) \xrightarrow{f} (S, \sigma)$ are those $Q_{\mathbf{B}}$-continuous maps $(|R|, \tau) \xrightarrow{f} (|S|, \sigma)$ that are also r-homomorphisms (called *r-$Q_{\mathbf{B}}$-morphisms*). The underlying functor to the ground category $Q_{\mathbf{B}}$-**Top** is defined by $|(R, \tau) \xrightarrow{f} (S, \sigma)| = (|R|, \tau) \xrightarrow{f} (|S|, \sigma)$.

In the language of enriched category theory of G. M. Kelly [40], $Q_{\mathbf{B}}$-**RTop** is nothing else than the category $Q_{\mathbf{B}}$-**Top** enriched in an r-variety $\mathbf{R}$.

Example 18. The category **Top** enriched in the r-variety **Pos** yields the category **PoTop** of partially-ordered topological spaces and order-preserving continuous maps.

The reader may remember that dualities of [15] consist of a dual equivalence between the algebraic quasi-variety $\boldsymbol{\mathcal{A}}$ generated by M_A (algebraic personality of the schizophrenic object in question) and the respective topological quasi-variety $\boldsymbol{\mathfrak{X}}$ induced by M_T (topological personality of M). We propose the following generalization of the machinery, motivated by the approach to the Stone representation theories of P. T. Johnstone [39].

Step 1.: Replace $\boldsymbol{\mathfrak{X}}$ (resp. $\boldsymbol{\mathcal{A}}$) with $Q_{\mathbf{B}}$-**RTop** (resp. a variety $\mathbf{C}$).

Step 2.: Construct two functors $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{E} \mathbf{LoC}$ and $\mathbf{LoC} \xrightarrow{D} Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ with D a right adjoint to E.

Step 3.: Single out particular subcategories of $Q_{\mathbf{B}}$-**RTop** (resp. **LoC**), the restriction to which of the adjunction obtained provides an equivalence.

Having outlined briefly the forthcoming developments, we proceed to the explicit construction.

We begin by fixing a variety $\mathbf{C}$, with the ultimate goal to construct a functor $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{E} \mathbf{LoC}$. Two preliminary concepts are necessary for the completion of the task. The first one is a modification of the notion of reduct, already encountered by the reader in the paper (Definitions 1, 14).

Definition 19. An *r-reduct* of a variety $\mathbf{C}$ is a pair $(\| - \|, \mathbf{S})$, where $\mathbf{S}$ is an r-variety and $\mathbf{C} \xrightarrow{\|-\|} \mathbf{S}$ is a concrete functor. An r-reduct is *algebraic* provided that for every algebra C and every $\upsilon \in \Upsilon_{\mathbf{S}}$, $\varpi_{\upsilon}^{\|C\|}$ is a subalgebra of $C^{m_{\upsilon}}$.

Notice the notation $\Upsilon_{\mathbf{S}}$ for the signature of $\mathbf{S}$ in Definition 19, which will be used frequently in the subsequent developments. Also important is the fact that Definition 19 never assumes any connection between the signatures of $\mathbf{C}$ and $\mathbf{S}$. The next example gives the intuition for the new concept.

Example 20. The functor $\mathbf{DSQuant} \xrightarrow{\|-\|} \mathbf{Pos}$ defined by $\|(A \xrightarrow{\varphi} B)\| = (A, \leqslant) \xrightarrow{\varphi} (B, \leqslant)$ produces an algebraic r-reduct. Similarly, $(\| - \|, \mathbf{Pos})$ is an r-reduct of **SQuant** which is not algebraic.

To get the intuition for the second concept recall that sometimes topologies are defined on sets already equipped with an algebraic structure. It is natural then to ask for some compatibility between algebra and topology. One of the simplest requirements is the continuity of the algebraic operations. The next definition shows a catalg modification of the concept (recall that $Q_{\mathbf{B}}$-**Top** has concrete products by Corollary 13).

Definition 21. A $Q_{\mathbf{B}}$*-continuous algebra* is a pair (D, τ), where D is an algebra of some variety $\mathbf{D}$ and $(|D|, \tau)$ is a $Q_{\mathbf{B}}$-space such that $|D|^{n_\lambda} \xrightarrow{\omega_\lambda^D} |D|$ is $Q_{\mathbf{B}}$-continuous for every $\lambda \in \Lambda_{\mathbf{D}}$.

The next lemmas suggest two important examples of continuous algebras.

Lemma 22. *Let $\Omega_{\mathbf{B}}$ induce the structure of* **SFrm** *on $\|Q\|$ and let* **D** *be a finitary variety. Every* **D***-algebra D equipped with the* discrete $Q_{\mathbf{B}}$-topology $\tau^d = Q^{|D|}$ *provides a $Q_{\mathbf{B}}$-continuous algebra (D, τ^d).*

Proof. Given $\lambda \in \Lambda_{\mathbf{D}}$, we show that $|D|^{n_\lambda}$ has the discrete $Q_{\mathbf{B}}$-topology. Every $(d, q) \in |D| \times |Q|$ gives rise to a map $D \xrightarrow{\alpha_d^q} Q$ defined by $\alpha_d^q(e) = q$, if $e = d$; otherwise, $\alpha_d^q(e) = \bot$. Given $\beta \in Q^{|D|^{n_\lambda}}$, every $\langle d_i \rangle_{n_\lambda} \in |D|^{n_\lambda}$ induces a map (n_λ is finite) $\alpha_{\langle d_i \rangle_{n_\lambda}}^{\beta(\langle d_i \rangle_{n_\lambda})} = \wedge_{i \in n_\lambda} (\alpha_{d_i}^{\beta(\langle d_i \rangle_{n_\lambda})} \circ \pi_i) = \wedge_{i \in n_\lambda} (\pi_i)_Q^{\leftarrow}(\alpha_{d_i}^{\beta(\langle d_i \rangle_{n_\lambda})}) \in \tau_{|D|^{n_\lambda}}$ ($\Omega_{\mathbf{SFrm}} \subseteq \Omega_{\mathbf{B}}$ on $\|Q\|$), yielding $\beta = \bigvee_{\langle d_i \rangle_{n_\lambda} \in |D|^{n_\lambda}} \alpha_{\langle d_i \rangle_{n_\lambda}}^{\beta(\langle d_i \rangle_{n_\lambda})} \in \tau_{|D|^{n_\lambda}}$. $\square$

Corollary 23. *In the framework of the category* **Top***, the lattice $\mathbf{2} = \{\bot, \top\}$ equipped with the discrete topology $\tau^d = \{\varnothing, \{\bot\}, \{\top\}, \mathbf{2}\}$ provides a continuous algebra.*

Lemma 24. *Let* **D** *be a variety such that $\Omega_{\mathbf{D}} \subseteq \Omega_{\mathbf{B}}$ and let D be a* **D***-algebra with an $\Omega_{\mathbf{D}}$-homomorphism $D \xrightarrow{\varphi} Q$. The* Sierpinski $Q_{\mathbf{B}}$-topology $\tau^s = \langle \varphi \rangle$ *on D provides a $Q_{\mathbf{B}}$-continuous algebra (D, τ^s).*

Proof. Given $\lambda \in \Lambda_{\mathbf{D}}$, $(\omega_\lambda^D)_Q^{\leftarrow}(\varphi) = \varphi \circ \omega_\lambda^D = \omega_\lambda^Q \circ \varphi^{n_\lambda} = \omega_\lambda^{Q^{|D|^{n_\lambda}}}(\langle \varphi \circ \pi_i \rangle_{n_\lambda}) = \omega_\lambda^{Q^{|D|^{n_\lambda}}}(\langle (\pi_i)_Q^{\leftarrow}(\varphi) \rangle_{n_\lambda}) \in \tau_{|D|^{n_\lambda}}$. The desired result now follows from Corollary 11. $\square$

Corollary 25. *In the framework of the category* **Top***, the frame $\mathbf{2} = \{\bot, \top\}$ equipped with the Sierpinski topology $\tau^s = \{\varnothing, \{\top\}, \mathbf{2}\}$ provides a continuous algebra.*

All preliminaries on their places, we proceed to the definition of the desired functor $Q_{\mathbf{B}}$-**RTop** $\xrightarrow{E}$ **LoC**. Following the line of schizophrenic object of D. Clark and B. Davey [15], we fix a **C**-algebra $\mathbb{C}$ and equip it with a $Q_{\mathbf{B}}$-topology $\boldsymbol{\delta}$. It appears that sufficient conditions for the existence of the functor in question can be formulated as follows (notice that $\boldsymbol{\delta}$ is never assumed to be discrete):

($\mathfrak{R}$) **R** is an algebraic r-reduct of **C**.

($\mathfrak{C}$) $(\mathbb{C}, \boldsymbol{\delta})$ is a $Q_{\mathbf{B}}$-continuous algebra.

The next lemma constructs the functor explicitly (notice the use of the vbp-theory $\mathcal{S}_{\mathbf{C}}^{\mathbf{S}_{\mathbb{C}}}$ in the action on morphisms).

Lemma 26. *If ($\mathfrak{R}$), ($\mathfrak{C}$) hold, then there exists a functor $Q_{\mathbf{B}}$-***RTop** $\xrightarrow{E}$ **LoC** *given by $E((R, \tau) \xrightarrow{f} (S, \sigma)) = Q_{\mathbf{B}}$-***RTop***$(R, \|\mathbb{C}\|) \xrightarrow{(f_{\mathbb{C}}^{\leftarrow})^{op}} Q_{\mathbf{B}}$-***RTop***$(S, \|\mathbb{C}\|)$.*

Proof. It will be enough to check the correctness of E on both objects and morphisms. For the first claim, we show that $Q_{\mathbf{B}}$-**RTop**$(R, \|\mathbb{C}\|)$ is a subalgebra of $\mathbb{C}^{|R|}$. Fix $\lambda \in \Lambda_{\mathbf{C}}$ and $\alpha_i \in Q_{\mathbf{B}}$-**RTop**$(R, \|\mathbb{C}\|)$ for $i \in n_\lambda$. Given

$\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle r_j \rangle_{m_\upsilon} \in \varpi_\upsilon^R$, $\langle \alpha_i(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every $i \in n_\lambda$ and therefore $\langle (\omega_\lambda^{\mathbb{C}^{|R|}}(\langle \alpha_i \rangle_{n_\lambda}))(r_j) \rangle_{m_\upsilon} = \langle \omega_\lambda^{\mathbb{C}}(\langle \alpha_i(r_j) \rangle_{n_\lambda}) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$, yielding $\omega_\lambda^{\mathbb{C}^{|R|}}(\langle \alpha_i \rangle_{n_\lambda}) \in \mathbf{R}(R, \|\mathbb{C}\|)$. To show that the map is also $Q_{\mathbf{B}}$-continuous, notice that the existence of products of $Q_{\mathbf{B}}$-spaces (Corollary 13) induces a $Q_{\mathbf{B}}$-continuous map $R \xrightarrow{\alpha} |\mathbb{C}|^{n_\lambda}$ making the diagram

$$\begin{tikzcd}
R \arrow[d, dotted, "\alpha"'] \arrow[dr, "\alpha_i"] & \\
{|\mathbb{C}|^{n_\lambda}} \arrow[r, "\pi_i"'] & \mathbb{C}
\end{tikzcd}$$

commute for every $i \in n_\lambda$. The desired $Q_{\mathbf{B}}$-continuity follows then from the facts that $\omega_\lambda^{\mathbb{C}^{|R|}}(\langle \alpha_i \rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{C}} \circ \alpha$ and $\omega_\lambda^{\mathbb{C}}$ is $Q_{\mathbf{B}}$-continuous.

To show correctness of E on morphisms, notice that given some $\alpha \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(S, \|\mathbb{C}\|)$, $f_{\mathbb{C}}^{\leftarrow}(\alpha) = \alpha \circ f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|)$ since both α and f are r-$Q_{\mathbf{B}}$-morphisms. The rest follows from the fact that the assignment $(-)^{\leftarrow}$ at the beginning of Example 4 defines a functor, or, more particularly, given $\lambda \in \Lambda_{\mathbf{C}}$ and $\alpha_i \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(S, \|\mathbb{C}\|)$ for $i \in n_\lambda$, $f_{\mathbb{C}}^{\leftarrow}(\omega_\lambda^{\mathbb{C}^{|S|}}(\langle \alpha_i \rangle_{n_\lambda})) = \omega_\lambda^{\mathbb{C}^{|S|}}(\langle \alpha_i \rangle_{n_\lambda}) \circ f = \omega_\lambda^{\mathbb{C}^{|R|}}(\langle \alpha_i \circ f \rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{C}^{|R|}}(\langle f_{\mathbb{C}}^{\leftarrow}(\alpha_i) \rangle_{n_\lambda})$. $\square$

It is important to notice that algebraicity of $\mathbf{R}$ w.r.t. $\mathbf{C}$ was exploited on $\mathbb{C}$ only.

Theorem 27. *If* $(\mathcal{R})$, $(\mathcal{C})$ *hold, then* $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{E} \mathbf{LoC}$ *has a right adjoint.*

Proof. We show that every localic algebra C has a E-co-universal arrow, i.e., a localic homomorphism $ED(C) \xrightarrow{\varepsilon_C^{op}} C$ such that every localic homomorphism $E(R) \xrightarrow{\varphi^{op}} C$ has a unique r-$Q_{\mathbf{B}}$-morphism $R \xrightarrow{f} D(C)$ making the diagram

$$\begin{tikzcd}
E(R) \arrow[d, dotted, "Ef"'] \arrow[dr, "\varphi^{op}"] & \\
ED(C) \arrow[r, "\varepsilon_C^{op}"'] & C
\end{tikzcd}$$

commute.

Let the underlying set of $D(C)$ be $\mathbf{C}(C, \mathbb{C})$. Given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle \varphi_j \rangle_{m_\upsilon} \in (\mathbf{C}(C, \mathbb{C}))^{m_\upsilon}$ let $\langle \varphi_j \rangle_{m_\upsilon} \in \varpi_\upsilon^{\mathbf{C}(C,\mathbb{C})}$ iff $\langle \varphi_j(c) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|C\|}$ for every $c \in C$ (pointwise relational structure induced on $\mathbf{C}(C, \mathbb{C})$ by the product $\|\mathbb{C}\|^{|C|}$). Given $c \in C$ and $\alpha \in \boldsymbol{\delta}$, define $\mathbf{C}(C, \mathbb{C}) \xrightarrow{t_{c\alpha}} Q$ by $t_{c\alpha}(\varphi) = \alpha \circ \varphi(c) = ev_c((\varphi_Q^{\leftarrow})(\alpha))$ and set $\tau = \langle \{ t_{c\alpha} \mid c \in C,\ \alpha \in \boldsymbol{\delta} \} \rangle$. It follows that $D(C)$ is an r-$Q_{\mathbf{B}}$-space (notice the use of closure of r-varieties under products and subobjects). The desired map $C \xrightarrow{\varepsilon_C} (ED(C) = Q_{\mathbf{B}}\text{-}\mathbf{RTop}(\mathbf{C}(C, \mathbb{C}), \|\mathbb{C}\|))$ is now given by $\varepsilon_C(c) = ev_c$.

Two points are the subject to verification at once. Firstly, $\varepsilon_C(c)$ should be in $ED(C)$ for every $c \in C$. Given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle \varphi_j \rangle_{m_\upsilon} \in \varpi_\upsilon^{D(C)}$, $\langle (\varepsilon_C(c))(\varphi_j) \rangle_{m_\upsilon} = \langle \varphi_j(c) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ and therefore $\varepsilon_C(c)$ is an r-homomorphism. To show $Q_{\mathbf{B}}$-continuity, notice that given $\alpha \in \boldsymbol{\delta}$, $((\varepsilon_C(c))^{\leftarrow}_Q(\alpha))(\varphi) = \alpha \circ \varphi(c) = t_{c\alpha}(\varphi)$ for every $\varphi \in \mathbf{C}(C, \mathbb{C})$, yields $(\varepsilon_C(c))^{\leftarrow}_Q(\alpha) = t_{c\alpha} \in \tau$ and use Corollary 11. Secondly, ε_C should be a homomorphism. Given $\lambda \in \Lambda_{\mathbf{C}}$ and $c_i \in C$ for $i \in n_\lambda$, $(\varepsilon_C(\omega_\lambda^C(\langle c_i \rangle_{n_\lambda})))(\varphi) = \varphi(\omega_\lambda^C(\langle c_i \rangle_{n_\lambda})) = \omega_\lambda^{\mathbb{C}}(\langle \varphi(c_i) \rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{C}}(\langle (\varepsilon_C(c_i))(\varphi) \rangle_{n_\lambda}) = (\omega_\lambda^{ED(C)}(\langle \varepsilon_C(c_i) \rangle_{n_\lambda}))(\varphi)$ for every $\varphi \in \mathbf{C}(C, \mathbb{C})$.

It remains to show that ε_C^{op} has the properties of an E-co-universal arrow. Given a localic homomorphism $E(R) \xrightarrow{\varphi^{op}} C$, define $R \xrightarrow{f} D(C)$ by $(f(r))(c) = (\varphi(c))(r)$. To check that $f(r)$ is a homomorphism, notice that given $\lambda \in \Lambda_{\mathbf{C}}$ and $c_i \in C$ for $i \in n_\lambda$, $f(r)(\omega_\lambda^C(\langle c_i \rangle_{n_\lambda})) = (\varphi(\omega_\lambda^C(\langle c_i \rangle_{n_\lambda})))(r) = (\omega_\lambda^{\mathbb{C}^{|R|}}(\langle \varphi(c_i) \rangle_{n_\lambda}))(r) = \omega_\lambda^{\mathbb{C}}(\langle (\varphi(c_i))(r) \rangle_{n_\lambda}) = \omega_\lambda^{\mathbb{C}}(\langle (f(r))(c_i) \rangle_{n_\lambda})$. To verify that f is an r-homomorphism, notice that given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle r_j \rangle_{m_\upsilon} \in \varpi_\upsilon^R$, $\langle (f(r_j))(c) \rangle_{m_\upsilon} = \langle (\varphi(c))(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every $c \in C$, yields $\langle f(r_j) \rangle_{n_\lambda} \in \varpi_\upsilon^{D(C)}$. To show $Q_{\mathbf{B}}$-continuity, take any $t_{c\alpha} \in \tau$ and get $(f^{\leftarrow}_Q(t_{c\alpha}))(r) = (\alpha \circ f(r))(c) = (\alpha \circ \varphi(c))(r) = ((\varphi(c))^{\leftarrow}_Q(\alpha))(r)$ for every $r \in R$ and therefore $f^{\leftarrow}_Q(t_{c\alpha}) = (\varphi(c))^{\leftarrow}_Q(\alpha) \in \tau_R$. It is time to use Corollary 11 again.

Equality $\varepsilon_C^{op} \circ Ef = \varphi^{op}$ comes the fact that for $c \in C$, $((Ef)^{op} \circ \varepsilon_C(c))(r) = (\varepsilon_C(c) \circ f)(r) = (f(r))(c) = (\varphi(c))(r)$ for every $r \in R$. Suppose $R \xrightarrow{g} D(C)$ is another r-$Q_{\mathbf{B}}$-morphism with $\varepsilon_C^{op} \circ Eg = \varphi^{op}$. Given $r \in R$ and $c \in C$, $(g(r))(c) = (\varepsilon_C(c))(g(r)) = (g^{\leftarrow}_{\mathbb{C}}(\varepsilon_C(c)))(r) = (((Eg)^{op} \circ \varepsilon_C)(c))(r) = (\varphi(c))(r) = (f(r))(c)$. $\square$

Corollary 28. *If* $(\mathcal{R})$, $(\mathcal{C})$ *hold, then there exists an adjoint situation* $(\eta, \varepsilon) : E \dashv D : \mathbf{LoC} \to Q_{\mathbf{B}}\text{-}\mathbf{RTop}$.

Proof. We use the standard scheme of obtaining an adjunction from the existence of co-universal arrows [1]. Given a localic homomorphism $C_1 \xrightarrow{\varphi^{op}} C_2$, $D(C_1 \xrightarrow{\varphi^{op}} C_2) = D(C_1) \xrightarrow{D\varphi^{op}} D(C_2)$, with $D\varphi^{op}$ defined by commutativity of the diagram

$$\begin{tikzcd}
ED(C_1) \arrow[r, "\varepsilon_{C_1}^{op}"] \arrow[d, dotted, "ED\varphi^{op}"'] & C_1 \arrow[d, "\varphi^{op}"] \\
ED(C_2) \arrow[r, "\varepsilon_{C_2}^{op}"'] & C_2
\end{tikzcd}$$

and therefore $D\varphi^{op} = \varphi^{\leftarrow}_{\mathbb{C}}$. Given an r-$Q_{\mathbf{B}}$-space R, $R \xrightarrow{\eta_R} (DE(R) = \mathbf{C}(Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|), \mathbb{C}))$ is defined by commutativity of the diagram

$$\begin{tikzcd} E(R) \arrow[d, dotted, "E\eta_R"'] \arrow[dr, "1_{E(R)}"] & \\ EDE(R) \arrow[r, "\varepsilon^{op}_{E(R)}"'] & E(R) \end{tikzcd}$$

and therefore $(\eta_R(r))(f) = f(r)$. $\square$

It is worthwhile to underline once more that the action on morphisms of the obtained adjoint situation is based on the functor $\mathbf{Set} \times \mathbf{S}_{\mathbb{C}} \xrightarrow{(-)^{\leftarrow}_{\mathbb{C}}} \mathbf{LoC}$, which in general is different from the underlying cvt-theory $\mathbf{Set} \times \mathbf{S}_Q \xrightarrow{\|-\|\circ(-)^{\leftarrow}_Q} \mathbf{LoB}$ of the category $Q_{\mathbf{B}}$-$\mathbf{Top}$.

3.2. **Catalg sobriety and spatiality.** Having succeeded in the construction of the desired adjunction (or a *preduality* in terms of [15]), we proceed to the last stage of our plan, i.e., to singling out particular subcategories of $Q_{\mathbf{B}}$-$\mathbf{RTop}$ (resp. $\mathbf{LoC}$) such that the restriction to them of the adjunction produces an equivalence. Simple as it looks, the task has the drawback of the (potential) multitude of the possible solutions. There is, however, a "maximal" equivalence between a pair of full subcategories induced by an adjunction [50] and that will be our choice. For convenience of the reader, we start with the necessary categorical preliminaries.

Lemma 29. *Let $(\eta, \varepsilon) : F \dashv G : \mathbf{A} \to \mathbf{X}$ be an adjunction and let $\bar{\mathbf{A}}$ (resp. $\bar{\mathbf{X}}$) be the full subcategory of $\mathbf{A}$ (resp. $\mathbf{X}$) of those objects A (resp. X) for which $FG(A) \xrightarrow{\varepsilon_A} A$ (resp. $X \xrightarrow{\eta_X} GF(X)$) is an isomorphism in $\mathbf{A}$ (resp. $\mathbf{X}$).*

(1) There exists the restriction $(\bar{\eta}, \bar{\varepsilon}) : \bar{F} \dashv \bar{G} : \bar{\mathbf{A}} \to \bar{\mathbf{X}}$ which is an equivalence, maximal in the sense that every other equivalence $(\bar{\bar{\eta}}, \bar{\bar{\varepsilon}}) : \bar{\bar{F}} \dashv \bar{\bar{G}} : \bar{\bar{\mathbf{A}}} \to \bar{\bar{\mathbf{X}}}$ provides subcategories $\bar{\bar{\mathbf{A}}}$ (resp. $\bar{\bar{\mathbf{X}}}$) of $\bar{\mathbf{A}}$ (resp. $\bar{\mathbf{X}}$).
(2) An $\mathbf{A}$-object A is in $\bar{\mathbf{A}}$ iff $A \cong F(X)$ for some $\mathbf{X}$-object X such that η_X is an $\mathbf{X}$-epimorphism.
(3) An $\mathbf{X}$-object X is in $\bar{\mathbf{X}}$ iff $X \cong G(A)$ for some $\mathbf{A}$-object A such that ε_A is an $\mathbf{A}$-monomorphism.
(4) Let $\mathbf{X}_e$ be the full subcategory of $\mathbf{X}$ of all objects X such that η_X is an $\mathbf{X}$-epimorphism. The full embedding $\bar{\mathbf{X}} \xhookrightarrow{M_{\bar{\mathbf{X}}}} \mathbf{X}_e$ has a left adjoint $\mathbf{X}_e \xrightarrow{GF} \bar{\mathbf{X}}$.
(5) Let $\mathbf{A}_m$ be the full subcategory of $\mathbf{A}$ of all objects A such that ε_A is an $\mathbf{A}$-monomorphism. The full embedding $\bar{\mathbf{A}} \xhookrightarrow{M_{\bar{\mathbf{A}}}} \mathbf{A}_m$ has a right adjoint $\mathbf{A}_m \xrightarrow{FG} \bar{\mathbf{A}}$.

Proof. *Ad* (1). Is is enough to show the existence of the restrictions $\bar{\mathbf{A}} \xrightarrow{\bar{G}} \bar{\mathbf{X}}$ and $\bar{\mathbf{X}} \xrightarrow{\bar{F}} \bar{\mathbf{A}}$. Given $A \in \mathcal{O}b(\bar{\mathbf{A}})$, commutativity of the diagram

$$\begin{tikzcd}
G(A) \arrow[r, "\eta_{G(A)}"] \arrow[dr, "1_{G(A)}"'] & GFG(A) \arrow[d, "G\varepsilon_A"] \\
& G(A)
\end{tikzcd}$$

and the assumption on A yield, $\eta_{G(A)}$ is an $\mathbf{X}$-isomorphism. The case of $\bar{F}$ is similar.

Ad (2). For the necessity notice that $A \in \mathcal{O}b(\bar{\mathbf{A}})$ implies $FG(A) \xrightarrow{\varepsilon_A} A$ is an isomorphism and $G(A) \in \mathcal{O}b(\bar{\mathbf{X}})$ by Item (1). It follows that $\eta_{G(A)}$ is an isomorphism and therefore an epimorphism. For the sufficiency let $A \xrightarrow{\varphi} F(X)$ be the isomorphism in question. On the first step, we show that $\varepsilon_{F(X)}$ is an isomorphism. The first part of the result follows from commutativity of the diagram

$$\begin{tikzcd}
F(X) \arrow[d, "F\eta_X"'] \arrow[dr, "1_{F(X)}"] & \\
FGF(X) \arrow[r, "\varepsilon_{F(X)}"'] & F(X)
\end{tikzcd}$$

Moreover, it implies $F\eta_X \circ \varepsilon_{F(X)} \circ F\eta_X = F\eta_X \circ 1_{F(X)} = 1_{FGF(X)} \circ F\eta_X$. Since left adjoint functors preserve epimorphisms, $F\eta_X$ is an epimorphism and therefore $F\eta_X \circ \varepsilon_{F(X)} = 1_{FGF(X)}$, yielding the desired result. Since ε is a natural transformation, the diagram

$$\begin{tikzcd}
FG(A) \arrow[r, "\varepsilon_A"] \arrow[d, "FG\varphi"'] & A \arrow[d, "\varphi"] \\
FGF(X) \arrow[r, "\varepsilon_{F(X)}"'] & F(X)
\end{tikzcd}$$

commutes and therefore $\varepsilon_A = \varphi^{-1} \circ \varepsilon_{F(X)} \circ FG\varphi$, the morphism on the right being an isomorphism.

Ad (3). Dual to *Ad* (2).

Ad (4). Given an $\mathbf{X}_e$-object X, $F(X)$ is in $\bar{\mathbf{A}}$ by Item (2) and therefore $GF(X)$ is in $\bar{\mathbf{X}}$ by Item (1). It follows that $X \xrightarrow{\eta_X} E_{\bar{\mathbf{X}}}GF(X)$ is an $M_{\bar{\mathbf{X}}}$-universal arrow for X.

Ad (5). Dual to *Ad* (4). $\square$

Applying Lemma 29 to the adjunction of Corollary 28, we get the category $\overline{\mathbf{LoC}}$ (resp. $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$) and the desired equivalence seems to be in hand. It appears, however, that in the current setting, the categories in question have a more explicit description, motivated by the respective one of P. T. Johnstone [39]. We begin with the case of the category $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$, which requires an additional definition.

Definition 30. An r-$Q_{\mathbf{B}}$-space (R,τ) is called

- $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0 provided that
  (1) every distinct $r, s \in R$ have an r-$Q_{\mathbf{B}}$-morphism $R \xrightarrow{f} \|\mathbb{C}\|$ such that $f(r) \neq f(s)$;
  (2) given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle r_j\rangle_{m_\upsilon} \in R^{m_\upsilon}$, if $\langle f(r_j)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every r-$Q_{\mathbf{B}}$-morphism $R \xrightarrow{f} \|\mathbb{C}\|$, then $\langle r_j\rangle_{m_\upsilon} \in \varpi_\upsilon^R$.
- $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-S_0 provided that
  (1) every homomorphism $Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|) \xrightarrow{\varphi} \mathbb{C}$ has some $r \in R$ such that $\varphi(f) = f(r)$ for every r-$Q_{\mathbf{B}}$-morphism $R \xrightarrow{f} \|\mathbb{C}\|$;
  (2) $\tau = \langle\{\alpha \circ f \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|),\ \alpha \in \boldsymbol{\delta}\}\rangle$.
- $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-*sober* provided that it is both r-$Q_{\mathbf{B}}$-T_0 and r-$Q_{\mathbf{B}}$-S_0.

The first and the last items of Definition 30 were suggested by the classical topological notions of T_0 *separation axiom* (every two distinct points have an open set containing only one of them) and *sobriety* (every irreducible closed subset is the closure of a unique point), whereas the middle one is a modified version of the respective notion of [57, Definition 5.3]. The intuition for the new concepts is given by the following example from Priestley duality [19].

Example 31. A **PoTop**-space $(X, \leqslant, \tau)$ is called *totally order-disconnected* provided that for every $x, y \in X$ such that $x \nleqslant y$, there exists a *clopen* (closed and open) *up-set* $U \subseteq X$ ($z \in U$ and $z \leqslant w$ yield $w \in U$) such that $x \in U$ and $y \notin U$. Given the lattice $\mathbf{2} = \{\bot, \top\}$ equipped with the discrete topology, a **PoTop**-space X is $r_{\mathbf{2}}$-T_0 iff X is totally order-disconnected (**PoTop**-morphisms $X \xrightarrow{f} \mathbf{2}$ are in one-to-one correspondence with clopen up-sets $U \subseteq X$).

Lemma 32. *An r-$Q_{\mathbf{B}}$-space R is $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-sober iff η_R is an isomorphism.*

Proof. For the necessity, we show that η_R is bijective and its inverse η_R^{-1} is an r-$Q_{\mathbf{B}}$-morphism. Since R is r$_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0, Item (1) and the definition of η_R in Corollary 28 imply its injectivity. Similarly, r$_{\mathbb{C}}$-$Q_{\mathbf{B}}$-S_0 implies surjectivity and therefore η_R is bijective. To show that η_R^{-1} is an r-homomorphism, fix $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle\varphi_j\rangle_{m_\upsilon} \in \varpi_\upsilon^{DE(R)}$. By the bijectivity of η_R, $\varphi_j = \eta_R(r_j)$ for every $j \in m_\upsilon$. Given an r-$Q_{\mathbf{B}}$-morphism $R \xrightarrow{f} \|\mathbb{C}\|$, $\varpi_\upsilon^{\|\mathbb{C}\|} \ni \langle\varphi_j(f)\rangle_{m_\upsilon} = \langle(\eta_R(r_j))(f)\rangle_{m_\upsilon} = \langle f(r_j)\rangle_{m_\upsilon}$ and therefore $\langle\eta_R^{-1}(\varphi_j)\rangle_{m_\upsilon} = \langle r_j\rangle_{m_\upsilon} \in \varpi_\upsilon^R$ by Item (2) of r$_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0. To show $Q_{\mathbf{B}}$-continuity of η_R^{-1}, notice that given $f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|)$ and $\alpha \in \boldsymbol{\delta}$, $((\eta_R^{-1})_Q^{\leftarrow}(\alpha \circ f))(\eta_R(r)) = \alpha \circ f(r) = \alpha \circ (\eta_R(r))(f) = t_{f\alpha}(\eta_R(r))$ for every

$r \in R$ and therefore $(\eta_R^{-1})_Q^{\leftarrow}(\alpha \circ f) = t_{f\alpha} \in \tau_{DE(R)}$. Corollary 11 yields the desired result.

For the sufficiency, notice that bijectivity of η_R implies Item (1) of both $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0 and $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-S_0. To show Item (2) of $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-T_0, notice that given $\upsilon \in \Upsilon_{\mathbf{R}}$ and $\langle r_j \rangle_{m_\upsilon} \in R^{m_\upsilon}$ such that $\langle f(r_j)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every r-$Q_{\mathbf{B}}$-morphism $R \xrightarrow{f} \|\mathbb{C}\|$, $\langle (\eta_R(r_j))(f)\rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every $f \in E(R)$, and therefore $\langle \eta_R(r_j)\rangle_{m_\upsilon} \in \varpi_\upsilon^{DE(R)}$. It follows that $\langle r_j \rangle_{m_\upsilon} = \langle \eta_R^{-1} \circ \eta_R(r_j)\rangle_{m_\upsilon} \in \varpi_\upsilon^R$ since η_R^{-1} is an r-homomorphism. To show Item (2) of $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-S_0, notice that the inclusion "$\supseteq$" follows from the fact that every f in question is $Q_{\mathbf{B}}$-continuous. Notice as well that given $f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|)$ and $\alpha \in \boldsymbol{\delta}$, $((\eta_R)_Q^{\leftarrow}(t_{f\alpha}))(r) = \alpha \circ (\eta_R(r))(f) = \alpha \circ f(r)$ for every $r \in R$ and therefore $\alpha \circ f = (\eta_R)_Q^{\leftarrow}(t_{f\alpha})$. On the other hand, $((\eta_R^{-1})_Q^{\leftarrow})^{\rightarrow}(\tau_R) \subseteq \tau_{DE(R)}$ by $Q_{\mathbf{B}}$-continuity of η_R^{-1} and therefore $\tau_R = ((1_R)_Q^{\leftarrow})^{\rightarrow}(\tau_R) = ((\eta_R^{-1} \circ \eta_R)_Q^{\leftarrow})^{\rightarrow}(\tau_R) = ((\eta_R)_Q^{\leftarrow} \circ (\eta_R^{-1})_Q^{\leftarrow})^{\rightarrow}(\tau_R) = ((\eta_R)_Q^{\leftarrow})^{\rightarrow} \circ ((\eta_R^{-1})_Q^{\leftarrow})^{\rightarrow}(\tau_R) \subseteq ((\eta_R)_Q^{\leftarrow})^{\rightarrow}(\tau_{DE(R)}) = ((\eta_R)_Q^{\leftarrow})^{\rightarrow}(\langle\{t_{f\alpha} \mid f \in E(R), \alpha \in \boldsymbol{\delta}\}\rangle) = \langle\{(\eta_R)_Q^{\leftarrow}(t_{f\alpha}) \mid f \in E(R),\ \alpha \in \boldsymbol{\delta}\}\rangle = \langle\{\alpha \circ f \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|),\ \alpha \in \boldsymbol{\delta}\}\rangle$ using functorial properties of the image (resp. preimage) operators $(-)^{\rightarrow}$ (resp. $(-)_Q^{\leftarrow}$) and Item (2) of Lemma 10. □

Corollary 33. $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$ *is the full subcategory* $Q_{\mathbf{B}}$-$\mathbb{C}$**RSob** *of* $Q_{\mathbf{B}}$-**RTop** *comprising precisely the* $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-sober r-*$Q_{\mathbf{B}}$*-spaces.*

Having characterized the category $\overline{Q_{\mathbf{B}}\text{-}\mathbf{RTop}}$, we do the same job for $\overline{\mathbf{LoC}}$.

Definition 34. A **LoC**-object C is called $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-spatial* provided that

(1) every distinct $c, d \in C$ have some homomorphism $C \xrightarrow{\varphi} \mathbb{C}$ such that $\varphi(c) \neq \varphi(d)$;

(2) every r-$Q_{\mathbf{B}}$-morphism $\mathbf{C}(C, \mathbb{C}) \xrightarrow{f} \|\mathbb{C}\|$ has some $c \in C$ such that $f(\varphi) = \varphi(c)$ for every homomorphism $C \xrightarrow{\varphi} \mathbb{C}$.

Lemma 35. *A* **LoC***-object* C $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-spatial iff* ε_C *is an isomorphism.*

Proof. The result follows from the definition of ε_C in Theorem 27 and the fact that bijective homomorphisms are isomorphisms. □

Corollary 36. $\overline{\mathbf{LoC}}$ *is the full subcategory* $Q_{\mathbf{B}}$-$\mathbb{C}$**RSpat** *of* **LoC** *comprising precisely the* $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-spatial localic algebras.*

Corollaries 28, 33, 36 and Lemma 29 imply the main result of the section, which provides a generalization of the respective one for the Stone representation theories.

Theorem 37. *Suppose* $(\mathcal{R})$, $(\mathcal{C})$ *hold.*

(1) There exists an equivalence $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat} \to Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}$.

(2) The full embedding $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob} \overset{M_{Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}}}{\hookrightarrow} Q_{\mathbf{B}}\text{-}\mathbf{RTop}_e$ *has a left adjoint* $Q_{\mathbf{B}}\text{-}\mathbf{RTop}_e \xrightarrow{DE} Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}$.

(3) The full embedding $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat} \overset{M_{Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat}}}{\hookrightarrow} \mathbf{LoC}_m$ *has a right adjoint* $\mathbf{LoC}_m \xrightarrow{ED} Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat}$.

By analogy with the case of the Stone representation theories considered by P. T. Johnstone [39], $Q_{\mathbf{B}}\text{-}\mathbf{RTop}_e \xrightarrow{DE} Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}$ is called the $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$*-soberification functor*.

3.3. **Composite setting.** The reader should be aware of the fact that there exists no direct generalization of Theorem 37 for the framework of composite topology, providing an equivalence involving a subcategory of some product category $\prod_{i\in I}\mathbf{LoC}_i$. To show the sticking point, we provide a possible approach to such a modification.

Start with the category $\mathbf{CTop}((\mathcal{S}_{\mathbf{A}_i}^{\mathbf{S}_{Q_i}}, \mathbf{B}_i)_{i\in I})$, for the sake of convenience (as well as to fit the just considered framework) denoted by $(Q_{i\mathbf{B}_i})_I$-$\mathbf{CTop}$. Relational enrichment requires then a more rigid formulation, where a common underlying set for a family of relational structures should be stated explicitly.

Definition 38. Given a family of r-varieties $(\mathbf{R}_i)_{i\in I}$, $(Q_{i\mathbf{B}_i})_I$-$\mathbf{CRTop}$ is the category, whose objects (called *r-*$(Q_{i\mathbf{B}_i})_I$*-spaces*) are triples $(X,(R_i)_{i\in I},(\tau_i)_{i\in I})$, where $(R_i)_{i\in I}$ is a $\prod_{i\in I}\mathbf{R}_i$-object and $(X,(\tau_i)_{i\in I})$ is a $(Q_{i\mathbf{B}_i})_I$-space with $|R_i|= X$ for every $i\in I$, and whose morphisms $(X,(R_i)_{i\in I},(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(S_i)_{i\in I},(\sigma_i)_{i\in I})$ are $(Q_{i\mathbf{B}_i})_I$-continuous maps $(X,(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(\sigma_i)_{i\in I})$ such that $(R_i)_{i\in I} \xrightarrow{(f)_{i\in I}} (S_i)_{i\in I}$ is a $\prod_{i\in I}\mathbf{R}_i$-morphism (called *r-*$(Q_{i\mathbf{B}_i})_I$*-morphisms*). The underlying functor to the ground category $(Q_{i\mathbf{B}_i})_I$-$\mathbf{CTop}$ is defined by $|(X,(R_i)_{i\in I},(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(S_i)_{i\in I},(\sigma_i)_{i\in I})| = (X,(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(\sigma_i)_{i\in I})$.

On the next step, we fix a family $(\mathbf{C}_i)_{i\in I}$ of varieties and the respective set of schizophrenic objects $(\mathbb{C}_i)_{i\in I}$, each equipped with a topology $\boldsymbol{\delta}_i$. Moreover, we stipulate an additional coherence condition, namely, the existence of a set X such that $|\mathbb{C}_i| = X$ for every $i\in I$. Requirement $(\mathfrak{R})$ is converted to the family $((\mathfrak{R}_i))_{i\in I}$. On the other hand, requirement $(\mathfrak{C})$ uses the following modification of Definition 21.

Definition 39. A $(Q_{i\mathbf{B}_i})_I$*-continuous family of algebras* is a triple $(X,(D_i)_{i\in I},(\tau_i)_{i\in I})$, where $(X,(\tau_i)_{i\in I})$ is a $(Q_{i\mathbf{B}_i})_I$-space and for every $i\in I$, (D_i,τ_i) is a $Q_{i\mathbf{B}_i}$-continuous algebra (in the sense of Definition 21) of some variety $\mathbf{D}_i$ such that $|D_i| = X$.

Modification of Lemma 26 in the new setting is straightforward (the explicit details are left to the reader).

Lemma 40. *If* $((\mathcal{R}_i))_{i\in I}$, $(\mathcal{C})$ *hold, then there exists a functor* $(Q_{i\mathbf{B}_i})_I$-$\mathbf{CRTop} \xrightarrow{E_I} \prod_{i\in I}\mathbf{LoC}_i$ *given by*

$$E_I((X,(R_i)_{i\in I},(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(S_i)_{i\in I},(\sigma_i)_{i\in I})) =$$
$$(Q_{i\mathbf{B}_i}\text{-}\mathbf{RTop}(R_i,\|\mathbb{C}_i\|))_{i\in I} \xrightarrow{((f^{\leftarrow}_{\mathbb{C}_i})^{op})_{i\in I}} (Q_{i\mathbf{B}_i}\text{-}\mathbf{RTop}(S_i,\|\mathbb{C}_i\|))_{i\in I}.$$

It is the result of Theorem 27 that causes the main problem. An attentive reader will recall that the right adjoint to E was based on the hom-set $\mathbf{C}(C,\mathbb{C})$ for a given localic algebra C. Our framework will translate the single hom-set into a family $(\mathbf{C}_i(C_i,\mathbb{C}_i))_{i\in I}$, which should produce an r-$(Q_{i\mathbf{B}_i})_I$-space. The sticking point is the requirement of Definition 38 on the common underlying set of the elements of the family obtained. The question on whether E_I has a right adjoint for I having more than one element is still open.

4. Beyond the framework

We have already noticed in Introduction that it is not the topological duality itself, but its consequences that constitute its real worth. In particular, there are numerous procedures for obtaining new representations from the already existing ones. In the following, we provide a catalg foundations for some of them.

4.1. **Representations induced by subcategories.** The reader is probably aware that the classical Stone representation theorems are consequences of the equivalence $\mathbf{Sob} \sim \mathbf{Spat}$ for the variety $\mathbf{Frm}$ of frames [39]. The variety $\mathbf{BDLat}$ of bounded distributive lattices is dually equivalent to the subcategory of $\mathbf{Spat}$ comprising *coherent locales*, the image of which under the equivalence in question is the category of *coherent spaces*, providing the famous Stone representation theorem for distributive lattices. Since Boolean algebras constitute a subcategory $\mathbf{Bool}$ of $\mathbf{BDLat}$, one obtains the second Stone representation theorem, singling out a particular subcategory of coherent spaces consisting of the *Stone* (compact, Hausdorff, totally disconnected) ones. The following shows catalg foundations for the procedure, yielding a simple (but extremely useful) machinery for obtaining new dualities from old. For the sake of flexibility, we take a rather general standpoint from the beginning of Section 3.2.

Definition 41. A subcategory $\mathbf{S}$ of a category $\mathbf{X}$ is said to be *strongly isomorphism-closed* in $\mathbf{X}$ provided that given an $\mathbf{S}$-morphism $S_1 \xrightarrow{f} S_2$ and two $\mathbf{X}$-isomorphisms $X_1 \xrightarrow{g} S_1$ and $S_2 \xrightarrow{h} X_2$, $X_1 \xrightarrow{h\circ f\circ g} X_2$ is an $\mathbf{S}$-morphism.

Notice that given a category, every its full isomorphism-closed [1] subcategory is strongly isomorphism-closed, but not vise versa (the fullness fails).

Definition 42. Given a functor $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ and a subcategory $\mathbf{S}$ of $\mathbf{Y}$, $F^{\leftarrow}(\mathbf{S})$ is the subcategory of $\mathbf{X}$ of all morphisms f such that Ff is an $\mathbf{S}$-morphism.

Lemma 43. *Let $(\bar{\eta}, \bar{\varepsilon}) : \bar{F} \dashv \bar{G} : \bar{\mathbf{A}} \to \bar{\mathbf{X}}$ be an equivalence and let $\mathfrak{A}$ be a strongly isomorphism-closed subcategory of $\bar{\mathbf{A}}$. If $\mathfrak{X} = \bar{F}^{\leftarrow}(\mathfrak{A})$, then there exists the restriction $\mathfrak{A} \xrightarrow{\bar{G}} \mathfrak{X}$, providing the equivalence $(\bar{\bar{\eta}}, \bar{\bar{\varepsilon}}) : \bar{\bar{F}} \dashv \bar{\bar{G}} : \mathfrak{A} \to \mathfrak{X}$.*

Proof. Given $A \xrightarrow{\varphi} B$ in $\mathfrak{A}$, $\bar{F}\bar{G}\varphi = \bar{\varepsilon}_B^{-1} \circ \varphi \circ \bar{\varepsilon}_A$ yields $\bar{F}\bar{G}\varphi$ is an $\mathfrak{A}$-morphism and thus, $\bar{G}\varphi$ lies in $\mathfrak{X}$. □

Notice that the proposed machinery can be reversed, in the sense that given a strongly isomorphism-closed subcategory $\mathfrak{X}$ of $\bar{\mathbf{X}}$, one obtains the category $\mathfrak{A} = \bar{G}^{\leftarrow}(\mathfrak{X})$ and the equivalence $\mathfrak{X} \sim \mathfrak{A}$. Applying the new concepts to our setting, we get the following result.

Corollary 44. *If $\mathfrak{A}$ is a strongly isomorphism-closed subcategory of the category $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}$ and $\mathfrak{T} = \bar{E}^{\leftarrow}(\mathfrak{A})$, then there exists the equivalence $(\bar{\bar{\eta}}, \bar{\bar{\varepsilon}}) : \bar{\bar{E}} \dashv \bar{\bar{D}} : \mathfrak{A} \to \mathfrak{T}$. In particular, if $\mathbf{D}$ is a variety such that $\mathbf{LoD} \sim \mathfrak{A}$, then $\mathbf{LoD} \sim \mathfrak{T}$.*

4.2. **Representations induced by reducts.** In the previous section we considered the case, when a new representation is induced by a particular subcategory of the variety in question. A more common occurrence, however, is to have a reduct instead of a subcategory. More particularly, fix a variety $\mathbf{C}$ (resp. $\mathbf{C}'$) and its algebraic r-reduct $(\| - \|, \mathbf{R})$ (resp. $(\| - \|, \mathbf{R}')$). Moreover, assume that $\mathbf{C}'$ (resp. $\mathbf{R}'$) is a reduct of $\mathbf{C}$ (resp. $\mathbf{R}$) such that the diagram

(1)
$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\|-\|"] \arrow[d, "\|-\|"'] & \mathbf{C}' \arrow[d, "\|-\|"] \\
\mathbf{R} \arrow[r, "\|-\|"'] & \mathbf{R}'
\end{tikzcd}$$

commutes. The problem of the previous section translates into the new setting as follows: given a catalg duality for $\mathbf{C}$ and $\mathbf{R}$ (resp. $\mathbf{C}'$ and $\mathbf{R}'$), is it possible to obtain a duality for $\mathbf{C}'$ and $\mathbf{R}'$ (resp. $\mathbf{C}$ and $\mathbf{R}$). It is the purpose of the next three subsections to give a partial answer to the problem.

4.2.1. *From variety to its reduct.* Suppose there exists an equivalence $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat} \to Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$ based on $\mathbf{C}$ and $\mathbf{R}$. This subsection investigates the question on whether some parts of it can be used in the setting of $\mathbf{C}'$ and $\mathbf{R}'$. To begin with, notice that the new framework satisfies requirement $(\mathcal{R})$. Moreover, $\|\mathbb{C}\|$ (for the sake of shortness denoted by $\mathbb{C}'$) is a $\mathbf{C}'$-algebra, and since $(\mathbb{C}, \boldsymbol{\delta})$ is $Q_{\mathbf{B}}$-continuous, $(\mathbb{C}', \boldsymbol{\delta})$ must be as well. By Corollary 28, there exists an adjoint situation $(\eta', \varepsilon') : E' \dashv D' : \mathbf{LoC}' \to Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$. A relation between the adjunctions obtained is established by the functor $Q_{\mathbf{B}}$-$\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$

given by $\|(R,\tau) \xrightarrow{f} (S,\sigma)\| = (\|R\|,\tau) \xrightarrow{\|f\|} (\|S\|,\sigma)$, which provides two (in general, non-commutative) diagrams:

$$\text{(2)}\qquad \begin{tikzcd} Q_{\mathbf{B}}\text{-}\mathbf{RTop} \arrow[r, "E"] \arrow[d, "\|-\|"'] & \mathbf{LoC} \arrow[d, "\|-\|^{op}"] \\ Q_{\mathbf{B}}\text{-}\mathbf{R'Top} \arrow[r, "E'"'] & \mathbf{LoC'} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{LoC} \arrow[r, "D"] \arrow[d, "\|-\|^{op}"'] & Q_{\mathbf{B}}\text{-}\mathbf{RTop} \arrow[d, "\|-\|"] \\ \mathbf{LoC'} \arrow[r, "D'"'] & Q_{\mathbf{B}}\text{-}\mathbf{R'Top}. \end{tikzcd}$$

It appears that the non-commutativity in question can be replaced by a suitable 2-cell structure [6]:

$$\text{(3)}\qquad \begin{tikzcd} Q_{\mathbf{B}}\text{-}\mathbf{RTop} \arrow[r, bend left, "E'\|-\|"] \arrow[r, bend right, "\|-\|^{op}E"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{LoC'} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{LoC} \arrow[r, bend left, "\|-\|D"] \arrow[r, bend right, "D'\|-\|^{op}"'] \arrow[r, phantom, "\Downarrow\beta"] & Q_{\mathbf{B}}\text{-}\mathbf{R'Top}, \end{tikzcd}$$

where $(\|E(R)\| = \|Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R,\|\mathbb{C}\|)\|) \xrightarrow{\alpha_R} (E'(\|R\|) = Q_{\mathbf{B}}\text{-}\mathbf{R'Top}(\|R\|,\|\mathbb{C}'\|))$ is given by $\alpha_R(f) = f$ and $(\|D(C)\| = \|\mathbf{C}(C,\mathbb{C})\|) \xrightarrow{\beta_C} (D'(\|C\|) = \mathbf{C}'(\|C\|,\mathbb{C}'))$ is defined by $\beta_C(\varphi) = \varphi$ (notice that we have omitted some $(-)^{op}$ indexing for the sake of clearness). Moreover, straightforward computations show that the following diagrams commute (e.g., for the left one, notice that $((E'\beta_C)^{op} \circ \varepsilon'_{\|C\|}(c))(\varphi) = (\varepsilon'_{\|C\|}(c)) \circ \beta_C(\varphi) = \varphi(c) = (\|\varepsilon_C\|(c))(\varphi) = (\alpha_{D(C)} \circ \|\varepsilon_C\|(c))(\varphi)$ for every $C \in \mathcal{O}b(\mathbf{LoC})$, $c \in C$ and $\varphi \in \|\mathbf{C}(C,\mathbb{C})\|$):

$$\text{(4)}\qquad \begin{tikzcd} E'\| - \|D \arrow[r, "E'\beta"] \arrow[d, "\alpha D"'] & E'D'\| - \|^{op} \arrow[d, "\varepsilon'\|-\|^{op}"] \\ \| - \|^{op}ED \arrow[r, "\|-\|^{op}\varepsilon"'] & \| - \|^{op} \end{tikzcd} \qquad \begin{tikzcd} \| - \| \arrow[r, "\|-\|\eta"] \arrow[d, "\eta'\|-\|"'] & \| - \|DE \arrow[d, "\beta E"] \\ D'E'\| - \| \arrow[r, "D'\alpha"'] & D'\| - \|^{op}E. \end{tikzcd}$$

It appears that there exists a nice relation between sobriety (resp. spatiality) of both settings.

Lemma 45. *For an $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-sober space (R,τ), the following are equivalent:*
(1) $(\|R\|,\tau)$ is $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-sober;
(2) $\eta'_{\|R\|}$ is surjective.

Proof. Since the implication (1) $\Rightarrow$ (2) is clear, it will be enough to show the converse one. We will use Definition 30 for the purpose. By the right-hand rectangle of Diagram (4), $D'\alpha_R \circ \eta'_{\|R\|} = \beta_{E(R)} \circ \|\eta_R\|$. Since both $\beta_{E(R)}$ and $\|\eta_R\|$ are injective, $\eta'_{\|R\|}$ must be as well and therefore $\eta'_{\|R\|}$ is bijective. Thus, Item (1) of both $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-T_0 and $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-S_0 hold. To show Item (2) of $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-T_0, notice that given $\upsilon \in \Upsilon_{\mathbf{R}'}$ and $\langle r_j \rangle_{m_\upsilon} \in R^{m_\upsilon}$, if $\langle f(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}'\|}$ for every $f \in Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}(\|R\|, \|\mathbb{C}'\|)$, then $\langle f(r_j) \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|\mathbb{C}\|}$ for every $f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|)$ and therefore $\langle r_j \rangle_{m_\upsilon} \in \varpi_\upsilon^{\|R\|}$ by the assumption. Item (2) of $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-S_0 follows from $\tau = \langle \{\alpha \circ f \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|),\ \alpha \in \boldsymbol{\delta}\} \rangle \subseteq \langle \{\alpha \circ f \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}(\|R\|, \|\mathbb{C}'\|),\ \alpha \in \boldsymbol{\delta}\} \rangle \subseteq \tau$. □

Lemma 46. *For an $\mathrm{r}_{\mathbb{C}}$-$Q_{\mathbf{B}}$-spatial localic algebra C, the following are equivalent:*
(1) $\|C\|$ is $\mathrm{r}_{\mathbb{C}'}$-$Q_{\mathbf{B}}$-spatial;
(2) $\varepsilon'_{\|C\|}$ is surjective.

Proof. The implication (1) $\Rightarrow$ (2) being clear, we show the converse one. By the left-hand rectangle of Diagram (4), $\alpha_{D(C)} \circ \|\varepsilon_C\| = (E'\beta_C)^{op} \circ \varepsilon'_{\|C\|}$. Since both $\alpha_{D(C)}$ and $\|\varepsilon_C\|$ are injective, $\varepsilon'_{\|C\|}$ must be as well and therefore $\varepsilon'_{\|C\|}$ is bijective. □

Define $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}_s$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}_s$) to be the full subcategory of $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}$) of all r-$Q_{\mathbf{B}}$-spaces (R, τ) (resp. localic algebras C) such that $\eta'_{\|R\|}$ (resp. $\varepsilon'_{\|C\|}$) is surjective.

Lemma 47. *There exist the restrictions $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSob}_s \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$, $Q_{\mathbf{B}}\text{-}\mathbb{C}\mathbf{RSpat}_s \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat}$ of the functors $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}$, $\mathbf{LoC} \xrightarrow{\|-\|^{op}} \mathbf{LoC}'$.*

Proof. Follows from Lemmas 45 and 46. □

The reader should notice the important point that (in general) there is no restriction of the functors $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}$ (resp. $\mathbf{LoC} \xrightarrow{\|-\|^{op}} \mathbf{LoC}'$) to the categories $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$, $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}$, $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Spat}$).

4.2.2. *From reduct to its generating variety through an algebraic r-reduct.* With Diagram (1) in mind, suppose there exists an equivalence $(\bar{\eta}', \bar{\varepsilon}') : \bar{E}' \dashv \bar{D}' : Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat} \to Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ based on $\mathbf{C}'$ and $\mathbf{R}'$. The question is how it relates to the setting of $\mathbf{C}$ and $\mathbf{R}$. Unlike the just considered framework, the current one needs some additional requirements:

($\mathcal{A}$) There exists a $\mathbf{C}$-algebra $\mathbb{C}$ such that $\|\mathbb{C}\| = \mathbb{C}'$.

($\mathfrak{T}$) There exists a $Q_{\mathbf{B}}$-topology $\boldsymbol{\delta}$ on $\mathbb{C}$ such that
- (1) $(\mathbb{C}, \boldsymbol{\delta})$ is a $Q_{\mathbf{B}}$-continuous algebra;
- (2) $\boldsymbol{\delta}' \subseteq \boldsymbol{\delta}$.

By Corollary 28, there exists an adjoint situation $(\eta, \varepsilon) : E \dashv D : \mathbf{LoC} \to Q_{\mathbf{B}}\text{-}\mathbf{RTop}$. A relation between the adjunctions in question is established again by the functor $Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}$, providing (non-commutative) Diagram (2). The 2-cell structure of Diagram (3) is guaranteed by Item (2) of ($\mathfrak{T}$).

Straightforward computations provide Diagram (4). Moreover, there exists a relation between sobriety (resp. spatiality) of both settings.

Lemma 48. *For an $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-sober space (R, τ), the following are equivalent:*
(1) $(\|R\|, \tau)$ *is* $r_{\mathbb{C}'}$-$Q_{\mathbf{B}}$*-sober;*
(2) *(i)* $\eta'_{\|R\|}$ *is surjective;*
(ii) $\langle\{\alpha \circ f \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{RTop}(R, \|\mathbb{C}\|),\ \alpha \in \boldsymbol{\delta}\}\rangle \subseteq \langle\{\alpha \circ f \mid f \in Q_{\mathbf{B}}\text{-}\mathbf{R}'\mathbf{Top}(\|R\|, \|\mathbb{C}'\|),\ \alpha \in \boldsymbol{\delta}'\}\rangle$.

Proof. Use the machinery of the proof of Lemma 45. Notice that Item (ii) of (2) holds in case of $\boldsymbol{\delta} \subseteq \boldsymbol{\delta}'$ (and therefore $\boldsymbol{\delta} = \boldsymbol{\delta}'$ by Item (2) of $(\mathfrak{T})$). □

Lemma 49. *For an $r_{\mathbb{C}}$-$Q_{\mathbf{B}}$-spatial localic algebra C, the following are equivalent:*
(1) $\|C\|$ *is* $r_{\mathbb{C}'}$-$Q_{\mathbf{B}}$*-spatial;*
(2) $\varepsilon'_{\|C\|}$ *is surjective.*

Proof. Use the proof of Lemma 46. □

By analogy with the previous section, one can define the categories $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}_s$ and $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}_s$, the former one having an additional condition on its objects induced by Lemma 48. Lemmas 48, 49 yield the restriction $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}_s \xrightarrow{\|-\|}$ $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}_s \xrightarrow{\|-\|} Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Spat}$) of the functor $Q_{\mathbf{B}}$-$\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$ (resp. $\mathbf{LoC} \xrightarrow{\|-\|^{op}} \mathbf{LoC}'$). The reader should notice that again (in general) $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}_s$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}_s$) could not be changed to $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$ (resp. $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat}$).

4.2.3. *From reduct to its generating variety through a non-algebraic r-reduct.* The last subsection dealt with a relation between dualities for a given variety and its reduct. Motivated by various representations in the literature, this subsection considers a more general setting, presenting the just mentioned problem in a different light. Consider once more Diagram (1) and suppose that $(\| - \|, \mathbf{R})$ is an r-reduct of $\mathbf{C}$ which is not necessarily algebraic. Due to the assumption, it may be not possible to obtain an equivalence of the type $Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSpat} \sim Q_{\mathbf{B}}$-$\mathbb{C}\mathbf{RSob}$. On the other hand, numerous examples (see Introduction) clearly show that a $\mathbf{C}'$-$\mathbf{R}'$-equivalence $(\bar{\eta}', \bar{\varepsilon}') : \bar{E}' \dashv \bar{D}' : Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Spat} \to Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ can provide a $\mathbf{C}$-$\mathbf{R}$-representation theorem. In the following, we show a catalg approach to the challenge. Start by introducing two additional categories, serving as a cornerstone of the approach (notice that we reverse slightly the setting of Definition 17 and consider the category $Q_{\mathbf{B}}$-$\mathbf{RTop}$ as the r-variety $\mathbf{R}$ enriched in the category $Q_{\mathbf{B}}$-$\mathbf{Top}$, with the respective underlying functor to $\mathbf{R}$ defined by $|(R, \tau) \xrightarrow{f} (S, \sigma)| = R \xrightarrow{f} S$).

Definition 50. $\mathbb{T}$ is the category, whose objects are triples (R, τ, C), where (R, C) is in $\mathbf{R} \times \mathbf{LoC}$ and $(\|R\|, \tau)$ is in $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ such that $\|C\| = E'(\|R\|, \tau)$, and whose morphisms $(R_1, \tau_1, C_1) \xrightarrow{f} (R_2, \tau_2, C_2)$ are $Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$-morphisms $(\|R_1\|, \tau_1) \xrightarrow{f} (\|R_2\|, \tau_2)$ such that $\|C_1\| \xrightarrow{E'f} \|C_2\|$ is a $\mathbf{LoC}$-morphism.

Definition 51. $\mathbb{A}$ is the category, whose objects are pairs (C, R), where (C, R) is in $\mathbf{LoC} \times \mathbf{R}$ and $\|C\|$ is in $Q_{\mathbf{B}}$-$\mathbb{C}'\mathbf{R}'\mathbf{Spat}$ such that $\|R\| = |D'(\|C\|)|$ (notice the use of the aforesaid underlying functor to $\mathbf{R}'$), and whose morphisms $(C_1, R_1) \xrightarrow{\varphi} (C_2, R_2)$ are $\mathbf{C}$-morphisms $C_1 \xrightarrow{\varphi} C_2$.

On the first step, we construct a functor $\mathbb{T} \xrightarrow{E} \mathbb{A}^{op}$. Given a $\mathbb{T}$-object (R, τ, C), $(\|R\|, \tau) \xrightarrow{\eta'_{\|R\|}} (D'E'(\|R\|, \tau) = D'(\|C\|))$ is a $Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$-isomorphism and thus, a bijective map. Given $v \in \Upsilon_{\mathbf{R}} \backslash \Upsilon_{\mathbf{R}'}$ and $\langle r_j \rangle_{m_v} \in R^{m_v}$, let $\langle \eta'_{\|R\|}(r_j) \rangle_{m_v} \in \varpi_v^{|D'(\|C\|)|}$ iff $\langle r_j \rangle_{m_v} \in \varpi_v^R$, and obtain an $\mathbf{R}$-isomorphism $R \xrightarrow{\eta'_{\|R\|}} |D'(\|C\|)|$. That gives an $\mathbf{R}$-object $\hat{R}$ such that $\|\hat{R}\| = |D'(\|C\|)|$ (notice that R and $\hat{R}$ have different underlying sets). The considerations, backed by the definition of $\mathbb{T}$-morphisms, suggest the next lemma.

Lemma 52. *There exists a functor* $\mathbb{T} \xrightarrow{E} \mathbb{A}^{op}$, $E((R_1, \tau_1, C_1) \xrightarrow{f} (R_2, \tau_2, C_2)) = (C_1, \hat{R}_1) \xrightarrow{E'f} (C_2, \hat{R}_2)$.

On the second step, we obtain a functor $\mathbb{A}^{op} \xrightarrow{D} \mathbb{T}$. Given an $\mathbb{A}^{op}$-morphism $(C_1, R_1) \xrightarrow{\varphi} (C_2, R_2)$, $\|C_i\| \xrightarrow{\varepsilon'_{\|C_i\|}} (E'D'(\|C_i\|) = E'(\|R_i\|, \tau_i))$ is a $\mathbf{LoC}'$-isomorphism and thus, a bijective map. Given $\lambda \in \Lambda_{\mathbf{C}} \backslash \Lambda_{\mathbf{C}'}$ and $\langle c_j \rangle_{n_\lambda} \in C^{n_\lambda}$, let $\omega_\lambda^{E'(\|R_i\|, \tau_i)}(\langle \varepsilon'_{\|C_i\|}(c_j) \rangle_{n_\lambda}) = \varepsilon'_{\|C_i\|}(\omega_\lambda^{C_i}(\langle c_j \rangle_{n_\lambda}))$, and obtain a $\mathbf{C}$-isomorphism $C_i \xrightarrow{\varepsilon'_{\|C_i\|}} E'(\|R_i\|, \tau_i)$. That gives a $\mathbf{C}$-algebra $\hat{C}_i$ such that $\|\hat{C}_i\| = E'(\|R_i\|, \tau_i)$ (notice again that C_i and $\hat{C}_i$ have different underlying sets). Moreover, commutativity of the diagram

$$\begin{tikzcd}
(\|\hat{C}_1\| = E'D'(\|C_1\|)) \arrow[r, "(\varepsilon'_{\|C_1\|})^{op}"] \arrow[d, "E'D'\varphi"'] & C_1 \arrow[d, "\varphi"] \\
(\|\hat{C}_2\| = E'D'(\|C_2\|)) \arrow[r, "(\varepsilon'_{\|C_2\|})^{op}"'] & C_2
\end{tikzcd}$$

and our definition of $\hat{C}_i$ imply $E'D'\varphi = ((\varepsilon'_{\|C_2\|})^{op})^{-1} \circ \varphi \circ (\varepsilon'_{\|C_1\|})^{op}$, the right-hand side of the equality being a $\mathbf{LoC}$-morphism. Altogether, one gets the following result.

Lemma 53. *There exists a functor* $\mathbb{A}^{op} \xrightarrow{D} \mathbb{T}$, $D((C_1, R_1) \xrightarrow{\varphi} (C_2, R_2)) = (R_1, \tau_1, \hat{C}_1) \xrightarrow{D'\varphi} (R_2, \tau_2, \hat{C}_2)$.

Having the functors in hand, we proceed to constructing two natural transformations. Given a $\mathbb{T}$-object (R,τ,C), define $(R,\tau,C) \xrightarrow{\eta_{(R,\tau,C)}} (DE(R,\tau,C) = (\hat{R},\tau_{|D'(\|C\|)|},\hat{C}))$ by $\eta_{(R,\tau,C)}(r) = \eta'_{\|R\|}(r)$. It follows that $(\|R\|,\tau) \xrightarrow{\eta_{(R,\tau,C)}} ((\|\hat{R}\|,\tau_{|D'(\|C\|)|}) = D'E'(\|R\|,\tau))$ is a $Q_{\mathbf{B}}$-**RTop**-isomorphism. Moreover, it appears that a stronger result holds.

Lemma 54. *$(R,\tau,C) \xrightarrow{\eta_{(R,\tau,C)}} DE(R,\tau,C)$ is a $\mathbb{T}$-isomorphism.*

Proof. Following Definition 50, it will be enough to show that $\|C\| \xrightarrow{E'\eta_{(R,\tau,C)}} \|\hat{C}\|$ is a **LoC**-isomorphism. Consider the following commutative triangle:

$$\begin{tikzcd}
(\|C\| = E'(\|R\|,\tau)) \arrow[d, "E'\eta_{(R,\tau,C)}=E'\eta'_{(\|R\|,\tau)}"'] \arrow[dr, "1_{E'(\|R\|,\tau)}"] & \\
(\|\hat{C}\| = E'(\|\hat{R}\|,\tau_{|D'(\|C\|)|}) = E'D'(\|C\|) = E'D'E'(\|R\|,\tau)) \arrow[r, "(\varepsilon'_{\|C\|})^{op}=(\varepsilon'_{E'(\|R\|,\tau)})^{op}"'] & (\|C\| = E'(\|R\|,\tau)).
\end{tikzcd}$$

By the construction of E and D, $(\varepsilon'_{\|C\|})^{op}$ is a **LoC**-isomorphism and then $E'\eta_{(R,\tau,C)}$ must be as well. □

Corollary 55. *$1_{\mathbb{T}} \xrightarrow{\eta} DE$ is a natural isomorphism.*

Proof. The statement in question follows from the fact that both D and E as well as η are based on the functors D' and E' as well as the natural transformation η'. □

The second natural transformation can be obtained equally easy. Given an $\mathbb{A}^{op}$-object (C,R), define $(C,R) \xrightarrow{\varepsilon_{(C,R)}} (ED(C,R) = (\hat{C},\hat{R}))$ by $\varepsilon_{(C,R)}(c) = \varepsilon'_{\|C\|}(c)$. It follows that $C \xrightarrow{\varepsilon_{(C,R)}} (\|\hat{C}\| = E'D'(\|C\|))$ is a **C**-isomorphism. Moreover, similar to Corollary 55, one can show the following lemma.

Lemma 56. *$EQ \xrightarrow{\varepsilon} 1_{\mathbb{A}^{op}}$ is a natural isomorphism.*

It is possible now to state the first important result of this section.

Theorem 57. *There exists an equivalence $(\eta,\varepsilon) : E \dashv D : \mathbb{A}^{op} \to \mathbb{T}$.*

Proof. Follows from Lemmas 52, 53, 56 and Corollary 55. □

Theorem 57, being interesting by itself, gives rise to a procedure of obtaining new dualities from old, running as follows. Let $\mathcal{F}_T$ (resp. $\mathcal{F}_A$) be a set of axioms (in the obvious sense; for a particular example, see the next section) which can be satisfied by $Q_{\mathbf{B}}$-**RTop**-spaces (resp. **LoC**-algebras). Satisfaction relation for an r-$Q_{\mathbf{B}}$-space (R,τ) (resp. localic algebra C) will be denoted by $(R,\tau) \models \mathcal{F}_T$ (resp. $C \models \mathcal{F}_A$). Define

- $\mathcal{T} = \{(R,\tau) \in \mathcal{O}b(Q_{\mathbf{B}}\text{-}\mathbf{RTop}) \mid (R,\tau) \models \mathcal{F}_T \text{ and } (\|R\|,\tau) \in \mathcal{O}b(Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob})\}$;

- $\mathcal{A} = \{C \in \mathcal{O}b(\mathbf{LoC}) \mid C \models \mathcal{F}_A \text{ and } \|C\| \in \mathcal{O}b(Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat})\}$.

and suppose there exist two maps (notice that both their domains and codomains can be proper classes):

- $\mathcal{T} \xrightarrow{F_T} \mathcal{O}b(\mathbf{C})$ such that $\|F_T(R,\tau)\| = E'(\|R\|,\tau)$;
- $\mathcal{A} \xrightarrow{F_A} \mathcal{O}b(\mathbf{R})$ such that $\|F_A(C)\| = |D'(\|C\|)|$.

The new definitions give rise to a particular subcategory $\bar{\mathbb{T}}$ (resp. $\bar{\mathbb{A}}$) of $\mathbb{T}$ (resp. $\mathbb{A}$).

Definition 58. $\bar{\mathbb{T}}$ is the full subcategory of $\mathbb{T}$ of all triples (R,τ,C) such that $(R,\tau) \in \mathcal{T}$ and $C = F_T(R,\tau)$.

Definition 59. $\bar{\mathbb{A}}$ is the full subcategory of $\mathbb{A}$ of all pairs (C,R) such that $C \in \mathcal{A}$ and $R = F_A(C)$.

We would like to restrict the equivalence $\mathbb{A}^{op} \sim \mathbb{T}$ of Theorem 57 to the new setting and therefore introduce the following requirements:

($\mathcal{C}_T$) If $(R,\tau) \in \mathcal{T}$, then $F_T(R,\tau) \models \mathcal{F}_A$.
($\mathcal{C}_A$) If $C \in \mathcal{A}$, then $(F_A(C), \tau_{|D'(\|C\|)|}) \models \mathcal{F}_T$.
($\mathcal{I}_T$) If $(R,\tau) \in \mathcal{T}$, then $\|R\| \xrightarrow{\eta'_{\|R\|}} \|F_A F_T(R,\tau)\|$ is an $\mathbf{R}$-isomorphism.
($\mathcal{I}_A$) If $C \in \mathcal{A}$, then $\|C\| \xrightarrow{\varepsilon'_{\|C\|}} \|F_T F_A(C)\|$ is a $\mathbf{C}$-isomorphism.

Lemma 60. *There exist the restrictions $\bar{\mathbb{T}} \xrightarrow{\bar{E}} \bar{\mathbb{A}}^{op}$ and $\bar{\mathbb{A}}^{op} \xrightarrow{\bar{D}} \bar{\mathbb{T}}$ of the functors $\mathbb{T} \xrightarrow{E} \mathbb{A}^{op}$ and $\mathbb{A}^{op} \xrightarrow{D} \mathbb{T}$.*

Proof. Given $(R,\tau,F(R,\tau))$ in $\bar{\mathbb{T}}$, $E(R,\tau,F(R,\tau)) = (F_T(R,\tau),\hat{R})$, where $\|R\| \xrightarrow{\eta'_{\|R\|}} \|\hat{R}\|$ is an $\mathbf{R}$-isomorphism. Since $|\hat{R}| = |D'E'(\|R\|,\tau)| = |F_A F_T(R,\tau)|$ (recall that $|-|$ denotes the underlying set of the structure in question), ($\mathcal{I}_T$) implies $\hat{R} = F_A F_T(R,\tau)$ and thus, $(F_T(R,\tau),\hat{R}) = (F_T(R,\tau), F_A F_T(R,\tau))$ is in $\bar{\mathbb{A}}^{op}$ by ($\mathcal{C}_T$). On the other hand, given $(C, F_A(C))$ in $\bar{\mathbb{A}}^{op}$, $D(C,F_A(C)) = (F_A(C), \tau_{|D'(\|C\|)|}, \hat{C})$, where $\|C\| \xrightarrow{\varepsilon'_{\|C\|}} \|\hat{C}\|$ is a $\mathbf{C}$-isomorphism. Since $|\hat{C}| = |E'D'(\|C\|)| = |F_T F_A(C)|$, ($\mathcal{I}_A$) yields $\hat{C} = F_T F_A(C)$ and thus, $(F_A(C), \tau_{|D'(\|C\|)|}, \hat{C}) = (F_A(C), \tau_{|D'(\|C\|)|}, F_T F_A(C))$ is in $\bar{\mathbb{T}}$ by ($\mathcal{C}_A$). ☐

Corollary 61. *There exists an equivalence $(\bar{\eta},\bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \bar{\mathbb{A}}^{op} \to \bar{\mathbb{T}}$.*

To bring more clarity in the machinery developed, we add two more definitions.

Definition 62. $\mathcal{T}Q_{\mathbf{B}}$-**RTop** is the category, whose objects are the elements of $\mathcal{T}$, and whose morphisms $(R_1,\tau_1) \xrightarrow{f} (R_2,\tau_2)$ are those $Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$-morphisms $(\|R_1\|,\tau_1) \xrightarrow{f} (\|R_2\|,\tau_2)$ for which $\|F_T(R_1,\tau_1)\| \xrightarrow{E'f} \|F_T(R_2,\tau_2)\|$ is a **LoC**-morphism.

Definition 63. $\mathbf{LoC}_{\mathcal{A}}$ is the full subcategory of $\mathbf{LoC}$, whose objects are the elements of $\mathcal{A}$.

It is possible now to state the second important result of this section.

Theorem 64. *There exists an equivalence* $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{LoC}_{\mathcal{A}} \to \mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$.

Proof. It is easy to see that $\mathbf{LoC}_{\mathcal{A}}$ (resp. $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$) is isomorphic to $\bar{\mathbb{A}}^{op}$ (resp. $\bar{\mathbb{T}}$). $\square$

The reader should be aware that the category $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ (resp. $\mathbf{LoC}_{\mathcal{A}}$) is not a subcategory of $Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ (resp. $Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat}$). More precisely, the following diagrams commute:

(5)

$$\begin{tikzcd}
\mathbf{LoC}_{\mathcal{A}} \arrow[r, "\bar{D}"] \arrow[d, "\|-\|^{op}"'] & \mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop} \arrow[d, "\|-\|"] \\
Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat} \arrow[r, "D'"'] & Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop} \arrow[r, "\bar{E}"] \arrow[d, "\|-\|"'] & \mathbf{LoC}_{\mathcal{A}} \arrow[d, "\|-\|^{op}"] \\
Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob} \arrow[r, "E'"'] & Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat},
\end{tikzcd}$$

where $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop} \xrightarrow{\|-\|} Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Sob}$ is given by $\|(R_1, \tau_1) \xrightarrow{f} (R_2, \tau_2)\|$ $= (\|R_1\|, \tau_1) \xrightarrow{f} (\|R_2\|, \tau_2)$. Also notice that we do not give an explicit description of the axioms $\mathcal{F}_T$ (resp. $\mathcal{F}_A$) and their induced maps F_T (resp. F_A), the duality is based upon. Our goal (motivated by category theory itself) is to provide a common framework for such procedures, and that stands in contrast to *piggyback dualities* of D. Clark and B. Davey [15], where the authors try to show an explicit construction of the duality in question.

The reader has probably noticed (the fact was underlined by our notations as well) that the category $\mathbf{LoC}_{\mathcal{A}}$ is a (full) subcategory of $\mathbf{LoC}$, whereas $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ (in general) is not a subcategory of $Q_{\mathbf{B}}\text{-}\mathbf{RTop}$ since its morphisms are just $\mathbf{R}'$-morphisms. Concrete examples (see the next section) show that often the category obtained is indeed a subcategory of $Q_{\mathbf{B}}\text{-}\mathbf{RTop}$, the result, however, being rather dependant on the particular setting employed. In our current general one, it is possible to state the following lemma.

Lemma 65. *Given a* $\mathcal{T}Q_{\mathbf{B}}\text{-}\mathbf{RTop}$*-morphism* $(R_1, \tau_1) \xrightarrow{f} (R_2, \tau_2)$*, equivalent are:*

(1) $\|R_1\| \xrightarrow{f} \|R_2\|$ *is an* $\mathbf{R}$*-morphism;*

(2) $\|F_A F_T(R_1, \tau_1)\| \xrightarrow{D'E'f} \|F_A F_T(R_2, \tau_2)\|$ *is an* $\mathbf{R}$*-morphism.*

Proof. The statement follows from commutativity of the diagram

$$\begin{tikzcd} \|R_1\| \arrow[r, "\eta'_{\|R_1\|}"] \arrow[d, "f"'] & \|F_A F_T(R_1,\tau_1)\| \arrow[d, "D'E'f"] \\ \|R_2\| \arrow[r, "\eta'_{\|R_2\|}"'] & \|F_A F_T(R_2,\tau_2)\| \end{tikzcd}$$

and requirement $(\mathcal{I}_T)$. $\square$

With the just obtained result in view, an additional requirement seems to be advisable:

($\mathcal{H}$) Given $(R_1,\tau_1),(R_2,\tau_2) \in \mathcal{T}$ and a $Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$-morphism $(\|R_1\|,\tau_1) \xrightarrow{f} (\|R_2\|,\tau_2)$ with the property that $\|F_T(R_1,\tau_1)\| \xrightarrow{E'f} \|F_T(R_2,\tau_2)\|$ is a $\mathbf{LoC}$-morphism, $\|F_AF_T(R_1,\tau_1)\| \xrightarrow{D'E'f} \|F_AF_T(R_2,\tau_2)\|$ is an $\mathbf{R}$-morphism.

The new assumption allows one to make a modification in the definition of the category $\mathcal{T}Q_{\mathbf{B}}$-$\mathbf{RTop}$.

Definition 66. $Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$ is the subcategory of $Q_{\mathbf{B}}$-$\mathbf{RTop}$, with objects those of $\mathcal{T}Q_{\mathbf{B}}$-$\mathbf{RTop}$, and morphisms $(R_1,\tau_1) \xrightarrow{f} (R_2,\tau_2)$ having the property of $\|F_T(R_1,\tau_1)\| \xrightarrow{E'f} \|F_T(R_2,\tau_2)\|$ being in $\mathbf{LoC}$.

Lemma 67. *If* ($\mathcal{H}$) *holds, then the categories* $\mathcal{T}Q_{\mathbf{B}}$-$\mathbf{RTop}$ *and* $Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$ *are isomorphic.*

As an immediate consequence of Lemma 67 and Theorem 64, one gets the main result of this section.

Theorem 68. *There exists an equivalence* $\mathbf{LoC}_{\mathcal{A}} \sim Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$.

In other words, we have obtained an equivalence between particular subcategories of $\mathbf{LoC}$ and $Q_{\mathbf{B}}$-$\mathbf{RTop}$ ($\mathbf{LoC}_{\mathcal{A}}$ and $Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathcal{T}}$), based entirely on a catalg duality in the framework of $\mathbf{LoC}'$ and $Q_{\mathbf{B}}$-$\mathbf{R}'\mathbf{Top}$.

5. Examples of representations

In the previous sections we presented a catalg approach to natural dualities in the sense of D. Clark and B. Davey [15]. This section illustrates the obtained machinery by the famous representation theorem for bounded distributive lattices of H. Priestley [53] and its application to topological representations of *J-distributive lattices* of A. Petrovich [49] and $\neg$-*lattices* of S. Celani [8], providing a better insight into their properties. The case of distributive lattices has already been considered by us in [70] and will be recalled here once more for the convenience of the reader, whereas its application to the structures of A. Petrovich and S. Celani was motivated by the results on relations between topological representations of a given variety and its reduct, dealt extensively upon in the previous section.

5.1. **Representation theorem for distributive lattices of H. Priestley.** Start with the classical vbp-theory $\mathcal{P}$ (Example 4(1)) and obtain the category **Top** of topological spaces and continuous maps (Example 7(1)), where $Q = \mathbf{2}$ and $\mathbf{B} = \mathbf{Frm}$. Enrichment of **Top** in the r-variety **Pos** (Example 15) provides the category **PoTop** of partially-ordered topological spaces and order-preserving, continuous maps (Example 18). Choose the variety **BLat** of bounded lattices as the required one $\mathbf{C}$, which has the algebraic r-reduct $(\| - \|, \mathbf{Pos})$ given by $\|(A \xrightarrow{\varphi} B)\| = (A, \leqslant) \xrightarrow{\varphi} (B, \leqslant)$. The lattice $\mathbf{2} = \{\perp, \top\}$ with the discrete topology $\tau^d = \{\varnothing, \{\perp\}, \{\top\}, \mathbf{2}\}$ produces a continuous algebra (Corollary 23) and therefore take $(\mathbb{C}, \boldsymbol{\delta}) = (\mathbf{2}, \tau^d)$. Since requirements $(\mathcal{R})$, $(\mathcal{C})$ are satisfied, Corollary 28 gives the adjoint situation $(\eta, \varepsilon) : E \dashv D : \mathbf{LoBLat} \to \mathbf{PoTop}$, the explicit form of which is as follows:

- $\mathbf{PoTop} \xrightarrow{E} \mathbf{LoBLat}$ is defined by $E(X) = (\mathcal{COU}(X), \cap, \cup, \varnothing, X)$ and $Ef = (f^{\leftarrow})^{op}$, where $\mathcal{COU}(X)$ is the set of clopen up-sets of X (Example 31);
- $\mathbf{LoBLat} \xrightarrow{D} \mathbf{PoTop}$ is defined by $D(C) = (\mathcal{PF}(C), \subseteq, \tau)$ and $D\varphi = (\varphi^{op})^{\leftarrow}$, where $\mathcal{PF}(C)$ is the set of *prime filters* of C ($c_1 \vee c_2 \in F$ implies $c_1 \in F$ or $c_2 \in F$) and $\tau = \langle \{\rho_c \,|\, c \in C\} \cup \{\hat{\rho}_c \,|\, c \in C\} \rangle$, with $F \in \rho_c$ (resp. $F \in \hat{\rho}_c$) iff $c \in F$ (resp. $c \notin F$);
- $C \xrightarrow{\varepsilon} ED(C)$ is defined by $\varepsilon_C(c) = \rho_c$;
- $X \xrightarrow{\eta_X} DE(X)$ is defined by $\eta_X(x) = \{U \in \mathcal{COU}(X) \,|\, x \in U\}$.

The obtained framework is that of Priestley duality, with the exception of the target categories, i.e., **BLat** (resp. **PoTop**) instead of the variety **BDLat** of bounded distributive lattices (resp. the category **PrSpc** of *Priestley* (compact, totally order-disconnected) *spaces*). Theorem 37 gives the equivalence $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{2}_d\mathbf{Spat} \to \mathbf{2}_d\mathbf{Sob}$.

Lemma 69. *A bounded lattice C is $\mathbf{2}_d$-spatial iff it is distributive.*

Proof. For the necessity, notice that given a $\mathbf{2}_d$-spatial lattice C, $C \cong ED(C) = \mathcal{COU}(\mathcal{PF}(C))$, the latter lattice being a sublattice of $\mathcal{P}(\mathcal{PF}(C))$ and therefore distributive. For the sufficiency, one can use the technique applied in the proof of the Priestley representation theorem of, e.g., [19]. $\square$

Lemma 70. *An ordered topological space X is $\mathbf{2}_d$-sober iff it is a Priestley space.*

Proof. For the necessity notice that given a $\mathbf{2}_d$-sober space X, $X \cong DE(X) = \mathcal{PF}(\mathcal{COU}(X))$. Since $\mathcal{COU}(X)$ is a sublattice of $\mathcal{P}(X)$, it is distributive and then the result follows from the technique used in the proof of Priestley duality. The same technique can be applied to obtain the sufficiency. $\square$

All preliminaries done, we can finally state the well-known result.

Theorem 71 (Representation Theorem of H. Priestley)**.** *There exists an equivalence* $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{LoBDLat} \to \mathbf{PrSpc}$.

It is shown in [70] that the case of the representation theorems of M. Stone [75, 76] can be incorporated in the framework using the two-element frame $\mathbf{2}$ with the Sierpinski topology $\tau^s = \{\varnothing, \{\top\}, \mathbf{2}\}$ (Corollary 25).

5.2. **Representation theorem for J-distributive lattices of A. Petrovich.** Motivated by recent interest of numerous researchers in J-distributive lattices of A. Petrovich [49], in this section we incorporate his topological representation theorem for the structure into our catalg framework. As will be seen later on, the machinery for the procedure is based on the above-mentioned representation theorem of H. Priestley and the fact that the new concept has the variety **BDLat** as a reduct. For convenience of the reader, we begin with the necessary preliminaries, modifying the original notations of A. Petrovich (replacing the symbol "J" by "∇"), to fit our current framework more conveniently.

Definition 72. A ∇*-lattice* is a bounded lattice C equipped with a unary operation ∇ such that $\nabla(\bot) = \bot$ and $\nabla(c_1 \vee c_2) = \nabla(c_1) \vee \nabla(c_2)$ for every c_1, $c_2 \in C$. $\nabla\,\mathbf{BLat}$ is the variety of ∇-lattices.

Definition 73. **RPos** is the category, whose objects are triples $(X, \leqslant, R)$, where $(X, \leqslant)$ is a partially ordered set and R is a binary relation on X, and whose morphisms $(X, \leqslant, R) \xrightarrow{f} (Y, \leqslant, S)$ are order-preserving maps which also preserve the relation in question.

In the language of enriched category theory [40], **RPos** is just the category **Pos** enriched in the category **Rel**(2) (cf. Example 15). Moreover, it is easy to see that **BLat** (resp. **Pos**) is a reduct of $\nabla\,\mathbf{BLat}$ (resp. **RPos**). Slightly more sophisticated is the proof that **RPos** is an algebraic r-reduct of $\nabla\,\mathbf{BLat}$. The concrete functor in question $\nabla\,\mathbf{BLat} \xrightarrow{\|-\|} \mathbf{RPos}$ can be defined by (cf. Example 16) $\|(C_1, \nabla_1) \xrightarrow{\varphi} (C_2, \nabla_2)\| = (|C_1|, \leqslant, \langle \mathrm{Grph}\,\nabla_1 \rangle) \xrightarrow{\varphi} (|C_2|, \leqslant, \langle \mathrm{Grph}\,\nabla_2 \rangle)$, providing an r-reduct. Since ∇_i is order-preserving, the order-relation "$\leqslant$" is a subalgebra of $C_i \times C_i$, yielding algebraicity of the reduct obtained. Altogether, the considerations give the commutative diagram

$$\begin{tikzcd}
\nabla\,\mathbf{BLat} \arrow[r, "\|-\|"] \arrow[d, "\|-\|"'] & \mathbf{BLat} \arrow[d, "\|-\|"] \\
\mathbf{RPos} \arrow[r, "\|-\|"'] & \mathbf{Pos}.
\end{tikzcd}$$

The current framework fits the setting of Section 4.2.3, since the above-mentioned Priestley duality is available for the right-hand side of the diagram. Motivated by Definitions 50, 51, we introduce their particular instances for the current framework (notice that the index $(-)_P$ comes from "Petrovich").

Definition 74. $\mathbb{T}_P$ is the category, whose objects are quintuples $(X, \leqslant, R, \tau, \nabla)$ such that $(X, \leqslant, R)$ is in **RPos**, $(X, \leqslant, \tau)$ is in **PrSpc** and $(\mathcal{COU}(X), \nabla)$ is in $\nabla\,$**BLat**, and whose morphisms $(X_1, \leqslant, R_1, \tau_1, \nabla_1) \xrightarrow{f} (X_2, \leqslant, R_2, \tau_2, \nabla_2)$ are **PoTop**-morphisms $(X_1, \leqslant, \tau_1) \xrightarrow{f} (X_2, \leqslant, \tau_2)$, making the following diagram commute (notice the above-mentioned functor $\bar{E}$ from Priestley duality):

$$\begin{tikzcd}
\mathcal{COU}(X_2) \arrow[r, "\nabla_2"] \arrow[d, "(\bar{E}f)^{op}"'] & \mathcal{COU}(X_2) \arrow[d, "(\bar{E}f)^{op}"] \\
\mathcal{COU}(X_1) \arrow[r, "\nabla_1"'] & \mathcal{COU}(X_1).
\end{tikzcd}$$

Definition 75. $\mathbb{A}_P$ is the category, whose objects are triples (C, ∇, R) such that (C, ∇) is in $\nabla\,$**BDLat** and $(\mathcal{PF}(C), \subseteq, R)$ is in **RPos**, and whose morphisms $(C_1, \nabla_1, R_1) \xrightarrow{\varphi} (C_2, \nabla_2, R_2)$ are $\nabla\,$**BLat**-morphisms $(C_1, \nabla_1) \xrightarrow{\varphi} (C_2, \nabla_2)$.

By Theorem 57, there exists an equivalence $\mathbb{A}_P^{op} \sim \mathbb{T}_P$, which can be developed further using the technique of Definitions 58, 59. For the sake of convenience, we denote the enrichment of **Top** in **RPos** by **RPosTop**. Start by introducing topological axioms $\mathcal{F}_T$, suitable for the occasion. Notice that we are working with the objects of **RPosTop**, i.e., tuples $(X, \leqslant, R, \tau)$. Thus, we let $\mathcal{F}_T$ consist of the following two axioms [49]:

($\mathcal{A}_1$) Given $x \in X$, $R(x) = \{y \in X \mid xRy\}$ is a closed down-set (cf. Example 31).

($\mathcal{A}_2$) Given $U \in \mathcal{COU}(X)$, $R^{\uparrow}(U) = \{x \in X \mid R(x) \bigcap U \neq \varnothing\} \in \mathcal{COU}(X)$.

The set $\mathcal{F}_A$ of algebraic axioms is supposed to be empty. With these preliminaries in hand, we can introduce the two required maps:

- $(\mathcal{T}_P = \{(X, \leqslant, R, \tau) \in \mathcal{O}b(\mathbf{RPosTop}) \mid (X, \leqslant, R, \tau) \models \{(\mathcal{A}_1), (\mathcal{A}_2)\}$ and $(X, \leqslant, \tau) \in \mathcal{O}b(\mathbf{PrSpc})\}) \xrightarrow{F_T} \mathcal{O}b(\nabla\mathbf{BLat})$ defined by $F_T(X, \leqslant, R, \tau) = (\mathcal{COU}(X), R^{\uparrow})$;

- $(\mathcal{A}_P = \{(C,\nabla) \in \mathcal{O}b(\mathbf{Lo\nabla BLat}) \,|\, C \in \mathcal{O}b(\mathbf{BDLat})\}) \xrightarrow{F_A} \mathcal{O}b(\mathbf{RPos})$, $F_A(C,\nabla) = (\mathcal{PF}(C), \subseteq, R_\nabla)$, where R_∇ is defined as follows: given $F_1, F_2 \in \mathcal{PF}(C)$, $F_1 R_\nabla F_2$ iff $(C \backslash \nabla^{\leftarrow}(F_1)) \bigcap F_2 = \varnothing$.

The next categories are particular versions of Definitions 58, 59, suitable for our current framework.

Definition 76. $\bar{\mathbb{T}}_P$ is the full subcategory of $\mathbb{T}_P$ of all quintuples $(X, \leqslant, R, \tau, \nabla)$ such that $(X, \leqslant, R, \tau) \in \mathcal{T}_P$ and $\nabla = R^{\uparrow}$.

Definition 77. $\bar{\mathbb{A}}_P$ is the full subcategory of $\mathbb{A}_P$ of all triples (C, ∇, S) such that $(C, \nabla) \in \mathcal{A}_P$ and $S = R_\nabla$.

Requirements $(\mathcal{C}_T) - (\mathcal{I}_A)$ have already been checked in [49] (actually were motivated by the results of the article and the respective one of S. Celani [8]). By Corollary 61, there exists an equivalence $\bar{\mathbb{A}}_P^{op} \sim \bar{\mathbb{T}}_P$. It appears that $(\mathcal{H})$ also holds, and since the requirement is slightly off the framework of A. Petrovich, we deem it advisable to give its simple proof.

Given a continuous, order-preserving map $(X, \leqslant, R, \tau) \xrightarrow{f} (Y, \leqslant, S, \sigma)$ such that

$$\begin{tikzcd} \mathcal{COU}(Y) \arrow[r, "S^{\uparrow}"] \arrow[d, "f^{\leftarrow}"'] & \mathcal{COU}(Y) \arrow[d, "f^{\leftarrow}"] \\ \mathcal{COU}(X) \arrow[r, "R^{\uparrow}"'] & \mathcal{COU}(X), \end{tikzcd} \tag{6}$$

it will be enough to verify that $(\mathcal{PF}(\mathcal{COU}(X)), T_{R^{\uparrow}}) \xrightarrow{(f^{\leftarrow})^{\leftarrow}} (\mathcal{PF}(\mathcal{COU}(Y)), T_{S^{\uparrow}})$ is in $\mathbf{Rel}(2)$ (cf. Example 15). If $F_1, F_2 \in \mathcal{PF}(\mathcal{COU}(X))$ are such that $F_1 T_{R^{\uparrow}} F_2$, then $F_2 \subseteq (R^{\uparrow})^{\leftarrow}(F_1)$ and therefore $U \in F_2$ implies $R^{\uparrow}(U) \in F_1$. On the other hand, $(f^{\leftarrow})^{\leftarrow}(F_i) = \{U \in \mathcal{COU}(Y) \,|\, f^{\leftarrow}(U) \in F_i\}$ for $i \in \{1, 2\}$. It follows that for $U \in (f^{\leftarrow})^{\leftarrow}(F_2)$, $f^{\leftarrow}(U) \in F_2$ and therefore $f^{\leftarrow} \circ S^{\uparrow}(U) = R^{\uparrow} \circ f^{\leftarrow}(U) \in F_1$, yielding $S^{\uparrow}(U) \in (f^{\leftarrow})^{\leftarrow}(F_1)$. Altogether, $(f^{\leftarrow})^{\leftarrow}(F_2) \subseteq (S^{\uparrow})^{\leftarrow}((f^{\leftarrow})^{\leftarrow}(F_1))$ and thus, $((f^{\leftarrow})^{\leftarrow}(F_1)) T_{S^{\uparrow}} ((f^{\leftarrow})^{\leftarrow}(F_2))$.

The final touch is now made by Definition 66 and Theorem 68.

Definition 78. $\nabla\,\mathbf{PrSpc}$ is the subcategory of $\mathbf{RPosTop}$, whose objects are the elements of $\mathcal{T}_P$, and whose morphisms $(X, \leqslant, R, \tau) \xrightarrow{f} (Y, \leqslant, S, \sigma)$ make Diagram (6) commute.

Theorem 79 (Representation Theorem of A. Petrovich)**.** *There exists an equivalence* $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{Lo\nabla\,BDLat} \to \nabla\,\mathbf{PrSpc}$.

It is easy to show a simple (but extremely useful) characterization of $\nabla\,\mathbf{PrSpc}$-morphisms [49].

Lemma 80. *Given two* $\nabla\,\mathbf{PrSpc}$*-objects* $(X, \leqslant, R, \tau)$, $(Y, \leqslant, S, \sigma)$, *a map* $X \xrightarrow{f} Y$ *is a* $\nabla\,\mathbf{PrSpc}$*-morphism iff the following hold:*

(1) f *is an* **RPosTop***-morphism;*
(2) if $x \in X$, $y \in Y$ *and* $f(x)Sy$, *then there exists* $x' \in X$ *such that* $x' \in R(x)$ *and* $y \leqslant f(x')$.

Using the machinery of Section 4.1 and the equivalence of Theorem 79, one can obtain the topological representation theorem for *Q-distributive lattices* of R. Cignoli [12], which are $\nabla\,\mathbf{BDLat}$-lattices (C, ∇) satisfying for every $c, c' \in C$ two additional conditions:

($\mathcal{D}_1$) $c \wedge \nabla(c) = c$;
($\mathcal{D}_2$) $\nabla(c \wedge \nabla(c')) = \nabla(c) \wedge \nabla(c')$.

The respective result has already been considered by A. Petrovich [49], the corresponding relations being *quasiequivalences*, i.e., relations $R \subseteq X \times X$ which are reflexive, transitive (the so-called *preorders*) and satisfy the following condition:

($\mathcal{S}$) For every $x, y \in X$ with xRy, there exists $z \in X$ such that $y \leqslant z$, xRz and zRx.

Notice that if the partial order "$\leqslant$" is given by equality, quasiequivalences reduce to equivalence relations.

5.3. **Representation theorem for ¬-lattices of S. Celani.** The results of this section were motivated by our research on *quasi-Stone algebras* introduced by N. H. Sankappanavar and H. P. Sankappanavar [65] and studied later on by various researchers [8, 9, 10, 26, 27]. In particular, there exists a topological representation theorem for the structure proved by H. Gaitán [26] and induced by Priestley duality. From the applicational point of view, however, a much more transparent result of S. Celani [8] seems to be advisable. The representation in question is based on the concept of *¬-lattice* [8, 9, 10], which is similar to the notion of ∇-lattice considered in the previous section. It is our current purpose to incorporate the duality in the catalg framework. We begin again with the necessary algebraic preliminaries.

Definition 81. A *¬-lattice* is a bounded lattice C equipped with a unary operation $\neg$ such that $\neg(\bot) = \top$ and $\neg(c_1 \vee c_2) = \neg(c_1) \wedge \neg(c_2)$ for every c_1, $c_2 \in C$. ¬**BLat** is the variety of ¬-lattices.

It is easy to see that **BLat** is a reduct of ¬**BLat**. Moreover, using the underlying functor from the previous section, one can show that **RPos** is an r-reduct of ¬**BLat**. Unlike the results for ∇-lattices, the reduct in question is not algebraic, since given a ¬-lattice $(C, \neg)$, $\neg$ is order-reversing and therefore "$\leqslant$" is not a subalgebra of $C \times C$. Thus, we have a commutative diagram

$$\begin{tikzcd} \neg\mathbf{BLat} \arrow[r, "\|-\|"] \arrow[d, "\|-\|"'] & \mathbf{BLat} \arrow[d, "\|-\|"] \\ \mathbf{RPos} \arrow[r, "\|-\|"'] & \mathbf{Pos}, \end{tikzcd}$$

where the left-hand side never satisfies requirement ($\mathcal{R}$) and therefore the procedure of catalg duality of Section 3 is not applicable. On the other hand, the machinery of Section 4.2.3 is fruitful even in this setting. By analogy with Definitions 74, 75, one can introduce the categories $\mathbb{T}_C$ and $\mathbb{A}_C$ (notice that the index $(-)_C$ comes from "Celani") and obtain an equivalence $\mathbb{A}_C^{op} \sim \mathbb{T}_C$. The set of topological (resp. algebraic) axioms is similar to the already considered one, with $(\mathcal{A}_2)$ changed as follows:

($\mathcal{A}_2'$) Given $U \in \mathcal{COU}(X)$, $R^{\downarrow}(U) = \{x \in X \mid R(x) \bigcap U = \varnothing\} \in \mathcal{COU}(X)$.

The map $\mathcal{T}_C \xrightarrow{F_T} \mathcal{O}b(\neg\mathbf{BLat})$ has the respective modification of $(-)^{\uparrow}$ to $(-)^{\downarrow}$, whereas in the map $\mathcal{A}_C \xrightarrow{F_A} \mathcal{O}b(\mathbf{RPos})$, the relation $R_{\neg}$ is defined by $F_1 R_{\neg} F_2$ iff $\neg^{\leftarrow}(F_1) \bigcap F_2 = \varnothing$. All the other proceedings are similar to the already considered ones, resulting in the following theorem.

Theorem 82 (Representation Theorem of S. Celani)**.** *There exists an equivalence* $(\bar{\eta}, \bar{\varepsilon}) : \bar{E} \dashv \bar{D} : \mathbf{Lo}\neg\mathbf{BDLat} \to \neg\mathbf{PrSpc}$.

Moreover, characterization Lemma 80 is applicable in the new setting. Using the machinery of Section 4.1 and the equivalence of Theorem 82, one can obtain the topological representation theorem for *quasi-Stone algebras* [65], which are $\neg$**BDLat**-lattices $(C, \neg)$ satisfying for every $c, c' \in C$ four additional conditions:

($\mathcal{Q}_1$) $\neg(\top) = \bot$;
($\mathcal{Q}_2$) $\neg(c \wedge \neg(c')) = \neg(c) \vee \neg(\neg(c'))$;
($\mathcal{Q}_3$) $c \wedge \neg(\neg(c)) = c$;
($\mathcal{Q}_4$) $\neg c \vee \neg(\neg(c)) = \top$.

The respective result has already been considered by S. Celani in [8], with the corresponding binary relations appearing to be equivalences. An important point, however, should be noticed at once. In [8, 10] S. Celani shows that some axioms in the definition of quasi-Stone algebras are dependant on the others. In particular, axioms ($\mathcal{Q}_1$) and ($\mathcal{Q}_2$) are supposed to superfluous. While the latter statement is true, the former one contains a flaw. A possible counterexample is quite simple, i.e., consider the lattice $\mathbf{2} = \{\bot, \top\}$ with $\neg(\bot) = \top = \neg\top$. It follows that the structure satisfies all the required axioms with the exception of ($\mathcal{Q}_1$).

In [10] S. Celani went even further, introducing a particular generalization of equivalence relations, i.e., considering a relation $R \subseteq X \times X$ which is *serial*

($R(x) \neq \varnothing$ for every $x \in X$), *euclidean* ($R^{-1} \circ R \subseteq R$) and transitive. Every equivalence relation satisfies the properties, but not vice versa. Using the duality of Theorem 82 and the reversed technique of Section 4.1 (from topology to algebra), one obtains the subcategory $\mathfrak{A}$ of $\neg$**BDLat** equivalent to the subcategory of $\neg$**PrSpc**, where the relations on objects satisfy the above-mentioned three properties. It appears [10] that the objects C of $\mathfrak{A}$ are characterized by two axioms:

($\mathcal{W}_1$) $\neg c \wedge \neg(\neg(c)) = \bot$;
($\mathcal{W}_2$) $\neg c \vee \neg(\neg(c)) = \top$.

The new structure was coined by S. Celani *weak-quasi-Stone algebra* and studied extensively in [10].

6. Conclusion and open problems

In the paper we have presented a *categorically-algebraic* (*catalg*) approach to the natural dualities of D. Clark and B. Davey [15], motivated by the outlook on the Stone representation theorems of P. T. Johnstone [39] and our recent attempt [70] on a generalization of the topological representation theorem for bounded distributive lattices of H. Priestley [53]. The new setting underlines catalg properties of the dualities in question, on one hand, and serves as a tool for generating new topological representation theorems for algebraic structures, on the other. In particular, we have presented several procedures for obtaining new dualities from the already existing ones, based on relations between a given variety and its reduct, and motivated by the multitude of techniques encountered in the literature. The results obtained were illustrated by two examples relying on Priestley duality, the first one employing a modal operator of *possibility* and the second one providing analogous results for a *negation* operator. The examples extend their influence from the realm of *algebraic logic* (Q-distributive lattices of R.Cignoli [12]) to the setting of *pseudocomplemented lattices* ((weak-) quasi-Stone algebras of N. H. Sankappanavar and H. P. Sankappanavar [65] and S Celani [10]). The most important property of the machinery proposed is its applicability to both crisp and fuzzy developments, making another step towards our ultimate goal of erasing the border between traditional and fuzzy approaches in mathematics. Moreover, the results of this paper show once again the advantage of our *catalg* approach over the *poslat* one of S. E. Rodabaugh [56], where one is tied to varieties of *lattices*, being unable to shift to those of *arbitrary algebras*. In conclusion of the paper, we would to draw the attention of the reader to several open problems related to the topic.

Lemma 80 shows a characterization of morphisms of both the category ∇**PrSpc** and $\neg$**PrSpc**. Since the characterization appears to be extremely useful in applications, the first problem is immediate.

Problem 83. Is it possible to generalize Lemma 80 to the general setting of the category $Q_{\mathbf{B}}$-$\mathbf{RTop}_{\mathfrak{J}}$?

In [13] R. Cignoli uses Priestley duality to construct free Q-distributive lattices from bounded distributive lattices, whereas H. Gaitán [26] does the same job for quasi-Stone algebras. In our catalg framework, the result is equivalent to the functor $\mathbf{D} \xrightarrow{\|-\|} (Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat})^{op}$ from Diagram (5) having a left adjoint in those particular two cases, where $\mathbf{D}$ is the respective full subcategory of $\mathbf{LoC}_{\mathcal{A}}$, suitable for the occasion. The second problem can be thus stated as follows.

Problem 84. Under what conditions the functor $\mathbf{LoC}_{\mathcal{A}} \xrightarrow{\|-\|} (Q_{\mathbf{B}}\text{-}\mathbb{C}'\mathbf{R}'\mathbf{Spat})^{op}$ from Diagram (5) has a left adjoint?

The next problem stems from [70], being still actual. The current manuscript presented several examples illustrating the new approach, all of which are *crisp* (based on the standard vbp-theory $\mathcal{P}$). Since the fruitfulness of every new theory is measured by the amount of useful applications arising of it, the next problem springs into mind immediately.

Problem 85. Find other examples of catalg dualities based on both crisp and fuzzy topological spaces.

Since our approach incorporates natural dualities of [15], possible candidates can be found in [15, Chapter 4]. It will be the topic of our further research to translate them into catalg language as well as to find new ones.

The last problem is a reiteration of the one touched in Section 3.3, when trying to switch the developed theory to composite topological spaces.

Problem 86. Under what conditions the functor $(Q_{i\mathbf{B}_i})_I$-$\mathbf{CRTop} \xrightarrow{E_I} \prod_{i\in I}\mathbf{LoC}_i$ of Lemma 40 has a right adjoint?

Notice that by Theorem 27, the sufficient condition is the set I being a singleton. It is (probably) a nice challenge to answer the question on whether the condition is also a necessary one.

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