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Prof. RNDr. Beloslav Riečan, DrSc. – One of the last Spiritus Mathematicus Slovakiensis

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Dedicated to the 75th birthday of Beloslav Riečan

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On November 10, 1936, in Žilina, Slovakia, a small boy, called by his parents Belko, was born, and who for many years is known as Prof. RNDr. Beloslav Riečan, DrSc. The Fates gave him many gifts but also many stickers: being four and half his beloved mother passed away, being six his eight-year brother Horislav died, too, and in autumn 1994, his firstborn daughter Hanka, a very gifted mathematician, died in the result of a car accident at one Austria highway. Fortunately, other Fates gave him also very scarce gifts: love to Maths, love to music, and the most important gift — love to people, which helped him very deeply during his whole life — and such Belo is known by the most among us, his students, colleagues and friends.

After the mother death the family moved from Púchov to Banská Bystrica where he attended the primary school, and a very famous high school, Gymnázium of Andrej Sládkovič, which had a very great influence to the young soul of Belo. Many important personalities of the science, culture and social life of the Slovakia attended this high school. Young Riečan absorbed through all his pores a unique atmosphere of Banská Bystrica. In the persons of Š. Moyzes (catholic bishop) and K. Kuzmány (evangelical superintedent), he is discovering a fascinating possibility of the coexistence for the development of Slovakia, and from which he is able always successfully to draw, which is very significant in his full age. Belo was one of the best students, he was

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excellent in Maths and in playing piano and organ; he became even an organist in an evangelical church in Banská Bystrica (1950–53, and also now since 2001). Young students used to meet in his father's flat where they discussed with a great ignition about everything, declaimed verses, and an old duck was hearing them in still. As a student, Belo won an All-Slovakian competition for young pianists which caused him a big head ache: it was necessary to choose between a career of a pianist or of a mathematician. He preferred mathematics which have had a great influence for Slovak Maths.

In 1953, Belo is going to Bratislava, where he attended the Faculty of Natural Sciences of Comenius University. Here he had excellent teachers who belong to the first generation of the Slovak Mathematicians: Acad. Jur Hronec, Profs O. Borůvka, M. Greguš, A. Huťa, M. Kolibiar, T. Neubrunn, J. Srb, M. Sypták, T. Šalát, V. Šeda and others. His school-class was one of the best in the history; his student fellows became a decoration of the Slovak mathematical society: Profs P. Brunovský, J. Černý, A. Dávid, O. Erdélská-Klaučová, M. Franek, J. Gruska, P. Kluvánek, J. Moravčík, Z. Petrovičová-Riečanová, Z. Zalabai, etc.

During his studies he wired to scientific activity under the guidance of Prof. M. Kolibiar and immediately his first paper *On axiomatic of modular lattices*, Acta Fac. Rer. Nat. Univ. Comenianae, Math. **2** (1957), 257–262 (in Slovak), was a top hit. This paper was quoted in monographs of G. Birkhoff, L. A. Skornjakov and G. Grätzer. Every specialist of lattice theory knows very well how a great distinction for the author are such quotations in these three fundamental books. Today, when takes one's stand on CC-publications and SCI-quotations, it is wonderful how these mathematical giants could quote this paper, although it was written in Slovak and in a non-current journal.

After finishing his studies in 1958, he started to work at the Department of Mathematics of the Slovak Technical University, Bratislava. In 1962-64 he was a PhD-student of Prof. Š. Schwarz, another giant of the Slovak mathematics. In 1966 he was the Associated Prof., and he started to give lectures also at Faculty of Natural Sciences of Comenius Univ., where later he started to read lectures also to the author of these lines. In 1979 he defended the scientific degree DrSc., and in 1981 he was appointed as the University Professor. From 1972 he worked at this Faculty, and in 1985 he moved to Liptovský Mikuláš, to the Department of Mathematics of the Military Academy to come back in 1989 to Faculty of Mathematics and Physics, Bratislava, as the first willingly elected Dean. Since 1992 he was the Director of Institute of Mathematics of the Slovak Academy of Sciences, and since 1998 he is back in the city of his youth, Banská Bystrica, where since October 2001 he is at the Institute of Mathematics and Informatics, the joint institute of the Institute of Mathematics, Slovak Academy of Sciences, and the Faculty of Natural Sciences of the University of Matej Bel.

Prof. Riečan belongs to the most significant mathematicians of Slovakia. He is the author or coauthor of 7 monographs (the last one [M3] appeared in 2009) plus 2 chapters in books, 240 papers published in scientific journals, over 80 technical papers, 4 university textbooks, 30 high-school textbooks and text tools, 8 scripts plus 3 in electronic form, 8 books on mathematics (one book on probability had 6 editions), 8 tv-scripts, over 500 publicists articles.[1] His papers were quoted more than 500-times.

[1] The list of his first 122 publications and the list of monographs can be found in [3]. The updated list of publications from 2005–2010 is at the end of this article.
This note is based on the article [7] where is a list of his papers [118–192]. Other articles on B. Riečan: [1, 2, 4, 5, 6].

He was a PhD-supervisor of 30 PhD-students, which is a Slovak unique, and he was a supervisor of over 55 diploma-theses. Many of his former students are nowadays leading personalities of our universities and of the Slovak Academy of Sciences. His professional activity is probability theory, measure theory and integration, fuzzy sets, and quantum structures. His activities are very large and reach even besides of Maths. He is a member of 6 international scientific societies.

Among his most important mathematical contributions we can surely insert establishing unifying theory of measure and integration in ordered spaces. He extended the notion on entropy of dynamical systems. He initiated study of quantum structures and fuzzy sets in Slovakia. He developed probability theory of fuzzy sets. That has a very important connection to probability theory on MV-algebras [Ch1]. He is very often invited to address his talks on many conferences in home as well as in abroad.

He is a tireless organizer of many scientific events; many of traditional scientific conferences in Slovakia or seminars have arose due to his direct personal stimulus. Thanks to him, in Liptovský Ján Valley an important mathematical congress centrum have appeared which is already very well entered in awareness of foreign colleagues; as a rule, a concert of conference participants is regularly organized which is always highly welcome. He was a long-standing president of the Union of Slovak Mathematicians and Physicists, he is the Head of the Slovak Association of Rome Club. As an outstanding musician and musical expert, since 1984 together with an important Slovak musician Prof. Roman Berger are guiding the Seminar Mathematics and Music, and around this seminar a circle of Slovak, Czech as well as foreign intellectuals is concerned. He is a fanatic propagator of Slovak books, namely of mathematical ones, and he is a mathematical modern version of Matej Hrebenda (M. Hrebenda, 1796–1880, was a famous blind propagator of Slovak books). When in the late 1980s, the famous Prague Vičichlo Library of the Czech Technical University announced that their old books would discard, he saved them and organized the transport of the books to Liptovský Mikuláš as well as to Institute of Mathematics, Slovak Academy of Sciences, Bratislava; for example we have an old original monograph by J.C.F. Gauss on number theory, *Disqvisitiones Arithmeticae*, 1801.

On pages of daily press, on TV-screen and radio he is trying on uplift of the education in Slovakia, he is voicing to momentous questions of education, science, culture and clergy in our society, and to acute questions of collaboration between universities and Slovak Academy of Sciences. His scientific, pedagogical, organizing activity was many time awarded on many important national and international platforms. I mention only the latest ones: Honorary Medal of Bernard Bolzano of the Academy of Sciences of the Czech republic (1998), the Silver Medal of University of Milano (2000), and Medal of the Slovak Academy of Sciences for support for science (2001), Order of Ľ. Štúr of the First Grade (2002) (the highest estimation in Slovakia for scholars awarded by the state president), member of the Learning Society of the Slovak Academy of Sciences 2005, Dr.h.c. of the Military Academy, Liptovský Mikuláš, 2006, the grants for the most successful PhD tutors.

All these outstanding scientific degrees which Prof. Riečan achieved are very important and needful, however they don't reflect the main feature of his own. And this is his interest for the man, for the pupil and student, which very often borders on self-sacrificing, and which is very typical for him. Not once I had opportunity to see him how he already as a known professor was near with a young adept of Maths. Or how he was carrying on his back the books and offering them to people. Thanks for that he induced interest for Maths in many young novices while his sparks are very susceptible. In addition, he has a gift to put people together for the well of matters and he is not shaming to let enlighten himself by younger colleagues. I have not yet understood where he takes so much energy and so many ideas.

Dear Belo, you are a genuine and one of the last spiritus mathematicus slovakiensis, and therefore we wish you on the occasion of your important life jubilee good health, happiness and many new interesting mathematical results and new ideas on organizing mathematical life.

Ad multos annos!

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On a new approach towards defining intuitionistic fuzzy subtractions

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To my friend Prof. Beloslav Riečan

Abstract

A new set of operations subtraction over intuitionistic fuzzy sets are defined and some of their basic properties are studied.

Keywords Intuitionistic fuzzy set, Operation, Subtraction

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1 Introduction

In a series of papers, part of which written together with Prof. Beloslav Riečan, the concept of "substraction" operation over an Intitionistic Fuzzy Set (IFS, see [1]), was introduced for the first time (see, [2, 3, 4, 5, 6, 7, 8]).

In the first two papers [5, 6], we offered direct definitions of subtractions. Later, an approach providing a series of definitions was introduced and 67 different instances of the "substraction" operation were constructed and their properties were studied. B. Riečan participated actively in this research [7, 8].

Now, a new approach to defining different "substraction" operations is constructed and some of the basic properties of the derived new instances will be studied.

2 Some preliminary results

Up to now, different operations have been defined over IFS. Let

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ stand for the degrees of membership and non-membership of the element x from a fixed universe E to the set $A \subset E$, respectively, and every x satisfies that: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Let for every $x \in E$:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Therefore, function π determines the degree of uncertainty.

Below, for brevity, we write A instead of A^*. When the IFSs A and B are given, we can construct the IFS $A - B$. The currently existing forms of this operation are given below. The first two forms are taken, respectively, from [5] and [6] and we will denote them as $BR1$ and $BR2$:

$$A -_{BR1} B = \{\langle x, \mu_{A-B}(x), \nu_{A-B}(x)\rangle | x \in E\},$$

where

$$\mu_{A-B}(x) = \begin{cases} \dfrac{\mu_A(x) - \mu_B(x)}{1 - \mu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \\ & \text{and } \nu_B(x) > 0 \\ & \text{and } \nu_A(x)\pi_B(x) \leq \pi_A(x)\nu_B(x) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{A-B}(x) = \begin{cases} \dfrac{\nu_A(x)}{\nu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \\ & \text{and } \nu_B(x) > 0 \\ & \text{and } \nu_A(x)\pi_B(x) \leq \pi_A(x)\nu_B(x) \\ 1, & \text{otherwise} \end{cases}$$

and

$$A -_{BR2} B = \{\langle \min(\mu_A(x), \nu_B(x)), \max(\mu_B(x), \nu_A(x))\rangle | x \in E\}.$$

In some definitions below, we use functions sg and $\overline{\text{sg}}$, defined by

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \qquad \overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

The next definitions of instances of the “substraction” operation are based on the well-known formula from set theory:

$$A - B = A \cap \neg B$$

where A and B are given sets. In the IFS case, if the IFSs A and B are given, we define the following versions of “substraction” operation:

$$A -'_i B = A \cap \neg_i B, \text{ and } A -''_i B = \neg_i \neg_i A \cap \neg_i B,$$

where $i = 1, 2, \ldots, 34$.

Of course, for every two IFSs A and B, it is valid that

$$A -'_1 B = A -''_1 B,$$

because the first negation will satisfy the Law of Excluded Middle, but in the other cases this equality is not valid.

All new subtractions are given in Table 1.

Table 1: List of intuitionistic fuzzy subtractions.

$-'_1$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle \mid x \in E\}$
$-'_2$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(\mu_B(x))), \max(\nu_A(x), \text{sg}(\mu_B(x)))\rangle \mid x \in E\}$
$-'_3$	$\{\langle x, \min(\mu_A(x), \nu_B(x)),$ $\max(\nu_A(x), \mu_B(x).\nu_B(x) + \mu_B(x)^2)\rangle \mid x \in E\}$
$-'_4$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), 1 - \nu_B(x))\rangle \mid x \in E\}$
$-'_5$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))),$ $\max(\nu_A(x), \text{sg}(1 - \nu_B(x)))\rangle \mid x \in E\}$
$-'_6$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))), \max(\nu_A(x), \text{sg}(\mu_B(x)))\rangle \mid x \in E\}$
$-'_7$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))), \max(\nu_A(x), \mu_B(x))\rangle \mid x \in E\}$
$-'_8$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle \mid x \in E\}$
$-'_9$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(\mu_B(x))), \max(\nu_A(x), \mu_B(x))\rangle \mid x \in E\}$
$-'_{10}$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))), \max(\nu_A(x), 1 - \nu_B(x))\rangle \mid x \in E\}$
$-'_{11}$	$\{\langle x, \min(\mu_A(x), \text{sg}(\nu_B(x))), \max(\nu_A(x), \overline{\text{sg}}(\nu_B(x)))\rangle \mid x \in E\}$
$-'_{12}$	$\{\langle x, \min(\mu_A(x), \nu_B(x).(\mu_B(x) + \nu_B(x))),$ $\max(\nu_A(x), \mu_B(x).(\nu_B(x)^2 + \mu_B(x) + \mu_B(x).\nu_B(x)))\rangle \mid x \in E\}$
$-'_{13}$	$\{\langle x, \min(\mu_A(x), \text{sg}(1 - \mu_B(x))),$ $\max(\nu_A(x), \overline{\text{sg}}(1 - \mu_B(x)))\rangle \mid x \in E\}$
$-'_{14}$	$\{\langle x, \min(\mu_A(x), \text{sg}(\nu_B(x))), \max(\nu_A(x), \overline{\text{sg}}(1 - \mu_B(x)))\rangle \mid x \in E\}$
$-'_{15}$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))),$ $\max(\nu_A(x), \overline{\text{sg}}(1 - \mu_B(x)))\rangle \mid x \in E\}$
$-'_{16}$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(\mu_B(x))), \max(\nu_A(x), \overline{\text{sg}}(1 - \mu_B(x)))\rangle \mid x \in E\}$
$-'_{17}$	$\{\langle x, \min(\mu_A(x), \overline{\text{sg}}(1 - \nu_B(x))), \max(\nu_A(x), \overline{\text{sg}}(\nu_B(x)))\rangle \mid x \in E\}$
$-'_{18}$	$\{\langle x, \min(\mu_A(x), \nu_B(x), \text{sg}(\mu_B(x))),$ $\max(\nu_A(x), \min(\mu_B(x), \text{sg}(\nu_B(x))))\rangle \mid x \in E\}$
$-'_{19}$	$\{\langle x, \min(\mu_A(x), \nu_B(x), \text{sg}(\mu_B(x))), \nu_A(x)\rangle \mid x \in E\}$
$-'_{20}$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \nu_A(x)\rangle \mid x \in E\}$
$-'_{21}$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x), \text{sg}(\mu_B(x))),$ $\max(\nu_A(x), \min(\mu_B(x), \text{sg}(1 - \mu_B(x))))\rangle \mid x \in E\}$
$-'_{22}$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x), \text{sg}(\mu_B(x))), \nu_A(x)\rangle \mid x \in E\}$
$-'_{23}$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x)), \nu_A(x)\rangle \mid x \in E\}$
$-'_{24}$	$\{\langle x, \min(\mu_A(x), \nu_B(x), \text{sg}(1 - \nu_B(x))),$ $\max(\nu_A(x), \min(1 - \nu_B(x), \text{sg}(\nu_B(x))))\rangle \mid x \in E\}$
$-'_{25}$	$\{\langle x, \min(\mu_A(x), \nu_B(x), \text{sg}(1 - \nu_B(x))), \nu_A(x)\rangle \mid x \in E\}$
$-'_{26}$	$\{\langle x, \min(\mu_A(x), \nu_B(x)),$ $\max(\nu_A(x), \mu_B(x).\nu_B(x) + \overline{\text{sg}}(1 - \mu_B(x)))\rangle \mid x \in E\}$
$-'_{27}$	$\{\langle x, \min(\mu_A(x), 1 - \mu_B(x)),$ $\max(\nu_A(x), \mu_B(x).(1 - \mu_B(x)) + \overline{\text{sg}}(1 - \mu_B(x)))\rangle \mid x \in E\}$

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Table 1 – continued from previous page

$-'_{28}$	$\{\langle x, \min(\mu_A(x), \nu_B(x)),$ $\max(\nu_A(x), (1-\nu_B(x)).\nu_B(x) + \overline{\text{sg}}(\nu_B(x)))\rangle \| x \in E\}$
$-'_{29}$	$\{\langle x, \min(\mu_A(x), \max(0, \mu_B(x).\nu_B(x) + \overline{\text{sg}}(1-\nu_B(x)))),$ $\max(\nu_A(x), \mu_B(x).(\mu_B(x).\nu_B(x)$ $+\overline{\text{sg}}(1-\nu_B(x))) + \overline{\text{sg}}(1-\mu_B(x)))\rangle \| x \in E\}$
$-'_{30}$	$\{\langle x, \min(\mu_A(x), \mu_B(x).\nu_B(x)),$ $\max(\nu_A(x), \mu_B(x).(\mu_B(x).\nu_B(x)$ $+\overline{\text{sg}}(1-\nu_B(x))) + \overline{\text{sg}}(1-\mu_B(x)))\rangle \| x \in E\}$
$-'_{31}$	$\{\langle x, \min(\mu_A(x), (1-\mu_B(x)).\mu_B(x) + \overline{\text{sg}}(\mu_B(x))),$ $\max(\nu_A(x), \mu_B(x).((1-\mu_B(x)).\mu_B(x)$ $+\overline{\text{sg}}(\mu_B(x))) + \overline{\text{sg}}(1-\mu_B(x)))\rangle \| x \in E\}$
$-'_{32}$	$\{\langle x, \min(\mu_A(x), (1-\mu_B(x)).\mu_B(x)),$ $\max(\nu_A(x), \mu_B(x).((1-\mu_B(x)).\mu_B(x)$ $+\overline{\text{sg}}(\mu_B(x))) + \overline{\text{sg}}(1-\mu_B(x)))\rangle \| x \in E\}$
$-'_{33}$	$\{\langle x, \min(\mu_A(x), \nu_B(x).(1-\nu_B(x)) + \overline{\text{sg}}(1-\nu_B(x))),$ $\max(\nu_A(x), (1-\nu_B(x)).(\nu_B(x).(1-\nu_B(x))$ $+\overline{\text{sg}}(1-\nu_B(x))) + \overline{\text{sg}}(\nu_B(x)))\rangle \| x \in E\}$
$-'_{34}$	$\{\langle x, \min(\mu_A(x), \nu_B(x).(1-\nu_B(x))),$ $\max(\nu_A(x), (1-\nu_B(x)).(\nu_B(x).(1-\nu_B(x))$ $+\overline{\text{sg}}(1-\nu_B(x))) + \overline{\text{sg}}(\nu_B(x)))\rangle \| x \in E\}$
$-''_1$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$-''_2$	$\{\langle x, \min(\text{sg}(\mu_A(x)), \overline{\text{sg}}(\mu_B(x))),$ $\max(\overline{\text{sg}}(\mu_A(x)), \text{sg}(\mu_B(x)))\rangle \| x \in E\}$
$-''_3$	$\{\langle x, \min(\mu_A(x).\nu_A(x) + \mu_A(x)^2, \nu_B(x)),$ $\max(\nu_A(x).(\mu_A(x).\nu_A(x) + \mu_A(x)^2) + \nu_A(x)^2,$ $\mu_B(x).\nu_B(x) + \mu_B(x)^2)\rangle \| x \in E\}$
$-''_4$	$\{\langle x, \min(1-\nu_A(x), \nu_B(x)), \max(\nu_A(x), 1-\nu_B(x))\rangle \| x \in E\}$
$-''_5$	$\{\langle x, \min(\text{sg}(1-\nu_A(x)), \overline{\text{sg}}(1-\nu_B(x))),$ $\max(\overline{\text{sg}}(1-\nu_A(x)), \text{sg}(1-\nu_B(x)))\rangle \| x \in E\}$
$-''_6$	$\{\langle x, \min(\text{sg}(\mu_A(x)), \overline{\text{sg}}(1-\nu_B(x))),$ $\max(\overline{\text{sg}}(1-\nu_A(x)), \text{sg}(\mu_B(x)))\rangle \| x \in E\}$
$-''_7$	$\{\langle x, \min(\overline{\text{sg}}(1-\mu_A(x)), \overline{\text{sg}}(1-\nu_B(x))),$ $\max(\overline{\text{sg}}(1-\nu_A(x)), \mu_B(x))\rangle \| x \in E\}$
$-''_8$	$\{\langle x, \min(\mu_A(x), 1-\mu_B(x)), \max(1-\mu_A(x), \mu_B(x))\rangle \| x \in E\}$
$-''_9$	$\{\langle x, \min(\text{sg}(\mu_A(x)), \overline{\text{sg}}(\mu_B(x))), \max(\overline{\text{sg}}(\mu_A(x)), \mu_B(x))\rangle \| x \in E\}$
$-''_{10}$	$\{\langle x, \min(\overline{\text{sg}}(\nu_A(x)), \overline{\text{sg}}(1-\nu_B(x))),$ $\max(\nu_A(x), 1-\nu_B(x))\rangle \| x \in E\}$
$-''_{11}$	$\{\langle x, \min(\overline{\text{sg}}(\nu_A(x)), \text{sg}(\nu_B(x))),$ $\max(\text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_B(x)))\rangle \| x \in E\}$
$-''_{12}$	$\{\langle x, \min(\mu_A(x).(\nu_A(x)^2 + \mu_A(x) + \mu_A(x).\nu_A(x)).(\mu_A(x).(\nu_A(x)^2$ $+\mu_A(x) + \mu_A(x).\nu_A(x)) + (\nu_A(x).(\mu_A(x) + \nu_A(x)))),$ $\nu_B(x).(\mu_B(x) + \nu_B(x))),$ $\max(\nu_A(x).(\mu_A(x) + \nu_A(x)).(\mu_A(x)^2.(\nu_A(x)^2 + \mu_A(x)$ $+\mu_A(x).\nu_A(x))^2 + \nu_A(x).(\mu_A(x) + \nu_A(x))) + \mu_A(x).\nu_A(x)$ $.(\nu_A(x)^2 + \mu_A(x) + \mu_A(x).\nu_A(x)).(\mu_A(x) + \nu_A(x)),$ $\mu_B(x).(\nu_B(x)^2 + \mu_B(x) + \mu_B(x).\nu_B(x)))\rangle \| x \in E\}$

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$-''_{13}$	$\{\langle x, \min(\overline{\mathrm{sg}}(1-\mu_A(x)), \mathrm{sg}(1-\mu_B(x))),$ $\max(\mathrm{sg}(1-\mu_A(x)), \overline{\mathrm{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{14}$	$\{\langle x, \min(\overline{\mathrm{sg}}(1-\mu_A(x)), \mathrm{sg}(\nu_B(x))),$ $\max(\mathrm{sg}(\nu_A(x)), \overline{\mathrm{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{15}$	$\{\langle x, \min(\overline{\mathrm{sg}}(1-\mu_A(x)), \overline{\mathrm{sg}}(1-\nu_B(x))),$ $\max(\overline{\mathrm{sg}}(1-\nu_A(x)), \overline{\mathrm{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{16}$	$\{\langle x, \min(\mathrm{sg}(\mu_A(x)), \overline{\mathrm{sg}}(\mu_B(x))),$ $\max(\overline{\mathrm{sg}}(\mu_A(x)), \overline{\mathrm{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{17}$	$\{\langle x, \min(\overline{\mathrm{sg}}(\nu_A(x)), \overline{\mathrm{sg}}(1-\nu_B(x))),$ $\max(\mathrm{sg}(\nu_A(x)), \overline{\mathrm{sg}}(\nu_B(x)))\rangle \vert x \in E\}$
$-''_{18}$	$\{\langle x, \min(\mu_A(x), \mathrm{sg}(\nu_A(x)), \nu_B(x), \mathrm{sg}(\mu_B(x))),$ $\max(\min(\nu_A(x), \mathrm{sg}(\mu_A(x))), \min(\mu_B(x), \mathrm{sg}(\nu_B(x))))\rangle \vert x \in E\}$
$-''_{19}$	$\{\langle x, 0, 0\rangle \vert x \in E\}$
$-''_{20}$	$\{\langle x, 0, 0\rangle \vert x \in E\}$
$-''_{21}$	$\{\langle x, \mu_A(x).\mathrm{sg}(1-\mu_A(x)),$ $\max((1-\mu_A(x)).\mathrm{sg}(\mu_A(x)), \min(\mu_B(x),$ $\mathrm{sg}(1-\mu_B(x))))\rangle \vert x \in E\}$
$-''_{22}$	$\{\langle x, \min(\mu_A(x).\mathrm{sg}(\mu_A(x)), 1-\mu_B(x), \mathrm{sg}(\mu_B(x))), 0\rangle \vert x \in E\}$
$-''_{23}$	$\{\langle x, \min(\mu_A(x), 1-\mu_B(x)), 0\rangle \vert x \in E\}$
$-''_{24}$	$\{\langle x, \min(1-\nu_A(x), \mathrm{sg}(\nu_A(x)), \nu_B(x), \mathrm{sg}(1-\nu_B(x))),$ $\max(\nu_A(x).\mathrm{sg}(1-\nu_A(x))), \min(1-\nu_B(x), \mathrm{sg}(\nu_B(x)))\rangle \vert x \in E\}$
$-''_{25}$	$\{\langle x, 0, 0\rangle \vert x \in E\}$
$-''_{26}$	$\{\langle x, \min(\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\mu_A(x)), \nu_B(x)),$ $\max(\nu_A(x).(\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\mu_A(x))) + \overline{\mathrm{sg}}(1-\nu_A(x)),$ $\mu_B(x).\nu_B(x) + \overline{\mathrm{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{27}$	$\{\langle x, \min(\mu_A(x), 1-\mu_B(x)),$ $\max(((1-\mu_A(x)).\mu_A(x)) + \overline{\mathrm{sg}}(\mu_A(x)),$ $\mu_B(x).(1-\mu_B(x)) + \overline{\mathrm{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{28}$	$\{\langle x, \min((1-\nu_A(x)).\nu_A(x) + \overline{\mathrm{sg}}(\nu_A(x)), \nu_B(x)),$ $\max((1-(1-\nu_A(x)).\nu_A(x)) - \overline{\mathrm{sg}}(\nu_A(x))).((1-\nu_A(x)).\nu_A(x)$ $+\overline{\mathrm{sg}}((1-\nu_A(x)).\nu_A(x) + \overline{\mathrm{sg}}(\nu_A(x))),$ $(1-\nu_B(x)).\nu_B(x) + \overline{\mathrm{sg}}(\nu_B(x)))\rangle \vert x \in E\}$
$-''_{29}$	$\{\langle x, \min((\mu_A(x).(\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\nu_A(x))) + \overline{\mathrm{sg}}(1-\mu_A(x)))$ $.(\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\nu_A(x))) + \overline{\mathrm{sg}}(1-\mu_A(x).(\mu_A(x).\nu_A(x)$ $+\overline{\mathrm{sg}}(1-\nu_A(x))) - \overline{\mathrm{sg}}(1-\mu_A(x))), \mu_B(x).\nu_B(x)$ $+\overline{\mathrm{sg}}(1-\nu_B(x))), \max((\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\nu_A(x))).((\mu_A(x)$ $.(\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\nu_A(x))) + \overline{\mathrm{sg}}(1-\mu_A(x))).(\mu_A(x).\nu_A(x)$ $+\overline{\mathrm{sg}}(1-\nu_A(x))) + \overline{\mathrm{sg}}(1-\mu_A(x).(\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\nu_A(x)))$ $-\overline{\mathrm{sg}}(1-\mu_A(x)))) + \overline{\mathrm{sg}}(1-\mu_A(x).\nu_A(x) - \overline{\mathrm{sg}}(1-\nu_A(x))), \mu_B(x)$ $.(\mu_B(x).\nu_B(x) + \overline{\mathrm{sg}}(1-\nu_B(x))) + \overline{\mathrm{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
$-''_{30}$	$\{\langle x, \min(((\mu_A(x).(\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\nu_A(x))) + \overline{\mathrm{sg}}(1-\mu_A(x)))$ $.\mu_A(x).\nu_A(x)), \mu_B(x).\nu_B(x)),$ $\max(\mu_A(x).\nu_A(x).((\mu_A(x).(\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\nu_A(x)))$ $+\overline{\mathrm{sg}}(1-\mu_A(x))).\mu_A(x).\nu_A(x) + \overline{\mathrm{sg}}(1-\mu_A(x).(\mu_A(x).\nu_A(x)$ $+\overline{\mathrm{sg}}(1-\nu_A(x))) - \overline{\mathrm{sg}}(1-\mu_A(x)))) + \overline{\mathrm{sg}}(1-(\mu_A(x).\nu_A(x))),$ $\mu_B(x).(\mu_B(x).\nu_B(x) + \overline{\mathrm{sg}}(1-\nu_B(x))) + \overline{\mathrm{sg}}(1-\mu_B(x)))\rangle \vert x \in E\}$
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$-''_{31}$	$\{\langle x, \min((1-(1-\mu_A(x)).\mu_A(x)-\overline{\text{sg}}(\mu_A(x))).((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x)))+\overline{\text{sg}}(((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x)))),(1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x))),\max(((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x))).((1-(1-\mu_A(x)).\mu_A(x))-\overline{\text{sg}}(\mu_A(x)).((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x)))+\overline{\text{sg}}((1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}(\mu_A(x))))+\overline{\text{sg}}(1-(1-\mu_A(x)).\mu_A(x)-\overline{\text{sg}}(\mu_A(x))),\mu_B(x).((1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x)))+\overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{32}$	$\{\langle x, \min((1-(1-\mu_A(x)).\mu_A(x)).(1-\mu_A(x)).\mu_A(x),(1-\mu_B(x)).\mu_B(x)),\max(((1-\mu_A(x)).\mu_A(x).((1-(1-\mu_A(x)).\mu_A(x)).(1-\mu_A(x)).\mu_A(x)+\overline{\text{sg}}((1-\mu_A(x)).\mu_A(x)))+\overline{\text{sg}}(1-(1-\mu_A(x)).\mu_A(x))),\mu_B(x).((1-\mu_B(x)).\mu_B(x)+\overline{\text{sg}}(\mu_B(x)))+\overline{\text{sg}}(1-\mu_B(x)))\rangle \mid x \in E\}$
$-''_{33}$	$\{\langle x, \min(((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))).(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x)))+\overline{\text{sg}}(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))),\nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x))),\max((1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))).(((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))).(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x)))+\overline{\text{sg}}(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))))+\overline{\text{sg}}((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))),(1-\nu_B(x)).(\nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x)))+\overline{\text{sg}}(\nu_B(x)))\rangle \mid x \in E\}$
$-''_{34}$	$\{\langle x, \min(((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))).(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))),\nu_B(x).(1-\nu_B(x))),\max(((1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x))).(((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))).(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x)))+\overline{\text{sg}}(1-(1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))-\overline{\text{sg}}(\nu_A(x)))))+\overline{\text{sg}}((1-\nu_A(x)).(\nu_A(x).(1-\nu_A(x))+\overline{\text{sg}}(1-\nu_A(x)))+\overline{\text{sg}}(\nu_A(x))),(1-\nu_B(x)).(\nu_B(x).(1-\nu_B(x))+\overline{\text{sg}}(1-\nu_B(x)))+\overline{\text{sg}}(\nu_B(x)))\rangle \mid x \in E\}$

We immediately see that operation $-_{BR1}$ does not occur in Table 1, while operations $-_{BR2}$, $-'_1$ and $-''_2$ coincide.

3 Main results

Initially, we give the list of all intuitionistic fuzzy implications (see Table 2). They generate 34 different negations, given in Table 3. The relations between the implications and negatiions are shown in Table 4.

Table 2: List of the first 14 intuitionistic fuzzy implications.

$\to_1$	$\{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))), \min(\mu_A(x), \nu_B(x))\rangle \| x \in E\}$
$\to_2$	$\{\langle x, \mathrm{sg}(\mu_A(x) - \mu_B(x)), \nu_B(x).\mathrm{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\to_3$	$\{\langle x, 1 - (1 - \mu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x)))),$ $\nu_B(x).\mathrm{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$\to_4$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \| x \in E\}$
$\to_5$	$\{\langle x, \min(1, \nu_A(x) + \mu_B(x)), \max(0, \mu_A(x) + \nu_B(x) - 1)\rangle \| x \in E\}$
$\to_6$	$\{\langle x, \nu_A(x) + \mu_A(x)\mu_B(x), \mu_A(x)\nu_B(x)\rangle \| x \in E\}$
$\to_7$	$\{\langle x, \min(\max(\nu_A(x), \mu_B(x)), \max(\mu_A(x), \nu_A(x)),$ $\max(\mu_B(x), \nu_B(x))),\ \max(\min(\mu_A(x), \nu_B(x)),$ $\min(\mu_A(x), \nu_A(x)), \min(\mu_B(x), \nu_B(x)))\rangle \| x \in E\}$
$\to_8$	$\{\langle x, 1 - (1 - \min(\nu_A(x), \mu_B(x))).\mathrm{sg}(\mu_A(x) - \mu_B(x)),$ $\max(\mu_A(x), \nu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x)),$ $\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$
$\to_9$	$\{\langle x, \nu_A(x) + \mu_A(x)^2\mu_B(x), \mu_A(x)\nu_A(x) + \mu_A(x)^2\nu_B(x)\rangle \| x \in E\}$
$\to_{10}$	$\{\langle x, \mu_B(x).\overline{\mathrm{sg}}(1 - \mu_A(x))$ $+\mathrm{sg}(1 - \mu_A(x)).(\overline{\mathrm{sg}}(1 - \mu_B(x)) + \nu_A(x).\mathrm{sg}(1 - \mu_B(x))),$ $\nu_B(x).\overline{\mathrm{sg}}(1 - \mu_A(x)) + \mu_A(x).\mathrm{sg}(1 - \mu_A(x))$ $.\mathrm{sg}(1 - \mu_B(x))\rangle \| x \in E\}$
$\to_{11}$	$\{\langle x, 1 - (1 - \mu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x)),$ $\nu_B(x).\mathrm{sg}(\mu_A(x) - \mu_B(x)).\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$
$\to_{12}$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), 1 - \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$\to_{13}$	$\{\langle x, \nu_A(x) + \mu_B(x) - \nu_A(x).\mu_B(x), \mu_A(x).\nu_B(x)\rangle \| x \in E\}$
$\to_{14}$	$\{\langle x, 1 - (1 - \mu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x))$ $-\nu_B(x).\overline{\mathrm{sg}}(\mu_A(x) - \mu_B(x)).\mathrm{sg}(\nu_B(x) - \nu_A(x)),$ $\nu_B(x).\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle \| x \in E\}$

Table 3: List of the first 5 intuitionistic fuzzy negations.

$\neg_1$	$\{\langle x, \nu_A(x), \mu_A(x)\rangle \| x \in E\}$
$\neg_2$	$\{\langle x, \overline{\mathrm{sg}}(\mu_A(x)), \mathrm{sg}(\mu_A(x))\rangle \| x \in E\}$
$\neg_3$	$\{\langle x, \nu_A(x), \mu_A(x).\nu_A(x) + \mu_A(x)^2\rangle \| x \in E\}$
$\neg_4$	$\{\langle x, \nu_A(x), 1 - \nu_A(x)\rangle \| x \in E\}$
$\neg_5$	$\{\langle x, \overline{\mathrm{sg}}(1 - \nu_A(x)), \mathrm{sg}(1 - \nu_A(x))\rangle \| x \in E\}$

Table 4: Correspondence between intuitionistic fuzzy negations and implications.

$\neg_1$	$\rightarrow_1, \rightarrow_4, \rightarrow_5, \rightarrow_6, \rightarrow_7, \rightarrow_{10}, \rightarrow_{13}$
$\neg_2$	$\rightarrow_2, \rightarrow_3, \rightarrow_8, \rightarrow_{11}$
$\neg_3$	$\rightarrow_9$
$\neg_4$	$\rightarrow_{12}$
$\neg_5$	$\rightarrow_{14}$

Now, we introduce the definitions of the new "substraction" operations. As a basis of the new instances of this operation, we use the formula from classical set theory

$$A - B = A \cap \neg B = \neg(\neg A \cup B) = \neg(A \rightarrow B),$$

where A and B are two IFSs. Hence, for $i = 1, 2, ..., 134$ (or in the present case, for $i = 1, 2, ..., 14$)

$$A -_i B = \neg_{\delta(i)}(A \rightarrow_i B),$$

where $\delta(i)$ is the number of the negation that corresponds to i-th implication (see Table 4). Therefore, 134 new "substraction" operations can originate. This process is difficult, having in mind the very complex forms of some implications and negations from Tables 2 and 3. By this reason, here we introdice the definition of the first 14 new instances of the "substraction" operation (see Table 5) and the rest definitions will be given in future.

Table 5: List of the first 14 new intuitionistic fuzzy subtractions.

$-_1$	$\{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \min(\mu_A(x), \mu_B(x)))\rangle \| x \in E\}$
$-_2$	$\{\langle x, \text{sg}(\mu_A(x) - \mu_B(x)), \overline{\text{sg}}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$-_3$	$\{\langle x, \overline{\text{sg}}(\mu_A(x) - \mu_B(x)), \overline{\text{sg}}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$-_4$	$\{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \| x \in E\}$
$-_5$	$\{\langle x, \max(0, \mu_A(x) + \nu_B(x) - 1), \min(1, \nu_A(x) + \mu_B(x))\rangle \| x \in E\}$
$-_6$	$\{\langle x, \mu_A(x)\nu_B(x), \nu_A(x) + \mu_A(x)\mu_B(x)\rangle \| x \in E\}$
$-_7$	$\{\langle x, \max(\min(\mu_A(x), \nu_B(x)), \min(\mu_A(x), \nu_A(x)),$ $\min(\mu_B(x), \nu_B(x))),$ $\min(\max(\nu_A(x), \mu_B(x)), \max(\mu_A(x), \nu_A(x)),$ $\max(\mu_B(x), \nu_B(x)))\rangle \| x \in E\}$
$-_8$	$\{\langle x, ((1 - \text{sg}(\min(\nu_A(x), \mu_B(x)))).\text{sg}(\mu_A(x) - \mu_B(x)),$ $\overline{\text{sg}}(\mu_A(x) - \mu_B(x)) + \text{sg}(\min(\nu_A(x), \mu_B(x))))$ $.\text{sg}(\mu_A(x) - \mu_B(x))\rangle \| x \in E\}$
$-_9$	$\{\langle x, \mu_A(x).\nu_A(x) + \mu_A(x)^2\nu_B(x),$ $\mu_A(x)\nu_A(x)^2 + \mu_A(x)^2\nu_A(x)\nu_B(x) + \mu_A(x)^3\nu_A(x)\mu_B(x)$ $+\mu_A(x)^4\mu_B(x)\nu_B(x) + nu_A(x)^2 + 2 + \mu_A(x)^2\nu_A(x)\mu_B(x)$ $+\mu_A(x)^4\mu_B(x)^2\rangle \| x \in E\}$
	Continued on next page

Table 5 – continued from previous page

$-_{10}$	$\{\langle x, \nu_B(x).\overline{\text{sg}}(1-\mu_A(x))+\mu_A(x).\text{sg}(1-\mu_A(x)).\text{sg}(1-\mu_B(x)), \mu_B(x).\overline{\text{sg}}(1-\mu_A(x))+\text{sg}(1-\mu_A(x)).(\overline{\text{sg}}(1-\mu_B(x))+\nu_A(x) .\text{sg}(1-\mu_B(x)))\rangle \| x \in E\}$
$-_{11}$	$\{\langle x, \overline{\text{sg}}(\mu_A(x)-\mu_B(x)), \overline{\text{sg}}(\mu_A(x)-\mu_B(x))\rangle \| x \in E\}$
$-_{12}$	$\{\langle x, 1-\max(\nu_A(x), \mu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle \| x \in E\}$
$-_{13}$	$\{\langle x, \mu_A(x).\nu_B(x), \nu_A(x)+\mu_B(x)-\nu_A(x).\mu_B(x)\rangle \| x \in E\}$
$-_{14}$	$\{\langle x, \overline{\text{sg}}(1-\nu_B(x).\text{sg}(\nu_B(x)-\nu_A(x)), \text{sg}(1-\nu_B(x).\text{sg}(\nu_B(x)-\nu_A(x))\rangle \| x \in E\}$

Some of the most important properties of the subtractions are:
(a) $A - E^* = O^*$,
(b) $A - O^* = A$,
(c) $E^* - A = \neg A$,
(d) $O^* - A = O^*$,
(e) $(A - B) \cap C = (A \cap C) - B = A \cap (C - B)$,
(f) $(A \cap B) - C = (A - C) \cap (B - C)$,
(g) $(A \cup B) - C = (A - C) \cup (B - C)$,
(h) $(A - B) - C = (A - C) - B$,
(i) $(A - C) \cap B = A \cap (B - C)$,
(j) $O^* - U^* = O^*$,
(k) $O^* - E^* = O^*$,
(l) $U^* - O^* = U^*$,
(m) $U^* - E^* = O^*$,
(n) $E^* - O^* = E^*$,
(o) $E^* - U^* = O^*$.

In Table 6 are given these subtractions that satisfy these properties.

Table 6: Properties of the "subtraction" operations.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
$\rightarrow_1$	-	+	+	+	-	-	-	-	-	+	+	+	-	+	-
$\rightarrow_2$	+	-	-	+	-	+	+	-	-	+	+	-	+	+	-
$\rightarrow_3$	+	-	+	+	-	+	+	+	-	+	+	-	+	+	-
$\rightarrow_4$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-
$\rightarrow_5$	+	+	+	+	-	+	+	+	-	+	+	+	+	+	-
$\rightarrow_6$	-	+	+	+	-	-	-	-	-	+	+	+	-	+	-
$\rightarrow_7$	-	+	+	-	-	-	-	-	-	-	+	+	-	+	-
$\rightarrow_8$	+	-	-	+	-	+	-	-	-	+	+	-	+	+	-
$\rightarrow_9$	-	-	+	+	-	-	-	-	-	+	+	+	-	+	-
$\rightarrow_{10}$	+	+	+	+	-	-	-	-	-	+	+	+	+	+	-
$\rightarrow_{11}$	+	-	+	+	-	+	+	+	-	+	+	-	+	+	-
$\rightarrow_{12}$	+	-	-	+	-	+	+	+	-	+	+	-	+	+	-
$\rightarrow_{13}$	+	+	+	+	-	+	+	+	-	+	+	+	+	+	-
$\rightarrow_{14}$	+	-	+	+	-	+	+	+	-	+	+	-	+	+	+

In a next research we will continue to study the definitions and properties of the new subtractions based on the intuitionistic fuzzy implications.

An **OPEN PROBLEM** is to find another approach to introducing variants of the "subtraction" operation over IFSs. If this is possible, the behaviour of the new operations must be studied, also.

4 Final remarks: Beloslav Riečan's group and intuitionistic fuzzy sets

In the beginning of the 21^{st} century, Prof. Beloslav Riečan established in the Matej Bel University, Banská Bystrica one of the most active research groups in the world in the area of intitionistc fuzzy set theory. After the two annual conferences on IFSs, organized in Sofia (since 1998) and Warsaw (since 2000), Banská Bystrica became the third place, where such regular meetings are being held ever since 2006.

Prof. Riečan participates actively in the organization of the Bulgarian conferences (since 2006) and in the edition of the specialized journal "Notes on Intuitionistic Fuzzy Sets". He and his PhD students and collaborators developed whole areas of IFSs theory, related to intuitionistic fuzzy integrals, probabilities, etc. To this end, Prof. Riečan has the largest number of successfully defended PhD students with theses on IFSs in the world.

On the behalf of his Bulgarian friends and colleagues, I wish him to keep up his research activity for a long years in future.

5 Acknowledgement

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On number of interior periodic points of a Lotka-Volterra map

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Dedicated to the 75th birthday of Beloslav Riečan

Abstract

Given the plane triangle $\Delta = \{ [x, y] : 0 \le x,\ 0 \le y,\ x + y \le 4 \}$ and the transformation $F : \Delta \to \Delta$, $[x, y] \mapsto [x(4 - x - y),\ xy]$ we give a lower estimate of the number of interior periodic orbits with period $n \le 36$.

Keywords Periodic point, Jacobi matrix, saddle fixed point, itinerary, Brouwer theorem

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1 Introduction

We study periodic points of the map $F : [x, y] \mapsto [x(4 - x - y), xy]$ lying inside the triangle

$$\Delta = \{ [x, y] : 0 \le x,\ 0 \le y,\ x + y \le 4 \} .$$

The map F maps the triangle Δ onto itself. This map has been studied in the papers [8], [4],[5],[6] and is sometimes called Lotka–Volterra. Y. Avishai and D. Berend in [1] (see also [2] and [3]) studied a discrete system related with the dynamics of the map $F : \Delta \to \Delta$. The basic transformation considered in [1] is $H[x, y] = [y,\ x^2y - 2x^2 + 2]$ defined on $\mathbb{R}^2$. The system (Δ, F) was obtained from $(\mathbb{R}^2, H)$ employing some conjugacy reductions. A. N. Sharkovskiĭ in [7] stated some open problems on the dynamics of the map F. It is easy to find three fixed points of the map F, namely $[0, 0]$, $[3, 0]$ and $[1, 2]$. (Periodic points on the lower side of Δ are well known, because the restriction of F to the lower side is the logistic map $f : x \mapsto x(4 - x)$ which is conjugate with the tent map.)

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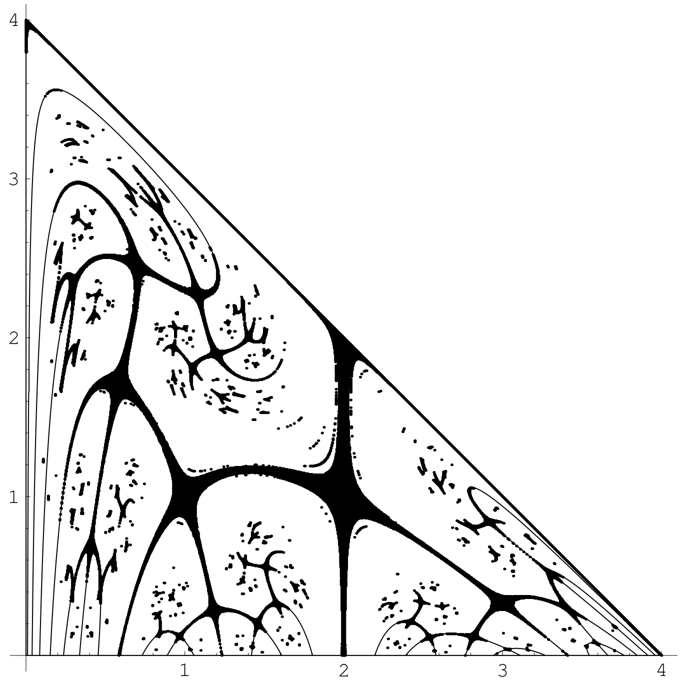

Figure 1: Interior periodic points of the map F whose periods are ≤ 36. Such interior periodic points are white. The black part of the triangle does not contain those points.

Until recently nothing has been known on the existence of interior periodic points different from $[1, 2]$. Only in 2006, in [4], the interior point $[1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}]$ with period 4 was found. Trying to find other interior periodic points, we started to study periodic points by numerical experiments and soon we found the point $[1, \frac{3+\sqrt{5}}{2}]$ with period 6 and numerically also many other periodic points. We omit these numerical experiments because they are not necessary for reading the present paper. In fact, after a careful analysis of them we were able to prove an *exact result*, Theorem 4.3, which was proved in [6]. It implies the existence of interior periodic points of all periods $n \geq 4$ inside Δ. The results of our numerical experiments are illustrated on Fig. 1. It contains about $5.4 \cdot 10^{10}$ periodic points with period $n \leq 36$.

The present paper is a continuation of [6]. Our main result is Theorem 3.3 and Table 1.

2 Notations and preliminary results

We denote by $[x, y]$ a point in the plane, while (α, β) and $\langle \alpha, \beta \rangle$ are open and closed intervals on the real line. Throughout the paper we denote by F the map of the plane $\mathbb{R}^2$ given by $F[x, y] = [x(4 - x - y), xy]$. Let $\Delta = \{ [x, y] : 0 \leq x, 0 \leq y, x + y \leq 4 \}$. The sides of the triangle Δ are denoted by a, b and c as it is shown in Fig. 2. It is easy to see that $F(\Delta) = \Delta$. Note that $F[x, 0] = [f(x), 0]$, where

$$f : \langle 0, 4 \rangle \to \langle 0, 4 \rangle, \; f(x) = x(4 - x)$$

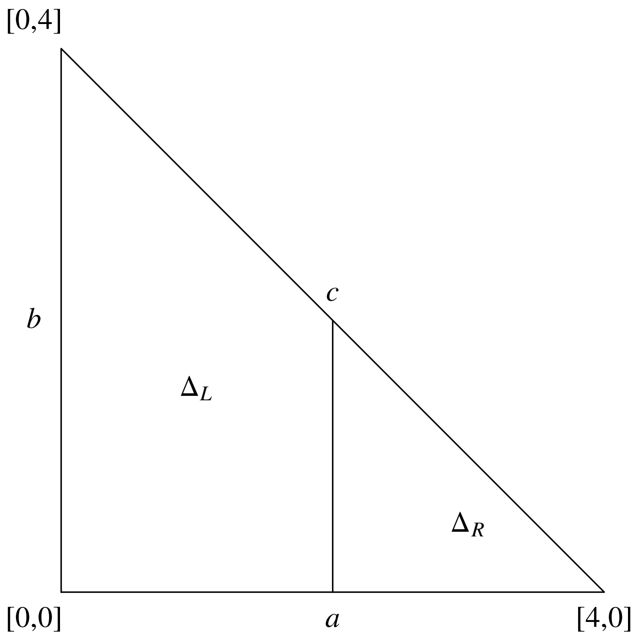

Figure 2: Notations concerning the triangle Δ.

is the logistic map. Note that any point $x \in \langle 0, 4\rangle$ may be written in the form $x = 4\sin^2 t$ with $t \in \langle 0, \frac{\pi}{2}\rangle$ and in this case

$$\begin{aligned} f(x) &= f(4\sin^2 t) = 4\sin^2 t(4 - 4\sin^2 t) = 16\sin^2 t\cos^2 t \\ &= 4\sin^2 2t = 4\sin^2(\pi - 2t)\ . \end{aligned} \tag{2.1}$$

The logistic map f is conjugate with the tent map $g : \langle 0, 1\rangle \to \langle 0, 1\rangle,\ g(t) = 1 - |1 - 2t|$ via the conjugation $h : \langle 0, 1\rangle \to \langle 0, 4\rangle,\ h(t) = 4\sin^2(\pi t/2)$. Since any fixed point of the map g^n is of the form $\frac{2k}{2^n \pm 1}$, any lower fixed point of the map F^n is of the form $\left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$ where n and k are integers such that $0 < n$ and $0 \le 2k < 2^n \pm 1$. It is easy to see that the Jacobi matrix of the map F at the point $[x, y]$ has the form

$$\begin{pmatrix} 4 - 2x - y & -x \\ y & x \end{pmatrix}.$$

Therefore the Jacobi matrix of the map F at the point $[x, 0]$ has the form

$$\begin{pmatrix} 4 - 2x & -x \\ 0 & x \end{pmatrix}.$$

It means that the Jacobi matrix of the map F^n at the point $[x_0, 0]$ has the form

$$\begin{pmatrix} \prod_{i=0}^{n-1}(4 - 2x_i) & \mu \\ 0 & \prod_{i=0}^{n-1} x_i \end{pmatrix},$$

where $x_i = f^i(x_0)$. As we shall see, the value of μ is unimportant. Clearly, the Jacobi matrix of the map F^n at the point $[0,0]$ has the form

$$\begin{pmatrix} 4^n & 0 \\ 0 & 0 \end{pmatrix}.$$

(As we shall see it is an exception. For the other lower fixed points of the map F^n we have the eigenvalue 2^n instead of 4^n). Let $x_0 > 0$ and $P = [x_0, 0] \in \Delta$ be a fixed point of the map F^n. So $x_0 = 4\sin^2 \frac{k\pi}{2^n \pm 1}$ where $k \geq 1$ and

$$\begin{aligned}
x_i &= 4\sin^2 \frac{2^i k\pi}{2^n \pm 1}, \\
4 - 2x_i &= 4\cos \frac{2^{i+1} k\pi}{2^n \pm 1}, \\
\sin \frac{2^n k\pi}{2^n \pm 1} &= \mp(-1)^k \sin \frac{k\pi}{2^n \pm 1}, \\
\cos \frac{2^n k\pi}{2^n \pm 1} &= (-1)^k \cos \frac{k\pi}{2^n \pm 1}, \\
\sin \frac{2^n k\pi}{2^n \pm 1} &= 2^n \sin \frac{k\pi}{2^n \pm 1} \prod_{i=0}^{n-1} \cos \frac{2^i k\pi}{2^n \pm 1}, \\
\prod_{i=0}^{n-1} \cos \frac{2^i k\pi}{2^n \pm 1} &= \frac{\mp(-1)^k}{2^n}, \\
\prod_{i=0}^{n-1} (4 - 2x_i) &= 4^n \prod_{i=0}^{n-1} \cos \frac{2^{i+1} k\pi}{2^n \pm 1} = (-1)^k 4^n \prod_{i=0}^{n-1} \cos \frac{2^i k\pi}{2^n \pm 1} = \mp 2^n.
\end{aligned}$$

Hence the Jacobi matrix of the map F^n at the point P has the form

$$\begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \mp 2^n & \mu \\ 0 & \prod_{i=0}^{n-1} x_i \end{pmatrix} = \begin{pmatrix} \mp 2^n & \mu \\ 0 & \prod_{i=0}^{n-1} 4\sin^2 \frac{2^i k\pi}{2^n \pm 1} \end{pmatrix}. \tag{2.2}$$

So,

$$\lambda_2 = \prod_{i=0}^{n-1} 4\sin^2 \frac{2^i k\pi}{2^n \pm 1}.$$

For λ_2 we have the possibilities

(i) $0 \leq \lambda_2 < 1$, i.e. $[x_0, 0]$ is a saddle point, e.g. $x_0 = 4\sin^2 \frac{\pi}{17}$,

(ii) $\lambda_2 = 1$, i.e. $[x_0, 0]$ is a non-hyperbolic point, e.g. $x_0 = 4\sin^2 \frac{\pi}{15}$,

(iii) $1 < \lambda_2$, i.e. $[x_0, 0]$ is a repulsive point, e.g. $x_0 = 4\sin^2 \frac{3\pi}{17}$.

Remark 2.1. All the chosen points $[x_0, 0]$ in (i)-(iii) have period 4. Lower periodic points with period n and $0 < \lambda_2 < 1$ appear for all $n \geq 4$. Lower periodic points with period n and $\lambda_2 = 1$ appear for infinitely many n, e.g. $n = 4 \cdot 3^i \cdot 5^j$, where $i \geq 0$, $j \geq 0$. Lower periodic points with period n and $1 < \lambda_2$ appear for all $n \geq 1$.

3 Estimates of the number of lower saddle periodic points.

In connection with saddle points and the main result, Theorem 4.3, it is necessary to have at least a sufficient condition for a fixed point of F^n to be saddle. Therefore we include the following theorem.

Theorem 3.1 ([6]). *Let $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$ where $n > 0$ and k are integers such that*

$$1 \le k \le \frac{\sqrt{2}(2^n \pm 1)}{\pi \cdot 2^{\sqrt{2n+1/4}}} . \tag{3.1}$$

Then P is a saddle fixed point of F^n.

Remark 3.2. Note that for $4 \le n \le 13$ all points $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$, where k satisfies (3.1), have period n (and not less). If $n = 14$ and $k = 127$ or 129 then (3.1) is satisfied (with the sign $-$) and the point $P = \left[4\sin^2 \frac{k\pi}{2^n - 1}, 0\right]$ has period 7.

Unfortunately, the previous theorem gives only sufficient condition for a saddle point. So fix an integer n, the choice of signs $\pm$ and an integer k such that $1 \le k < \frac{2^n \pm 1}{2}$. We want to decide whether the point $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$ is a saddle point of the map F^n and whether its period is n (because it is a divisor of n in general). So we need to decide whether $\lambda_2 < 1$, $\lambda_2 = 1$ or $\lambda_2 > 1$, where

$$\lambda_2 = \prod_{i=0}^{n-1} 4\sin^2 \frac{2^i k\pi}{2^n \pm 1} .$$

Put $k_0 = k$ and

$$k_{i+1} = \begin{cases} 2k_i & \text{if } 2k_i < \frac{2^n \pm 1}{2} , \\ 2^n \pm 1 - 2k_i & \text{otherwise.} \end{cases}$$

Then $\sqrt{\lambda_2} = \prod_{i=0}^{n-1} 2\sin \frac{k_i\pi}{2^n \pm 1}$. If $k_i = k$ for $0 < i < n-1$ then the period of the point $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$ is less than n. If $k_i < k$ for $0 < i < n-1$ than the point P belongs to the orbit of the point $\left[4\sin^2 \frac{k_i\pi}{2^n \pm 1}, 0\right]$ and this point has been already considered (we assume that we consider k from 1 to $\frac{2^n \pm 1}{2} - \frac{1}{2}$ with the step 1). So, if $k_i \le k$ the evaluation of $\sqrt{\lambda_2} = \prod_{i=0}^{n-1} 2\sin \frac{k_i\pi}{2^n \pm 1}$ is not necessary and this evaluation may be interrupted. To find the number of saddle periodic points of the map F with period n it is sufficient to find the number of saddle periodic orbits and multiply this number by n. For any lower saddle periodic orbit it is sufficient to find that point which has the smallest x–coordinate.

Theorem 3.3. *Consider integers $n \ge 1$ and $1 \le k < \frac{2^n \pm 1}{2}$ for a fixed choice of $\pm$. Let*

$$\lambda_2 = \prod_{i=0}^{n-1} 4\sin^2 \frac{2^i k\pi}{2^n \pm 1} < 1$$

and

$$4\sin^2 \frac{k\pi}{2^n \pm 1} < 4\sin^2 \frac{2^i k\pi}{2^n \pm 1} \ \textit{for } 1 \le i \le n-1 .$$

Then k is odd and

$$\frac{k}{2^n \pm 1} < \frac{1}{12} . \tag{3.2}$$

Proof. If $k = 2j$, then $4\sin^2 \frac{2^{n-1}k\pi}{2^n \pm 1} = 4\sin^2 \frac{j\pi}{2^n \pm 1} < 4\sin^2 \frac{k\pi}{2^n \pm 1}$. Put $x_i = 4\sin^2 \frac{2^i k\pi}{2^n \pm 1}$. Clearly, $x_{i+1} = f(x_i)$ and $f^n(x_0) = x_0$. Assume that $\lambda_2 = \prod_{i=0}^{n-1} x_i < 1$, and

$x_i > x_0$ for $0 < i < n$. If $\frac{k}{2^n \pm 1} > \frac{1}{6}$ then $x_0 > 1$, $x_i > x_0 > 1$ for $1 \le i \le n-1$ and $\lambda_2 > 1$. So we obtain a contradiction. If $\frac{k}{2^n \pm 1} = \frac{1}{6}$ then $x_0 = 1$ and $x_i = 3$ for $i > 0$ and we have again a contradiction. We shall show that the assumption $\frac{1}{12} < \frac{k}{2^n \pm 1} < \frac{1}{6}$ leads to a contradiction. Let I be the set of all integers i such that $0 \le i < n$ and $x_i < 1$. Let $i_0 < i_1 \cdots < i_j$ be all elements of I. Put also $i_{j+1} = n$. Then $\lambda_2 = \prod_{s=0}^{j} \prod_{i=i_s}^{i_{s+1}-1} x_i$. Since $\lambda_2 < 1$ then $\prod_{i=i_s}^{i_{s+1}-1} x_i < 1$ at least for one $s = 0, \cdots, j$. Since $2 - \sqrt{3} = 4\sin^2 \frac{\pi}{12} < x_0 < 4 \sin^2 \frac{\pi}{6} = 1$ and $x_0 \le x_{i_s}$ we have

$$\begin{aligned} 2 - \sqrt{3} = 4\sin^2 \frac{\pi}{12} &< x_{i_s} < 4\sin^2 \frac{\pi}{6} = 1 , \\ 1 < f(x_{i_s}) &= x_{i_s+1} < 3 , \\ f(x_{i_s+1}) &= x_{i_s+2} > 3 . \end{aligned}$$

If $i_{s+1} = i_s + 3$ then $2 - \sqrt{3} = 4\sin^2 \frac{\pi}{12} < x_{i_s+3} < 4\sin^2 \frac{\pi}{6} = 1$. It is possible only for $x_{i_s+2} > 2+\sqrt{3}$, because f is decreasing on $\langle 2, 4\rangle$, $x_{i_s+3} = f(x_{i_s+2})$ and $f(2+\sqrt{3}) = 1$. We obtain $x_{i_s} \cdot x_{i_s+1} \cdot x_{i_s+2} > (2-\sqrt{3}) \cdot 1 \cdot (2-\sqrt{3}) = 1$ what is a contradiction. If $i_{s+1} - i_s > 3$ then the difference $i_{s+1} - i_s$ is odd, $x_{i_s+2j} > 3$ and $1 < x_{i_s+2j+1} < 3$ for $2j < i_{s+1} - i_s - 1$. Therefore $\prod_{i=i_s}^{i_{s+1}-1} x_i > (2-\sqrt{3}) \cdot 9 > 1$ what is a contradiction. So $\frac{k}{2^n \pm 1} \le \frac{1}{12}$. If $\frac{k}{2^n \pm 1} = \frac{1}{12}$ then $x_0 = 2 - \sqrt{3}$, $x_1 = 1$ and $x_i = 3$ for $i \ge 2$ which is impossible. We have $\frac{k}{2^n \pm 1} < \frac{1}{12}$. $\square$

Remark 3.4. The previous theorem shows that it is not necessary to consider all possible k but only odd k which satisfy (3.2). It shortens the computation of saddle periodic orbits and points 12 times. In fact, with a little care but essentially in the same way, for $n \ge 5$ the inequality

$$\frac{k}{2^n \pm 1} < \frac{1}{17}$$

can be proved. (For $n = 4$ we have 3 periodic orbits. Only one of them is a saddle orbit, see Remark 2.1.) Thus for $n \ge 5$ the computation can be shortened 17 times.

We denote by s_n the number of lower saddle periodic orbits and by $p_n = n \cdot s_n$ the number of lower saddle periodic points of the map F with period n. Table 1 contains values s_n and p_n for $1 \le n \le 36$.

4 Relationship between lower and interior periodic points

Let $P = [x, y] \in \Delta$ be a periodic point of the map F and $F^i[x_0, y_0] = [x_i, y_i]$. Then $x_i \ne 2$, because otherwise we would have $F[x_i, y_i] = [4 - 2y_i, 2y_i]$, $F^2[x_i, y_i] = [0, 8y_i - 4y_i^2]$, $F^3[x_i, y_i] = [0, 0]$, $F^j[x_i, y_i] = [0, 0]$ for $j \ge 3$ and $F^m[x_0, y_0] = [0, 0]$ for all $m \ge i + 3$ which is a contradiction. For any fixed point P of the map F^n we define its *itinerary* as a sequence $W = (w_i)_{i=0}^{n-1}$, where

$$w_i = \begin{cases} L \text{ if } x_i < 2 \\ R \text{ if } x_i > 2 . \end{cases}$$

More generally, any sequence $W = (w_i)_{i=0}^{n-1}$ of letters L and R will also be called an *itinerary*. Such an itinerary is said to be *aperiodic* if for any proper divisor k of n there is $j < n - k$ such that $w_j \ne w_{j+k}$.

n	s_n	$p_n = n \cdot s_n$	$\frac{n \cdot s_n}{2^n}$
1	1	1	0.5
2	0	0	0
3	0	0	0
4	1	4	0.250000
5	2	10	0.312500
6	3	18	0.281250
7	5	35	0.273438
8	11	88	0.343750
9	18	162	0.316406
10	37	370	0.361328
11	72	792	0.386719
12	122	1464	0.357422
13	223	2899	0.353882
14	418	5852	0.357178
15	793	11895	0.363007
16	1500	24000	0.366211
17	2903	49351	0.376518
18	5477	98586	0.376076
19	10412	197828	0.377327
20	19890	397800	0.379372
21	38090	799890	0.381417
22	72892	1603624	0.382334
23	140345	3227935	0.384800
24	270239	6485736	0.386580
25	520870	13021750	0.388078
26	1005368	26139568	0.389510
27	1945782	52536114	0.391425
28	3766954	105474712	0.392924
29	7298398	211653542	0.394235
30	14159124	424773720	0.395601
31	27492108	852255348	0.396862
32	53415336	1709290752	0.397975
33	103871727	3427766991	0.399045
34	202193966	6874594844	0.400154
35	393867993	13785379755	0.401207
36	767755134	27639184824	0.402203

Table 1: Number of saddle orbits and saddle periodic points with period n for $1 \leq n \leq 36$.

Remark 4.1. Itineraries are usually defined as infinite sequences. In this paper we consider only itineraries of fixed points of the iterates F^n and so finite sequences are sufficient.

Proposition 4.2 ([6])**.** *For any itinerary $W = (w_i)_{i=0}^{n-1}$ there is a unique lower fixed point P of the map F^n with itinerary W. The period of P is n if and only if W is aperiodic.*

Now we are ready to formulate the main result on periodic point of the map F.

Theorem 4.3 ([6])**.** *Let P be a lower saddle periodic point of the map F. Then there is an interior periodic point Q of F with the same itinerary and period.*

Let $\mathrm{Fix}_{\mathrm{Int}}(F^n)$ be the set of all interior fixed points of the map F^n and $\mathrm{Per}_{\mathrm{Int}}(F, n)$ be the set of all interior n-periodic points of the map F.

Theorem 4.4 ([6])**.** *For cardinalities of $\mathrm{Fix}_{\mathrm{Int}}(F^n)$ and $\mathrm{Per}_{\mathrm{Int}}(F, n)$ we have the estimates*

(i) $\#\,\mathrm{Fix}_{\mathrm{Int}}(F^n) \geq \frac{2\sqrt{2}}{\pi} \cdot 2^{n-\sqrt{2n+1/4}} - 2$

(ii) $\#\,\mathrm{Per}_{\mathrm{Int}}(F, n) \geq \frac{2\sqrt{2}}{\pi} 2^{n-\sqrt{2n+1/4}} - 2^{1+\frac{n}{2}} + 1$

(iii) $\#\,\mathrm{Per}_{\mathrm{Int}}(F, n) \geq (2-\varepsilon)^n$ *for $0 < \varepsilon < 1$ and sufficiently large n.*

Remark 4.5. The estimate given in (ii) is useless for $n \leq 12$. In such a case it may be used that $\#\,\mathrm{Per}_{\mathrm{Int}}(F, n) \geq \frac{2\sqrt{2}}{\pi} 2^{n-\sqrt{2n+1/4}} - 2$. Moreover, for small n the number of lower saddle n-periodic points of F may be easily found.

The points $[4\sin^2 \frac{\pi}{2^n \pm 1}, 0]$ have period n. It follows from Theorem 3.1 that they are saddle fixed points of F^n for $n \geq 4$ and $n \geq 5$ provided we choose the sign $+$ and $-$, respectively. So we obtain the following theorem.

Theorem 4.6 ([6])**.** *For any $n \geq 4$ there is an interior point Q of the map F with period n.*

The following theorem is also a consequence of Theorem 4.3.

Theorem 4.7. *For $1 \leq n \leq 36$ the third column of Table 1 gives a lower estimate of $\#\,\mathrm{Per}_{\mathrm{Int}}(F, n)$.*

Note that these estimates, for $1 \leq n \leq 36$, are much better than those from Theorem 4.4.

5 Existence and nonexistence of periodic points with prescribed itineraries

Proposition 4.2 says that the lower periodic points may be described by their itineraries. In this section we prove that for some itineraries interior periodic points need not exist. It is sufficient to consider itineraries $W = (w_i)_{i=0}^{n-1}$ with $w_0 = L$ and $w_{n-1} = R$, see the proof of Theorem 4.3. We shall write such itineraries in the form $W = L^{j_1} R^{k_1} \ldots L^{j_m} R^{k_m}$, where all j_i and k_i are positive integers and $n = j_1 + k_1 + \ldots + j_m + k_m$.

Now we show that interior fixed points of the map F^n with itineraries containing too many R's do not exist.

Theorem 5.1 ([6]). *Let $W = L^{j_1}R^{k_1}\cdots L^{j_m}R^{k_m}$ be an itinerary such that $j_i > 0$, $k_i > 0$ and $\sum_{i=1}^{m}(j_i + k_i) = n$. If*

$$\sum_{i=1}^{m} k_i \geq \frac{\ln 2}{\ln 3}\sum_{i=1}^{m} j_i^2 - \frac{\ln(4-2\sqrt{2})}{\ln 3}\sum_{i=1}^{m} j_i + m\,, \tag{5.1}$$

then there is no interior fixed point of the map F^n with the itinerary W.

The following theorem can be sometimes more useful than the previous one.

Theorem 5.2 ([6]). *Let $W = L^{j_1}R^{k_1}\cdots L^{j_m}R^{k_m}$ be an itinerary such that $j_i > 0$, $k_i > 0$ and $\sum_{i=1}^{m}(j_i + k_i) = n$. If*

$$\sum_{i=1}^{m} k_i \geq \sum_{i=1}^{m} j_i^2 - \frac{\ln(4-2\sqrt{2})}{\ln 2}\sum_{i=1}^{m} j_i\,, \tag{5.2}$$

then there is no interior fixed point of the map F^n with the itinerary W.

On the other hand, the following theorem shows that if an itinerary W of length n contains sufficiently many L's then the map F^n has an interior fixed point with this itinerary.

Theorem 5.3 ([6]). *Let $W = L^{j_1}R^{k_1}\cdots L^{j_m}R^{k_m}$ be an itinerary such that $j_i > 0$, $k_i > 0$ and $\sum_{i=1}^{m}(j_i + k_i) = n$. If*

$$\sum_{i=1}^{m} k_i \leq \frac{\ln 2}{\ln 3}\sum_{i=1}^{m} j_i^2 - \frac{\ln\frac{\pi^2}{2}}{\ln 3}\sum_{i=1}^{m} j_i - \frac{\ln\frac{32}{3\pi^2}}{\ln 3}m\,, \tag{5.3}$$

then there exists an interior fixed point of the map F^n with itinerary W.

6 Conclusion and future directions

Many problems concerning periodic points of the Lotka–Volterra map remain open. On the base of our numerical experiments and Table 1 we formulate the following conjectures.

Conjecture 6.1. *If $P \in \Delta$ is a lower repulsive (non-hyperbolic) fixed point of the map F^n, then there is no interior fixed point of F^n with the same itinerary.*

Conjecture 6.2. *If $P \in \Delta$ is a lower saddle fixed point of F^n, then there is a unique interior fixed point of F^n with the same itinerary.*

Conjecture 6.3.

$$\liminf_{n\to\infty} \frac{\#\,\mathrm{Fix}_{\mathrm{Int}}(F^n)}{2^n} > 0\,.$$

It turns out that the eigenvalue λ_2 is related to some open problems in number theory. In the near future we plan to publish corresponding results.

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Probability measures on interval–valued fuzzy events*

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Dedicated to Prof. B. Riečan

Abstract

Probability measures on intuitionistic fuzzy events were axiomatically characterized by B. Riečan in 2004 and subsequently studied in several papers. All these results can be straightforwardly transformed for the case of probability measures on interval–valued fuzzy sets. We give an alternative representation of all such probability measures. Comparison with the previous results of Riečan with co–authors is also included.

Keywords IFS event, IFS–probability, T_L–tribe, interval–valued event

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1 Introduction

Fuzzy sets were introduced by Zadeh in 1965 [9]. Recall that each fuzzy set in the universe X is characterized by its membership function $A : X \to [0,1]$ (we will not distinguish in notation fuzzy sets and their respective membership functions). In 1968, Zadeh has introduced probability measures on fuzzy events. Note that, for a measurable universe $(X, \mathcal{A})$, $\mathcal{A}$ being a σ–algebra of subsets of X, fuzzy events are just measurable fuzzy sets. For any classical probability measure P on $(X, \mathcal{A})$, the induced fuzzy probability measure $\mathcal{P}_P(A) = E_P(A)$, where E_P is the classical P–based expected value. An axiomatic approach to fuzzy probability measures was proposed by Butnariu [4], where the additivity was modelled by means of the Lukasiewicz t–norm $\odot : [0,1]^2 \to [0,1]$, $a \odot b = max\,\{a + b - 1, 0\}$ and of the Lukasiewicz t–conorm $\oplus : [0,1]^2 \to [0,1]$, $a \oplus b = min\,\{a + b, 1\}$. The main result of [4] shows

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that Zadeh's fuzzy probability measures coincide with axiomatically defined fuzzy probability measures.

Atanassov in [2], see also [3], has introduced intuitionistic fuzzy set $A : X \to [0,1]^2$ as a couple of fuzzy sets, $A = (B,\, C)$, such that $B \odot C = 1$ (i.e., $B(x) \odot C(x) \leq 1$ for all $x \in X$). Observe that $B \odot C = 0$ if and only if $B \leq \mathbf{1} - C$, and thus the intuitionistic fuzzy set A can be isomorphically seen as an interval valued fuzzy set $\tilde{A} = [B,\, \mathbf{1} - C] = \left[\underline{A},\, \overline{A}\right]$, where $\underline{A},\, \overline{A}$ are fuzzy sets satisfying $\underline{A} \leq \overline{A}$. Clearly, fuzzy sets can be embedded into interval fuzzy sets, supposing $\underline{A} = \overline{A}$. Grzegorzewski and Mrówka in 2002 [6] have proposed probability measures on intuitionistic fuzzy sets generalizing the original Zadeh's approach from 1968 [10]. Based on a probability measure P on $(X, \mathcal{A})$, intuitionistic fuzzy probability $\mathcal{P}_P$ was given as an interval–valued mapping by

$$\mathcal{P}_P\left([B, C]\right) = \left[E_P(B), 1 - E_P(C)\right]. \tag{1.1}$$

Transforming formula (1.1) for interval–valued fuzzy events, we get

$$\mathcal{P}_P\left(A\right) = \mathcal{P}_P\left(\left[\underline{A},\, \overline{A}\right]\right) = \left[E_P(\underline{A}),\, E_P(\overline{A})\right]. \tag{1.2}$$

Riečan in 2004 [7] has proposed an axiomatic characterization of intuitionistic fuzzy probability measures, and later in [8, 5] has studied the structure of these mappings.

All these results can be easily reformulated for interval–valued fuzzy sets, yielding more transparent look (event values and probability values are then in both cases intervals, and thus one can look on these probabilities as a kind of expected values).

The aim of this contribution is an alternative characterization of interval–valued probability measures of interval–valued fuzzy events on a general measurable space $(X, \mathcal{A})$. The paper is organized as follows. In the next section, Riečan's results are transformed for the interval–valued case. Section 3 brings the main result – complete characterization of interval–valued probability measures of interval–valued fuzzy events. In Section 4 results of Riečan are compared with our results and the convex structure of discussed probabilities is completely determined. Finally, some concluding remarks are added.

2 Riečan's results on probability measures on interval–valued fuzzy events

For a measurable space $(X, \mathcal{A})$, denote by $\mathcal{J}$ the class of all interval–valued fuzzy events, $\mathcal{J} = \left\{\left[\underline{A},\, \overline{A}\right] \mid \underline{A},\, \overline{A} : X \to [0,1] \text{ are } \mathcal{A} - \text{measurable},\ \underline{A} \leq \overline{A}\right\}$. Let $\mathcal{J} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$.

The next definition is a version of the original Riečan's definition from [7].

Definition 2.1. A mapping $\mathcal{P} : \mathcal{J} \to I$ is called an interval probability measure if the next axioms are satisfied:

(i) $\mathcal{P}\left([\mathbf{1},\, \mathbf{1}]\right) = [1,\, 1], \quad \mathcal{P}\left([\mathbf{0},\, \mathbf{0}]\right) = [0,\, 0]$;

(ii) for any $A = \left[\underline{A},\, \overline{A}\right], \quad B = \left[\underline{B},\, \overline{B}\right]$, if $\overline{A} \odot \overline{B} = 0$ then $\mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B)$, where $A \oplus B(x) = \left[\underline{A}(x) \oplus \underline{B}(x),\, \overline{A}(x) \oplus \overline{B}(x)\right]$, and $+$ is the standard addition of intervals;

(iii) if $A_n \nearrow A$ then $\mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.

In [8], Riečan has shown the existence of interval probability measures differing from those given by (1.2), i.e., he has shown that the probabilistic environment for interval–valued events is much more richer than that for fuzzy events.

Theorem 2.2. *Let $P : \mathcal{A} \to [0,1]$ be a probability measure and let $f, g : [0,1]^2 \to [0,1]$ be functions. Then the mapping $\mathcal{P} : \mathcal{J} \to I$ given by*

$$\mathcal{P}\left(\left[\underline{A}, \overline{A}\right]\right) = \left[f\left(E_P(\underline{A}), E_P(\overline{A}\right), g\left(E_P(\underline{A}), E_P(\overline{A})\right]\right. \tag{2.1}$$

is an interval probability measure if and only if $f(u,v) = (1-\alpha)\,u + \alpha\,v$ and $g(u,v) = (1-\beta)\,u + \beta\,v$ for some $\alpha, \beta \in [0,1],\ \alpha \le \beta$, i.e., if

$$\mathcal{P}\left(\left[\underline{A}, \overline{A}\right]\right) = \left[(1-\alpha)\,E_P(\underline{A}) + \alpha\,E_P(\overline{A}), (1-\beta)\,E_P(\underline{A}) + \beta\,E_P(\overline{A})\right]. \tag{2.2}$$

Evidently, Grzegorzewski and Mrówka's proposal (1.2) corresponds to the case $\alpha = 0$ and $\beta = 1$, i.e., it is the largest solution of the problem (2.1).

A complete characterization of IFS–probabilities was shown by Theorem 3.1 in Ciungu and Riečan, 2010 [5]. In interval–valued approach this result can be reformulated as follows.

Theorem 2.3. *A mapping $\mathcal{P} : \mathcal{J} \to I$ is an interval probability measure if and only if there are probability measures $P_1, R_1, R_2 : \mathcal{A} \to [0,1]$ and constants β, γ such that $0 \le \beta \le \gamma \le 1,\ \beta\,R_1 \le \gamma\,R_2$, so that*

$$\mathcal{P}\left(\left[\underline{A}, \overline{A}\right]\right) = \left[E_{P_1}(\underline{A}) + \beta\,E_{R_1}(\overline{A} - \underline{A}), E_{P_1}(\underline{A}) + \gamma\,E_{R_2}(\overline{A} - \underline{A})\right]. \tag{2.3}$$

3 An alternative approach to probability measures on interval–valued fuzzy events

We introduce now an alternative approach how to characterize interval–valued probability measures. Due to their continuity (axiom (iii) in Definition 2.1) it is enough to consider finite spaces $X = \{1, \dots, n\}$ and $\mathcal{A} = 2^X$ only.

Theorem 3.1. *Let $X = \{1, \dots, n\}$ for some $n \in N$ and $\mathcal{A} = 2^X$. Then a mapping $\mathcal{P} : \mathcal{J} \to I$ is an interval–valued probability measure if and only if there are probability measures $P, R, Q : \mathcal{A} \to [0,1]$ and constants $u, v \in [0,1]$ such that $v\,Q \le u\,R$, so that*

$$\begin{aligned}\mathcal{P}\left(\left[\underline{A}, \overline{A}\right]\right) = \big[&(1-u)\,E_P(\underline{A}) + u\,E_R(\overline{A}) - v\,E_Q(\overline{A} - \underline{A})\ ,\\ &(1-u)\,E_P(\underline{A}) + u\,E_R(\overline{A})\big].\end{aligned} \tag{3.1}$$

Proof. Observe first that each interval–valued fuzzy event $A = \left[\underline{A}, \overline{A}\right] \in \mathcal{J}$ can be seen as an $n \times 2$ matrix $A = (a_{ij})$ such that $0 \le a_{i1} \le a_{i2} \le 1,\ i = 1, \dots, n$.

Suppose that $\mathcal{P} : \mathcal{J} \to I$ is an interval–valued probability measure on X. Due to the additivity (axiom ii in Definition 1) and due to the classical Cauchy equation [1] it holds

$$\mathcal{P}(A) = \left[\sum_{i,j} \lambda_{ij}\,a_{ij}, \sum_{i,j} \mu_{ij}\,a_{ij}\right]$$

for some non–negative constants λ_{ij}, μ_{ij} independently of $A \in \mathcal{J}$. The boundary condition $\mathcal{P}([\mathbf{1}, \mathbf{1}]) = [1,1]$ forces $\sum_{i,j} \lambda_{ij} = 1 = \sum_{i,j} \mu_{ij}$. On the other hand, denoting $B_k = \left(b_{ij}^{(k)}\right)$ the matrix given by $b_{ij}^{(k)} = \delta_i(k)$ (Dirac function), we have $[\mathbf{1}, \mathbf{1}] = B_1 \oplus \dots \oplus B_n$ and $[1,1] = \mathcal{P}([\mathbf{1}, \mathbf{1}]) = \mathcal{P}(B_1) \oplus \dots \oplus \mathcal{P}(B_n)$. Then necessarily $\mathcal{P}(B_k)$ is a trivial singleton interval and hence $\lambda_{k1} + \lambda_{k2} = \mu_{k1} + \mu_{k2},\ k = 1, \dots, , n$. Further,

$$\mathcal{P}\left(\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}\right)\right) = [\lambda_{12}, \mu_{12}],$$

i.e., $\lambda_{12} \le \mu_{12}$ (and hence $\lambda_{11} \ge \mu_{11}$). Similarly, $\lambda_{k2} \le \mu_{k2}$ for $k = 2, \dots, , n$. Denote $u = \sum_{i=1}^{n} \mu_{i2}$. Then $\sum_{i=1}^{n} \mu_{i1} = 1 - u$.

If $u = 0$, $(\lambda_{11}, \dots, \lambda_{n1}) = (\mu_{11}, \dots, \mu_{n1})$ is a probability vector linked to a probability measure P, $v = 0$ and $\mathcal{P}(A) = [E_P(\underline{A}), E_P(\underline{A})]$, i.e., (3.1) holds.

If $u = 1$ then the probability vector $(\mu_{12}, \dots, \mu_{n2})$ is linked to a probability measure R, and $v = \sum_{i=1}^{n} \lambda_{i1}$. If $v = 0$, $\mathcal{P}(A) = \left[E_R(\overline{A}), E_R(\overline{A})\right]$ and (3.1) holds. If $v > 0$, the probability vector $\left(\frac{\lambda_{i1}}{v}, \dots, \frac{\lambda_{in}}{v}\right)$ is linked to a probability measure Q, and clearly $v\,Q \le R$, i.e., (3.1) holds.

Finally, let $0 < u < 1$. Then the probability vectors $\left(\frac{\mu_{11}}{1-u}, \dots, \frac{\mu_{n1}}{1-u}\right)$ and $\left(\frac{\mu_{12}}{u}, \dots, \frac{\mu_{n2}}{u}\right)$ are linked to probability measures P and R, respectively. If we denote $v = \sum_{i=1}^{n}(\mu_{i2} - \lambda_{i2})$, evidently $v \in [0, u]$. If $v = 0$, $\mathcal{P}(A) = \left[(1-u)\,E_P(\underline{A}) + u\,E_R(\overline{A}), (1-u)\,E_P(\underline{A}) + u\,E_R(\overline{A})\right]$. Finally, if $v > 0$, we define a probability measure Q by means of a probability vector $\left(\frac{\mu_{12}-\lambda_{12}}{v}, \dots, \frac{\mu_{n2}-\lambda_{n2}}{v}\right)$, so that evidently $v\,Q \le R$, and (3.1) holds.

On the other hand, if (3.1) holds, it is an easy verification to see that $\mathcal{P}$ is an interval–valued probability measure on X. $\square$

As already mentioned, formula (3.1) applies also in the case of a general measurable space $(X, \mathcal{A})$.

4 Comparison of two different representations of probability measures and their convex structure on interval–valued fuzzy events

To see the coincidence of formulas (2.3) and (3.1), one should verify the validity (for all $A \in \mathcal{J}$) of the equality

$$\begin{aligned}&\left[E_{P_1}(\underline{A}) + \beta E_{R_1}(\overline{A} - \underline{A}), E_{P_1}(\underline{A}) + \gamma E_{R_2}(\overline{A} - \underline{A})\right] = \\ &= \left[(1-u)E_P(\underline{A}) + uE_R(\overline{A}) - vE_Q(\overline{A} - \underline{A}), (1-u)E_P(\underline{A}) + uE_R(\overline{A})\right].\end{aligned}$$

Evidently, $u = \gamma$ and $v = \gamma - \beta$. Moreover, $R_2 = R$ (it is enough to consider $\underline{A} = \mathbf{0}$ if $u > 0$). Putting $\underline{A} = \frac{1}{2}\,1_S$, $\overline{A} = 1_S$, $S \subseteq X$, we see that $P_1 = (1-u)\,P + u\,R$. If $v = u$, i.e., $\beta = 0$, R_1 can be chosen arbitrarily. Otherwise, again applying $\underline{A} = \frac{1}{2}\,1_S$, $\overline{A} = 1_S$, $S \subseteq X$, we see that $R_1 = \frac{u\,R - v\,Q}{u - v}$.

Summarizing, we can conclude that the formulas (2.3) and (3.1) fully describe the same class of all probability measures on interval–valued fuzzy events.

Due to Definition 2.1, it is evident that the class $\mathcal{I}_P$ of all probability measures of interval–valued events on a measurable space $(X, \mathcal{A})$ is convex.

Using formula (3.1), each $\mathcal{P} \in \mathcal{I}_P$ can be characterized by a pentuple (P, Q, R, u, v), such that P, Q, R are classical probability measures on $(X, \mathcal{A})$, $u, v \in [0, 1]$, $v\,Q \le u\,R$. Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{I}_P$, $\mathcal{P}_i \sim (P_i, Q_i, R_i, u_i, v_i)$, $i = 1, 2$. Then $\mathcal{P} = \lambda\,\mathcal{P}_1 + (1-\lambda)\,\mathcal{P}_2$ is characterized by (P, Q, R, u, v), where

$$P = \frac{\lambda\,(1-u_1)\,\mathcal{P}_1 + (1-\lambda)\,(1-u_2)\,\mathcal{P}_2}{1 - \lambda u_1 - (1-\lambda)u_2},$$

$$R = \frac{\lambda u_1 R_1 + (1-\lambda) u_2 R_2}{\lambda u_1 + (1-\lambda) u_2},$$

$$Q = \frac{\lambda v_1 Q_1 + (1-\lambda) v_2 Q_2}{\lambda v_1 + (1-\lambda) v_2},$$

$$u = \lambda u_1 + (1-\lambda) u_2,$$

$$v = \lambda v_1 + (1-\lambda) v_2,$$

with convention that if $u = 0$ then $R = \lambda R_1 + (1-\lambda) R_2$, and if $v = 0$ then $Q = \lambda Q_1 + (1-\lambda) Q_2$. Obviously, P, Q, R are probability measures on $(X, \mathcal{A})$, and $v\,Q \leq u\,R$, and thus $\mathcal{P} \in \mathcal{I}_P$.

In the case of a finite space $X = \{1, \ldots, n\}$, the convex class $\mathcal{I}_P$ has the next vertices (here D_i is the Dirac measure concentrated in point $\{i\}$):

$$\underline{\mathcal{P}}_i \sim (D_i, D_j, D_k, 0, 0),\ j, k \text{ can be chosen arbitrarily, } \underline{\mathcal{P}}_i(A) = [a_{i1}, a_{i1}];$$

$$\underline{\mathcal{P}}_{ik} \sim (D_i, D_j, D_k, 1, 0),\ j \text{ can be chosen arbitrarily, } \underline{\mathcal{P}}_{ik}(A) = [a_{i1}, a_{i1} + a_{k2} - a_{k1}];$$

$$\overline{\mathcal{P}}_{ij} \sim (D_i, D_j, D_k, 1, 1),\ k \text{ can be chosen arbitrarily, } \overline{\mathcal{P}}_{ij}(A) = [a_{i1}+a_{j2}-a_{j1}, a_{i1}+a_{j2}-a_{j1}],$$

where $i, j, k \in X$. Hence $\mathcal{I}_P$ has exactly $2n^2 + n$ vertices.

Note that the convex closure of vertices $\underline{\mathcal{P}}_{11}, \ldots, \underline{\mathcal{P}}_{nn}$yields just the class of Grzegorzewski and Mrówka's probabilities given by formula (1.2). Riečan's probabilities given by formula (2.2) are a convex closure of the vertices $\underline{\mathcal{P}}_{11}, \ldots, \underline{\mathcal{P}}_{nn}, \underline{\mathcal{P}}_1, \ldots, \underline{\mathcal{P}}_n, \overline{\mathcal{P}}_{11}, \ldots, \overline{\mathcal{P}}_{nn}$.

5 Concluding remarks

We have given an alternative look on representation of probability measures on interval–valued fuzzy events by means of 3 classical probability measures, confirming the original results of Riečan with co-author. Note that there are several classical set functions closely related to probability measures, such as belief and plausibility measures, k–additive capacities, etc. In the further investigation of the measure theory on interval–valued (or intuitionistic–valued) fuzzy events we propose to consider the above mentioned generalizations of probability measures.

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The Change-of-Variables Theorem for the Lebesgue Integral*

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Dedicated to the 75th birthday of Beloslav Riečan

Abstract

We present a short proof of the change-of-variables theorem for diffeomorphic mappings. This is a modification of the proof given in [3].

Keywords change-of-variables, Lebesgue integral

MSC (2010) 28A25, 28-02, 28-01

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The following change-of-variables theorem for the Lebesgue integral is standard[1].

Theorem 1. *Let $V \subset \mathbb{R}^d$ be an open set and $\varphi : V \to \mathbb{R}^d$ be a one-to-one $\mathcal{C}^1$-mapping with non-vanishing Jacobian J_φ. Then*

$$\int_{\varphi(V)} h \, d\lambda_d = \int_V (h \circ \varphi)|J_\varphi| \, d\lambda_d, \quad h \in C_c\big(\varphi(V)\big). \tag{1}$$

The proof that we shall describe is based on a *smearing technique* and uses the following standard result on the transformation of Lebesgue measure by linear mappings: *Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be a one-to-one linear mapping. Then, for every Lebesgue measurable set $A \subset \mathbb{R}^d$,*

$$\lambda_d\big(\psi(A)\big) = |J_\psi| \cdot \lambda_d(A). \tag{2}$$

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[1] We use the usual terminology and notation: our assumption says that φ is a diffeomorphism of V onto $\varphi(V)$; λ_d stands for d-dimensional Lebesgue measure; for $U \subset \mathbb{R}^d$ open, $C_c(U)$ is the set of all continuous functions $g : U \to \mathbb{R}$ such that their support $S(g) := \overline{\{x \in U : g(x) \neq 0\}}$ is a compact subset of U.

Here, of course, $J_\psi = \det\psi$. It follows immediately from (2) (by integration with respect to the image measure) that

$$\int_{\mathbb{R}^d} (g\circ\psi)\,d\lambda_d = \bigl(1/|J_\psi|\bigr)\int_{\mathbb{R}^d} g\,d\lambda_d, \quad g\in C_c(\mathbb{R}^d). \tag{3}$$

Let us fix a positive function $\omega \in C_c(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}\omega\,d\lambda_d = 1$, and, for $r>0$, define

$$\omega_r(y) := r^{-d}\omega(y/r), \quad y\in\mathbb{R}^d.$$

For $f\in C_c(\mathbb{R}^d)$ and $r>0$, the convolution of f and ω_r is defined by[2]

$$f*\omega_r : x\mapsto \int_{\mathbb{R}^d} f(x-y)\,\omega_r(y)\,dy.$$

Then $f*\omega_r\in C_c(\mathbb{R}^d)$ and, using (3),

$$(f*\omega_r)(x) - f(x) = \int_{\mathbb{R}^d}\bigl(f(x-rz)-f(x)\bigr)\,\omega(z)\,dz, \quad x\in\mathbb{R}^d.$$

Since f is uniformly continuous, it follows that

$$f*\omega_r\to f \quad \text{uniformly on } \mathbb{R}^d \text{ for } r\to 0+\,. \tag{4}$$

Let V and φ be as in the theorem. The following result will be useful.

Lemma 2. *For $r>0$, let us define*

$$g_r : y\mapsto \int_V \omega_r\bigl(\varphi(z)-y\bigr)\,dz, \quad y\in\varphi(V).$$

If $K\subset\varphi(V)$ is a compact set, then

$$\lim_{r\to 0+} g_r(y) = 1/|J_\varphi\bigl(\varphi^{-1}(y)\bigr)|, \quad y\in K. \tag{5}$$

Proof. Let us fix a ball B centered at 0 and containing $S(\omega)$. Since $\varphi^{-1}(K)$ is compact, there exists $\rho>0$ such that $\varphi^{-1}(K)+\rho B\subset V$. Then, for every $r\in(0,\rho)$ and every $y\in K$,

$$V_r(y) := \frac{1}{r}\bigl(V-\varphi^{-1}(y)\bigr)\supset B\supset S(\omega).$$

An affine change of variables (cf. (3)) yields

$$g_r(y) = \int_{S(\omega)} \omega\Bigl(\frac{1}{r}\bigl(\varphi(\varphi^{-1}(y)+rt)-\varphi(\varphi^{-1}(y))\bigr)\Bigr)\,dt, \quad y\in K,\ r\in(0,\rho).$$

(Here we replaced the integration over $V_r(y)$ by integration over $S(\omega)$. For later use, let us notice that $\{g_r : r\in(0,\rho)\}$ is a uniformly bounded family of continuous functions on K.) By Lebesgue's Dominated Convergence Theorem,

$$\lim_{r\to 0+} g_r(y) = \int_{S(\omega)} \omega\bigl(\varphi'(\varphi^{-1}(y))(t)\bigr)\,dt,$$

which, in view of (3), yields (5). □

[2]Sometimes we write, for instance, $\int_A g(y)\,dy$ instead of $\int_A g\,d\lambda_d$.

Now we shall prove the equality (1). Let $h \in C_c\big(\varphi(V)\big)$ and $K := S(h)$. We may suppose that h is positive. For $r > 0$, let us define

$$I_r := \int_{V\times\varphi(V)} h(y)\,\big|J_\varphi\big(\varphi^{-1}(y)\big)\big|\,\omega_r\big(\varphi(z)-y\big)\,dz\,dy$$

(integration with respect to the product measure $\lambda_d \times \lambda_d$). By the Fubini theorem,

$$I_r = \int_K h(y)\big|J_\varphi\big(\varphi^{-1}(y)\big)\big|g_r(y)\,dy.$$

Obviously, $\{h|J_\varphi \circ \varphi^{-1}|\,g_r : r \in (0,\rho)\}$ is a uniformly bounded family of continuous functions on K. Applying Lebesgue's Dominated Convergence Theorem and using (5) we see that

$$\lim_{r\to 0+} I_r = \int_K h(y)\,dy = \int_{\varphi(V)} h\,d\lambda_d. \tag{6}$$

Let us define $f := h|J_\varphi \circ \varphi^{-1}|$ on $\varphi(V)$ and $f = 0$ elsewhere on $\mathbb{R}^d$. Then $f \in C_c(\mathbb{R}^d)$ and there exist $\xi > 0$ and a compact set $L \subset \varphi(V)$ such that $S(f * \omega_r) \subset L$ for every $r \in (0,\xi)$. The Fubini theorem yields

$$I_r = \int_{\varphi^{-1}(L)} (f*\omega_r)\big(\varphi(z)\big)\,dz,\ \ r\in(0,\xi).$$

By (4) and Lebesgue's Dominated Convergence Theorem it follows that

$$\lim_{r\to 0+} I_r = \int_{\varphi^{-1}(L)} f\big(\varphi(z)\big)\,dz = \int_V f\big(\varphi(z)\big)\,dz = \int_V (h\circ\varphi)|J_\varphi|\,d\lambda_d, \tag{7}$$

since $S(f) \subset L$. Now (1) follows from (6) and (7) and this finishes the proof.

Comments.

1. The proof of the integral calculus version of the change-of-variables formula is based on smearing of the value of $f\big(\varphi(z)\big)$ on small neighbourhoods. I learned this approach from Professor A. Cornea[3] some twenty years ago; cf. [3]. It seems that this method of proof does not appear in textbooks on integration and, in my opinion, deserves to be better known. Cornea's proof provides the *inequality* $\le$ in (1), which, of course, is sufficient in view of the symmetry of φ and φ^{-1}. In our proof we establish the *equality* (6) instead of the inequality

$$\int_{\varphi(V)} h\,d\lambda_d \le \liminf_{r\to 0+} I_r$$

obtained by Fatou's Lemma.

[3] Aurel Cornea (1933–2005) was born in Romania. At the age of 14 he had an accident during a chemical experiment and lost his eyesight. He studied mathematics and completed his Ph.D. thesis under S. Stoilow. He worked at the University of Bucharest and the Academy of Sciences. In 1978 he left Romania. After short stays in Canada and USA, he spent the rest of his life in Germany. In 1980, he was appointed as a professor at the Katholische Universität Eichstätt. Aurel Cornea was a distinguished specialist in potential theory, an excellent scientist, and an exceptional man. For further information, see [32].

2. Clearly, (1) holds for much more general functions h. To see this, one can first deduce from (1) the equality

$$\lambda_d\big(\varphi(A)\big) = \int_A |J_\varphi|\, d\lambda_d \tag{8}$$

for every Lebesgue measurable set $A \subset V$. (In particular, the mapping φ has the N-property, which means that the image of every set of zero measure is also of zero measure.) The equality (8) can be shown, for instance, using a Lusin's Theorem type of argument; see Corollary to Theorem 2.24 in [26]. Then integration with respect to the image measure shows that *the integral of a function h over $\varphi(V)$ exists if and only if the integral of $(h \circ \varphi)|J_\varphi|$ over V exists, and we have the equality* (1).

3. The equality (2) is usually proved using a factorization of the linear mapping and the multiplicative property for determinants; see, for instance, [5], [6], [9], [10], [18], [21], [26]. Group theoretical arguments are used in [4]. An approach based on Fubini's theorem is employed in [29].

4. The calculus version of the change-of-variables formula has a long history and is connected with names such as L. Euler, J.-L. Lagrange, S. Laplace, C. F. Gauss, M. Ostrogradski, C. Jacobi and others; see [16]. For various methods of proofs one may consult [2], [3], [9], [10], [15], [18], [19], [20], [21], [27], [28], [29].

5. Of course, the conditions imposed on φ may be substantially weakened in various respects. This issue is discussed in numerous textbooks as well as articles. Let us mention at least some references: [12], [14], [24], [25], [26], [31].

6. It turns out that the change-of-variables formula is a very special case of the so-called *area theorem*, which has been extensively studied in various degrees of generality in the setting of geometric measure theory. As a sample, we list the following books and papers: [1], [7], [8], [10], [11], [13], [17], [22], [23], [30], [33].

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Probability Limit Identification Functions Revisited*

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To my colleague and friend professor Beloslav Riečan

Abstract

The concept of probability limit identification function introduced by G. Simons in 1970 is revisited and some recent advances on the topic presented. A new proof to the existence of the function under the hypothesis continuum is presented in Section 3.

Keywords Convergence in Probability, Probability Limit Identification Function

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1 Introduction

Consider a parametric family of probability distributions

$$\{P_x,\, x \in E\}$$

on a sample space $(\Omega, \mathcal{F})$, where E will be a **separable metric space**, if not stated otherwise. Its Borel σ-algebra is denoted as $\mathcal{B}(E)$. A sequence of random variables $X_n : \Omega \to E$ such that

$$X_n \to x \quad \text{in} \quad P_x \quad \text{probability for all} \quad x \in E$$

and

$$X_n \to x \quad P_x - \text{almost surely for all} \quad x \in E$$

is called a **weak and strong estimator** of the parameter x, respectively.

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Recall that a sequence of E-valued random variables X_n converge to an E-valued random variable X in P-probability if

$$\lim_n P[d(X_n, X) \geq \epsilon] = 0 \quad \forall \epsilon > 0, \tag{1.1}$$

where d is an equivalent metric for E. A useful equivalent definition is stated by

Theorem 1.1 (Riesz theorem [3, Lemma 4.2]). *The convergence* (1.1) *holds if and only if every increasing sequence of natural numbers* (n_k) *has an subsequence* (n_{k_j}) *such that*

$$\lim_{j\to\infty} X_{n_{k_j}} = X \quad P - \textit{almost surely.}$$

Hence, for arbitrary $x \in E$ there exists an increasing subsequence of natural numbers $n_k(x)$ such that

$$X_{n_k(x)} \to x \quad P_x - \text{almost surely.}$$

Provided that $S \subset E$ is **at most countable set** we are able to apply Riesz theorem and the diagonal Helly's procedure to construct a subsequence (n_k) such that

$$\lim_{k\to\infty} X_{n_k} = x \quad P_x - \text{almost surely for all} \quad x \in S.$$

Unfortunately, such an universal subsequence (n_k) is not available generally as we shall see later on. However, we might be inclined to believe that for arbitrary weak estimator X_n there are transformations $g_n(x_1, \dots, x_n) : E^n \to E$ such that

$$g_n(X_1, X_2, \dots, X_n) \to x \quad P_x - \text{almost surely} \quad x \in E.$$

Unfortunately, this is not a true statement, again (see Example 2.5 in Section 2).

We prefer to state our problem in a probabilistic setting. Assume for a while that E is **a Borel set in a Polish metric space**, especially a separable metric space. Given a family of probability measures $\{P_x, x \in E\}$ and a sequence of estimators $\mathbb{X} = (X_1, X_2 \dots)$ then by Borel isomorphism theorem (see [3, Theorem 8.3.6]) there is an universal probability space $(\Omega, \mathcal{F}, P)$ and $E^{\mathbb{N}}$-valued random sequences

$$\mathbb{X}^x = (X_1^x, X_2^x, \dots) : (\Omega, \mathcal{F}) \to (E^{\mathbb{N}}, \mathcal{B}(\mathbb{E}^{\mathbb{N}})), \quad x \in E$$

such that

$$\mathcal{L}(\mathbb{X} \mid P_x) \quad = \mathcal{L}(\mathbb{X}^x \mid P) \quad \forall x \in E, \tag{1.2}$$

where $\mathcal{L}(\mathbb{X} \mid P)$ denotes the Borel probability distribution of a sequence $\mathbb{X}$ on the space $E^{\mathbb{N}}$ w.r.t. the measure P. Thus, we may, in this case, operate with one probability measure P and a set of sequences of random variables that are convergent in P-probability because (1.2) implies that $X_n \to x$ in P_x-probability iff $X_n^x \to x$ in P-probability.

Remark that the Lebesgue interval $[0, 1]$ is a space rich enough to be suitable as an universal probability space to allow the above construction.

Denote by $\mathcal{E}(E)$ the space of all sequences $\mathbb{X} = (X_1, X_2, \dots)$ of E-valued random variables X_n that are convergent in probability to a limit denoted as $p(\mathbb{X})$. Our problem is then stated as follows:

Which are the subsets $\mathcal{F}$ of $\mathcal{E}(E)$ such that it is possible to identify the probability limit $p(\mathbb{X})$ for all $\mathbb{X} \in \mathcal{F}$ almost surely?

An adapted (Borel) identification for $\mathcal{F}$ (a concept perhaps more suitable for the needs of statistics) requires the existence of (Borel) $g_n(x_1, x_2, \dots, x_n) : E^n \to E$ transforms such that

$$g_n(X_1, X_2, \dots, X_n) \to p(\mathbb{X}) \quad \text{almost surely} \quad \forall\, \mathbb{X} = (X_1, X_2 \dots) \in \mathcal{F}.$$

A humble probabilist might be perhaps satisfied with the existence of a (Borel) map $f(x_1, x_2, \dots) : E^{\mathbb{N}} \to E$ such that

$$f(X_1, X_2, \dots) \;=\; p(\mathbb{X}) \quad \text{almost surely} \quad \forall\, \mathbb{X} = (X_1, X_2 \dots) \in \mathcal{F},$$

the map f being called a **(Borel) probability limit identification function (PLIF)** for $\mathcal{F}$.

Observe that if $g_n(x_1, x_2, \dots, x_n)$ define an adapted (Borel) identification, then

$$f(x_1, x_2, \dots) \;=\; \limsup_n g_n(x_1, x_2, \dots, x_n)$$

is a (Borel) PLIF for $\mathcal{F}$.

2 PLIFs - the present state of art

If $\mathcal{F}$ is a countable subset of $\mathcal{E}(E)$ (E a separable metric space) , then Riesz theorem yields an universal subsequence of natural numbers $n_1 < n_2 < \dots$ such that

$$X_{n_k} \to p(\mathbb{X}) \quad \text{almost surely} \quad \forall \mathbb{X} = (X_1, X_2, \dots) \in \mathcal{F}.$$

and therefore there is an adapted Borel identification for $\mathcal{F}$, hence also a Borel PLIF for $\mathcal{F}$.

P. Kříž in [4] proved a deeper statement

Theorem 2.1. *Let E be a separable metric space and $\mathcal{F}$ a subset of $\mathcal{E}(E)$ such that there is a Borel σ-finite measure μ on E that dominates all probability distributions in*

$$\mathcal{P}(\mathcal{F}) = \{P_{\mathbb{X}},\, \mathbb{X} \in \mathcal{F}\}, \quad i.e. \quad P_{\mathbb{X}} \ll \mu \quad \forall\, \mathbb{X} \in \mathcal{F}. \tag{2.1}$$

Then there exists a Borel adapted identification, hence a Borel PLIF for $\mathcal{F}$.

We have denoted by $P_{\mathbb{X}}$ the probability distribution of the sequence $\mathbb{X}$ defined as

$$P_{\mathbb{X}}(B) = P[\mathbb{X} \in B], \quad B \subset E^{\mathbb{N}} \quad \text{a Borel set.}$$

Proof is an application of a well known theorem that states that any set $\mathcal{P}$ of probability measures on a countably generated σ-algebra ($\mathcal{B}(E^{\mathbb{N}})$ in our case) that is dominated in sense of (2.1) is separable w.r.t. the total variation metric (see [7, Theorem A.4.1]).

To construct sets $\mathcal{F} \subset \mathcal{E}(E)$ that satisfy (2.1) might not be a very easy task.

Example 2.2. Consider $E = \{0, 1\}$ that provides $E^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ as a compact Abel group with the Haar probability measure $\mu = P_{\mathbb{Y}}$, where $\mathbb{Y} = (Y_1, Y_2, \dots)$ is a sequence of i.i.d. random variables with $P[Y_n = 0] = P[Y_n = 1] = 1/2$. Then

$$P_{\mathbb{X}} \quad \perp \quad \mu \quad \forall \mathbb{X} \in \mathcal{E}(\{0,1\}).$$

Indeed, having a sequence $\mathbb{X} = (X_1, X_2, \dots)$ of $0-1$ random variables that is convergent in probability then there is a subsequence $r = (r_1 < r_2 < \dots)$ such that $P_{\mathbb{X}}(A_r) = 1$, where

$$A_r := \{x = (x_1, x_2 \dots) \in \{0,1\}^{\mathbb{N}} : \lim_{k\to\infty} x_{r_k} \quad \text{exists}\}.$$

On the other hand, $\mu(A_r) = 0$ by Kolomogoroff 0-1 law.

A good space to start with is $\mathcal{E}(\{0,1\})$ of all sequences of 0-1 random variables that converge in probability. Its important subset is

$$\mathcal{H} = \{\mathbb{X} \in \mathcal{E}(\{0,1\}) : p(\mathbb{X}) = 1 \quad \text{or} \quad 0\}.$$

It is a good space because of

Theorem 2.3 (G. Simons (1971))**.** *If there is a (Borel) PLIF for $\mathcal{H}$ then there is a (Borel) PLIF for $\mathcal{E}([0,1])$.*

We recapitulate shortly **the Simon's proof**. Consider $\mathbb{X} \in \mathcal{E}([0,1])$ and $a \in [0,1]$. Put

$$\mathbb{X}^a = \left(I_{[X_1<a]}, I_{[X_2<a]}, \dots\right) \quad \text{and} \quad \mathbb{Y}^a = \left(I_{[X_1<a,\, p(\mathbb{X})>a]}, I_{[X_2<a,\, p(\mathbb{X})>a]}, \dots\right).$$

Note that

$$\mathbb{Y}^a \in \mathcal{H} \quad \text{with} \quad p(\mathbb{Y}) = 0 \quad \text{almost surely.}$$

So if $f(x_1, x_2, \dots)$ is a PLIF for $\mathcal{H}$, then outside a P-null set and all rational numbers $a \in [0,1]$

$$p(\mathbb{X}) > a \quad \Rightarrow \quad f(\mathbb{X}^a) = f(\mathbb{Y}^a) = p(\mathbb{Y}^a) = 0$$

and by a symmetry,

$$p(\mathbb{X}) < a \quad \Rightarrow \quad f(\mathbb{X}^a) = 1$$

hold. Define a $g : [0,1]^{\mathbb{N}} \to [0,1]$ by

$$g(x) := \sup\{a \in \mathbb{Q} \cap [0,1] \quad \text{such that} \quad f(x^a) = 0\}, \quad x = (x_1, x_1, \dots) \in [0,1]^{\mathbb{N}},$$

where $x^a = (x_1^a, x_2^a, \dots)$ and $x_n^a = I_{[x_n<a]}$. It follows that outside a P-null set and for all rational numbers $0 \le a < b \le 1$

$$a < p(\mathbb{X}) < b \quad \Rightarrow \quad a < g(\mathbb{X}) < b.$$

Hence, $p(\mathbb{X}) = g(\mathbb{X})$ almost surely. If f is a (Borel) PLIF for $\mathcal{H}$, then g is a (Borel) PLIF for $\mathcal{E}([0,1])$.

We complement slightly the above result.

Corollary 2.4. *If there is a (Borel) PLIF for $\mathcal{H}$ then there is a (Borel) PLIF for $\mathcal{E}(E)$ where E is arbitrary separable metric space.*

Proof. By Theorem 2.3 we may assume that there is a (Borel) PLIF $f(x_1, x_2, \dots)$ for $\mathcal{E}([0,1])$. It follows that

$$f^{\mathbb{N}}(x_1, x_2, \dots) := (f(x_1), f(x_2), \dots), \quad \text{where} \quad x_n \in [0,1]^{\mathbb{N}}$$

defines a (Borel) PLIF for $\mathcal{E}([0,1]^{\mathbb{N}})$. Hence, there exists a (Borel) PLIF for $\mathcal{E}(F)$ where $F \subset [0,1]^{\mathbb{N}}$ is arbitrary subset. The proof is concluded by Urysohn Theorem that states that any separable metric space E is homeomorphic to a subset of the Urysohn's cube $[0,1]^{\mathbb{N}}$. $\square$

Obviously, the space $\mathcal{H}$ deserves our attention. We construct its subset $\mathcal{H}^*$ as follows:

Example 2.5 (G.Simons (1971))**.** Pick up a sequence

$$\mathbb{Y} = (Y_1, Y_2, \dots) \quad \text{of independent 0-1 random variables such that}$$
$$\lim_n P[Y_n = 1] = 0 \quad \text{or} \quad 1 \quad (\Rightarrow \quad \mathbb{Y} \in \mathcal{H}).$$

Also, choose a sequence of natural numbers $r = (r_1 < r_2 < \dots)$ and define $\mathbb{X} = \mathbb{X}(\mathbb{Y}, r)$ as

$$\begin{array}{cccccccccccc} & 1 & 2 & \dots\dots & r_1 & r_1+1 & r_1+2 & \dots\dots & r_2 & r_2+1 & r_2+2 & \dots\dots \\ \mathbb{X} = & Y_1 & Y_1 & \dots\dots & Y_1 & Y_2 & Y_2 & \dots\dots & Y_2 & Y_3 & Y_3 & \dots\dots \end{array}$$

Finally, denote by $\mathcal{H}^*$ the set of all sequences constructed in the above manner. We state that **there is no adapted identification of probability limit for $\boldsymbol{\mathcal{H}^*}$**. G. Simons reasons as follows. Assume that there are $g_n(x_1, x_2, \dots, x_n) : \{0,1\}^n \to \{0,1\}$ such that

$$g_n(X_1, X_2, \dots, X_n) \to p(\mathbb{X}) \quad \text{almost surely} \quad \forall\, \mathbb{X} = (X_1, X_2, \dots) \in \mathcal{H}^*.$$

For a fixed $j \in \mathbb{N}$ choose a sequence of independent r.v.'s $\mathbb{Y} = (Y_1, Y_2, \dots)$ such that

$$P[Y_k = 1] \in (0,1) \quad \text{if} \quad k \le j \quad \text{and} \quad P[Y_k = 1] = 0 \quad \text{if} \quad k > j.$$

Note that

$$g_1(Y_1), \dots, g_j(Y_1, \dots, Y_j), \dots, g_m(Y_1, \dots, Y_j, Y_{j+1}, \dots, Y_m), \dots$$

is a sequence that converges to $p(\mathbb{Y}) = 0$ almost surely and is almost surely one of the 2^j sequences

$$g_1(x_1), \dots, g_j(x_1, \dots, x_j), \dots, g_m(x_1, \dots, x_j, 0, \dots, 0), \quad (x_1, \dots, x_j) \in \{0,1\}^j.$$

Hence, there is an m_j such that for all $m \ge m_j$ and $(x_1, \dots, x_j) \in \{0,1\}^j$

$$g_m(x_1, \dots, x_j, 0, \dots, 0) = 0 \tag{2.2}$$

and by a symmetry

$$g_m(x_1, \dots, x_j, 1, \dots, 1) = 1. \tag{2.3}$$

holds.

Finally, choose a sequence of independent r.v.'s $\mathbb{Y} = (Y_1, Y_2, \dots) \in \mathcal{H}^*$ such that

$$\sum_1^\infty P[Y_n = 0] = \infty \quad \text{and} \quad \sum_1^\infty P[Y_n = 1] = \infty$$

and put $\mathbb{X} = \mathbb{X}(\mathbb{Y}, r)$, where $r = (r_1 < r_2 < \dots)$. The Borel-Cantelli 0-1 law states that the $\mathbb{X}$ owns infinitely many strings of zeros and infinitely many strings of ones. If the sequence $r = (r_1, r_2, \dots)$ grows rapidly enough (perhaps as $r_{k+1} \geq m_{r_k}$), then according to (2.2) and (2.3).

$$P[g_{r_{k+1}}(X_1, \dots, X_{r_k}, X_{r_k+1}, \dots X_{r_{k+1}}) = 0, \quad \infty \times k] = 1, \tag{2.4}$$
$$P[g_{r_{k+1}}(X_1, \dots, X_{r_k}, X_{r_k+1}, \dots X_{r_{k+1}}) = 1, \quad \infty \times k] = 1. \tag{2.5}$$

Hence, a contradiction.

Example 2.6 (G. Simons (1971))**.** An interesting subset of $\mathcal{H}^*$ is $\mathcal{H}^{**} = \{\mathbb{X}(\mathbb{Y}, r)\}$ defined by

$$\begin{array}{lcllr} p(\mathbb{Y}) = 1 & \Rightarrow & \text{almost all} & r_k & \text{even} \\ p(\mathbb{Y}) = 0 & \Rightarrow & \text{almost all} & r_k & \text{odd.} \end{array} \tag{2.6}$$

We state that **there is no adapted identification for the probability limit for $\mathcal{H}^{**}$** (to be applied the same reasoning as in Example 2.5), but **there is a Borel PLIF for $\mathcal{H}^{**}$**. For the proof of the latter statement we are indebted to P. Kříž (2012).

We denote by $\mathcal{N}_0$ the set of sequences in $\{0,1\}^{\mathbb{N}}$ that converge to zero and by $\mathcal{N}_1$ the set of those that converge to one. Note that $\mathcal{N} := \{0,1\}^{\mathbb{N}} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1)$ is the set of all sequences with an infinite number of changes. Put

$$\mathcal{N}^{even} := \{\text{the sequences in } \mathcal{N} \text{ with at most finite changes in the even coordinates}\}$$
$$\mathcal{N}^{odd} := \{\text{the sequences in } \mathcal{N} \text{ with at most finite changes in the odd coordinates}\}.$$

Note that the sets $\mathcal{N}_0$, $\mathcal{N}_1$, $\mathcal{N}^{odd}$, $\mathcal{N}^{even}$ are paire-wise disjoint.

It follows by Borel-Cantelli 0-1 law that a sequence $\mathbb{Y}$ of independent 0-1 random variables is either convergent almost surely or owns infinitely many ones and infinitely zeros. Hence, for any $\mathbb{X} = \mathbb{X}(\mathbb{Y}, r) \in \mathcal{H}^{**}$

$$\begin{array}{lcl} p(\mathbb{X}) = 1 & \Rightarrow & \mathbb{X} \in \mathcal{N}_1 \cup \mathcal{N}^{odd} \\ p(\mathbb{X}) = 0 & \Rightarrow & \mathbb{X} \in \mathcal{N}_0 \cup \mathcal{N}^{even} \end{array}$$

holds almost surely. It follows that

$$\begin{array}{l} f(x) = 1 \quad \text{if} \quad x \in \mathcal{N}_1 \cup \mathcal{N}^{odd}, \quad f(x) = 0 \quad \text{if} \quad x \in \mathcal{N}_0 \cup \mathcal{N}^{even}, \\ f(x) = 0 \quad \text{else} \end{array}$$

defines a Borel PLIF for $\mathcal{H}^{**}$.

A very deep and rather conclusive is a result of D. Blackwell (1980).

Theorem 2.7. *There is no Borel PLIF for the set*

$$\mathcal{H} := \{\mathbb{X} \in \mathcal{E}(0,1) : p(\mathbb{X}) = 1 \quad OR \quad p(\mathbb{X}) = 0\}.$$

A straightforward implication is

Corollary 2.8. *There is no Borel PLIF for $\mathcal{E}(E)$ whenever E is a separable metric space with card$(E) \geq 2$.*

The Blackwell's sophisticated **proof** goes along the following lines:

Assume that $f(x_1, x_2, \dots)$ is a Borel PLIF for $\mathcal{H}$. We may assume w.l.g. that f is the indicator of a tail Borel set $C \subset \{0,1\}^{\mathbb{N}}$ which means that it is a Borel set such that

$$\begin{aligned} x = (x_1, x_2, \dots) \in C, \quad y = (y_1, y_2, \dots) \in \{0,1\}^{\mathbb{N}}, \\ x_k = y_k, \quad k \geq n \quad \text{for some} \quad n \in \mathbb{N} \qquad \Rightarrow \quad y \in C \end{aligned}$$

holds. By Oxtoby's category 0-1 law [8, Theorem 21.4] either C or $\{0,1\}^{\mathbb{N}} \setminus C$ is a set of the first category which means that the other set contains a G_δ set H such that $\overline{H} = \{0,1\}^{\mathbb{N}}$. Assume that $H \subset C$. The corner stone of the proof is a lemma that says that for arbitrary dense G_δ-set H there is a sequence of 0-1 r.v.'s $\mathbb{X} = (X_1, X_2, \dots)$ such that

$$P[\mathbb{X} \in H] = 1 \quad \text{and} \quad p(\mathbb{X}) = 0 \quad \text{almost surely,}$$

hence, a contradiction with $f(\mathbb{X}) = 1$.

Surprisingly, skipping the measurability requirement for PLIF's and allowing the continuum hypothesis we have all the PLIFs we might need.

Theorem 2.9. *Let E be a separable metric space. Then, under the continuum hypothesis, there exists a PLIF for $\mathcal{E}(E)$. In other words, there is a map $f : E^{\mathbb{N}} \to E$ such that*

$$f(\mathbb{X}) = p(\mathbb{X}) \quad \textit{almost surely}$$

for all sequences $\mathbb{X}$ of E-valued r.v.'s that converge in probability.

Theorem was first proved in 1973 in [10] for $E = \mathbb{R}$ and generalized to separable metric spaces in 2010 in [4]. We shall propose a new and more straightforward proof of the theorem in the coming Section.

3 Proof of Theorem 2.9

First, we employ the continuum hypothesis to prove that there is a PLIF for the set

$$\mathcal{H} = \{\mathbb{X} \in \mathcal{E}(\{0,1\}) : p(\mathbb{X}) = 1 \quad \text{or} \quad 0\}.$$

What we have to prove is that there exits a set $C \subset \{0,1\}^{\mathbb{N}}$ such that

$$\mathbb{X} \in \mathcal{H}, \quad p(\mathbb{X}) = 1 \quad \Rightarrow \quad \mathbb{X} \in C \quad \text{almost surely}$$

and

$$\mathbb{X} \in \mathcal{H}, \quad p(\mathbb{X}) = 0 \quad \Rightarrow \quad \mathbb{X} \notin C \quad \text{almost surely.}$$

This is as to say that that there exists a set $C \subset \{0,1\}^{\mathbb{N}}$ such that

$$\mu \in \mathcal{M}_1 \quad \Rightarrow \quad \mu_*(C) = 1, \qquad \mu \in \mathcal{M}_0 \quad \Rightarrow \quad \mu_*(\{0,1\}^{\mathbb{N}} \setminus C) = 1, \tag{3.1}$$

where we have denoted

$$\mathcal{M}_1 = \{P_{\mathbb{X}},\, \mathbb{X} \in \mathcal{H},\, p(\mathbb{X}) = 1\}, \quad \mathcal{M}_0 = \{P_{\mathbb{X}},\, \mathbb{X} \in \mathcal{H},\, p(\mathbb{X}) = 0\}.$$

Note that $\mathrm{card}(\mathcal{M}_1) = \mathrm{card}(\mathcal{M}_0) = c$. Hence, the continuum hypothesis allows to enumerate the set $\mathcal{M}_1$ as

$$\mathcal{M}_1 = \{\mu^\alpha,\, \alpha < \Omega\},$$

where Ω denotes the first uncountable ordinal number. Further, by the transfinite construction we get a net

$$r(\alpha) = (r_1(\alpha) < r_2(\alpha) < \dots), \quad \alpha < \Omega$$

of sequences of natural numbers such that $\mu^\alpha(A_\alpha) = 1$ holds for all $\alpha < \Omega$, where

$$A_\alpha := \{x = (x_1, x_2, \dots) \in \{0,1\}^{\mathbb{N}} : \lim_{n\to\infty} x_{r_n(\alpha)} = 1\}$$

and such that

$$\alpha < \beta \quad \Rightarrow \quad r(\beta) \quad \text{is a subsequence of} \quad r(\alpha).$$

The steps of the transfinite construction are performed as follows:

If α is an isolated ordinal, then $\alpha = \beta + 1$ for some $\beta < \alpha$ and $r(\alpha)$ is constructed as a subsequence of $r(\beta)$ by means of Riesz theorem If α is a limit ordinal number, then there are $\beta_1 < \cdots < \beta_n \cdots < \alpha$ such that $\alpha = \sup_n \beta_n$. Then, $r(\alpha)$ is received as a subsequence of all $r(\beta_n)$ sequences by Riesz theorem, again. Observe that $\{A_\alpha,\, \alpha < \Omega\}$ is an increasing net and put $C := \bigcup_{\alpha<\Omega} A_\alpha$. Then

$$\mu^\alpha_*(C) = 1 \quad \forall \alpha < \Omega$$

and the former implication in (3.1) is satisfied.

Define by

$$T(x_1, x_2, \dots, x_n, \dots) = (1-x_1, 1-x_2, \dots, 1-x_n, \dots), \quad (x_1, x_2, \dots, x_n, \dots) \in \{0,1\}^{\mathbb{N}}$$

a homeomorphic map $\{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ and observe that if $\mathbb{X} \in \mathcal{H}$ then

$$p(\mathbb{X}) = 1 \quad \Longleftrightarrow \quad p(T\mathbb{X}) = 0.$$

Hence,

$$\mathcal{M}_0 = \{\nu^\alpha := T \circ \mu^\alpha, \quad \alpha < \Omega\},$$

where the measure $\mu \circ T$ is defined on the Borel σ-algebra of $\{0,1\}^{\mathbb{N}}$ by $(T \circ \mu)(B) = \mu(T^{-1}B)$. Obviously, $C \cap T(C) = \emptyset$ since

$$x \in A_\alpha, \quad \mathbb{T}(x) \in A_\beta \quad (\text{say that} \quad \alpha < \beta) \quad \Rightarrow \quad x \in A_\beta \cap T(A_\beta) = \emptyset.$$

Finally, note that

$$\nu^{\alpha}(TA_{\alpha}) = \mu^{\alpha}(A_{\alpha}) = 1 \quad \Rightarrow \quad \nu_*^{\alpha}\left(\{0,1\}^{\mathbb{N}} \setminus C\right) = 1, \quad \forall \alpha < \Omega$$

verifies the latter implication in (3.1) and $f = I_C$ defines a PLIF for the set $\mathcal{H}$.

To conclude the proof of Theorem 2.9 apply Corollary 2.4. □

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Intuitionistic fuzzy sets – two and three term representations in the context of a Hausdorff distance

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Dedicated to the 75th birthday of Beloslav Riečan

Abstract

We consider here the two term and three term representations of Atanassov's intuitionistic fuzzy sets (A-IFSs, for short) in the context of the Hausdorff distance based on the Hamming metric. Especially, we pay attention to the consistency of the metric used and the essence of the Hausdorff distances. We also consider the same problem for the interval-valued fuzzy sets. It is shown that the essence of solutions obtained is different for the case of the A-IFSs and interval-valued fuzzy sets. In other words, the two term representation of A-IFSs (which makes the A-IFSs to boil down to the interval-valued fuzzy sets) does not work here (on the contrary to three term representation).

Keywords Intuitionistic fuzzy sets, Hausdorff distance

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1 Introduction

One of the most important measures are distances which are widely used both in theoretical considerations and for practical purposes in many areas. It is not possible to overestimate their importance also in the context of fuzzy sets (Zadeh [45]) or their generalizations, eg., the A-IFSs. Distances are necessary in analyses related to the entropy, similarity, when making group decisions, calculating degrees of soft consensus, in classification, pattern recognition, etc.

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Distances between the A-IFSs are calculated in the literature in two ways, using two terms only, i.e. the degree of membership and non-membership (e.g., Atanassov [4]) or all three terms. i.e. the membership and non-membership degrees and the hesitation margin (e.g., Szmidt and Kacprzyk [28], [35], Tasseva et al. [43], Atanassov et al. [5], Szmidt and Baldwin [22], [23], Deng-Feng [8], Tan and Zhang [42], Narukawa and Torra [13])). Mathematically, both ways are correct from the point of view of just the formal conditions concerning distances (all properties are fulfilled in both cases). However, when semantics come to play, one cannot say that both ways are equal when assessing the results obtained by the two approaches. In this paper we will consider one of such situations related to the calculating a Hausdorff distance using the two approaches to represent the A-IFSs.

The Hausdorff distances (cf. Grünbaum [9]) play an important role in practical applications, notably in image matching, image analysis, motion tracking, visual navigation of robots, computer-assisted surgery and so on (cf. e.g., Huttenlocher et al. [10], Huttenlocher and Rucklidge [11], Olson [14], Peitgen et al. [15], Rucklidge [17]-[21]). The definition of the Hausdorff distances is simple but the calculations needed to solve the real problems are complex. As a result the efficiency of the algorithms for computing the Hausdorff distances may be crucial and the use of some approximations may be relevant and useful (e.g, Aichholzer [1], Atallah [2], Huttenlocher et al. [10], Preparata and Shamos [16], Rucklidge [21], Veltkamp [44]).

The formulas proposed for calculating the distances should first of all be formally correct. It is the motivation of this paper. Namely, we consider the results of using the Hamming distances between the A-IFSs calculated in two possible ways - taking into account the two term representation (the membership and non-membership values) of A-IFSs, and next - taking into account the three term representation (the membership, non-membership values, and hesitation margin) of A-IFSs. We will verify if the resulting distances fulfill the properties of the Hausdorff distances.

We also consider the problem of calculating the Hausdorff distance based on the Hamming metric for the interval-valued fuzzy sets. We prove that the formulas that are effective and efficient for interval-valued fuzzy sets do not work well in the case of A-IFSs.

2 Brief introduction to the A-IFSs

One of the generalizations of a fuzzy set in X (Zadeh [45]) , given by

$$A^{'} = \{< x, \mu_{A'}(x) > | x \in X\} \tag{2.1}$$

where $\mu_{A'}(x) \in [0,1]$ is the membership function of the fuzzy set $A^{'}$, is the intuitionistic fuzzy set, or A-IFS, for short (Atanassov [3], [4]) A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2.2}$$

where: $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{2.3}$$

and $\mu_A(x)$, $\nu_A(x) \in [0,1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively. These degrees may be specified in different ways, and a constructive approach is given by Szmidt and Baldwin [24].

Obviously, each fuzzy set may be represented by the following A-IFS

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\} \tag{2.4}$$

An additional concept for each A-IFS in X, that is not only an obvious result of (2.2) and (2.3) but which is also relevant for applications, is

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{2.5}$$

a *hesitation margin (an intuitionistic fuzzy index)* of $x \in A$ which expresses a lack of knowledge of whether x belongs to A or not (cf. Atanassov [4]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

The hesitation margin turns out to be important while considering the distances (Szmidt and Kacprzyk [26], [28], [35], entropy (Szmidt and Kacprzyk [31], [38]), similarities (Szmidt and Kacprzyk [39]), etc. i.e., measures that play a crucial role in virtually all information processing tasks.

Also, from the point of view of the applications, the hesitation margin is crucial in many areas exemplified by image processing (cf. Bustince et al. [6], [7]), classification of imbalanced and overlapping classes (cf. Szmidt and Kukier [37], [40], [41]), group decision making, negotiations, voting and other situations (cf. Szmidt and Kacprzyk [25], [27], [29], [30], [32], [33], [34], [36]).

In other words, the three term representation of the A-IFSs (taking into account the membership values, non-membership values, and hesitation margins) has already proved to play important role both from the theoretical point of view and applications.

2.1 Distances Between the A-IFSs

In Szmidt and Kacprzyk [28], [35], Szmidt and Baldwin [22], [23], it is shown why in the calculation of distances between the A-IFSs one should use all three terms describing them. Examples of the distances between any two A-IFSs A and B in $X = \{x_1, x_2, \ldots, x_n\}$ while using the three term representation (Szmidt and Kacprzyk [28], Szmidt and Baldwin [22], [23]) may be as follows:

- the normalized Hamming distance:

$$\begin{aligned} l_{IFS}(A,B) &= \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + \\ &+ |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|) \end{aligned} \tag{2.6}$$

- the normalized Euclidean distance:

$$\begin{aligned} e_{IFS}(A,B) = (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + \\ + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2)^{\frac{1}{2}} \end{aligned} \tag{2.7}$$

The values of both distances are from the interval $[0, 1]$.

The counterparts of the above distances while using the two term representation of A-IFSs are:

- the normalized Hamming distance:

$$l'(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|) \tag{2.8}$$

- the normalized Euclidean distance:

$$q'(A,B) = (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2)^{\frac{1}{2}} \tag{2.9}$$

3 The Hausdorff distance

The Hausdorff distance is *the maximum distance of a set to the nearest point in the other set.* More formal description is given by the following

Definition 3.1. Given two finite sets $A = \{a_1, ..., a_p\}$ and $B = \{b_1, ..., b_q\}$, the Hausdorff distance $H(A, B)$ is defined as:

$$H(A, B) = \max\{h(A, B), h(B, A)\} \tag{3.1}$$

where

$$h(A, B) = \max_{a \in A} \min_{b \in B} d(a, b) \tag{3.2}$$

where:
– a and b are elements of sets A and B respectively,
– $d(a, b)$ is any metric between these elements,
– the two distances $h(A, B)$ and $h(B, A)$ (3.2) are called the directed Hausdorff distances.

The function $h(A, B)$ (the directed Hausdorff distance from A to B) ranks each element of A based on its distance to the nearest element of B, and then the highest ranked element specifies the value of the distance. In general $h(A, B)$ and $h(B, A)$ can be different values (the directed distances are not symmetric).

From Definition 3.1 it follows, that if A and B contain one element each (a_1 and b_1, respectively), the Hausdorff distance is just equal to $d(a_1, b_1)$. In other words, if a formula which is expected to express the Hausdorff distance gives for separate elements the results not consistent with the used metric d (e.g., the Hamming distance, the Euclidean distance, etc.), the formula considered is not a proper definition of the Hausdorff distance.

3.1 The Hausdorff distance between the interval-valued fuzzy sets

The Hausdorff distance between two intervals: $U = [u_1, u_2]$ and $W = [w_1, w_2]$ is (Moore [12]):

$$h(U, W) = \max\{|u_1 - w_1|,\ |u_2 - w_2|\} \tag{3.3}$$

Assuming the two-term representation for the A-IFSs: $A = \{x, \mu_A(x), \nu_A(x)\}$ and $B = \{x, \mu_B(x), \nu_B(x)\}$, we may consider the two A-IFSs, A and B, as two intervals, namely:

$$[\mu_A(x), 1 - \nu_A(x)] \quad \text{and} \quad [\mu_B(x), 1 - \nu_B(x)] \tag{3.4}$$

then

$$h(A, B) = \max\{|\mu_A(x) - \mu_B(x)|,\ |\nu_A(x) - \nu_B(x)|\} \tag{3.5}$$

We will verify later if (3.5) is a properly calculated Hausdorff distance between the A-IFSs while using the Hamming metric.

3.2 Two term representation of A-IFSs and the Hausdorff distance (Hamming metric)

Due to the algorithm of calculating the directed Hausdorff distances, when applying the two term type distance (2.8) for the A-IFSs, we obtain:

$$d_h(A, B) \quad = \quad \frac{1}{n} \sum_{i=1}^{n} max\{|\mu_A(x_i) - \mu_B(x_i)|\,, |\nu_A(x_i) - \nu_B(x_i)|\} \tag{3.6}$$

If the above distance is a properly calculated Hausdorff distance, in the case of degenerated, i.e., one-element sets $A = \{< x, \mu_A(x), \nu_A(x) >\}$ and $B = \{< x, \mu_B(x), \nu_B(x) >\}$, it should give the same results as the the two term type Hamming distance. It means that in the case of the two term type Hamming distance, for degenerated, one element A-IFSs, the following equations should give just the same results:

$$l^{'}(A,B) = \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|) \tag{3.7}$$

$$d_h(A,B) = \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|\} \tag{3.8}$$

where (3.7) is the normalized two term type Hamming distance, and (3.8) should be its counterpart Hausdorff distance.

We will verify on a simple example if (3.7) and (3.8) give the same results as they should do following the essence of of the Hausdorff measures.

Example 1

Let consider the following one-element A-IFSs: A, B, $\in X = \{x\}$

$$A \;=\; \{< x, 1, 0 >\}, \quad B = \{< x, \frac{1}{4}, \frac{1}{4} >\} \tag{3.9}$$

The result obtained from (3.8) is:

$$d_h(A,B) \;=\; \max\{|1 - 1/4|, |0 - 1/4|\} = 0.75$$

The counterpart Hamming distance calculated from (3.7) is:

$$l^{'}(A,B) \;=\; 0.5(|1 - 1/4| + |0 - 1/4||) = 0.5$$

i.e. the value of the Hamming distances (3.7) used to propose the Hausdorff measure (3.8), and the value of the resulting Hausdorff distance (3.8) calculated for the separate elements are not consistent (as they should be).

Now we will show that the inconsistencies as shown above occur for an infinite number of other cases.

Let us verify the conditions under which the equation (3.7) and (3.8) give the consistent results, i.e., when for the separate elements we have

$$\begin{aligned} &\frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|) = \\ =\; &\max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|\} \end{aligned} \tag{3.10}$$

Having in mind that

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1 \tag{3.11}$$

$$\mu_B(x) + \nu_B(x) + \pi_B(x) = 1 \tag{3.12}$$

from (3.11) and (3.12) we obtain

$$(\mu_A(x) - \mu_B(x)) + (\nu_A(x) - \nu_B(x)) + (\pi_A(x) - \pi_B(x)) = 0 \tag{3.13}$$

It is easy to verify that (3.13) is not fulfilled for all elements belonging to an A-IFSs but for some elements only. The following conditions guarantee that (3.10) is fulfilled

- for $\pi_A(x) - \pi_B(x) = 0$, from (3.13) we have

$$|\mu_A(x) - \mu_B(x)| = |\nu_A(x) - \nu_B(x)| \tag{3.14}$$

and taking into account (3.14), we can express (3.10) in the following way:

$$\begin{aligned} & 0.5(|\mu_A(x) - \mu_B(x)| + |\mu_A(x) - \mu_B(x)|) = \\ = \quad & \max\{|\mu_A(x) - \mu_B(x)|, |\mu_A(x) - \mu_B(x)|\} \end{aligned} \tag{3.15}$$

- if $\pi_A(x) - \pi_B(x) \neq 0$ but the same time

$$\mu_A(x) - \mu_B(x) = \nu_A(x) - \nu_B(x) = -\frac{1}{2}(\pi_A(x) - \pi_B(x)) \tag{3.16}$$

guarantee that (3.10) boils down again to (3.15).

In other words, (3.10) is fulfilled (which means that the Hausdorff measure given by (3.8) is a natural counterpart of (3.7)) only for such elements belonging to an A-IFS, for which some additional conditions are given, like $\pi_A(x) - \pi_B(x) = 0$ or (3.16). But in general, for an infinite numbers of elements, (3.10) is not valid.

In the above context it seems to be a bad idea to try constructing the Hausdorff distance using the two term type Hamming distance between the A-IFSs.

An immediate conclusion is that, relating to the results from Section 3.1, the Hausdorff distance for the A-IFSs can not be constructed in the same way as for the interval-valued fuzzy sets.

3.3 A straightforward generalizations of the Hamming distance based on the Hausdorff metric

Now we will show that applying the three term type Hamming distance for the A-IFSs, we obtain correct (in the sense of Definition 3.1) Hausdorff distance.

Namely, if we calculate the three term type Hamming distance between two degenerated, i.e. one-element A-IFSs, A and B in the spirit of Szmidt and Kacprzyk [28], [35], Szmidt and Baldwin [22], [23], i.e., in the following way:

$$\begin{aligned} l_{IFS}(A,B) \quad = \quad & \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)| + \\ + \quad & |\pi_A(x) - \pi_B(x)|) \end{aligned} \tag{3.17}$$

we can give a counterpart of the above distance in terms of the max function:

$$\begin{aligned} H_3(A,B) \quad = \quad & \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, \\ , \quad & |\pi_A(x) - \pi_B(x)|\} \end{aligned} \tag{3.18}$$

If $H_3(A,B)$ (3.18) is a properly specified Hausdorff distance (in the sense that for two degenerated, one element A-IFS the result is equal to the metric used), the following condition should be fulfilled:

$$\begin{aligned} & \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|) + |\pi_A(x) - \pi_B(x)|) = \\ = \quad & \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} \end{aligned} \tag{3.19}$$

Let us verify if (3.19) is valid. Without loss of generality we can assume

$$\begin{aligned} \max \quad & \{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} = \\ = \quad & |\mu_A(x) - \mu_B(x)| \end{aligned} \tag{3.20}$$

For $|\mu_A(x) - \mu_B(x)|$ fulfilling (3.20), and because of (3.11) and (3.12), we conclude that both $\nu_A(x) - \nu_B(x)$, and $\pi_A(x) - \pi_B(x)$ are of the same sign (both values are either positive or negative). Therefore

$$|\mu_A(x) - \mu_B(x)| = |\nu_A(x) - \nu_B(x)| + |\pi_A(x) - \pi_B(x)| \tag{3.21}$$

Applying (3.21) we can verify that (3.19) always is valid as

$$\begin{aligned} & 0.5\{|\mu_A(x) - \mu_B(x)| + |\mu_A(x) - \mu_B(x)|\} = \\ = \; & \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} = \\ = \; & |\mu_A(x) - \mu_B(x)| \end{aligned} \tag{3.22}$$

Now we will use the above formulas (3.17) and (3.18) for the data used in Example 1. But now, as we also take into account the hesitation margins $\pi(x)$ (2.5), instead of (3.9) we use the three term, "full" description of the data $\{< x, \mu(x), \nu(x), \pi(x) >\}$, i.e. employing all three functions (the membership, non-membership and hesitation margin) describing the considered A-IFSs:

$$A = \{< x, 1, 0, 0 >\}, \quad B = \{< x, \frac{1}{4}, \frac{1}{4}, \frac{1}{2} >\} \tag{3.23}$$

and obtain from (3.18):

$$H_3(A, B) = \max(|1 - 1/4|, |0 - 1/4|, |0 - 1/2|) = 0.75$$

Now we calculate the counterpart Hamming distances using (3.17) (with all three functions). The results are

$$l_{IFS}(A, B) = 0.5(|1 - 1/4| + |0 - 1/4| + |0 - 1/2|) = 0.75$$

As we can see, the Hausdorff distance (3.18) proposed in this paper (using the memberships, non-memberships and hesitation margins) and the Hamming distance (3.17) give for one-element IFS sets fully consistent results. The same situation occurs in a general case too.

In other words, for the normalized Hamming distance expressed in the spirit of (Szmidt and Kacprzyk [28], [35]) given by (2.6) we can give the following equivalent representation in terms of the max function:

$$\begin{aligned} H_3(A, B) = \; & \frac{1}{n} \sum_{i=1}^{n} \max\{|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|, \\ & |\pi_A(x_i) - \pi_B(x_i)|\} \end{aligned} \tag{3.24}$$

Unfortunately, it can be easily verified that it is impossible to give the counterpart pairs of the formulas as (2.6) and (3.24) for $r > 1$ in the Minkowski r-metrics ($r = 1$ is the Hamming distance, $r = 2$ is the Euclidean distance, etc.)

For details on other distances between the A-IFSs we refer the interested reader to Szmidt and Kacprzyk [28] and especially [35]. More details are given in [5] and [43].

4 Conclusions

A method for the calculation of Hausdorff distances (based on the Hamming metric) between the A-IFSs is presented and analyzed. The method employs all three terms describing the A-IFSs. The proposed method is both mathematically valid and intuitively appealing (cf. [35]).

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The partially ordered metric semigroup valued lower integral

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Abstract
The lower integral defined on the lattice ordered group with values in a partially ordered metric semigroup and the integrable elements are defined in this paper.

Keywords lower integral, ℓ-group, partially ordered metric semigroup, integrability

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1 Introduction

The lower integral on the real-valued functions was defined by Topsoe in [7]. An axiomatic definition of the real-valued upper or lower integral and the fuzzy number valued lower integral defined on the ℓ-group and the notion of integrability were introduced in [6], [8], [9] and [10]. We generalize this theory for the partially ordered metric semigroup valued lower integral. The metric semigroup was used in the case of Kurzweil–Henstock integral in [2], [3], [4], [11] and also as the range space of BV mappings of two real variables in [1]. The aim of this paper is to define the lower integral and the integrable elements, and to prove that the upper limits of the countable set of integrable elements are integrable.

2 Preliminaries

Definition 2.1. A partially ordered metric semigroup is a structure $(X, \varrho, +, \leq)$, where $\varrho : X \times X \to \mathbb{R}$, $+ : X \times X \to X$ satisfy the following conditions:

(i) (X, ϱ) is a metric space

(ii) $(X, +)$ is a commutative semigroup endowed with a neutral element 0

(iii) $(X, \leq)$ is a partially ordered set

(iv) if $u, v, z \in X$ and $u \leq v$ then $u + z \leq v + z$.

(v) ϱ is translation invariant: $\varrho(u, v) = \varrho(u + w, v + w)$ for all $u, v, w \in X$

(vi) $\varrho(u + y, v + z) \leq \varrho(u, v) + \varrho(y, z)$ for all $u, v, y, z \in X$

Examples 2.2. 1. A simple example of the partially ordered metric semigroup is the set $\mathbb{R}$ with metric $\varrho(u, v) = |u - v|$ and the usual order.

2. Let X be a set of all pointwise ordered bounded real-valued functions defined on a compact metric space M with the metric $\varrho(f, g) = \sup\{|f(u) - g(u)|;\ u \in M\}$. Then $(X, \varrho, +, \leq)$ is a partially ordered metric semigroup.

3. An example of the partially ordered metric semigroup which is not a group is the set of fuzzy numbers $E = (E, D, +, \leq)$. The sum of fuzzy numbers u, v is a fuzzy number z such that

$$z = u + v \Leftrightarrow (z)^\alpha = (u)^\alpha + (v)^\alpha \text{ for every } \alpha \in (0, 1],$$

where $(u)^\alpha = \{x \in \mathbb{R},\ u(x) \geq \alpha\}$ and the sum of intervals $[a, b] + [c, d] = [a + c, b + d]$. The partial ordering on the set E is defined in the following way:

$$u \leq v \ \Leftrightarrow\ (u)^\alpha \leq (v)^\alpha \text{ for every } \alpha \in (0, 1],$$

where $[a, b] \leq [c, d] \ \Leftrightarrow\ (a \leq c \wedge b \leq d)$. The Hausdorff distance d of closed possibly degenerate intervals is defined by equation:

$$d([a, b], [c, d]) = \max\{|c - a|, |d - b|\}.$$

Then (E, D), where $D : E \times E \to [0, \infty)$,

$$D(u, v) = \sup\left\{d\left((u)^\alpha, (v)^\alpha\right);\ \alpha \in (0, 1]\right\}$$

is a complete metric space. The properties (v) and (vi) of the metric D can be found in [12].

3 The lower integral

Definition 3.1. Let G be an ℓ-group. The lower integral is a mapping $I : G^+ \to X$ which fulfills the following conditions:

1) $I(0) = 0$,

2) if $x \leq y$ then $I(x) \leq I(y)$ for all $x, y \in G^+$,

3) $I(x) + I(y) \leq I(x + y)$ for all $x, y \in G^+$,

4) if $x_n \downarrow x,\ x, x_n \in G^+$ $(n = 1, 2, \ldots)$ then $\lim_{n\to\infty} \varrho(I(x_n), I(x)) = 0$.

Definition 3.2. Let G be an ℓ-group and I be a lower integral on G^+. An element $x \in G^+$ is called I-integrable iff

$$I(a) = I(a \wedge x) + I(a - (a \wedge x))$$

for any $a \in G^+$. We denote $G_I^+ = \{x \in G^+;\ x \text{ is } I\text{-integrable}\}$.

Theorem 3.3. *(i) If $a \in G^+$ and $x \in G_I^+$ then $I(a + x) = I(a) + I(x)$.*
(ii) If $x, y \in G_I^+$ then $x + y, x \wedge y \in G_I^+$. Furthermore, if $x, y \in G_I^+, y \leq x$ then $x - y \in G_I^+$.
(iii) If $x, y \in G_I^+$ then $x \vee y \in G_I^+$.

Proof. (i) Let $a \in G^+$ and $x \in G_I^+$. Then from properties of ℓ-group we get

$$I(a+x) = I\left((a+x)\wedge x\right) + I\left(a+x-(a+x)\wedge x\right) = I(x)+I(a).$$

(ii) If $a, x, y \in G, y \geq 0$ then

$$a\wedge x + \left((a - a\wedge x)\wedge y\right) = a\wedge\left((a\wedge x)+y\right) = a\wedge(a+y)\wedge(x+y) = a\wedge(x+y)\,.$$

Let $x, y \in G_I^+,\ a \in G^+$. From the property 3) of the lower integral we have

$$\begin{aligned} I\left(a\right) &= I\left(a\wedge x\right) + I\left(a - a\wedge x\right) \\ &= I\left(a\wedge x\right) + I\left((a-a\wedge x)\wedge y\right) + I\left(a - a\wedge x - (a-a\wedge x)\wedge y\right) \\ &\leq I\left(a\wedge(x+y)\right) + I\left(a - a\wedge(x+y)\right) \leq I\left(a\right). \end{aligned}$$

Hence, $x+y \in G_I^+$. Similarly

$$\begin{aligned} I\left(a\right) &= I\left(a\wedge x\right) + I\left(a - a\wedge x\right) \\ &= I\left(a\wedge x\wedge y\right) + I\left(a\wedge x - a\wedge x\wedge y\right) + I\left(a - a\wedge x\right) \\ &\leq I\left(a\wedge(x\wedge y)\right) + I\left(a - a\wedge(x\wedge y)\right) \leq I\left(a\right). \end{aligned}$$

It follows $x\wedge y \in G_I^+$. It holds $(a+y)\wedge x = a\wedge(x-y)+y$ in every ℓ-group. Let $x, y \in G_I^+$, $y \leq x$, $a \in G^+$. By the proof of the assertion (i),

$$\begin{aligned} I\left(a\right)+I\left(y\right) &= I\left(a+y\right) = I\left((a+y)\wedge x\right) + I\left(a+y-(a+y)\wedge x\right) \\ &= I\left(a\wedge(x-y)+y\right) + I\left(a - a\wedge(x-y)\right) \\ &= I\left(y\right) + I\left(a\wedge(x-y)\right) + I\left(a - a\wedge(x-y)\right). \end{aligned}$$

Using the property (v) of the metric ϱ we can write

$$\begin{aligned} &\varrho\left(I\left(a\right), I\left(a\wedge(x-y)\right) + I\left(a - a\wedge(x-y)\right)\right) \\ = \ &\varrho\left(I\left(a\right)+I\left(y\right), I\left(a\wedge(x-y)\right) + I\left(a - a\wedge(x-y)\right) + I\left(y\right)\right) = 0. \end{aligned}$$

Because $\varrho\left(x,y\right)=0$ iff $x=y$ we get

$$I\left(a\right) = I\left(a\wedge(x-y)\right) + I\left(a - a\wedge(x-y)\right).$$

Hence, $x-y \in G_I^+$.
(iii) The assertion follows from the part (ii) and the equation $x\vee y = (x+y) - x\wedge y$. $\square$

Theorem 3.4. *Let $x_n \uparrow x, x_n \in G_I^+, n = 1, 2, \ldots, x \in G^+$. Then $x \in G_I^+$ and $I\left(x_n\right) \to I\left(x\right)$ on the metric ϱ.*

Proof. Let $x_n \uparrow x, x_n \in G_I^+, n = 1, 2, \ldots, x \in X^+$. Using the integrability of x_n and the property (vi) of the metric ϱ we get

$$\begin{aligned} &\varrho\left(I\left(a\wedge x\right) + I\left(a - a\wedge x\right), I\left(a\right)\right) \\ = \ &\varrho\left(I\left(a\wedge x\right) + I\left(a - a\wedge x\right), I\left(a\wedge x_n\right) + I\left(a - a\wedge x_n\right)\right) \\ \leq \ &\varrho\left(I\left(a\wedge x\right), I\left(a\wedge x_n\right)\right) + \varrho\left(I\left(a - a\wedge x\right), I\left(a - a\wedge x_n\right)\right) \end{aligned}$$

for every $n \in \mathbb{N}$, hence

$$\begin{aligned} &\varrho\left(I\left(a\wedge x\right) + I\left(a - a\wedge x\right), I\left(a\right)\right) \\ \leq \ &\lim_{n\to\infty} \varrho\left(I\left(a\wedge x\right), I\left(a\wedge x_n\right)\right) + \lim_{n\to\infty} \varrho\left(I\left(a - a\wedge x\right), I\left(a - a\wedge x_n\right)\right). \end{aligned}$$

From the property 4) of the lower integral the second limit equals zero. The first limit equals zero too, because

$$\begin{aligned}
&\lim_{n\to\infty} \varrho\left(I\left(a\wedge x\right), I\left(a\wedge x_n\right)\right) \\
&= \lim_{n\to\infty} \varrho\left(I\left(a\wedge x\wedge x_n\right) + I\left(a\wedge x - a\wedge x\wedge x_n\right), I\left(a\wedge x_n\right)\right) \\
&= \lim_{n\to\infty} \varrho\left(I\left(a\wedge x_n\right) + I\left(a\wedge x - a\wedge x_n\right), I\left(a\wedge x_n\right)\right) \\
&\le \lim_{n\to\infty} \varrho\left(I\left(a\wedge x_n\right), I\left(a\wedge x_n\right)\right) + \lim_{n\to\infty} \varrho\left(I\left(a\wedge x - a\wedge x_n\right), 0\right) \\
&= 0 + \lim_{n\to\infty} \varrho\left(I\left(a\wedge x - a\wedge x_n\right), I\left(0\right)\right) = 0
\end{aligned}$$

by the property (vi) of ϱ and 4) of I. So

$$\varrho\left(I\left(a\wedge x\right) + I\left(a - a\wedge x\right), I\left(a\right)\right) = 0,$$

that is $I\left(a\wedge x\right) + I\left(a - a\wedge x\right) = I\left(a\right)$ and $x \in G_I^+$. Now we prove $I\left(x_n\right) \to I\left(x\right)$ in the metric ϱ. It holds

$$\begin{aligned}
\lim_{n\to\infty} \varrho\left(I\left(x\right), I\left(x_n\right)\right) &= \lim_{n\to\infty} \varrho\left(I\left(x\wedge x_n\right) + I\left(x - x\wedge x_n\right), I\left(x_n\right)\right) \\
&= \lim_{n\to\infty} \varrho\left(I\left(x_n\right) + I\left(x - x_n\right), I\left(x_n\right)\right) \\
&\le \lim_{n\to\infty} \varrho\left(I\left(x_n\right), I\left(x_n\right)\right) + \lim_{n\to\infty} \varrho\left(I\left(x - x_n\right), 0\right) \\
&= 0.
\end{aligned}$$

□

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