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The Change-of-Variables Theorem for the Lebesgue Integral*

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Dedicated to the 75th birthday of Beloslav Riečan

Abstract

We present a short proof of the change-of-variables theorem for diffeomorphic mappings. This is a modification of the proof given in [3].

Keywords change-of-variables, Lebesgue integral

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The following change-of-variables theorem for the Lebesgue integral is standard[1].

Theorem 1. *Let $V \subset \mathbb{R}^d$ be an open set and $\varphi : V \to \mathbb{R}^d$ be a one-to-one $\mathcal{C}^1$-mapping with non-vanishing Jacobian J_φ. Then*

$$\int_{\varphi(V)} h\,d\lambda_d = \int_V (h \circ \varphi)|J_\varphi|\,d\lambda_d, \quad h \in C_c\big(\varphi(V)\big). \tag{1}$$

The proof that we shall describe is based on a *smearing technique* and uses the following standard result on the transformation of Lebesgue measure by linear mappings: *Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be a one-to-one linear mapping. Then, for every Lebesgue measurable set $A \subset \mathbb{R}^d$,*

$$\lambda_d\big(\psi(A)\big) = |J_\psi| \cdot \lambda_d(A). \tag{2}$$

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[1] We use the usual terminology and notation: our assumption says that φ is a diffeomorphism of V onto $\varphi(V)$; λ_d stands for d-dimensional Lebesgue measure; for $U \subset \mathbb{R}^d$ open, $C_c(U)$ is the set of all continuous functions $g : U \to \mathbb{R}$ such that their support $S(g) := \overline{\{x \in U : g(x) \neq 0\}}$ is a compact subset of U.

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Here, of course, $J_\psi = \det \psi$. It follows immediately from (2) (by integration with respect to the image measure) that

$$\int_{\mathbb{R}^d} (g \circ \psi)\, d\lambda_d = \big(1/|J_\psi|\big) \int_{\mathbb{R}^d} g\, d\lambda_d, \quad g \in C_c(\mathbb{R}^d). \tag{3}$$

Let us fix a positive function $\omega \in C_c(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \omega\, d\lambda_d = 1$, and, for $r > 0$, define

$$\omega_r(y) := r^{-d}\omega(y/r), \quad y \in \mathbb{R}^d.$$

For $f \in C_c(\mathbb{R}^d)$ and $r > 0$, the convolution of f and ω_r is defined by[2]

$$f * \omega_r : x \mapsto \int_{\mathbb{R}^d} f(x-y)\, \omega_r(y)\, dy.$$

Then $f * \omega_r \in C_c(\mathbb{R}^d)$ and, using (3),

$$(f * \omega_r)(x) - f(x) = \int_{\mathbb{R}^d} \big(f(x - rz) - f(x)\big)\, \omega(z)\, dz, \quad x \in \mathbb{R}^d.$$

Since f is uniformly continuous, it follows that

$$f * \omega_r \to f \quad \text{uniformly on } \mathbb{R}^d \text{ for } r \to 0+. \tag{4}$$

Let V and φ be as in the theorem. The following result will be useful.

Lemma 2. *For $r > 0$, let us define*

$$g_r : y \mapsto \int_V \omega_r\big(\varphi(z) - y\big)\, dz, \quad y \in \varphi(V).$$

If $K \subset \varphi(V)$ is a compact set, then

$$\lim_{r \to 0+} g_r(y) = 1/|J_\varphi\big(\varphi^{-1}(y)\big)|, \quad y \in K. \tag{5}$$

Proof. Let us fix a ball B centered at 0 and containing $S(\omega)$. Since $\varphi^{-1}(K)$ is compact, there exists $\rho > 0$ such that $\varphi^{-1}(K) + \rho B \subset V$. Then, for every $r \in (0, \rho)$ and every $y \in K$,

$$V_r(y) := \frac{1}{r}\big(V - \varphi^{-1}(y)\big) \supset B \supset S(\omega).$$

An affine change of variables (cf. (3)) yields

$$g_r(y) = \int_{S(\omega)} \omega\Big(\frac{1}{r}\big(\varphi(\varphi^{-1}(y) + rt) - \varphi(\varphi^{-1}(y))\big)\Big)\, dt, \quad y \in K,\ r \in (0, \rho).$$

(Here we replaced the integration over $V_r(y)$ by integration over $S(\omega)$. For later use, let us notice that $\{g_r : r \in (0, \rho)\}$ is a uniformly bounded family of continuous functions on K.) By Lebesgue's Dominated Convergence Theorem,

$$\lim_{r \to 0+} g_r(y) = \int_{S(\omega)} \omega\big(\varphi'(\varphi^{-1}(y))(t)\big)\, dt,$$

which, in view of (3), yields (5). $\square$

[2]Sometimes we write, for instance, $\int_A g(y)\, dy$ instead of $\int_A g\, d\lambda_d$.

Now we shall prove the equality (1). Let $h \in C_c\big(\varphi(V)\big)$ and $K := S(h)$. We may suppose that h is positive. For $r > 0$, let us define

$$I_r := \int_{V \times \varphi(V)} h(y) \left| J_\varphi\big(\varphi^{-1}(y)\big)\right| \omega_r\big(\varphi(z) - y\big)\, dz\, dy$$

(integration with respect to the product measure $\lambda_d \times \lambda_d$). By the Fubini theorem,

$$I_r = \int_K h(y) \big| J_\varphi\big(\varphi^{-1}(y)\big)\big| g_r(y)\, dy.$$

Obviously, $\{h|J_\varphi \circ \varphi^{-1}|\, g_r : r \in (0, \rho)\}$ is a uniformly bounded family of continuous functions on K. Applying Lebesgue's Dominated Convergence Theorem and using (5) we see that

$$\lim_{r \to 0+} I_r = \int_K h(y)\, dy = \int_{\varphi(V)} h\, d\lambda_d. \tag{6}$$

Let us define $f := h|J_\varphi \circ \varphi^{-1}|$ on $\varphi(V)$ and $f = 0$ elsewhere on $\mathbb{R}^d$. Then $f \in C_c(\mathbb{R}^d)$ and there exist $\xi > 0$ and a compact set $L \subset \varphi(V)$ such that $S(f * \omega_r) \subset L$ for every $r \in (0, \xi)$. The Fubini theorem yields

$$I_r = \int_{\varphi^{-1}(L)} (f * \omega_r)\big(\varphi(z)\big)\, dz, \;\; r \in (0, \xi).$$

By (4) and Lebesgue's Dominated Convergence Theorem it follows that

$$\lim_{r \to 0+} I_r = \int_{\varphi^{-1}(L)} f\big(\varphi(z)\big)\, dz = \int_V f\big(\varphi(z)\big)\, dz = \int_V (h \circ \varphi)|J_\varphi|\, d\lambda_d, \tag{7}$$

since $S(f) \subset L$. Now (1) follows from (6) and (7) and this finishes the proof.

Comments.

1. The proof of the integral calculus version of the change-of-variables formula is based on smearing of the value of $f\big(\varphi(z)\big)$ on small neighbourhoods. I learned this approach from Professor A. Cornea[3] some twenty years ago; cf. [3]. It seems that this method of proof does not appear in textbooks on integration and, in my opinion, deserves to be better known. Cornea's proof provides the *inequality* $\leq$ in (1), which, of course, is sufficient in view of the symmetry of φ and φ^{-1}. In our proof we establish the *equality* (6) instead of the inequality

$$\int_{\varphi(V)} h\, d\lambda_d \leq \liminf_{r \to 0+} I_r$$

obtained by Fatou's Lemma.

[3]Aurel Cornea (1933–2005) was born in Romania. At the age of 14 he had an accident during a chemical experiment and lost his eyesight. He studied mathematics and completed his Ph.D. thesis under S. Stoilow. He worked at the University of Bucharest and the Academy of Sciences. In 1978 he left Romania. After short stays in Canada and USA, he spent the rest of his life in Germany. In 1980, he was appointed as a professor at the Katholische Universität Eichstätt. Aurel Cornea was a distinguished specialist in potential theory, an excellent scientist, and an exceptional man. For further information, see [32].

2. Clearly, (1) holds for much more general functions h. To see this, one can first deduce from (1) the equality

$$\lambda_d(\varphi(A)) = \int_A |J_\varphi| \, d\lambda_d \tag{8}$$

for every Lebesgue measurable set $A \subset V$. (In particular, the mapping φ has the N-property, which means that the image of every set of zero measure is also of zero measure.) The equality (8) can be shown, for instance, using a Lusin's Theorem type of argument; see Corollary to Theorem 2.24 in [26]. Then integration with respect to the image measure shows that *the integral of a function h over $\varphi(V)$ exists if and only if the integral of $(h \circ \varphi)|J_\varphi|$ over V exists, and we have the equality* (1).

3. The equality (2) is usually proved using a factorization of the linear mapping and the multiplicative property for determinants; see, for instance, [5], [6], [9], [10], [18], [21], [26]. Group theoretical arguments are used in [4]. An approach based on Fubini's theorem is employed in [29].

4. The calculus version of the change-of-variables formula has a long history and is connected with names such as L. Euler, J.-L. Lagrange, S. Laplace, C. F. Gauss, M. Ostrogradski, C. Jacobi and others; see [16]. For various methods of proofs one may consult [2], [3], [9], [10], [15], [18], [19], [20], [21], [27], [28], [29].

5. Of course, the conditions imposed on φ may be substantially weakened in various respects. This issue is discussed in numerous textbooks as well as articles. Let us mention at least some references: [12], [14], [24], [25], [26], [31].

6. It turns out that the change-of-variables formula is a very special case of the so-called *area theorem*, which has been extensively studied in various degrees of generality in the setting of geometric measure theory. As a sample, we list the following books and papers: [1], [7], [8], [10], [11], [13], [17], [22], [23], [30], [33].

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