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Differential Inequalities Implying Starlikeness and Convexity

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Abstract

We study a differential inequality involving a multiplier transformation and consequently get some sufficient conditions in terms of certain simple differential inequalities for normalized analytic functions to be starlike and convex of order β , $0 \leq \beta < 1$.

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1 Introduction

Let \mathcal{H} be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, let $\mathcal{H}[a, n]$ be the subclass of \mathcal{H} consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let \mathcal{A} be the class of all functions f which are analytic in \mathbb{E} and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Denote by $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$, the classes of starlike functions of order β and convex functions of order β respectively, which are analytically defined as under:

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} : \Re \left(\frac{z f'(z)}{f(z)} \right) > \beta, 0 \leq \beta < 1, z \in \mathbb{E} \right\}$$

and

$$\mathcal{K}(\beta) = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > \beta, 0 \leq \beta < 1, z \in \mathbb{E} \right\}.$$

We shall use \mathcal{S}^* and \mathcal{K} to denote $\mathcal{S}^*(0)$ and $\mathcal{K}(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

Let \mathcal{A}_p denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, $p \in \mathbb{N}$, which are analytic and multivalent in the open unit disk \mathbb{E} . Note $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define

the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n, \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda} \right)^n a_k z^k, \quad (\lambda \geq 0, n \in \mathbb{Z}).$$

The operator $I_p(n, \lambda)$ has been recently studied by Aghalary et al. [1]. Earlier, the operator $I_1(n, \lambda)$ was investigated by Cho and Kim [2] and Cho and Srivastava [3], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [10]. $I_1(n, 0)$ is the well-known Sălăgean [9] derivative operator D^n , defined as:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

where $f \in \mathcal{A}$. In 1989, the operator $I_1(n, 0)$ has been studied by Owa, Shen and Obradović [7]. Recently, Li and Owa [5] studied the operator $I_1(n, 0)$.

For two analytic functions f and g in the unit disk \mathbb{E} , we say that a function f is subordinate to a function g in \mathbb{E} and write $f \prec g$ if there exists a Schwarz function w analytic in \mathbb{E} with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{E}$ such that $f(z) = g(w(z))$, $z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

In the present paper, we study the differential inequality defined by the multiplier transformation $I_p(n, \lambda)$ in the open unit disk \mathbb{E} . As special cases to our main result, we obtain starlikeness and convexity of members of the class \mathcal{A} in terms of certain simple differential inequalities. To prove our main result, we shall make use of following lemma of Hallenbeck and Ruscheweyh [4].

Lemma 1. *Let G be a convex function in \mathbb{E} , with $G(0) = a$ and let γ be a complex number, with $\Re(\gamma) > 0$. If $F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, is analytic in \mathbb{E} and $F \prec G$, then*

$$\frac{1}{z^\gamma} \int_0^z F(w) w^{\gamma-1} dw = \frac{1}{nz^{\gamma/n}} \int_0^z G(w) w^{\frac{\gamma}{n}-1} dw$$

2 Main Theorem

Theorem 2. *Let α, β be real numbers such that $\alpha > \frac{2}{1-\beta}$, $0 \leq \beta < 1$ and let*

$$0 < M \equiv M(\alpha, \beta, \gamma, p) = \frac{(\alpha + p + \lambda)[\alpha(1 - \beta) - 2]}{\alpha[1 + (1 - \beta)(p + \lambda)]}, \quad (2.1)$$

If $f \in \mathcal{A}_p$ satisfies the differential inequality

$$\left| (1 - \alpha) \frac{I_p(n, \lambda)f(z)}{z^p} + \alpha \frac{I_p(n+1, \lambda)f(z)}{z^p} - 1 \right| < M(\alpha, \beta, \lambda, p), \quad z \in \mathbb{E}, \quad (2.2)$$

then

$$\Re \left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right) > \beta, \quad z \in \mathbb{E}.$$

Proof. Let us define

$$\frac{I_p(n, \lambda)f(z)}{z^p} = u(z), \quad z \in \mathbb{E}.$$

Differentiate logarithmically, we obtain

$$\frac{zI'_p(n, \lambda)f(z)}{I_p(n, \lambda)f(z)} - p = \frac{zu'(z)}{u(z)} \quad (2.3)$$

In view of the equality

$$zI'_p(n, \lambda)f(z) = (p + \lambda)I_p(n + 1, \lambda)f(z) - \lambda I_p(n, \lambda)f(z),$$

(2.3) reduces to

$$\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} = 1 + \frac{zu'(z)}{(p + \lambda)u(z)}$$

or

$$\frac{1}{p + \lambda}zu'(z) = \frac{I_p(n + 1, \lambda)f(z)}{z^p} - \frac{I_p(n, \lambda)f(z)}{z^p}.$$

Therefore, in view of (2.2), we have

$$u(z) + \frac{\alpha}{p + \lambda}zu'(z) \prec 1 + Mz. \quad (2.4)$$

In view of Lemma 1 (selecting $\gamma = \frac{p + \lambda}{\alpha}$) from (2.4), we have

$$u(z) \prec 1 + \frac{(p + \lambda)Mz}{\alpha + p + \lambda},$$

or

$$|u(z) - 1| < \frac{(p + \lambda)M}{\alpha + p + \lambda} < 1,$$

therefore, we obtain

$$|u(z)| > 1 - \frac{(p + \lambda)M}{\alpha + p + \lambda} \quad (2.5)$$

Write $\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} = (1 - \beta)w(z) + \beta$, $0 \leq \beta < 1$ and therefore

$$\frac{I_p(n + 1, \lambda)f(z)}{z^p} = u(z)[(1 - \beta)w(z) + \beta].$$

Therefore (2.2) reduces to

$$|(1 - \alpha)u(z) + \alpha u(z)[(1 - \beta)w(z) + \beta] - 1| < M.$$

We need to show that $\Re(w(z)) > 0$, $z \in \mathbb{E}$. If possible, suppose that $\Re(w(z)) \not> 0$, $z \in \mathbb{E}$, then there must exist a point $z_0 \in \mathbb{E}$ such that $w(z_0) = ix$, $x \in \mathbb{R}$. To prove the required result, it is now sufficient to prove that

$$|(1 - \alpha)u(z_0) + \alpha u(z_0)[(1 - \beta)ix + \beta] - 1| \geq M. \quad (2.6)$$

By making use of (2.5), we have

$$|(1 - \alpha)u(z_0) + \alpha u(z_0)[(1 - \beta)ix + \beta] - 1|$$

$$\begin{aligned}
&\geq |[1 - \alpha(1 - \beta) + \alpha(1 - \beta)ix] u(z_0)| - 1 \\
&= \sqrt{[1 - \alpha(1 - \beta)]^2 + \alpha^2(1 - \beta)^2 x^2} |u(z_0)| - 1 \\
&\geq |1 - \alpha(1 - \beta)| |u(z_0)| - 1 \\
&\geq |1 - \alpha(1 - \beta)| \left(1 - \frac{(p + \lambda)M}{\alpha + p + \lambda}\right) - 1 \geq M.
\end{aligned} \tag{2.7}$$

Now (2.7) is true in view of (2.1) and therefore, (2.6) holds. Hence $\Re(w(z)) > 0$ or

$$\Re \left[\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right] > \beta, \quad 0 \leq \beta < 1, \quad z \in \mathbb{E}.$$

□

3 Deductions

On writing $p = 1$ and $\lambda = 0$ in Theorem 2, we obtain:

Corollary 3. *Let α, β be real numbers such that $\alpha > \frac{2}{1-\beta}$, $0 \leq \beta < 1$ and let $f \in \mathcal{A}$ satisfy the differential inequality*

$$\left| (1 - \alpha) \frac{D^n f(z)}{z} + \alpha \frac{D^{n+1} f(z)}{z} - 1 \right| < \frac{(1 + \alpha)[\alpha(1 - \beta) - 2]}{\alpha(2 - \beta)}, \quad z \in \mathbb{E},$$

then

$$\Re \left(\frac{D^{n+1} f(z)}{D^n f(z)} \right) > \beta, \quad 0 \leq \beta < 1, \quad z \in \mathbb{E}.$$

Taking $p = 1$, $n = 0$ and $\lambda = 0$ in Theorem 2, we have the following result.

Corollary 4. *If α, β are real numbers such that $\alpha > \frac{2}{1-\beta}$, $0 \leq \beta < 1$ and $f \in \mathcal{A}$ satisfies*

$$\left| (1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) - 1 \right| < \frac{(1 + \alpha)[\alpha(1 - \beta) - 2]}{\alpha(2 - \beta)}, \quad z \in \mathbb{E},$$

then $f \in \mathcal{S}^*(\beta)$.

Setting $p = n = 1$ and $\lambda = 0$ in Theorem 2, we obtain:

Corollary 5. *Let α, β be real numbers such that $\alpha > \frac{2}{1-\beta}$, $0 \leq \beta < 1$ and let $f \in \mathcal{A}$ satisfy*

$$|f'(z) + \alpha z f''(z) - 1| < \frac{(1 + \alpha)[\alpha(1 - \beta) - 2]}{\alpha(2 - \beta)}, \quad z \in \mathbb{E},$$

then $f \in \mathcal{K}(\beta)$.

Write $p = 1$, $n = 0$ and $\lambda = 1$ in Theorem 2, to get the following result.

Corollary 6. *Let α, β be real numbers such that $\alpha > \frac{2}{1-\beta}$, $0 \leq \beta < 1$ and let $f \in \mathcal{A}$ satisfy*

$$\left| \left(1 - \frac{\alpha}{2}\right) \frac{f(z)}{z} + \frac{\alpha}{2} f'(z) - 1 \right| < \frac{(2 + \alpha)[\alpha(1 - \beta) - 2]}{\alpha(3 - 2\beta)}, \quad z \in \mathbb{E},$$

then

$$\Re \left(1 + \frac{z f'(z)}{f(z)} \right) > \beta, \quad z \in \mathbb{E}.$$

Remark 7. From Theorem 2, it follows, if $\alpha > \frac{2}{1-\beta}$, $0 \leq \beta < 1$ and $f \in \mathcal{A}_p$ satisfies

$$\left| \left(\frac{1}{\alpha} - 1 \right) \frac{I_p(n, \lambda)f(z)}{z^p} + \frac{I_p(n+1, \lambda)f(z)}{z^p} - \frac{1}{\alpha} \right| < \frac{(\alpha + p + \lambda)[\alpha(1-\beta) - 2]}{\alpha^2[1 + (1-\beta)(p + \lambda)]}, \quad z \in \mathbb{E},$$

then

$$\Re \left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right) > \beta, \quad z \in \mathbb{E}.$$

Letting $\alpha \rightarrow \infty$ in Remark 7, we have the following result.

Corollary 8. Let β ($0 \leq \beta < 1$) be a real number and let $f \in \mathcal{A}_p$ satisfy

$$\left| \frac{I_p(n+1, \lambda)f(z)}{z^p} - \frac{I_p(n, \lambda)f(z)}{z^p} \right| < \frac{1-\beta}{1 + (1-\beta)(p + \lambda)}, \quad z \in \mathbb{E},$$

then

$$\Re \left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right) > \beta, \quad z \in \mathbb{E}.$$

Setting $p = 1$ and $\lambda = 0$ in Corollary 8, we get:

Corollary 9. If $f \in \mathcal{A}$ satisfies

$$\left| \frac{D^{n+1}f(z)}{z} - \frac{D^n f(z)}{z} \right| < \frac{1-\beta}{2-\beta}, \quad z \in \mathbb{E},$$

then

$$\Re \left(\frac{D^{n+1}f(z)}{D^n f(z)} \right) > \beta, \quad 0 \leq \beta < 1, \quad z \in \mathbb{E}.$$

Writing $p = 1$ and $n = \lambda = 0$ in Corollary 8, we obtain the following result of Oros [6].

Corollary 10. If $f \in \mathcal{A}$ satisfies

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{1-\beta}{2-\beta}, \quad z \in \mathbb{E},$$

then $f \in \mathcal{S}^*(\beta)$, $0 \leq \beta < 1$.

Taking $p = n = 1$ and $\lambda = 0$ in Corollary 8, we get:

Corollary 11. If $f \in \mathcal{A}$ satisfies

$$|f''(z)| < \frac{1-\beta}{2-\beta}, \quad z \in \mathbb{E},$$

then $f \in \mathcal{K}(\beta)$, $0 \leq \beta < 1$.

Note that for $\beta = 0$, the above result was obtained by Mocanu [8].

Setting $p = \lambda = 1$ and $n = 0$ in Corollary 8, we obtain the following result.

Corollary 12. If $f \in \mathcal{A}$ satisfies

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{2(1-\beta)}{3-2\beta}, \quad 0 \leq \beta < 1, \quad z \in \mathbb{E},$$

then

$$\Re \left(1 + \frac{zf'(z)}{f(z)} \right) > \beta, \quad z \in \mathbb{E}.$$

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On cycles in graphs with specified radius and diameter

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Abstract

Let G be a graph of radius r and diameter d with $d \leq 2r - 2$. We show that G contains a cycle of length at least $4r - 2d$, i.e. for its circumference it holds $c(G) \geq 4r - 2d$. Moreover, for all positive integers r and d with $r \leq d \leq 2r - 2$ there exists a graph of radius r and diameter d with circumference $4r - 2d$.

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For a connected graph G , the *distance* $d_G(u, v)$ or briefly $d(u, v)$ between a pair of vertices u and v is the length of a shortest path joining them. The distance between a vertex $u \in V(G)$ and a subgraph H of G will be denoted by $d(u, H)$, i.e. $d(u, H) = \min\{d(u, v); v \in V(H)\}$. The *eccentricity* $e_G(u)$ (briefly $e(u)$) of a vertex u of G is the distance of u to a vertex farthest from u in G , i.e. $e_G(u) = \max\{d_G(u, v); v \in V(G)\}$. The *radius* $\text{rad } G$ of G is the minimum eccentricity among the vertices of G while the *diameter* $\text{diam } G$ of G is the maximum eccentricity. The *circumference* of a graph G , denoted $c(G)$, is the length of any longest cycle in G .

A path P (a cycle C) in G is called *geodesic* if for any two vertices of P (of C) their distance in P (in C) equals their distance in G . A nontrivial connected graph with no cut-vertices is called a *nonseparable graph*. A *block* of a graph G is a maximal nonseparable subgraph of G .

A connected unicyclic graph G with the cycle C is called a *sun-graph* (see [2]) if $\deg_G(u) \leq 3$ for $u \in V(C)$ and $\deg_G(u) \leq 2$ for $u \in V(G) \setminus V(C)$. A $u - v$ path P in a sun-graph G is called a *ray* if $V(P) \cap V(C) = \{u\}$ and $\deg_G(v) = 1$. A sun-graph with the cycle C_m of length m and with m rays of length k will be denoted by $S_{m,k}$.

In what follows we answer a question that was posed several decades ago in [3]:

“How large a cycle must there be in a graph of radius m and diameter n ? This question is also open. For radius 3 and diameter 4, the graph must have a cycle of length at least 4, which can be verified by brute force techniques The situation in general is unclear.”

Our main result is the following theorem. Note that in the case $d = 2r - 1$ or $d = 2r$ there are trees with radius r and diameter d .

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Theorem 1. *Let G be a graph of radius r and diameter d with $d \leq 2r - 2$. Then $c(G) \geq 4r - 2d$.*

Proof. Since $d \leq 2r - 2$, G is not a tree. Let C be a cycle of G and B be a block of G containing C . Suppose, contrary to our claim, that $c(G) < 4r - 2d$. Since B is a nonseparable subgraph of G , every two vertices of B lie on a common cycle (see [1, Theorem 1.6]) of length less than $4r - 2d$. Hence we get $\text{diam } B \leq 2r - d - 1 \leq r - 1$ (and so $B \neq G$).

Let u be a vertex such that $d(u, B) = \max\{d(v, B); v \in V(G)\}$ and let $u_B \in V(B)$ be a vertex with $d(u, u_B) = d(u, B)$. Evidently, u_B is a cut-vertex of G .

Let G_1 be a component of $G - u_B$ containing the vertex u . Put $d(u, u_B) = a$. We distinguish two cases.

(1) $a \leq r - 1$

Let v be a vertex of G . If $v \in V(G_1)$ then $d(v, u_B) \leq a \leq r - 1$. If $v \in V(B)$ then $d(v, u_B) \leq \text{diam } B \leq r - 1$. Let, finally, $v \in V(G) \setminus (V(G_1) \cup V(B))$ and $v_B \in V(B)$ be a vertex such that $d(v, v_B) = d(v, B)$. Evidently, v_B is a cut-vertex of G and $d_G(u_B, v_B) = d_B(u_B, v_B)$. Denote $d(v, v_B) = b$ and $d(u_B, v_B) = c$. Suppose first that $b + c \geq r$. We have $c \leq \text{diam } B \leq 2r - d - 1$ and $b \leq a$. Then $d(u, v) = d(u, u_B) + d(u_B, v_B) + d(v_B, v) = a + c + b \geq 2b + c \geq 2(r - c) + c \geq 2r - (2r - d - 1) = d + 1$. Since $\text{diam } G = d$, we get $b + c \leq r - 1$ and so $d(u_B, v) \leq r - 1$. Finally, we have $e(u_B) \leq r - 1$, a contradiction.

(2) $a \geq r$

Let u_1 be a vertex of a geodesic $u - u_B$ path P^1 with $d(u, u_1) = r - 1$. If w is a vertex from $V(G) \setminus V(G_1)$ then u_B is on a geodesic $w - u_1$ path and we get $d(w, u_1) \leq r - 1$ (since $d(w, u) \leq 2r - 2$). Since $e(u_1) \geq r$ (otherwise we have a contradiction), there is a vertex $v \in V(G_1)$ such that $d_{G_1}(v, u_1) = d_G(v, u_1) \geq r$. Let P^2 be a geodesic $v - u_B$ path and let v_1 be the first vertex of P^2 which is on P^1 . Since $d(v, u_B) \leq d(u, u_B)$, we get $d(u_B, v_1) < d(u_B, u_1)$. Let P^3 be a geodesic $v - u$ path and let v_2 be the first of its vertices which is on P^1 . It is obvious (since $d(v, u) \leq 2r - 2$) that $d(u_B, v_2) > d(u_B, u_1)$. Evidently, there is a cycle C' such that $\{v_1, v_2\} \subseteq V(C')$.

Let G_2 be a subgraph of G induced by the set $V(G_1) \cup \{u_B\}$. Let $w \in V(G) \setminus V(G_2)$ be such a vertex that $d(w, u_B) = \max\{d(x, u_B); x \in V(G) \setminus V(G_2)\}$ and P be a geodesic $w - u_B$ path. Consider a graph G' for which $V(G') = V(G_2) \cup V(P)$ and $E(G') = E(G_2) \cup E(P)$. It is obvious that $|V(G')| < |V(G)|$ and if there is a vertex $z \in V(G')$ with $e_{G'}(z) \leq r - 1$ then $e_G(z) \leq r - 1$, too.

We can repeat the previous considerations with the graph G' and its block B' containing the cycle C' . It is clear now that after a finite number of the described steps we get a contradiction. \square

Corollary 2. *If G is a graph with $\text{rad } G = r$ and $\text{diam } G \leq 2r - 2$, then G contains a cycle of length at least 4, i.e. $c(G) \geq 4$.*

Corollary 3. *If $c(G) = 3$ and $\text{rad } G = r$, then $\text{diam } G \in \{2r - 1, 2r\}$.*

For all positive integers r and d satisfying $r \leq d \leq 2r - 2$ there exists an infinite number of graphs of radius r , diameter d and circumference $4r - 2d$. One of these graphs is C_{2r} for $d = r$. If $d > r$, one of these graphs is $S_{4r-2d, d-r}$, i.e. a sun-graph with the cycle C_{4r-2d} and with $4r - 2d$ rays of length $d - r$ (see Figure 1a for $r = 3$, $d = 4$ and Figure 1b for $r = 5$, $d = 7$).

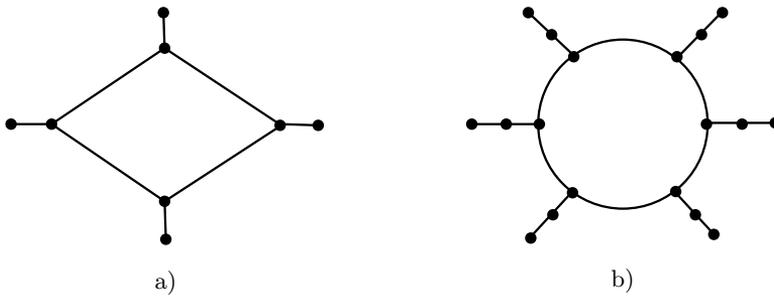


Figure 1

Now it is a simple matter to find infinite classes of graphs with mentioned properties (see Figure 2 for an inspiration).

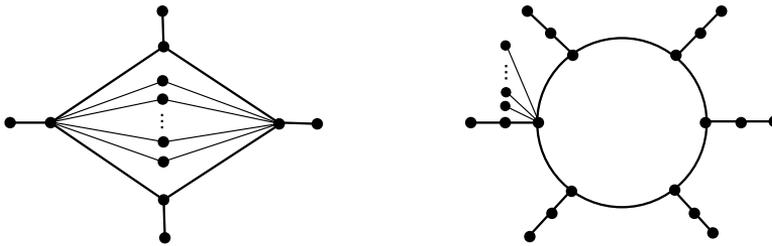


Figure 2

It is known that if a graph G with radius r and diameter $d \leq 2r - 2$ has at most $3r - 2$ vertices, then it holds $c(G) \geq 2r$. This fact is a consequence of the following theorem (see [2]).

Theorem 4 ([2]). *Let G be a graph with $\text{rad } G = r$, $\text{diam } G \leq 2r - 2$, on at most $3r - 2$ vertices. Then G contains a geodesic cycle of length $2r$ or $2r + 1$.*

Using Theorem 4 it is easy to find all nonisomorphic graphs of minimal order and specified radius and diameter (see [3],[2]).

Let G be a sun-graph with the cycle C_{2r-1} ($r \geq 3$), with r rays of length 1 and such that exactly two of its end-vertices have distance 3 (see Figure 3 for $r = 5$). It is easy to see that $\text{rad } G = r$, $|V(G)| = 3r - 1$ and $c(G) = 2r - 1$.

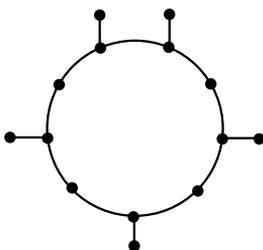


Figure 3

We can conclude that the bound $3r - 2$ in Theorem 4 is the best possible (for $r \geq 3$).

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Classification of regular maps whose automorphism groups are 2-groups of maximal class

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Abstract

It is proved by Malnič, Nedela and Škoviera [Regular maps with nilpotent automorphism groups, *European J. Combin.* **33** (2012), no. 8, 1974–1986] that regular maps with nilpotent automorphism groups can be decomposed into a direct product of two regular maps, a regular map whose automorphism group is a 2-group and a semistar of odd valency. This reduction theorem motivates the classification of regular maps whose automorphism groups are 2-groups. In this paper, we classify regular maps whose automorphism groups are 2-groups of maximal nilpotency class.

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1 Introduction

A map is a 2-cell embedding of a connected graph into a closed surface. A map on an orientable surface is regular if its group of orientation preserving automorphisms acts regularly on its darts. The best known examples of regular maps are the Platonic solids, which together with the dihedra and hosohedra give a full list of regular maps on the sphere. Thorough investigation was not carried out until in the 19th century the connection between modern topology, group theory, graph theory and the theory of complex functions was developed [13, 6]. Modern foundations of the theory of maps on orientable surfaces were built by Jones and Singerman [10], Gross and Tucker [7] and others. There are three different approaches to the classification of regular maps on orientable surfaces:

- (1) classification of regular maps with prescribed supporting surfaces;
- (2) classification of regular maps with prescribed automorphism groups;
- (3) classification of regular maps with prescribed underlying graphs.

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In the second direction, the classification of regular maps with automorphism group isomorphic to the group $\mathrm{PSL}(2, q)$, where q is a power of some prime p , was in principle done by Macbeath [14]. A finite group G is called a *Hurwitz group* if it is a finite nontrivial quotient of the infinite group

$$\langle x, y | x^3 = y^2 = (xy)^7 = 1 \rangle.$$

It is well-known that Hurwitz groups are corresponding to the automorphism groups of some Riemann surface with highest symmetry [9]. It is also obvious that every Hurwitz group gives rise to a regular map. Determination of the simple groups which are Hurwitz groups is an important problem. In this context, the symmetric and alternating groups [4], Suzuki groups [11], Ree groups [12], and various sporadic simple groups [21] have been investigated. A survey can be found in [1, 3]. In present paper, we consider the following problem:

Problem 1. *Determine and classify regular maps whose automorphism groups are nilpotent.*

This problem was studied by Malnič, Nedela and Škovič in [16], where the authors have proved that each such map can be decomposed into a direct product of two regular maps, a regular map whose automorphism group is a 2-group and a semistar of odd valency. Therefore, the above problem is reduced into the following

Problem 2. *Determine and classify regular maps whose automorphism groups are 2-groups.*

If the nilpotency class is small enough, Problem 2 was resolved [16, 5]. For convenience, we define a *regular 2-map* to be a regular map whose automorphism group is a 2-group. If the automorphism group of a regular 2-map has class c , we will refer to the map as a *regular 2-map of class c* . Regular 2-maps of class 1 and 2 have been classified in the aforementioned paper [16]. Regular 2-maps of class 3 have been recently classified by Du et al [5]. In present paper, we will classify regular 2-maps of maximal class.

2 Preliminaries

As mentioned before, a *topological map* \mathcal{M} is a 2-cell embedding $i : X \hookrightarrow S$ of a connected graph X into a closed surface S such that each component of $S - i(X)$ is homeomorphic to an open disc in \mathbb{C} . The vertices, edges of \mathcal{M} are inherited from the embedded graph X , whereas the faces are the components of $S - i(X)$. If the supporting surface S is orientable, the map \mathcal{M} is also called *orientable*, otherwise, \mathcal{M} is called *non-orientable*. As usual, we define a (*combinatorial oriented*) *map* to be a triple $(D; R, L)$, where D is a nonempty finite set whose elements are called *darts*, and R and L are two permutations on D such that $L^2 = 1$ and the group generated by R and L acts transitively on D . Here the permutation R describes the local orientation of darts around the vertices of \mathcal{M} and is called a *rotation*, whereas L inverts each pair of darts with the same underlying edge of \mathcal{M} . The group generated by R and L is called the *monodromy group* of \mathcal{M} and is denoted by $\mathrm{Mon}(\mathcal{M})$.

A *homomorphism* from a map $\mathcal{M}_1 = (D_1; R_1, L_1)$ to a map $\mathcal{M}_2 = (D_2; R_2, L_2)$ is a mapping $\phi : D_1 \rightarrow D_2$ such that

$$\phi R_1 = R_2 \phi \quad \text{and} \quad \phi L_1 = L_2 \phi.$$

The mapping ϕ is necessarily surjective due to the transitivity of monodromy groups. If ϕ is a bijection, then we say \mathcal{M}_1 is *isomorphic* to \mathcal{M}_2 and denote it by $\mathcal{M}_1 \cong \mathcal{M}_2$.

An isomorphism of a map \mathcal{M} to itself is called an *automorphism* of \mathcal{M} . The set of all automorphisms of \mathcal{M} forms the *automorphism group* of \mathcal{M} under the composition operation and is denoted by $\text{Aut}(\mathcal{M})$.

It follows from a classical result in the theory of maps that the action of $\text{Aut}(\mathcal{M})$ on D is semi-regular (namely, the stabiliser of $\text{Aut}(\mathcal{M})$ is trivial) [10]. If this action is transitive and hence regular, then \mathcal{M} is called a *regular map* as well. Therefore, regular maps exhibit highest possible symmetry.

In a regular map $\mathcal{M} = (D; R, L)$ with $G = \text{Aut}(\mathcal{M})$, we can identify D with G and regard the actions of $\text{Aut}(\mathcal{M})$ and $\text{Mon}(\mathcal{M})$ on D as the right and left multiplication by the elements of G , respectively. More precisely, if $G = \langle x, y \rangle$, $y^2 = 1$, we denote by (G, x, y) the regular map $(D; R, L)$ defined by setting $D = G$, $Rg = xg$, $Lg = yg$ for any $g \in G$. For a map $\mathcal{M} = (D; R, L)$, there are two associated maps: the *dual* $\mathcal{M}^* = (D; RL, L)$ and the *mirror map* $\mathcal{M}^{-1} = (D; R^{-1}, L)$ of \mathcal{M} . A map \mathcal{M} is called *self-dual* if $\mathcal{M} \cong \mathcal{M}^*$; a map \mathcal{M} is called *reflexible* if $\mathcal{M} \cong \mathcal{M}^{-1}$. Those maps which are not reflexible are called *chiral*.

To answer Problem 2 for a given finite 2-group G , we have first to see whether G is the automorphism group of a regular map \mathcal{M} , which is equivalent to decide whether G is generated by two elements, say x and y , and $y^2 = 1$. If this is the case, then such a group G will be called *admissible* and the associated generating pair (x, y) will be called *admissible* as well. Secondly, according to [18], for an admissible group G , two admissible generating pairs (x_i, y_i) ($i = 1, 2$) of G are called *equivalent* if there is an automorphism ϕ of G such that

$$x_1^\phi = x_2 \quad \text{and} \quad y_1^\phi = y_2. \quad (2.1)$$

The isomorphism classes of regular maps \mathcal{M} with $\text{Aut}(\mathcal{M}) = G$ are therefore in a one-to-one correspondence with the orbits of admissible generating pairs of G under the action of $\text{Aut}(G)$.

By definition, the automorphism groups of regular 2-maps of maximal class are 2-groups of maximal class. The classification of 2-groups of maximal class is known as Taussky's Theorem in group theory. We rephrase it as follows.

Proposition 3. [20][Taussky's Theorem] *Let G be a finite 2-group of maximal nilpotency class with $|G| = 2^n$, $n \geq 3$. Then, up to isomorphism, G belongs to one and only one of the following three classes of groups:*

1. *generalized quaternion groups defined by*

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle, n \geq 3; \quad (2.2)$$

2. *dihedral groups defined by*

$$D_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1} \rangle, n \geq 3; \quad (2.3)$$

3. *semidihedral groups defined by*

$$SD_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1+2^{n-2}} \rangle, n \geq 4. \quad (2.4)$$

3 Regular 2-maps of maximal class

In this section, we apply Taussky's Theorem to give a classification of regular 2-maps of maximal class. First, we define several families of regular maps. The *semistar* is defined to be a map $\mathcal{S}_n = (C_n, x, y)$ with

$$C_n = \langle x, y | x^n = y = 1 \rangle. \quad (3.1)$$

The *cycle of length n* is a map $\mathcal{C}_n = (D_{2n}, x, y)$ with

$$D_{2n} = \langle x, y | x^2 = y^2 = (xy)^n = 1 \rangle. \quad (3.2)$$

Moreover, the regular embeddings of *n-dipoles* are the maps $\mathcal{D}(n, e) = (G, x, y)$ with

$$G = \langle x, y | x^n = y^2 = 1, y^{-1}xy = x^e \rangle, \quad (3.3)$$

where $e^2 \equiv 1 \pmod{n}$ (Theorem 9.1 [17]). Note that $\mathcal{D}^*(n, -1) = \mathcal{C}_n$.

Moreover, by replacing each edge of a cycle C_n by m parallel edges incident to the same vertices, we get a *multicycle* $C_n^{(m)}$. The regular embeddings of multicycles $C_{2n}^{(2m)}$ are maps $\mathcal{C}(2n, 2m, e, f) = (G, x, y)$ with

$$G = \langle x, y | x^{4m} = y^2 = 1, y^{-1}x^2y = x^{2e}, (xy)^{2n} = x^{2f} \rangle, \quad (3.4)$$

where $e^2 \equiv 1 \pmod{2m}$ and $f \equiv (e+1)n/4 \pmod{2m}$ or $f \equiv ((e+1)n+4m)/4 \pmod{2m}$ (Theorem 2 [8]).

To present our classification of regular 2-maps of maximal class, we need to identify admissible 2-groups from the groups given in Proposition 3.

Lemma 4. *In Proposition 3, only the dihedral groups and semidihedral groups are admissible. Moreover, the admissible generating pairs of the dihedral groups have form $(a^r b^k, a^j b)$, where either $k = 0$, r is odd, or $k = 1$, $r - j$ is odd; the admissible generating pairs of the semidihedral groups have form $(a^r b^k, a^j b)$, where $k \in \{0, 1\}$, r is odd and j is even. Two admissible pairs $(a^r b^k, a^j b)$ and $(a^{r'} b^{k'}, a^j b)$ of D_{2^n} are equivalent if and only if $k = k'$ and two admissible pairs $(a^r b^k, a^j b)$ and $(a^{r'} b^{k'}, a^j b)$ of SD_{2^n} are equivalent if and only if $k = k'$.*

Proof. It is known that the generalised quaternion group Q_{2^n} has a unique involution $y = a^{2^{n-2}}$. Obviously, y is a central element. Since Q_{2^n} is non-abelian, Q_{2^n} has no admissible generating pairs. Furthermore, for both D_{2^n} and SD_{2^n} , it is clear the generators (a, b) given in Proposition 3 are admissible. To determine all admissible generating pairs (x, y) of D_{2^n} , we see that the involutions of D_{2^n} are either $a^{2^{n-2}}$ or of the form $a^j b$. But $a^{2^{n-2}} \in Z(D_{2^n})$ and D_{2^n} is non-abelian, which implies that $a^{2^{n-2}}$ can not be a generator in any admissible generating pair. On the other hand, let $y = a^j b$ and $x = a^r b^k$, where $0 \leq j, r \leq 2^{n-1} - 1$ and $0 \leq k \leq 1$. If $k = 0$, we show that $\langle x, y \rangle = D_{2^n}$ if and only if $(r, 2) = 1$. In fact, if $(r, 2) = 1$, then $\langle a^r \rangle = \langle a \rangle$. Hence,

$$\langle x, y \rangle = \langle a^r, a^j b \rangle = \langle a, a^j b \rangle = \langle a, b \rangle.$$

Consequently, $D_{2^n} = \langle x, y \rangle$. Conversely, assume $D_{2^n} = \langle x, y \rangle$, if r was an even number, let $m = o(a^r)$, then $m < o(a)$, we have

$$\langle x, y \rangle = \langle x, y | x^m = y^2 = 1, yxy = x^{-1} \rangle \cong D_{2m},$$

which implies that $\langle x, y \rangle$ is a proper subgroup of D_{2^n} , a contradiction. Similarly, if $k = 1$, we show that $\langle x, y \rangle = D_{2^n}$ if and only if $(r - j, 2) = 1$. In fact, since

$$\langle x, y \rangle = \langle a^r b, a^j b \rangle = \langle a^{r-j}, a^j b \rangle,$$

by the preceding case we have $\langle x, y \rangle = D_{2^n}$ if and only if $(r - j, 2) = 1$.

As concerns the admissible pairs of the semidihedral groups SD_{2^n} , we see the involutions of SD_{2^n} are either $a^{2^{n-2}}$ or of the form $a^j b$, where j is even. The former is a central element and hence can not be a generating involution in any admissible pair. Now let $y = a^j b$ and $x = a^r b^k$, where $0 \leq r, j \leq 2^{n-1} - 1$, j is even and $k \in \{0, 1\}$. If $k = 0$, then we show that $\langle x, y \rangle = SD_{2^n}$ if and only if r is odd. First assume r odd, then $\langle a \rangle = \langle a^r \rangle$. We have

$$\langle x, y \rangle = \langle a^r, a^j b \rangle = \langle a, a^j b \rangle = \langle a, b \rangle.$$

It follows that $\langle x, y \rangle = SD_{2^n}$. Conversely, assume $\langle x, y \rangle = SD_{2^n}$, if r was an even number, let $m = o(a^r)$, then $m < o(a)$. We have

$$\langle x, y | x^m = y^2 = 1, yxy = x^{-1} \rangle \cong D_{2m}.$$

Therefore, $|\langle x, y \rangle| < |SD_{2^n}|$, which implies that $\langle x, y \rangle$ is a proper subgroup of SD_{2^n} , a contradiction. Similarly, if $k = 1$, we have

$$\langle x, y \rangle = \langle a^r b, a^j b \rangle = \langle a^r a^{j(-1+2^{n-2})}, a^j b \rangle \stackrel{j \text{ even}}{=} \langle a^{r-j}, a^j b \rangle.$$

It follows the first case that $\langle x, y \rangle = SD_{2^n}$ if and only if $r - j$ is odd.

Finally, we decide the equivalence relationship of aforementioned admissible pairs. The admissible pair (x, y) of D_{2^n} has a defining relation

$$\langle x, y | x^{2^{n-1}} = y^2 = (xy)^2 = 1 \rangle,$$

if $x = a^r, y = a^j b$, where r is odd, or

$$\langle x, y | x^2 = y^2 = (xy)^{2^{n-1}} = 1 \rangle,$$

if $x = a^r b, y = a^j b$, where $r - j$ is odd. Therefore, two admissible pairs $(a^r b^k, a^j b)$ and $(a^{r'} b^{k'}, a^j b)$ of D_{2^n} are equivalent if and only if $k = k'$. Similarly, the admissible pair (x, y) of SD_{2^n} has a defining relation

$$\langle x, y | x^{2^{n-1}} = y^2 = (xy)^4 = 1, [x, y] = x^{-2+2^{n-2}} \rangle,$$

if $x = a^r, y = a^j b$, where r is odd, j is even, or

$$\langle x, y | x^4 = y^2 = (xy)^{2^{n-1}} = 1, [x, y] = (xy)^{2-2^{n-2}} \rangle,$$

if $x = a^r b, y = a^j b$, where r is odd, j is even. Therefore, two admissible pairs $(a^r b^k, a^j b)$ and $(a^{r'} b^{k'}, a^j b)$ of SD_{2^n} are equivalent if and only if $k = k'$, as claimed. \square

Now we are ready to formulate our classification theorem of regular 2-maps of maximal class.

Theorem 5. *Up to isomorphism, regular maps whose automorphism groups are non-abelian 2-groups of maximal nilpotency class are one of the following maps:*

- (1) a cycle $\mathcal{C}_{2^{n-1}}$ or its dual, a dipole $\mathcal{D}(2^{n-1}, -1)$, where $n \geq 3$, both of genus 0;
- (2) a dipole $\mathcal{D}(2^{n-1}, -1 + 2^{n-2})$ or its dual $\mathcal{C}(2^{n-2}, 2, 1, 1)$, the regular embedding of a multicycle, where $n \geq 4$, both of genus 2^{n-3} .

Moreover, the above maps are all reflexible.

Proof. Let \mathcal{M} be a regular 2-map of maximal class and $G = \text{Aut}(\mathcal{M})$, let $|G| = 2^n$. Since G is non-abelian, $n \geq 3$. By Taussky's Theorem, G is isomorphic to one of the groups in Proposition 3. By Lemma 4, the generalised quaternion group Q_{2^n} is not admissible, and every admissible pair (x, y) of D_{2^n} either has a form $x = a^r, y = a^j b$, where r is odd, or has a form $x = a^r b, y = a^j b$, where $r - j$ is odd. The former defines the map $\mathcal{D}(2^{n-1}, -1)$ and the latter defines the map $\mathcal{C}_{2^{n-1}}$, which is the dual of $\mathcal{D}(2^{n-1}, -1)$. Similarly, every admissible pair (x, y) of SD_{2^n} is of a form $x = a^r b^k, y = a^j b$, where r is odd, j is even, $k \in \{0, 1\}$. If $k = 0$, then (SD_{2^n}, x, y) defines the map $\mathcal{D}(2^{n-1}, -1 + 2^{n-2})$; if $k = 1$, we have its dual map $\mathcal{C}(2^{n-2}, 2, 1, 1)$, the regular embedding of a multicycle $\mathcal{C}_{2^{n-2}}^{(2)}$. It is clear that the maps are all reflexible, as claimed. \square

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Application of Taylor Expansion Method for the Volterra Fuzzy Integral Equations System

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Abstract

Since in some fuzzy mathematical problems finding an explicit solution is difficult, therefore it is convenient to employ numerical methods. The focus of this paper is to introduce an efficient computational method which can be applied to approximate solution of fuzzy linear Volterra integral equations system. The method is based on substitution of the Taylor polynomials instead of unknown functions in the given fuzzy system. This work reduces the present problem to a linear system in generalized case, therefore the solution of the result system yields the unknown Taylor coefficients of the solution functions. The proposed method is illustrated by an example and also results are compared with the exact solution by using computer simulations.

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1 Introduction

As a matter of fact, it might be said that many phenomena of almost all practical engineering and applied science problems like physical applications, potential theory and electrostatics are reduced to solve integral equations. Since these equations are usually difficult to solve explicitly, so it is required to obtain approximate solutions. There exist several numerical methods for approximating solution of integral equations. For example, Tricomi, in his book [20], introduced the classical method of successive approximations for nonlinear integral equations. Variational iteration method [13] and Adomian decomposition method [4] were effective and convenient for solving integral equations. Also the Homotopy analysis method (HAM) was proposed by Liao [14] and then has been applied in [1]. Moreover, some different valid methods for solving this kind of equations have been developed. First time, Taylor expansion approach was presented for solution of integral equations by Kanwal and Liu in [12] and then has been extended in [16, 17]. In addition, Babolian et al. [3] by using the orthogonal triangular basis functions solved some integral equations systems. Jafari et al [10] applied Legendre wavelets method to find numerical solution system of linear integral equations. Also Hilmi and Yalçınbaş [8] approximated a solution of linear Volterra integral equations system with the help of Taylor series.

In this paper we want to propose a new numerical approach to approximate solution of a fuzzy linear Volterra integral equations system. This method converts the given fuzzy system that supposedly has an unique fuzzy solution, into crisp linear system. For this scope, first the Taylor expansions of unknown functions are substituted in parametric form of the given fuzzy system. Then we differentiate both sides of the resulting integral equations of the system N times and also approximate the Taylor expansion by a suitable truncation limit. This work yields a linear system in crisp case, so the solution of the linear system yields the unknown Taylor coefficients of the solution functions. An interesting feature of this method is that we can get an approximate of the Taylor expansion in arbitrary point to any desired degree of accuracy. Here is an outline of the paper. In section 2, the basic notations and definitions of the integral equation and the Taylor polynomial method are briefly presented. Section 3 describes how to find an approximate solution of the given fuzzy Volterra integral equations system with using proposed approach. Finally in section 4, we apply the proposed method by an example to show the simplicity and efficiency of the method.

2 Preliminaries

In this section the most basic used notations in fuzzy calculus and integral equations are briefly introduced. We start by defining the fuzzy number.

Definition 1. A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \rightarrow I = [0, 1]$ such that:

- i. u is upper semi-continuous,
- ii. $u(x) = 0$ outside some interval $[a, d]$,
- iii. There are real numbers $b, c : a \leq b \leq c \leq d$, for which:
 1. $u(x)$ is increasing on $[a, b]$,
 2. $u(x)$ is decreasing on $[c, d]$,
 3. $u(x) = 1, b \leq x \leq c$.

The set of all fuzzy numbers (as given by Definition 1) is denoted by E^1 [7, 18]. An alternative definition which yields the same E^1 is given by Kaleva [11] and Ma et al. [15].

Definition 2. A fuzzy number v is a pair (\underline{v}, \bar{v}) of functions $\underline{v}(r)$ and $\bar{v}(r) : 0 \leq r \leq 1$, which satisfy the following requirements:

- i. $\underline{v}(r)$ is a bounded monotonically increasing, left continuous function on $(0, 1]$ and right continuous at 0,
- ii. $\bar{v}(r)$ is a bounded monotonically decreasing, left continuous function on $(0, 1]$ and right continuous at 0,
- iii. $\underline{v}(r) \leq \bar{v}(r) : 0 \leq r \leq 1$.

A popular fuzzy number is the triangular fuzzy number $v = (v_m, v_l, v_u)$ where v_m denotes the modal value and the real values $v_l \geq 0$ and $v_u \geq 0$ represent the left and right fuzziness, respectively. The membership function of a triangular fuzzy number is defined as follows:

$$\mu_v(x) = \begin{cases} \frac{x-v_m}{v_l} + 1, & v_m - v_l \leq x \leq v_m, \\ \frac{v_m-x}{v_u} + 1, & v_m \leq x \leq v_m + v_u, \\ 0, & \text{otherwise.} \end{cases}$$

Its parametric form is:

$$\underline{v}(r) = v_m + v_l(r - 1), \quad \bar{v}(r) = v_m + v_u(1 - r), \quad 0 \leq r \leq 1.$$

Triangular fuzzy numbers are fuzzy numbers in LR representation where the reference functions L and R are linear.

2.1 Operations on fuzzy numbers

We briefly mentioned fuzzy number operations that have had been defined by the extension principle [21, 22].

$$\begin{aligned} \mu_{A+B}(z) &= \max\{\mu_A(x) \wedge \mu_B(y) \mid z = x + y\}, \\ \mu_{AB}(z) &= \max\{\mu_A(x) \wedge \mu_B(y) \mid z = xy\}, \quad \mu_{f(Net)}(z) = \max\{\mu_{Net}(x) \mid z = f(x)\}, \end{aligned}$$

where A , B and Net are fuzzy numbers, $\mu_*(\cdot)$ denotes the membership function of each fuzzy number, \wedge is the minimum operator and f is a continuous function.

The above operations on fuzzy numbers are numerically performed on level sets (i.e. α -cuts). For $0 < \alpha \leq 1$, a α -level set of a fuzzy number A is defined as:

$$[A]^\alpha = \{x \mid \mu_A(x) \geq \alpha, x \in \mathbb{R}\},$$

and $[A]^0 = \overline{\bigcup_{\alpha \in (0,1)} [A]^\alpha}$. Since level sets of fuzzy numbers become closed intervals, we denote $[A]^\alpha$ by

$$[A]^\alpha = [[A]_l^\alpha, [A]_u^\alpha],$$

where $[A]_l^\alpha$ and $[A]_u^\alpha$ are the lower and the upper limits of the α -level set $[A]^\alpha$, respectively. From interval arithmetic [2], the above operations on fuzzy numbers are written for the α -level sets as follows:

$$[A]^\alpha + [B]^\alpha = [[A]_l^\alpha, [A]_u^\alpha] + [[B]_l^\alpha, [B]_u^\alpha] = [[A]_l^\alpha + [B]_l^\alpha, [A]_u^\alpha + [B]_u^\alpha], \quad (2.1)$$

$$f([Net]^\alpha) = f([Net]_l^\alpha, [Net]_u^\alpha) = [f([Net]_l^\alpha), f([Net]_u^\alpha)],$$

$$k[A]^\alpha = k[[A]_l^\alpha, [A]_u^\alpha] = [k[A]_l^\alpha, k[A]_u^\alpha], \quad \text{if } k \geq 0, \quad (2.2)$$

$$k[A]^\alpha = k[[A]_l^\alpha, [A]_u^\alpha] = [k[A]_u^\alpha, k[A]_l^\alpha], \quad \text{if } k < 0.$$

For arbitrary $u = (\underline{u}, \bar{u})$ and $v = (\underline{v}, \bar{v})$ we define addition $(u + v)$ and multiplication by k as [7, 18]:

$$\begin{aligned} \overline{(u + v)}(r) &= \bar{u}(r) + \bar{v}(r), \\ \underline{(u + v)}(r) &= \underline{u}(r) + \underline{v}(r), \end{aligned}$$

$$\begin{aligned} \overline{(ku)}(r) &= k \cdot \bar{u}(r), \quad \underline{(kv)}(r) = k \cdot \underline{v}(r), \quad \text{if } k \geq 0, \\ \overline{(ku)}(r) &= k \cdot \underline{u}(r), \quad \underline{(kv)}(r) = k \cdot \bar{v}(r), \quad \text{if } k < 0. \end{aligned}$$

Definition 3. For arbitrary fuzzy numbers $u, v \in E^1$ the quantity

$$D(u, v) = \sup_{0 \leq r \leq 1} \{\max[|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|]\}$$

is the distance between u and v . It is shown that (E^1, D) is a complete metric space [19].

Definition 4. Let $f : [a, b] \rightarrow E^1$. For each partition $P = \{t_0, t_1, \dots, t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i]$ ($1 \leq i \leq n$), suppose

$$R_P = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}),$$

$$\Delta := \max\{|t_i - t_{i-1}|, i = 1, \dots, n\}.$$

The definite integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t)dt = \lim_{\Delta \rightarrow 0} R_P$$

provided that this limit exists in the metric D . If the fuzzy function $f(t)$ is continuous in the metric D , its definite integral exists [7]. Also,

$$\overline{\left(\int_a^b f(t, r) dt \right)} = \int_a^b \bar{f}(t, r) dt,$$

$$\underline{\left(\int_a^b f(t, r) dt \right)} = \int_a^b \underline{f}(t, r) dt.$$

More details about properties of the fuzzy integral are given in [7, 11].

2.2 System of integral equations

The basic definition of integral equation is given in [9].

Definition 5. The Fredholm integral equation of the second kind is

$$F(t) = f(t) + \lambda(ku)(t), \quad (2.3)$$

where

$$(ku)(t) = \int_a^b k(s, t)F(s)ds, \quad a \leq t \leq b.$$

In Eq. (2.3), $k(s, t)$ is an arbitrary kernel function over the square $a \leq s, t \leq b$ and $f(t)$ is a function of $t : a \leq t \leq b$. If the kernel function satisfies $k(s, t) = 0, s > t$, we obtain the Volterra integral equation

$$F(t) = f(t) + \lambda \int_a^t k(s, t)F(s)ds. \quad (2.4)$$

In addition, if $f(t)$ is a crisp function then the solution of above equation is crisp as well. Also if $f(t)$ be a fuzzy function we have Fredholm fuzzy integral equation of the second kind which may only process fuzzy solutions. Sufficient conditions for the existence and uniqueness of the solution of the second kind equation where $f(t)$ is a fuzzy function, are given in [5, 6].

Definition 6. The second kind fuzzy linear Volterra integral equations system is in the form

$$\begin{cases} F_1(t) = f_1(t) + \sum_{j=1}^m \left(\lambda_{1j} \int_a^t k_{1j}(s, t)F_j(s)ds \right) \\ \vdots \\ F_i(t) = f_i(t) + \sum_{j=1}^m \left(\lambda_{ij} \int_a^t k_{ij}(s, t)F_j(s)ds \right) \\ \vdots \\ F_m(t) = f_m(t) + \sum_{j=1}^m \left(\lambda_{mj} \int_a^t k_{mj}(s, t)F_j(s)ds \right) \end{cases}, \quad (2.5)$$

where $a \leq s \leq t \leq b$ and $\lambda_{ij} \neq 0$ (for $i, j = 1, \dots, m$) are real constants. Moreover, in system (2.5), the fuzzy function $f_i(t)$ and kernel $k_{i,j}(s, t)$ are given and assumed to be sufficiently differentiable with respect to all their arguments on the interval $a \leq t, s \leq b$. Also we assume that the kernel function $k_{i,j}(s, t) \in L^2([a, b] \times [a, b])$ and $F(t) = [F_1(t), \dots, F_m(t)]^T$ is the solution to be determined.

Now let $(\underline{f}_i(t, r), \bar{f}_i(t, r))$ and $(\underline{F}_i(t, r), \bar{F}_i(t, r))$ ($0 \leq r \leq 1$; $a \leq t \leq b$) be parametric form of $f_i(t)$ and $F_i(t)$, respectively. For simplify, we assume that $\lambda_{ij} > 0$ (for $i, j = 1, \dots, m$). In order to design a numerical scheme for solving (2.5), we write the parametric form of the given fuzzy integral equations system as follows:

$$\begin{cases} \bar{F}_1(t, r) = \bar{f}_1(t, r) + \sum_{j=1}^m \left(\lambda_{1j} \int_a^t \bar{U}_{1,j}(s, r) ds \right) \\ \underline{F}_1(t, r) = \underline{f}_1(t, r) + \sum_{j=1}^m \left(\lambda_{1j} \int_a^t \underline{U}_{1,j}(s, r) ds \right) \\ \vdots \\ \bar{F}_m(t, r) = \bar{f}_m(t, r) + \sum_{j=1}^m \left(\lambda_{mj} \int_a^t \bar{U}_{m,j}(s, r) ds \right) \\ \underline{F}_m(t, r) = \underline{f}_m(t, r) + \sum_{j=1}^m \left(\lambda_{mj} \int_a^t \underline{U}_{m,j}(s, r) ds \right) \end{cases}, \quad (2.6)$$

where

$$\bar{U}_{i,j}(s, r) = \begin{cases} k_{i,j}(s, t) \bar{F}_j(s, r) & , k_{i,j}(s, t) \geq 0 \\ k_{i,j}(s, t) \underline{F}_j(s, r) & , k_{i,j}(s, t) < 0 \end{cases},$$

and

$$\underline{U}_{i,j}(s, r) = \begin{cases} k_{i,j}(s, t) \underline{F}_j(s, r) & , k_{i,j}(s, t) \geq 0 \\ k_{i,j}(s, t) \bar{F}_j(s, r) & , k_{i,j}(s, t) < 0 \end{cases}.$$

2.3 The general method

Let us first recall the basic principles of the Taylor polynomial method for solving Fredholm fuzzy integral equations system (2.5). Because these results are the key for our problems therefore we explain them. Without loss of generality, we assume that

$$\begin{cases} \lambda_{i,j} \cdot k_{i,j}(s, t) \geq 0 & , a \leq s \leq c_{i,j} \\ \lambda_{i,j} \cdot k_{i,j}(s, t) < 0 & , c_{i,j} \leq s \leq t \end{cases}.$$

With above supposition, the system (2.6) is transformed to following form:

$$\begin{cases} \bar{F}_1(t, r) = \bar{f}_1(t, r) + \\ \sum_{j=1}^m \lambda_{1j} \left(\int_a^{c_{1,j}} k_{1,j}(s, t) \bar{F}_j(s, r) ds + \int_{c_{1,j}}^t k_{1,j}(s, t) \underline{F}_j(s, r) ds \right) \\ \underline{F}_1(t, r) = \underline{f}_1(t, r) + \\ \sum_{j=1}^m \lambda_{1j} \left(\int_a^{c_{1,j}} k_{1,j}(s, t) \underline{F}_j(s, r) ds + \int_{c_{1,j}}^t k_{1,j}(s, t) \bar{F}_j(s, r) ds \right) \\ \vdots \\ \bar{F}_m(t, r) = \bar{f}_m(t, r) + \\ \sum_{j=1}^m \lambda_{mj} \left(\int_a^{c_{m,j}} k_{m,j}(s, t) \bar{F}_j(s, r) ds + \int_{c_{m,j}}^t k_{m,j}(s, t) \underline{F}_j(s, r) ds \right) \\ \underline{F}_m(t, r) = \underline{f}_m(t, r) + \\ \sum_{j=1}^m \lambda_{mj} \left(\int_a^{c_{m,j}} k_{m,j}(s, t) \underline{F}_j(s, r) ds + \int_{c_{m,j}}^t k_{m,j}(s, t) \bar{F}_j(s, r) ds \right) \end{cases}. \quad (2.7)$$

Now we want to obtain the solution of the above system in the form of

$$\overline{F}_{j,N}(t,r) = \sum_{i=0}^N \left(\frac{1}{i!} \cdot \frac{\partial^{(i)} \overline{F}_j(t,r)}{\partial t^i} \Big|_{t=z} \cdot (t-z)^i \right), \quad a \leq t, z \leq b, 0 \leq r \leq 1, \quad (2.8)$$

$$\underline{F}_{j,N}(t,r) = \sum_{i=0}^N \left(\frac{1}{i!} \cdot \frac{\partial^{(i)} \underline{F}_j(t,r)}{\partial t^i} \Big|_{t=z} \cdot (t-z)^i \right), \quad a \leq t, z \leq b, 0 \leq r \leq 1,$$

(for $j = 1, \dots, m$) which are the Taylor expansions of degree N at $t = z$ for the unknown functions $\overline{F}_j(t,r)$ and $\underline{F}_j(t,r)$, respectively. For this scope we calculate p -th (for $p = 0, \dots, N$) derivative of each equation in the system (2.7) with respect to t and get

$$\left\{ \begin{array}{l} \frac{\partial^{(p)} \overline{F}_1(t,r)}{\partial t^p} = \frac{\partial^{(p)} \overline{f}_1(t,r)}{\partial t^p} + \sum_{j=1}^m \lambda_{1j} \left(\int_a^{c_{1,j}} \frac{\partial^{(p)} k_{1,j}(s,t)}{\partial t^p} \cdot \overline{F}_j(s,r) ds \right. \\ \left. + \sum_{l=0}^{p-1} \sum_{q=0}^{p-l-1} \binom{p-q-1}{p-q-1-l} \left(\frac{\partial^{(q)} k_{1,j}(s,t)}{\partial t^q} \Big|_{s=t} \right)^{(p-q-1-l)} \cdot \left(\frac{\partial^{(l)} \underline{F}_j(t,r)}{\partial t^l} \right) \right. \\ \left. + \int_{c_{1,j}}^t \frac{\partial^{(p)} k_{1,j}(s,t)}{\partial t^p} \cdot \underline{F}_j(s,r) ds \right) \\ \\ \frac{\partial^{(p)} \underline{F}_1(t,r)}{\partial t^p} = \frac{\partial^{(p)} \underline{f}_1(t,r)}{\partial t^p} + \sum_{j=1}^m \lambda_{1j} \left(\int_a^{c_{1,j}} \frac{\partial^{(p)} k_{1,j}(s,t)}{\partial t^p} \cdot \underline{F}_j(s,r) ds \right. \\ \left. + \sum_{l=0}^{p-1} \sum_{q=0}^{p-l-1} \binom{p-q-1}{p-q-1-l} \left(\frac{\partial^{(q)} k_{1,j}(s,t)}{\partial t^q} \Big|_{s=t} \right)^{(p-q-1-l)} \cdot \left(\frac{\partial^{(l)} \overline{F}_j(t,r)}{\partial t^l} \right) \right. \\ \left. + \int_{c_{1,j}}^t \frac{\partial^{(p)} k_{1,j}(s,t)}{\partial t^p} \cdot \overline{F}_j(s,r) ds \right) \\ \\ \vdots \\ \\ \frac{\partial^{(p)} \overline{F}_m(t,r)}{\partial t^p} = \frac{\partial^{(p)} \overline{f}_m(t,r)}{\partial t^p} + \sum_{j=1}^m \lambda_{mj} \left(\int_a^{c_{m,j}} \frac{\partial^{(p)} k_{m,j}(s,t)}{\partial t^p} \cdot \overline{F}_j(s,r) ds \right. \\ \left. + \sum_{l=0}^{p-1} \sum_{q=0}^{p-l-1} \binom{p-q-1}{p-q-1-l} \left(\frac{\partial^{(q)} k_{m,j}(s,t)}{\partial t^q} \Big|_{s=t} \right)^{(p-q-1-l)} \cdot \left(\frac{\partial^{(l)} \underline{F}_j(t,r)}{\partial t^l} \right) \right. \\ \left. + \int_{c_{m,j}}^t \frac{\partial^{(p)} k_{m,j}(s,t)}{\partial t^p} \cdot \underline{F}_j(s,r) ds \right) \\ \\ \frac{\partial^{(p)} \underline{F}_m(t,r)}{\partial t^p} = \frac{\partial^{(p)} \underline{f}_m(t,r)}{\partial t^p} + \sum_{j=1}^m \lambda_{mj} \left(\int_a^{c_{m,j}} \frac{\partial^{(p)} k_{m,j}(s,t)}{\partial t^p} \cdot \underline{F}_j(s,r) ds \right. \\ \left. + \sum_{l=0}^{p-1} \sum_{q=0}^{p-l-1} \binom{p-q-1}{p-q-1-l} \left(\frac{\partial^{(q)} k_{m,j}(s,t)}{\partial t^q} \Big|_{s=t} \right)^{(p-q-1-l)} \cdot \left(\frac{\partial^{(l)} \overline{F}_j(t,r)}{\partial t^l} \right) \right. \\ \left. + \int_{c_{m,j}}^t \frac{\partial^{(p)} k_{m,j}(s,t)}{\partial t^p} \cdot \overline{F}_j(s,r) ds \right) \end{array} \right. \quad (2.9)$$

For brevity, we define below symbols as:

$$\overline{F}_{jN}^{(p)}(z,r) := \frac{\partial^{(p)} \overline{F}_{jN}(t,r)}{\partial t^p} \Big|_{t=z} \quad \text{and} \quad \underline{F}_{jN}^{(p)}(z,r) := \frac{\partial^{(p)} \underline{F}_{jN}(t,r)}{\partial t^p} \Big|_{t=z}, \quad j = 1, \dots, m.$$

The aim of this study is to determine of the coefficients $\overline{F}_j^{(p)}(z,r)$ and $\underline{F}_j^{(p)}(z,r)$, (for $p = 0, \dots, N; j = 1, \dots, m$) in system (2.9). For this intent, we expanded $\overline{F}_j(s,r)$ and $\underline{F}_j(s,r)$ in Taylor series at arbitrary point $z : a \leq z \leq b$ and substituted its N -th truncation in

(2.9). Now we can write:

$$\left\{ \begin{array}{l} \overline{F}_{1N}^{(p)}(z, r) = \frac{\partial^{(p)} \overline{f}_1(t, r)}{\partial t^p} \Big|_{t=z} + \sum_{j=1}^m (\sum_{l=0}^{p-1} v_{p,l}^{(1,j)} \cdot \underline{F}_{jN}^{(l)}(z, r) \\ \quad + \sum_{q=0}^N w_{p,q}^{(1,j)} \cdot \overline{F}_{jN}^{(q)}(z, r) + \sum_{q=0}^N w'_{p,q}{}^{(1,j)} \cdot \underline{F}_{jN}^{(q)}(z, r)) \\ \\ \underline{F}_{1N}^{(p)}(z, r) = \frac{\partial^{(p)} f_1(t, r)}{\partial t^p} \Big|_{t=z} + \sum_{j=1}^m (\sum_{l=0}^{p-1} v_{p,l}^{(1,j)} \cdot \overline{F}_{jN}^{(l)}(z, r) \\ \quad + \sum_{q=0}^N w_{p,q}^{(1,j)} \cdot \underline{F}_{jN}^{(q)}(z, r) + \sum_{q=0}^N w'_{p,q}{}^{(1,j)} \cdot \overline{F}_{jN}^{(q)}(z, r)) \\ \\ \vdots \\ \overline{F}_{mN}^{(p)}(z, r) = \frac{\partial^{(p)} \overline{f}_m(t, r)}{\partial t^p} \Big|_{t=z} + \sum_{j=1}^m (\sum_{l=0}^{p-1} v_{p,l}^{(m,j)} \cdot \underline{F}_{jN}^{(l)}(z, r) \\ \quad + \sum_{q=0}^N w_{p,q}^{(m,j)} \cdot \overline{F}_{jN}^{(q)}(z, r) + \sum_{q=0}^N w'_{p,q}{}^{(m,j)} \cdot \underline{F}_{jN}^{(q)}(z, r)) \\ \\ \underline{F}_{mN}^{(p)}(z, r) = \frac{\partial^{(p)} f_m(t, r)}{\partial t^p} \Big|_{t=z} + \sum_{j=1}^m (\sum_{l=0}^{p-1} v_{p,l}^{(m,j)} \cdot \overline{F}_{jN}^{(l)}(z, r) \\ \quad + \sum_{q=0}^N w_{p,q}^{(m,j)} \cdot \underline{F}_{jN}^{(q)}(z, r) + \sum_{q=0}^N w'_{p,q}{}^{(m,j)} \cdot \overline{F}_{jN}^{(q)}(z, r)) \end{array} \right. , \quad (2.10)$$

where

$$w_{p,q}^{(i,j)} = \frac{\lambda_{i,j}}{q!} \int_a^{c_{i,j}} \frac{\partial^{(p)} k_{i,j}(s, t)}{\partial t^p} \Big|_{t=z} \cdot (s - z)^q ds,$$

$$w'_{p,q}{}^{(i,j)} = \frac{\lambda_{i,j}}{q!} \int_{c_{i,j}}^b \frac{\partial^{(p)} k_{i,j}(s, t)}{\partial t^p} \Big|_{t=z} \cdot (s - z)^q ds, \quad p, q = 0, \dots, N$$

and

$$v_{p,l}^{(i,j)} = \sum_{q=0}^{p-l-1} \binom{p-q-1}{p-q-1-l} \left(\frac{\partial^{(q)} k_{i,j}(s, t)}{\partial t^q} \Big|_{s=t} \right)^{(p-q-1-l)} \Big|_{s=z}, \quad i, j = 1, \dots, m.$$

Consequently, the matrix form of expression (2.10) can be written as follows:

$$(W + V) Y = E, \quad (2.11)$$

where

$$Y = [\underline{F}_{1N}(a, r), \dots, \underline{F}_{1N}^{(N)}(a, r), \overline{F}_{1N}(a, r), \dots, \overline{F}_{1N}^{(N)}(a, r), \dots, \\ \underline{F}_{mN}(a, r), \dots, \underline{F}_{mN}^{(N)}(a, r), \overline{F}_{mN}(a, r), \dots, \overline{F}_{mN}^{(N)}(a, r)]',$$

$$E = [-\underline{f}_1(a, r), \dots, -\frac{\partial^{(N)} \underline{f}_1(t, r)}{\partial t^N} \Big|_{t=a}, -\overline{f}_1(a, r), \dots, -\frac{\partial^{(N)} \overline{f}_1(t, r)}{\partial t^N} \Big|_{t=a}, \dots, \\ -\underline{f}_m(a, r), \dots, -\frac{\partial^{(N)} \underline{f}_m(t, r)}{\partial t^N} \Big|_{t=a}, -\overline{f}_m(a, r), \dots, -\frac{\partial^{(N)} \overline{f}_m(t, r)}{\partial t^N} \Big|_{t=a}]'$$

and

$$W = \begin{bmatrix} W^{(1,1)} & \dots & W^{(1,m)} \\ \vdots & \ddots & \vdots \\ W^{(m,1)} & \dots & W^{(m,m)} \end{bmatrix}, \quad V = \begin{bmatrix} V^{(1,1)} & \dots & V^{(1,m)} \\ \vdots & \ddots & \vdots \\ V^{(m,1)} & \dots & V^{(m,m)} \end{bmatrix}.$$

Parochial matrices $W^{(i,j)}$, (for $i, j = 1, \dots, m$) are defined with following elements:

$$W^{(i,j)} = \begin{bmatrix} W_{1,1}^{(i,j)} & W_{1,2}^{(i,j)} \\ W_{2,1}^{(i,j)} & W_{2,2}^{(i,j)} \end{bmatrix}, \quad V^{(i,j)} = \begin{bmatrix} V_{1,1}^{(i,j)} & V_{1,2}^{(i,j)} \\ V_{2,1}^{(i,j)} & V_{2,2}^{(i,j)} \end{bmatrix}, \quad i, j = 1, \dots, m,$$

where

$$W_{1,1}^{(i,j)} = W_{2,2}^{(i,j)} = \begin{bmatrix} w_{0,0}^{(i,j)} - 1 & w_{0,1}^{(i,j)} & \dots & w_{0,N-1}^{(i,j)} & w_{0,N}^{(i,j)} \\ w_{1,0}^{(i,j)} & w_{1,1}^{(i,j)} - 1 & \dots & w_{1,N-1}^{(i,j)} & w_{1,N}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-1,0}^{(i,j)} & w_{N-1,1}^{(i,j)} & \dots & w_{N-1,N-1}^{(i,j)} - 1 & w_{N-1,N}^{(i,j)} \\ w_{N,0}^{(i,j)} & w_{N,1}^{(i,j)} & \dots & w_{N,N-1}^{(i,j)} & w_{N,N}^{(i,j)} - 1 \end{bmatrix},$$

$$W_{1,2}^{(i,j)} = W_{2,1}^{(i,j)} = \begin{bmatrix} w'_{0,0}^{(i,j)} & w'_{0,1}^{(i,j)} & \dots & w'_{0,N-1}^{(i,j)} & w'_{0,N}^{(i,j)} \\ w'_{1,0}^{(i,j)} & w'_{1,1}^{(i,j)} & \dots & w'_{1,N-1}^{(i,j)} & w'_{1,N}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w'_{N-1,0}^{(i,j)} & w'_{N-1,1}^{(i,j)} & \dots & w'_{N-1,N-1}^{(i,j)} & w'_{N-1,N}^{(i,j)} \\ w'_{N,0}^{(i,j)} & w'_{N,1}^{(i,j)} & \dots & w'_{N,N-1}^{(i,j)} & w'_{N,N}^{(i,j)} \end{bmatrix},$$

$$V_{1,2}^{(i,j)} = V_{2,1}^{(i,j)} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ v'_{1,0}^{(i,j)} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v'_{N-1,0}^{(i,j)} & v'_{N-1,1}^{(i,j)} & \dots & 0 & 0 \\ v'_{N,0}^{(i,j)} & v'_{N,1}^{(i,j)} & \dots & v'_{N,N-1}^{(i,j)} & 0 \end{bmatrix},$$

$$V_{1,1}^{(i,j)} = V_{2,2}^{(i,j)} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{(N+1) \times (N+1)}.$$

3 Convergence analysis

In this section we prove that the above numerical method converges to the exact solution of fuzzy system (2.5).

Theorem 7. Let $\overline{F}_{j,N}(t)$ and $\underline{F}_{j,N}(t)$ (for $j = 1, \dots, m$) be Taylor polynomials of degree N that their coefficients are produced by solving the linear system (2.11). Then these polynomials converge to the exact solution of the fuzzy Volterra integral equations system (2.5), when $N \rightarrow +\infty$.

Proof. Consider the system (2.5). Since, the series (2.8) converge to $\overline{F}_j(t, r)$ and $\underline{F}_j(t, r)$ (for $j = 1, \dots, m$) respectively, then we conclude that:

$$\left\{ \begin{array}{l} \bar{F}_{iN}(t, r) = \bar{f}_i(t, r) + \\ \sum_{j=1}^m \lambda_{ij} \left(\int_a^{c_{i,j}} k_{i,j}(s, t) \bar{F}_{jN}(s, r) ds + \int_{c_{i,j}}^t k_{i,j}(s, t) \underline{F}_{jN}(s, r) ds \right) \\ \underline{F}_{iN}(t, r) = \underline{f}_i(t, r) + \\ \sum_{j=1}^m \lambda_{ij} \left(\int_a^{c_{i,j}} k_{i,j}(s, t) \underline{F}_{jN}(s, r) ds + \int_{c_{i,j}}^t k_{i,j}(s, t) \bar{F}_{jN}(s, r) ds \right) \end{array} \right. , \quad (3.1)$$

(for $i = 1, \dots, m$) and it holds that

$$\bar{F}_j(t, r) = \lim_{N \rightarrow \infty} \bar{F}_{jN}(t, r), \text{ and } \underline{F}_j(t) = \lim_{N \rightarrow \infty} \underline{F}_{jN}(t, r).$$

We defined the error function $e_N(t, r)$ by subtracting Eqs. (3.1)–(2.7) as follows:

$$e_N(t, r) = \sum_{i=1}^m e_{iN}(t, r), \quad (3.2)$$

$$e_{iN}(t, r) = \bar{e}_{iN}(t, r) + \underline{e}_{iN}(t, r),$$

where

$$\begin{aligned} \bar{e}_{iN}(t, r) &= (\bar{F}_i(t, r) - \bar{F}_{iN}(t, r)) + \sum_{j=1}^m \lambda_{ij} \left(\int_a^{c_{i,j}} K_{i,j}(s, t) (\bar{F}_j(s, r) - \bar{F}_{jN}(s, r)) ds \right) \\ &\quad + \sum_{j=1}^m \lambda_{ij} \left(\int_{c_{i,j}}^t K_{i,j}(s, t) (\underline{F}_j(s, r) - \underline{F}_{jN}(s, r)) ds \right), \end{aligned}$$

and

$$\begin{aligned} \underline{e}_{iN}(t, r) &= (\underline{F}_i(t, r) - \underline{F}_{iN}(t, r)) + \sum_{j=1}^m \lambda_{ij} \left(\int_a^{c_{i,j}} K_{i,j}(s, t) (\underline{F}_j(s, r) - \underline{F}_{jN}(s, r)) ds \right) \\ &\quad + \sum_{j=1}^m \lambda_{ij} \left(\int_{c_{i,j}}^t K_{i,j}(s, t) (\bar{F}_j(s, r) - \bar{F}_{jN}(s, r)) ds \right), \end{aligned}$$

We must prove when $N \rightarrow +\infty$, the error function $e_N(t)$ becomes to zero. Hence we proceed as follows:

$$\begin{aligned} \|e_N\| &\leq \sum_{i=1}^m \|e_{iN}\| = \sum_{i=1}^m \|\bar{e}_{iN} + \underline{e}_{iN}\| \leq \sum_{i=1}^m (\|\bar{e}_{iN}\| + \|\underline{e}_{iN}\|) \leq \\ &\sum_{i=1}^m (\|(\bar{F}_i(t, r) - \bar{F}_{iN}(t, r))\| + \|(\underline{F}_i(t, r) - \underline{F}_{iN}(t, r))\|) + \\ &\sum_{i=1}^m \sum_{j=1}^m (|\lambda_{i,j}| \int_a^t \|k_{i,j}\| (\|\bar{F}_j(s, r) - \bar{F}_{jN}(s, r)\| + \|\underline{F}_j(s, r) - \underline{F}_{jN}(s, r)\|) ds). \end{aligned}$$

Since $\|k_{i,j}\|$ is bounded, therefore $\|(\bar{F}_j(s, r) - \bar{F}_{jN}(s, r))\| \rightarrow 0$ and $\|(\underline{F}_j(s, r) - \underline{F}_{jN}(s, r))\| \rightarrow 0$ imply that $\|e_N\| \rightarrow 0$ and proof is completed. \square

4 An example

In this section, in order to investigate the accuracy of the proposed method, we have chosen an example of fuzzy linear Volterra integral equations system. Also to show the efficiency of the present method for our problem, results will be compared with the exact solution.

Example 8. Consider the system of fuzzy linear Volterra integral equations with:

$$\begin{aligned}\bar{f}_1(t, r) &= \frac{t^2(r-2)}{4} + \frac{9(r^3-2)(t-1)^2}{10} + \frac{rt^2(t^2-1)^2}{4} \\ &\quad - t(r-2) + \frac{r(r^4+2)(t-1)^4(2t^3+4t^2+6t+3)}{10}, \\ \underline{f}_1(t, r) &= rt - \frac{rt^2}{4} - \frac{3(r^3-2)(t-1)^4(2t^3+4t^2+6t+3)}{10} \\ &\quad - \frac{3r(r^4+2)(t-1)^2}{10} - \frac{t^2(t^2-1)^2(r-2)}{4}, \\ \bar{f}_2(t, r) &= \frac{(r-2)(t+1)^4}{20} - t(3r^3-6) + t^2(r^3-2) \\ &\quad + \frac{r(4t+1)(t^2-1)^4}{20} + \frac{rt^2(r^4+2)(2t+1)(t-1)^2}{3}, \\ \underline{f}_2(t, r) &= t(r^5+2r) - \frac{rt^2(r^4+2)}{3} - \frac{(4t+1)(t^2-1)^4(r-2)}{20} \\ &\quad - \frac{r(t+1)^4}{20} - t^2(r^3-2)(2t+1)(t-1)^2,\end{aligned}$$

kernel functions

$$k_{1,1}(s, t) = t^2(1-s^2), \quad k_{1,2}(s, t) = (1-t)^2(1-s^3),$$

$$k_{2,1}(s, t) = (1+t)^4(1-s^3), \quad k_{2,2}(s, t) = 2t^2(1-s), \quad 0 \leq s \leq t \leq 2,$$

and $a = 0$, $b = 2$, $N = 1$, $\lambda_{i,j} = 1$ (for $i, j = 1, 2$). The exact solution in this case is given by

$$\bar{F}_1(t, r) = t(2-r), \quad \underline{F}_1(t, r) = tr,$$

$$\bar{F}_2(t, r) = t(6-3r^3) \text{ and } \underline{F}_2(t, r) = t(r^5+2r).$$

In this example we assume that $z = 0$. Using Eqs. (2.10)–(2.11), the coefficients matrix $W + V$ is calculated as following:

$$W + V = \begin{bmatrix} W^{1,1} + V^{1,1} & W^{1,2} + V^{1,2} \\ W^{2,1} + V^{2,1} & W^{2,2} + V^{2,2} \end{bmatrix},$$

where

$$W^{1,1} + V^{1,1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad W^{1,2} + V^{1,2} = \begin{bmatrix} \frac{3}{2} & \frac{3}{5} & -\frac{3}{4} & -\frac{3}{10} \\ -\frac{3}{5} & -\frac{3}{10} & \frac{3}{5} & \frac{3}{10} \\ -\frac{3}{5} & -\frac{3}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{3}{2} & \frac{3}{5} & -\frac{3}{4} & -\frac{3}{10} \end{bmatrix},$$

$$W^{2,1} + V^{2,1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{20} & -\frac{1}{4} & -\frac{1}{20} \\ 1 & \frac{1}{5} & 0 & -\frac{1}{5} \\ -\frac{1}{4} & -\frac{1}{20} & \frac{1}{4} & \frac{1}{20} \\ 0 & -\frac{1}{5} & 1 & \frac{1}{5} \end{bmatrix}, \quad W^{2,2} + V^{2,2} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

With using of above matrices, we can rewrite the linear system (2.11) as follows:

$$(W + V) \begin{bmatrix} \underline{F}_{1,1}(0, r) \\ \underline{F}'_{1,1}(0, r) \\ \overline{F}_{1,1}(0, r) \\ \overline{F}'_{1,1}(0, r) \\ \underline{F}_{2,1}(0, r) \\ \underline{F}'_{2,1}(0, r) \\ \overline{F}_{2,1}(0, r) \\ \overline{F}'_{2,1}(0, r) \end{bmatrix} = \begin{bmatrix} \frac{3r^5}{10} + \frac{9r^3}{10} + \frac{3r}{5} - \frac{9}{5} \\ -\frac{3r^5}{5} - \frac{9r^3}{5} - \frac{11r}{5} + \frac{18}{5} \\ -\frac{3r^5}{10} - \frac{9r^3}{10} - \frac{3r}{5} + \frac{9}{5} \\ \frac{3r^5}{5} + \frac{9r^3}{5} + \frac{11r}{5} - \frac{28}{5} \\ \frac{r}{10} - \frac{1}{10} \\ -r^5 - \frac{8r}{5} - \frac{2}{5} \\ \frac{1}{10} - \frac{r}{10} \\ 3r^3 - \frac{2r}{5} - \frac{28}{5} \end{bmatrix}.$$

The vector solution of above linear system is:

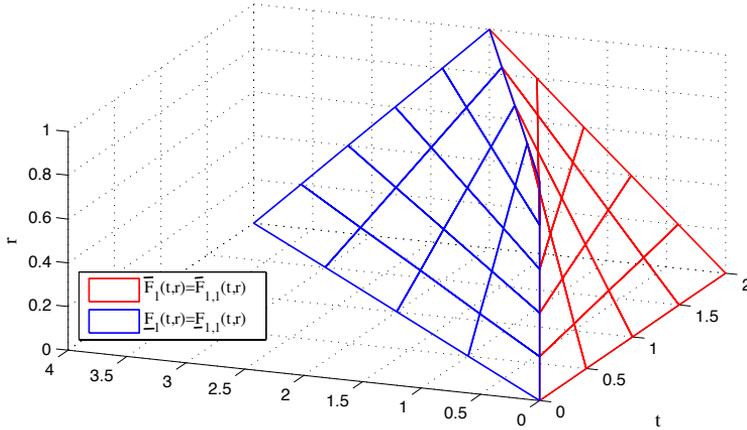
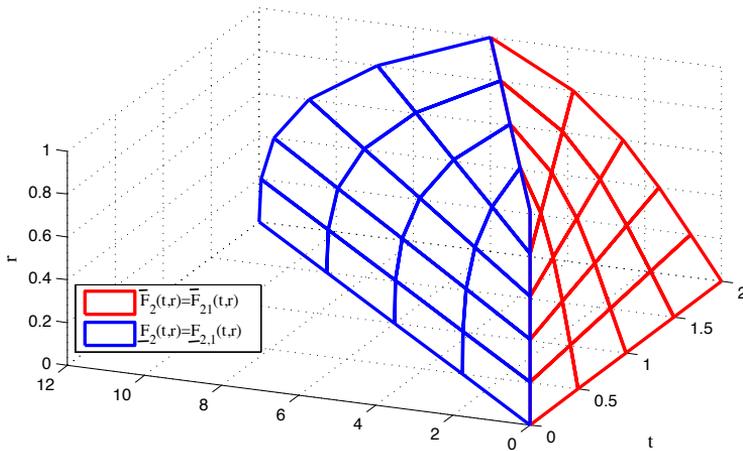
$$\begin{bmatrix} \underline{F}_{1,1}(0, r) \\ \underline{F}'_{1,1}(0, r) \\ \overline{F}_{1,1}(0, r) \\ \overline{F}'_{1,1}(0, r) \\ \underline{F}_{2,1}(0, r) \\ \underline{F}'_{2,1}(0, r) \\ \overline{F}_{2,1}(0, r) \\ \overline{F}'_{2,1}(0, r) \end{bmatrix} = \begin{bmatrix} 0 \\ r \\ 0 \\ 2 - r \\ 0 \\ r^5 + 2r \\ 0 \\ 6 - 3r^3 \end{bmatrix}.$$

After propagating this vector solution in Eq. (2.8), we have:

$$\overline{F}_{1,1}(t, r) = t(2 - r) = \overline{F}_1(t, r), \quad \underline{F}_{1,1}(t, r) = tr = \underline{F}_1(t, r),$$

$$\overline{F}_{2,1}(t, r) = t(6 - 3r^3) = \overline{F}_2(t, r), \quad \underline{F}_{2,1}(t, r) = t(r^5 + 2r) = \underline{F}_2(t, r).$$

As showing in Figs. 1 and 2, the present method gives the analytical solution for this kind of fuzzy equations system, if the exact solution be polynomial.

Figure 1. $\bar{F}_{1,1}(t, r)$ and $\underline{E}_{1,1}(t, r)$ for Example 4.1Figure 2. $\bar{F}_{2,1}(t, r)$ and $\underline{E}_{2,1}(t, r)$ for Example 4.1

5 Conclusions

Fuzzy integral equations systems which have a very important place in physics and engineering, are usually difficult to solve analytically, therefore it is required to obtain approximate solutions. In this study we have worked out a computational method to approximate solution of fuzzy linear Volterra integral equations systems. The present course is a method for approximating unknown function in terms of truncated sequences including Taylor polynomials. Consider that to get the best approximating solution of the given equation, the truncations degree N must be chosen large enough. An interesting feature of this method is finding the analytical solution for given equations, if the exact

solution be polynomial of degree N or less than N . The analyzed example illustrated the ability and reliability of the present approach. The obtained solution, in comparison with exact solution admit a remarkable accuracy. Extensions to the case of more general of integral equations systems are left for future studies.

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Generalized sequence spaces over n -normed spaces

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Abstract

In the present paper we introduce generalized sequence spaces over a n -normed space defined by Musielak-Orlicz function $\mathcal{M} = (M_k)$.

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1 Introduction and Preliminaries

The concept of 2-normed spaces was initially developed by Gähler [4] in the mid of 1960's, while that of n -normed spaces one can see in Misiak [11]. Since then, many others have studied this concept and obtained various results, see Gunawan [5, 6] and Gunawan and Mashadi [7]. Let $n \in \mathbb{N}$ and X be a linear space over the field \mathbb{K} , where \mathbb{K} is field of real or complex numbers of dimension d , where $d \geq n \geq 2$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four conditions:

1. $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent in X ;
2. $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{K}$, and
4. $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called a n -norm on X , and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called a n -normed space over the field \mathbb{K} .

For example, we may take $X = \mathbb{R}^n$ being equipped with the n -norm $\|x_1, x_2, \dots, x_n\|_E =$ the volume of the n -dimensional parallelopiped spanned by the vectors x_1, x_2, \dots, x_n which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$. Let $(X, \|\cdot, \dots, \cdot\|)$ be a n -normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \dots, a_n\}$ be linearly independent set in X . Then the following function $\|\cdot, \dots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \dots, x_n\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\}$$

defines an $(n-1)$ -norm on X with respect to $\{a_1, a_2, \dots, a_n\}$.

A sequence (x_k) in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \rightarrow \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

A sequence (x_k) in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k, p \rightarrow \infty} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n -norm. Any complete n -normed space is said to be n -Banach space. An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown in [9] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($p \geq 1$). The Δ_2 -condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see [10, 13]. A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \quad k = 1, 2, \dots$$

is called the complementary function of a Musielak-Orlicz function \mathcal{M} . For a given Musielak-Orlicz function \mathcal{M} , the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0 \right\},$$

$$h_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0 \right\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} \left(1 + I_{\mathcal{M}}(kx) \right) : k > 0 \right\}.$$

By w we denote the space of all real or complex valued sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=1}^{\infty}$. Also, we will use the conventions that $e = (1, 1, \dots)$. Any vector subspace of w is called a sequence space. We will write l_{∞} , c and c_0 for the sequence spaces of all bounded, convergent, and null sequences, respectively. Further, by l_p ($1 \leq p < \infty$), we denote the sequence space of all p -absolutely convergent series, that is, $l_p = \left\{ x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty \right\}$ for $1 \leq p < \infty$.

Throughout the article, $w(X)$, $l_{\infty}(X)$, and $l_p(X)$ denote, respectively, the spaces of all, bounded, and p -absolutely summable sequences with the elements in X , where (X, q) is a seminormed space. By $\theta = (0, 0, \dots)$, we denote the zero element in X . P_s denotes the set of all subsets of \mathbb{N} , that do not contain more than s elements. With (φ_s) , we will denote a non-decreasing sequence of positive real numbers such that $(s-1)\varphi_{s-1} \leq (s-1)\varphi_s$ and $\varphi_s \rightarrow \infty$, as $s \rightarrow \infty$. The class of all the sequences (φ_s) satisfying this property is denoted by φ .

In paper [12], the notion of λ -convergent and bounded sequences is introduced as follows: let $\lambda = (\lambda_k)$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \dots, \quad \lambda_k \rightarrow \infty \text{ as } k \rightarrow \infty.$$

We say that a sequence $x = (x_k) \in w$ is λ -convergent to the number $l \in \mathbb{C}$, called as the λ -limit of x , if $\Lambda_n(x) \rightarrow l$ as $n \rightarrow \infty$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k, \quad n \in \mathbb{N}.$$

In particular, we say that x is a λ -null sequence if $\Lambda_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Further, we say that x is λ -bounded if $\sup |\Lambda_n(x)| < \infty$. Here and in the sequel, we will use the convention that any term with a negative subscript is equal to naught, for example, $\lambda_{-1} = 0$ and $x_{-1} = 0$. Now, it is well known in [12] that if $\lim_n x_n = a$ in the ordinary sense of convergence, then

$$\lim_n \left(\frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.$$

This implies that

$$\lim_n |\Lambda_n(x) - a| = \lim_n \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) (x_k - a) \right| = 0,$$

which yields that $\lim_n \Lambda_n(x) = a$ and hence x is λ -convergent to a . We therefore deduce that the ordinary convergence implies the λ -convergence to the same limit. The notion

of difference sequence spaces was introduced by Kızmaz [8], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. The notion was further generalized by Et and Çolak [3] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$.

Let m, n be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^n (-1)^v \binom{n}{v} x_{k+mv}.$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$ studied by Et and Çolak [3]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$ introduced and studied by Kızmaz [8]. For more details about sequence spaces see [2, 14, 15, 16] and references therein.

The space $m(\phi)$ introduced and studied by Sargent [17] is defined as follows:

$$m(\phi) = \left\{ x = (x_k) \in s : \|x\|_{m(\phi)} = \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty \right\}.$$

Let M be an Orlicz function, then Tripathy and Mahanta [18] defined and studied the following sequence space:

$$m(M, \varphi) = \left\{ x = (x_k) \in s : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

Recently, Altun and Bilgin [1] defined and studied the following sequence spaces:

$$m(M, A, \varphi, p) = \left\{ x = (x_k) \in s : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M\left(\frac{|A_i x|}{\rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

We define the following sequence spaces which we shall discuss in the second section of the present paper:

$$m(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|) =$$

$$\left\{ x = (x_k) \in w : \lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} = 0, \text{ for some } \rho > 0 \right\}.$$

If we take $M_k(x) = x$, we get

$$m(\varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$$

$$= \left\{ x = (x_k) \in w : \lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} = 0, \text{ for some } \rho > 0 \right\}.$$

If we take $p = (p_k) = 1$, we get

$$m(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, \|\cdot, \dots, \cdot\|)$$

$$= \left\{ x = (x_k) \in w : \lim_k \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) = 0, \text{ for some } \rho > 0 \right\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \quad (1.1)$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper we study some topological properties and inclusion relation between spaces $m(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$ which we have defined above and the spaces $m(\mathcal{M}, A, \varphi, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$ which we shall discuss in the third section of the paper.

2 Sequence spaces defined by Musielak-Orlicz function

In this section we study some topological properties and inclusion relation between the spaces $m(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$.

Theorem 1. *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function and $p = (p_k)$ be a sequence of strictly positive real number, the sequence space $m(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$ is a linear space over the set of complex number \mathbb{C} .*

Proof. Let $x = (x_k)$ and $y = (y_k) \in m(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} = 0$$

and

$$\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot, \dots, \cdot\|$ is a n -norm on X and M_k are non-decreasing and convex function so by using inequality (1.1), we have

$$\begin{aligned} & \lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n (\alpha x_k + \beta y_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \\ & \leq \lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left[\left(q \left(\left\| \frac{\Lambda_k \Delta_m^n (\alpha x_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right) \right. \\ & \quad \left. + \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n (\beta y_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\ & \leq K \lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} \frac{1}{2^{p_k}} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \\ & \quad + K \lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} \frac{1}{2^{p_k}} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \\ & = 0 \end{aligned}$$

Thus, we have $\alpha x + \beta y \in m(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$. Hence $m(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$ is a linear space. \square

Theorem 2. For any Musielak Orlicz function $\mathcal{M} = (M_k)$ and a bounded sequence $p = (p_k)$ of strictly positive real numbers $m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ is a topological linear space paranormed by

$$g(x) = \inf \left\{ \rho^{\frac{pr}{M}} : \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1, r = 1, 2, 3, \dots \right\},$$

where $M = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_k) \in m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$.

Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x) = \inf \left\{ \rho^{\frac{pr}{M}} : \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exist some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho_\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1.$$

Thus

$$\begin{aligned} & \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\ & \leq \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho_\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\ & \leq 1, \end{aligned}$$

Suppose that $x_k \neq 0$ for each $k \in N$. this implies that $\Lambda_k \Delta_m^n x_k \neq 0$ for each $k \in N$. Let $\epsilon \rightarrow 0$, then $\left\| \frac{\Lambda_k \Delta_m^n x_k}{\epsilon}, z_1, \dots, z_{n-1} \right\| \rightarrow \infty$. It follows that

$$\left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \rightarrow \infty$$

Which is a contradiction. Therefore $\Lambda_n \Delta_m^n x_k = 0$ for each k and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1$$

and

$$\left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by Minkowski's inequality, we have

$$\begin{aligned}
& \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\
& \leq \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n (x_k + y_k)}{\rho_1 + \rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\
& \leq \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(\frac{\rho_1}{\rho_1 + \rho_2} \left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| + \frac{\rho_2}{\rho_1 + \rho_2} \left\| \frac{\Lambda_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\
& \leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\
& + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\
& \leq 1.
\end{aligned}$$

Since ρ' s are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf \left\{ \rho^{\frac{p_r}{M}} : \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n (x_k + y_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\} \\
&\leq \inf \left\{ \rho_1^{\frac{p_r}{M}} : \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\} \\
&+ \inf \left\{ \rho_2^{\frac{p_r}{M}} : \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\}.
\end{aligned}$$

Therefore, $g(x+y) \leq g(x) + g(y)$

Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x) = \inf \left\{ \rho^{\frac{p_r}{M}} : \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\mu \Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\}.$$

Then

$$g(\mu x) = \inf \left\{ (|\mu|t)^{\frac{p_r}{M}} : \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\mu \Lambda_k \Delta_m^n x_k}{t}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \right\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_r} \leq \max(1, |\mu| \sup p_r)$, we have

$$g(\mu x) = \max(1, |\mu| \sup p_r) \inf \left\{ (t)^{\frac{p_r}{M}} : \left(\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{t}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\}$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. \square

Theorem 3. If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then

$$m\left(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \subset m\left(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$$

if and only if $\sup_{s \geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} < \infty$.

Proof. Let $x \in m\left(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ and $N = \sup_{s \geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} < \infty$. Then we get

$$\begin{aligned} & \frac{1}{\varphi_s^{**}} \sum_{n \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \\ & \leq \sup_{s \geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} \frac{1}{\varphi_s^*} \sum_{n \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \\ & = N \frac{1}{\varphi_s^*} \sum_{n \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \\ & = 0. \end{aligned}$$

Thus $x \in m\left(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$. Conversely, suppose that

$$m\left(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \subset m\left(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$$

and $x \in m\left(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$. Then there exists a $\rho > 0$ such that

$$\frac{1}{\varphi_s^*} \sum_{n \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} < \epsilon,$$

for every $\epsilon > 0$. Suppose that $\sup_{s \geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} = \infty$, then there exists a sequence of numbers

(s_j) such that $\lim_{j \rightarrow \infty} \frac{\varphi_{s_j}^*}{\varphi_{s_j}^{**}} = \infty$. Hence, we have

$$\begin{aligned} & \frac{1}{\varphi_{s_j}^{**}} \sum_{n \in \sigma, \sigma \in P_{s_j}} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \\ & \leq \sup_{j \geq 1} \frac{\varphi_{s_j}^*}{\varphi_{s_j}^{**}} \frac{1}{\varphi_{s_j}^*} \sum_{n \in \sigma, \sigma \in P_{s_j}} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} = \infty. \end{aligned}$$

Therefore, $x \notin m\left(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$, which is a contradiction. This completes the proof of the theorem. \square

Corollary 4. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then*

$$m\left(\mathcal{M}, \varphi^*, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) = m\left(\mathcal{M}, \varphi^{**}, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$$

if and only if $\sup_{s \geq 1} \frac{\varphi_s^}{\varphi_s^{**}} < \infty$, $\sup_{s \geq 1} \frac{\varphi_s^{**}}{\varphi_s^*} > \infty$.*

Proof. It is easy to prove so we omit. \square

Theorem 5. *For Musielak-Orlicz functions $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ which satisfy Δ_2 -condition and q, q_1, q_2 are seminorms. Then the following relation holds:*

$$\begin{aligned}
(i) \quad & m\left(\mathcal{M}', \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \subset m\left(\mathcal{M} \circ \mathcal{M}', \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \\
(ii) \quad & m\left(\mathcal{M}', \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \cap m\left(\mathcal{M}'', \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \\
& \subset m\left(\mathcal{M}' + \mathcal{M}'', \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \\
(iii) \quad & m\left(\mathcal{M}, \varphi, q_1, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \cap m\left(\mathcal{M}, \varphi, q_2, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) m \\
& \subset \left(\mathcal{M}, \varphi, q_1 + q_2, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right).
\end{aligned}$$

Proof. The proof of theorem along the same lines as the proof of the Theorem 2.5 of [1]. \square

Corollary 6. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function which satisfy Δ_2 - condition. Then $m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \subset m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$.*

Theorem 7. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then the sequence space $m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ is solid.*

Proof. Let $x \in m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$. Then there exists $\rho > 0$ such that

$$\lim_k \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} < \epsilon,$$

for every $\epsilon > 0$. Let (α_n) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
& \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n (\alpha_k x_k)}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \\
& \leq \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} |\alpha_k| M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \\
& \leq \frac{1}{\varphi_s} \sum_{k \in \sigma, \sigma \in P_s} M_k \left(q \left(\left\| \frac{\Lambda_k \Delta_m^n x_k}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k},
\end{aligned}$$

which proves that $m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ is solid space. \square

Corollary 8. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. The sequence space $m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ is monotone.*

Proof. It is obvious. \square

3 Generalized sequence spaces

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. Then, we have

$$A(x) = (A_i(x)) \text{ if } A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$$

converges for each i . For more details see [1].

In this section we introduce the following sequence spaces which are actually the generalizations of sequence spaces defined by Altun and Bilgin [1]. Thus we have the spaces:

$$m(\mathcal{M}, A, \varphi, \Delta_m^n, p, \|\cdot, \dots, \cdot\|) = \left\{ x = (x_k) \in s : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

If $M_k(x) = x$, we have

$$m(A, \varphi, \Delta_m^n, p, \|\cdot, \dots, \cdot\|) = \left\{ x = (x_k) \in s : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

In this section of the present paper we shall also study some topological properties and inclusion relations between the spaces $m(\mathcal{M}, A, \varphi, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$.

Theorem 9. *For Musielak-Orlicz functions $\mathcal{M} = (M_k)$. Then the sequence space $m(\mathcal{M}, A, \varphi, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$ is a linear space over the set of complex number \mathbb{C} .*

Proof. Let $x = (x_k)$ and $y = (y_k) \in m(\mathcal{M}, A, \varphi, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} < \infty$$

and

$$\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n y_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot, \dots, \cdot\|$ is a n -norm on X and (M_k) are non-decreasing and convex function so by using inequality (1.1), we have

$$\begin{aligned} & \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n (\alpha x_k + \beta y_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \\ & \leq \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left[\left(\left\| \frac{A_k \Delta_m^n (\alpha x_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) + \left(\left\| \frac{A_k \Delta_m^n (\beta y_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq K \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} \frac{1}{2^{p_k}} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \\ & + K \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} \frac{1}{2^{p_k}} M_k \left(q \left(\left\| \frac{A_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \\ & < \infty. \end{aligned}$$

Thus, we have $\alpha x + \beta y \in m(\mathcal{M}, A, \varphi, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$. Hence $m(\mathcal{M}, A, \varphi, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$ is a linear space. \square

Theorem 10. For any Musielak Orlicz function $\mathcal{M} = (M_k)$ and a bounded sequence $p = (p_k)$ of strictly positive real numbers, the space $m\left(\mathcal{M}, \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ is a topological linear space paranormed by

$$g(x) = \inf \left\{ \rho^{\frac{pr}{M}} : \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \leq 1, r = 1, 2, 3, \dots \right\},$$

where $M = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_k) \in m\left(\mathcal{M}, \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x) = \inf \left\{ \rho^{\frac{pr}{M}} : \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \leq 1 \right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exist some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho_\epsilon} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \leq 1.$$

Thus

$$\begin{aligned} & \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\epsilon} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \\ & \leq \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho_\epsilon} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \\ & \leq 1, \end{aligned}$$

Suppose that $x_k \neq 0$ for each $k \in N$. this implies that $A_k \Delta_m^n x_k \neq 0$ for each $k \in N$. Let $\epsilon \rightarrow 0$, then $\left\| \frac{A_k \Delta_m^n x_k}{\epsilon} \right\|, z_1, \dots, z_{n-1} \rightarrow \infty$. It follows that

$$\left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\epsilon} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \rightarrow \infty$$

Which is a contradiction. Therefore, $A_k \Delta_m^n x_k = 0$ for each k and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho_1} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \leq 1$$

and

$$\left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n y_k}{\rho_2} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by Minkowski's inequality, we have

$$\begin{aligned}
& \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \right)^{\frac{1}{M}} \\
& \leq \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho_1 + \rho_2}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \right)^{\frac{1}{M}} \\
& \leq \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\frac{\rho_1}{\rho_1 + \rho_2} \left\| \frac{A_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| + \frac{\rho_2}{\rho_1 + \rho_2} \left\| \frac{A_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\
& \leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(q \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\
& + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(q \left(\left\| \frac{A_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right)^{\frac{1}{M}} \\
& \leq 1.
\end{aligned}$$

Since ρ 's are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf \left\{ \rho^{\frac{p_r}{M}} : \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n (x_k + y_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\} \\
&\leq \inf \left\{ \rho_1^{\frac{p_r}{M}} : \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\} \\
&+ \inf \left\{ \rho_2^{\frac{p_r}{M}} : \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n y_k}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\}.
\end{aligned}$$

Therefore, $g(x+y) \leq g(x) + g(y)$

Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x) = \inf \left\{ \rho^{\frac{p_r}{M}} : \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\}.$$

Then

$$g(\mu x) = \inf \left\{ (|\mu|t)^{\frac{p_r}{M}} : \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n (\mu x_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \right)^{\frac{1}{M}} \right\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_r} \leq \max(1, |\mu| \sup p_r)$, we have

$$g(\mu x) = \max(1, |\mu| \sup p_r) \inf \left\{ (t)^{\frac{p_r}{M}} : \left(\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{t}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \right)^{\frac{1}{M}} \leq 1 \right\}$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. \square

Theorem 11. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then*

$$m(\mathcal{M}, \varphi^*, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|) \subset m(\mathcal{M}, \varphi^{**}, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|)$$

if and only if $\sup_{s \geq 1} \frac{\varphi_s^}{\varphi_s^{**}} < \infty$.*

Proof. The proof is trivial so we omit it. □

Corollary 12. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then*

$$m\left(\mathcal{M}, \varphi^*, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) = m\left(\mathcal{M}, \varphi^{**}, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$$

if and only if $\sup_{s \geq 1} \frac{\varphi_s^*}{\varphi_s^{**}} < \infty, \quad \sup_{s \geq 1} \frac{\varphi_s^{**}}{\varphi_s^*} > \infty.$

Proof. It is easy to prove so we omit. □

Theorem 13. *For Musielak-Orlicz functions $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ which satisfy Δ_2 -condition. Then the following relation holds:*

$$\begin{aligned} (i) \quad & m\left(\mathcal{M}', \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \subset m\left(\mathcal{M} \circ \mathcal{M}', \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \\ (ii) \quad & m\left(\mathcal{M}', \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \cap m\left(\mathcal{M}'', \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \\ & \subset m\left(\mathcal{M}' + \mathcal{M}'', \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right). \end{aligned}$$

Proof. The proof is along the same lines as the proof of the Theorem 2.5 of [1]. □

Corollary 14. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function which satisfy Δ_2 -condition. Then $m\left(\mathcal{M}, \varphi, q, \Lambda, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right) \subset m\left(\mathcal{M}, \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right).$*

Theorem 15. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then the sequence space $m\left(\mathcal{M}, \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ is solid.*

Proof. Let $x \in m\left(\mathcal{M}, \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$. Then there exists $\rho > 0$ such that

$$\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} < \epsilon,$$

for every $\epsilon > 0$. Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} & \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n (\alpha_k x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \\ & \leq \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} |\alpha_k| M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \\ & \leq \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k \left(\left\| \frac{A_k \Delta_m^n x_k}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k}, \end{aligned}$$

which proves that $m\left(\mathcal{M}, \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ is solid space. □

Corollary 16. *If $\mathcal{M} = (M_k)$ be any Musielak Orlicz function. Then the sequence space $m\left(\mathcal{M}, \varphi, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\right)$ is monotone.*

Proof. It is obvious. □

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