Acta Universitatis Matthiae Belii, series Mathematics
Volume 20 (2012), 3–8, ISSN 1338-7111
Online version, http://actamath.savbb.sk

Differential Inequalities Implying Starlikeness and Convexity

Sukhwinder Singh Billing
Department of Applied Sciences, B.B.S.B.Engineering College,
Fatehgarh Sahib-140 407, Punjab, India
ssbilling@gmail.com

Abstract
We study a differential inequality involving a multiplier transformation and consequently get some sufficient conditions in terms of certain simple differential inequalities for normalized analytic functions to be starlike and convex of order $\beta,\ 0 \le \beta < 1$.

Received April 24, 2012
Revised October 24, 2012
Accepted in final form October 26, 2012
Communicated with Ľubomír Snoha.
Keywords multiplier transformation, analytic function, starlike function, convex function.
MSC(2000) 30C80, 30C45.

1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the class of all functions f which are analytic in $\mathbb{E}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Denote by $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$, the classes of starlike functions of order β and convex functions of order β respectively, which are analytically defined as under:

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ 0 \le \beta < 1,\ z \in \mathbb{E} \right\}$$

and

$$\mathcal{K}(\beta) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ 0 \le \beta < 1,\ z \in \mathbb{E} \right\}.$$

We shall use $\mathcal{S}^*$ and $\mathcal{K}$ to denote $\mathcal{S}^*(0)$ and $\mathcal{K}(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

Let $\mathcal{A}_p$ denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, p \in \mathbb{N}$, which are analytic and multivalent in the open unit disk $\mathbb{E}$. Note $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define

Copyright © 2012 Matej Bel University

the multiplier transformation $I_p(n,\lambda)$ as

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \ (\lambda \geq 0, n \in \mathbb{Z}).$$

The operator $I_p(n,\lambda)$ has been recently studied by Aghalary et al. [1]. Earlier, the operator $I_1(n,\lambda)$ was investigated by Cho and Kim [2] and Cho and Srivastava [3], whereas the operator $I_1(n,1)$ was studied by Uralegaddi and Somanatha [10]. $I_1(n,0)$ is the well-known Sălăgean [9] derivative operator D^n, defined as:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

where $f \in \mathcal{A}$. In 1989, the operator $I_1(n,0)$ has been studied by Owa, Shen and Obradovič [7]. Recently, Li and Owa [5] studied the operator $I_1(n,0)$.

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that a function f is subordinate to a function g in $\mathbb{E}$ and write $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1, \ z \in \mathbb{E}$ such that $f(z) = g(w(z)), \ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

In the present paper, we study the differential inequality defined by the multiplier transformation $I_p(n,\lambda)$ in the open unit disk $\mathbb{E}$. As special cases to our main result, we obtain starlikeness and convexity of members of the class $\mathcal{A}$ in terms of certain simple differential inequalities. To prove our main result, we shall make use of following lemma of Hallenbeck and Ruscheweyh [4].

Lemma 1. *Let G be a convex function in $\mathbb{E}$, with $G(0) = a$ and let γ be a complex number, with $\Re(\gamma) > 0$. If $F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, is analytic in $\mathbb{E}$ and $F \prec G$, then*

$$\frac{1}{z^\gamma} \int_0^z F(w) w^{\gamma-1} \, dw = \frac{1}{n z^{\gamma/n}} \int_0^z G(w) w^{\frac{\gamma}{n}-1} \, dw$$

2 Main Theorem

Theorem 2. *Let $\alpha, \ \beta$ be real numbers such that $\alpha > \dfrac{2}{1-\beta}, \ 0 \leq \beta < 1$ and let*

$$0 < M \equiv M(\alpha,\beta,\gamma,p) = \frac{(\alpha+p+\lambda)[\alpha(1-\beta)-2]}{\alpha[1+(1-\beta)(p+\lambda)]}, \tag{2.1}$$

If $f \in \mathcal{A}_p$ satisfies the differential inequality

$$\left|(1-\alpha)\frac{I_p(n,\lambda)f(z)}{z^p} + \alpha \frac{I_p(n+1,\lambda)f(z)}{z^p} - 1\right| < M(\alpha,\beta,\lambda,p), \ z \in \mathbb{E}, \tag{2.2}$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \beta, \ z \in \mathbb{E}.$$

Proof. Let us define

$$\frac{I_p(n,\lambda)f(z)}{z^p} = u(z),\ \ z \in \mathbb{E}.$$

Differentiate logarithmically, we obtain

$$\frac{zI'_p(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} - p = \frac{zu'(z)}{u(z)} \tag{2.3}$$

In view of the equality

$$zI'_p(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z),$$

(2.3) reduces to

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = 1 + \frac{zu'(z)}{(p+\lambda)u(z)}$$

or

$$\frac{1}{p+\lambda}zu'(z) = \frac{I_p(n+1,\lambda)f(z)}{z^p} - \frac{I_p(n,\lambda)f(z)}{z^p}.$$

Therefore, in view of (2.2), we have

$$u(z) + \frac{\alpha}{p+\lambda}zu'(z) \prec 1 + Mz. \tag{2.4}$$

In view of Lemma 1 $\left(\text{selecting } \gamma = \frac{p+\lambda}{\alpha}\right)$ from (2.4), we have

$$u(z) \prec 1 + \frac{(p+\lambda)Mz}{\alpha+p+\lambda},$$

or

$$|u(z) - 1| < \frac{(p+\lambda)M}{\alpha+p+\lambda} < 1,$$

therefore, we obtain

$$|u(z)| > 1 - \frac{(p+\lambda)M}{\alpha+p+\lambda} \tag{2.5}$$

Write $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = (1-\beta)w(z) + \beta,\ 0 \le \beta < 1$ and therefore

$$\frac{I_p(n+1,\lambda)f(z)}{z^p} = u(z)[(1-\beta)w(z)+\beta].$$

Therefore (2.2) reduces to

$$|(1-\alpha)u(z) + \alpha u(z)[(1-\beta)w(z)+\beta] - 1| < M.$$

We need to show that $\Re(w(z)) > 0,\ z \in \mathbb{E}$. If possible, suppose that $\Re(w(z)) \not> 0,\ z \in \mathbb{E}$, then there must exist a point $z_0 \in \mathbb{E}$ such that $w(z_0) = ix, x \in \mathbb{R}$. To prove the required result, it is now sufficient to prove that

$$|(1-\alpha)u(z_0) + \alpha u(z_0)[(1-\beta)ix+\beta] - 1| \ge M. \tag{2.6}$$

By making use of (2.5), we have

$$|(1-\alpha)u(z_0) + \alpha u(z_0)[(1-\beta)ix+\beta] - 1|$$

$$\begin{aligned}
&\geq |[1-\alpha(1-\beta)+\alpha(1-\beta)ix]\,u(z_0)|-1\\
&= \sqrt{[1-\alpha(1-\beta)]^2+\alpha^2(1-\beta)^2x^2}\;\;|u(z_0)|-1\\
&\geq |1-\alpha(1-\beta)|\,|u(z_0)|-1\\
&\geq |1-\alpha(1-\beta)|\left(1-\frac{(p+\lambda)M}{\alpha+p+\lambda}\right)-1\geq M. \qquad (2.7)
\end{aligned}$$

Now (2.7) is true in view of (2.1) and therefore, (2.6) holds. Hence $\Re(w(z))>0$ or

$$\Re\left[\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right]>\beta,\ 0\leq\beta<1,\ z\in\mathbb{E}.$$

□

3 Deductions

On writing $p=1$ and $\lambda=0$ in Theorem 2, we obtain:

Corollary 3. *Let $\alpha,\ \beta$ be real numbers such that $\alpha>\dfrac{2}{1-\beta}$, $0\leq\beta<1$ and let $f\in\mathcal{A}$ satisfy the differential inequality*

$$\left|(1-\alpha)\frac{D^nf(z)}{z}+\alpha\frac{D^{n+1}f(z)}{z}-1\right|<\frac{(1+\alpha)[\alpha(1-\beta)-2]}{\alpha(2-\beta)},\ z\in\mathbb{E},$$

then

$$\Re\left(\frac{D^{n+1}f(z)}{D^nf(z)}\right)>\beta,\ 0\leq\beta<1,\ z\in\mathbb{E}.$$

Taking $p=1,\ n=0$ and $\lambda=0$ in Theorem 2, we have the following result.

Corollary 4. *If $\alpha,\ \beta$ are real numbers such that $\alpha>\dfrac{2}{1-\beta}$, $0\leq\beta<1$ and $f\in\mathcal{A}$ satisfies*

$$\left|(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)-1\right|<\frac{(1+\alpha)[\alpha(1-\beta)-2]}{\alpha(2-\beta)},\ z\in\mathbb{E},$$

then $f\in\mathcal{S}^*(\beta)$.

Setting $p=n=1$ and $\lambda=0$ in Theorem 2, we obtain:

Corollary 5. *Let $\alpha,\ \beta$ be real numbers such that $\alpha>\dfrac{2}{1-\beta}$, $0\leq\beta<1$ and let $f\in\mathcal{A}$ satisfy*

$$|f'(z)+\alpha zf''(z)-1|<\frac{(1+\alpha)[\alpha(1-\beta)-2]}{\alpha(2-\beta)},\ z\in\mathbb{E},$$

then $f\in\mathcal{K}(\beta)$.

Write $p=1,\ n=0$ and $\lambda=1$ in Theorem 2, to get the following result.

Corollary 6. *Let $\alpha,\ \beta$ be real numbers such that $\alpha>\dfrac{2}{1-\beta}$, $0\leq\beta<1$ and let $f\in\mathcal{A}$ satisfy*

$$\left|\left(1-\frac{\alpha}{2}\right)\frac{f(z)}{z}+\frac{\alpha}{2}f'(z)-1\right|<\frac{(2+\alpha)[\alpha(1-\beta)-2]}{\alpha(3-2\beta)},\ z\in\mathbb{E},$$

then

$$\Re\left(1+\frac{zf'(z)}{f(z)}\right)>\beta,\ z\in\mathbb{E}.$$

Remark 7. From Theorem 2, it follows, if $\alpha > \frac{2}{1-\beta}$, $0 \le \beta < 1$ and $f \in \mathcal{A}_p$ satisfies

$$\left|\left(\frac{1}{\alpha}-1\right)\frac{I_p(n,\lambda)f(z)}{z^p}+\frac{I_p(n+1,\lambda)f(z)}{z^p}-\frac{1}{\alpha}\right| < \frac{(\alpha+p+\lambda)[\alpha(1-\beta)-2]}{\alpha^2[1+(1-\beta)(p+\lambda)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \beta,\ z \in \mathbb{E}.$$

Letting $\alpha \to \infty$ in Remark 7, we have the following result.

Corollary 8. *Let β $(0 \le \beta < 1)$ be a real number and let $f \in \mathcal{A}_p$ satisfy*

$$\left|\frac{I_p(n+1,\lambda)f(z)}{z^p}-\frac{I_p(n,\lambda)f(z)}{z^p}\right| < \frac{1-\beta}{1+(1-\beta)(p+\lambda)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \beta,\ z \in \mathbb{E}.$$

Setting $p = 1$ and $\lambda = 0$ in Corollary 8, we get:

Corollary 9. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{n+1}f(z)}{z}-\frac{D^n f(z)}{z}\right| < \frac{1-\beta}{2-\beta},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \beta,\ 0 \le \beta < 1,\ z \in \mathbb{E}.$$

Writing $p = 1$ and $n = \lambda = 0$ in Corollary 8, we obtain the following result of Oros [6].

Corollary 10. *If $f \in \mathcal{A}$ satisfies*

$$\left|f'(z)-\frac{f(z)}{z}\right| < \frac{1-\beta}{2-\beta},\ z \in \mathbb{E},$$

then $f \in \mathcal{S}^*(\beta),\ 0 \le \beta < 1$.

Taking $p = n = 1$ and $\lambda = 0$ in Corollary 8, we get:

Corollary 11. *If $f \in \mathcal{A}$ satisfies*

$$|f''(z)| < \frac{1-\beta}{2-\beta},\ z \in \mathbb{E},$$

then $f \in \mathcal{K}(\beta),\ 0 \le \beta < 1$.

Note that for $\beta = 0$, the above result was obtained by Mocanu [8].

Setting $p = \lambda = 1$ and $n = 0$ in Corollary 8, we obtain the following result.

Corollary 12. *If $f \in \mathcal{A}$ satisfies*

$$\left|f'(z)-\frac{f(z)}{z}\right| < \frac{2(1-\beta)}{3-2\beta},\ 0 \le \beta < 1,\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathbb{E}.$$

References

[1] Aghalary, R., Ali, R. M., Joshi, S. B. and Ravichandran, V., *Inequalities for analytic functions defined by certain linear operators*, Int. J. Math. Sci., **4**(2005) 267–274.

[2] Cho, N. E. and Kim, T. M., *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc., **40**(2003) 399–410.

[3] Cho, N. E. and Srivastava, H., *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, **37**(2003) 39–49.

[4] Hallenbeck, D. J. and Ruscheweyh, S., *Subordination by convex functions*, Proc. Amer. Math. Soc., **52**(1975) 191–195.

[5] Jian, L. and Owa, S., *Properties of the Sălăgean operator*, Georgian Math. J., **5**(4) (1998) 361–366.

[6] Oros, G., *On a condition for starlikeness*, in: "The Second International Conference on Basic Sciences and Advanced Technology" (Assiut, Egypt, November 2-8, 2000), 89–94.

[7] Owa, S., Shen, C. Y. and Obradovič, M., *Certain subclasses of analytic functions*, Tamkang J. Math., **20**(1989) 105–115.

[8] Mocanu, P.T., *Some simple criteria for starlikeness and convexity*, Libertas Mathematica, **13**(1993) 27–40.

[9] Sălăgean, G. S., *Subclasses of univalent functions*, Lecture Notes in Math., **1013** 362–372, Springer-Verlag, Heideberg, 1983.

[10] Uralegaddi, B. A. and Somanatha, C., *Certain classes of univalent functions*, in: "Current Topics in Analytic Function Theory, Srivastava", S. M. and Owa, S. (ed.),World Scientific, Singapore, (1992) 371–374.