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On cycles in graphs with specified radius and diameter

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Abstract

Let G be a graph of radius r and diameter d with $d \leq 2r-2$. We show that G contains a cycle of length at least $4r-2d$, i.e. for its circumference it holds $c(G) \geq 4r-2d$. Moreover, for all positive integers r and d with $r \leq d \leq 2r-2$ there exists a graph of radius r and diameter d with circumference $4r-2d$.

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For a connected graph G, the *distance* $d_G(u,v)$ or briefly $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The distance between a vertex $u \in V(G)$ and a subgraph H of G will be denoted by $d(u,H)$, i.e. $d(u,H) = \min\{d(u,v); v \in V(H)\}$. The *eccentricity* $e_G(u)$ (briefly $e(u)$) of a vertex u of G is the distance of u to a vertex farthest from u in G, i.e. $e_G(u) = \max\{d_G(u,v); v \in V(G)\}$. The *radius* $\operatorname{rad} G$ of G is the minimum eccentricity among the vertices of G while the *diameter* $\operatorname{diam} G$ of G is the maximum eccentricity. The *circumference* of a graph G, denoted $c(G)$, is the length of any longest cycle in G.

A path P (a cycle C) in G is called *geodesic* if for any two vertices of P (of C) their distance in P (in C) equals their distance in G. A nontrivial connected graph with no cut-vertices is called a *nonseparable graph.* A *block* of a graph G is a maximal nonseparable subgraph of G.

A connected unicyclic graph G with the cycle C is called a *sun-graph* (see [2]) if $\deg_G(u) \leq 3$ for $u \in V(C)$ and $\deg_G(u) \leq 2$ for $u \in V(G) \setminus V(C)$. A $u-v$ path P in a sun-graph G is called a *ray* if $V(P) \cap V(C) = \{u\}$ and $\deg_G(v) = 1$. A sun-graph with the cycle C_m of length m and with m rays of length k will be denoted by $S_{m,k}$.

In what follows we answer a question that was posed several decades ago in [3]:

"How large a cycle must there be in a graph of radius m and diameter n? This question is also open. For radius 3 and diameter 4, the graph must have a cycle of length at least 4, which can be verified by brute force techniques .... The situation in general is unclear."

Our main result is the following theorem. Note that in the case $d = 2r-1$ or $d = 2r$ there are trees with radius r and diameter d.

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Theorem 1. *Let G be a graph of radius r and diameter d with $d \le 2r-2$. Then $c(G) \ge 4r-2d$.*

Proof. Since $d \le 2r-2$, G is not a tree. Let C be a cycle of G and B be a block of G containing C. Suppose, contrary to our claim, that $c(G) < 4r-2d$. Since B is a nonseparable subgraph of G, every two vertices of B lie on a common cycle (see [1, Theorem 1.6]) of length less than $4r-2d$. Hence we get $\operatorname{diam} B \le 2r-d-1 \le r-1$ (and so $B \ne G$).

Let u be a vertex such that $d(u,B) = \max\{d(v,B); v \in V(G)\}$ and let $u_B \in V(B)$ be a vertex with $d(u,u_B) = d(u,B)$. Evidently, u_B is a cut-vertex of G.

Let G_1 be a component of $G - u_B$ containing the vertex u. Put $d(u,u_B) = a$. We distinguish two cases.

(1) $a \le r-1$

Let v be a vertex of G. If $v \in V(G_1)$ then $d(v,u_B) \le a \le r-1$. If $v \in V(B)$ then $d(v,u_B) \le \operatorname{diam} B \le r-1$. Let, finally, $v \in V(G) \setminus (V(G_1) \cup V(B))$ and $v_B \in V(B)$ be a vertex such that $d(v,v_B) = d(v,B)$. Evidently, v_B is a cut-vertex of G and $d_G(u_B,v_B) = d_B(u_B,v_B)$. Denote $d(v,v_B) = b$ and $d(u_B,v_B) = c$. Suppose first that $b+c \ge r$. We have $c \le \operatorname{diam} B \le 2r-d-1$ and $b \le a$. Then $d(u,v) = d(u,u_B) + d(u_B,v_B) + d(v_B,v) = a+c+b \ge 2b+c \ge 2(r-c)+c \ge 2r-(2r-d-1) = d+1$. Since $\operatorname{diam} G = d$, we get $b+c \le r-1$ and so $d(u_B,v) \le r-1$. Finally, we have $e(u_B) \le r-1$, a contradicton.

(2) $a \ge r$

Let u_1 be a vertex of a geodesic $u-u_B$ path P^1 with $d(u,u_1) = r-1$. If w is a vertex from $V(G) \setminus V(G_1)$ then u_B is on a geodesic $w-u_1$ path and we get $d(w,u_1) \le r-1$ (since $d(w,u) \le 2r-2$). Since $e(u_1) \ge r$ (otherwise we have a contradiction), there is a vertex $v \in V(G_1)$ such that $d_{G_1}(v,u_1) = d_G(v,u_1) \ge r$. Let P^2 be a geodesic $v-u_B$ path and let v_1 be the first vertex of P^2 which is on P^1. Since $d(v,u_B) \le d(u,u_B)$, we get $d(u_B,v_1) < d(u_B,u_1)$. Let P^3 be a geodesic $v-u$ path and let v_2 be the first of its vertices which is on P^1. It is obvious (since $d(v,u) \le 2r-2$) that $d(u_B,v_2) > d(u_B,u_1)$. Evidently, there is a cycle C' such that $\{v_1,v_2\} \subseteq V(C')$.

Let G_2 be a subgraph of G induced by the set $V(G_1) \cup \{u_B\}$. Let $w \in V(G) \setminus V(G_2)$ be such a vertex that $d(w,u_B) = \max\{d(x,u_B); x \in V(G) \setminus V(G_2)\}$ and P be a geodesic $w-u_B$ path. Consider a graph G' for which $V(G') = V(G_2) \cup V(P)$ and $E(G') = E(G_2) \cup E(P)$. It is obvious that $|V(G')| < |V(G)|$ and if there is a vertex $z \in V(G')$ with $e_{G'}(z) \le r-1$ then $e_G(z) \le r-1$, too.

We can repeat the previous considerations with the graph G' and its block B' containing the cycle C'. It is clear now that after a finite number of the described steps we get a contradiction. □

Corollary 2. *If G is a graph with $\operatorname{rad} G = r$ and $\operatorname{diam} G \le 2r-2$, then G contains a cycle of length at least 4, i.e. $c(G) \ge 4$.*

Corollary 3. *If $c(G) = 3$ and $\operatorname{rad} G = r$, then $\operatorname{diam} G \in \{2r-1, 2r\}$.*

For all positive integers r and d satisfying $r \le d \le 2r-2$ there exists an infinite number of graphs of radius r, diameter d and circumference $4r-2d$. One of these graphs is C_{2r} for $d = r$. If $d > r$, one of these graphs is $S_{4r-2d,d-r}$, i.e. a sun-graph with the cycle C_{4r-2d} and with $4r-2d$ rays of length $d-r$ (see Figure 1a for $r=3$, $d=4$ and Figure 1b for $r=5$, $d=7$).

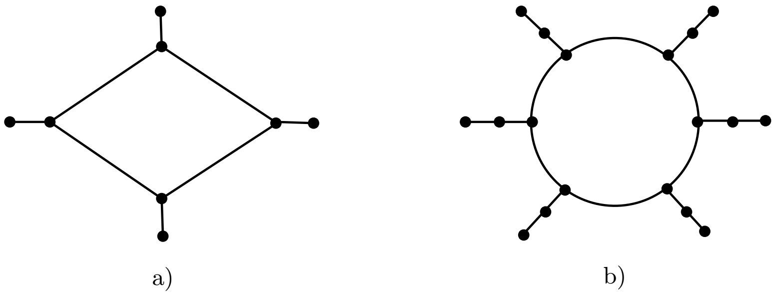

Figure 1

Now it is a simple matter to find infinite classes of graphs with mentioned properties (see Figure 2 for an inspiration).

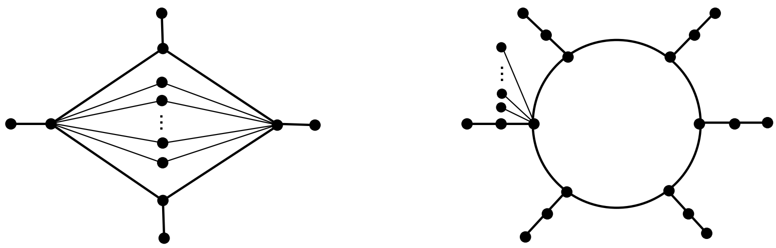

Figure 2

It is known that if a graph G with radius r and diameter $d \leq 2r-2$ has at most $3r-2$ vertices, then it holds $c(G) \geq 2r$. This fact is a consequence of the following theorem (see [2]).

Theorem 4 ([2])**.** *Let G be a graph with $\operatorname{rad} G = r$, $\operatorname{diam} G \leq 2r-2$, on at most $3r-2$ vertices. Then G contains a geodesic cycle of length $2r$ or $2r+1$.*

Using Theorem 4 it is easy to find all nonisomorphic graphs of minimal order and specified radius and diameter (see [3],[2]).

Let G be a sun-graph with the cycle C_{2r-1} ($r \geq 3$), with r rays of length 1 and such that exactly two of its end-vertices have distance 3 (see Figure 3 for $r = 5$). It is easy to see that $\operatorname{rad} G = r$, $|V(G)| = 3r-1$ and $c(G) = 2r-1$.

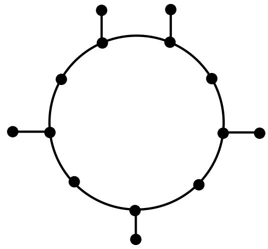

Figure 3

We can conclude that the bound $3r-2$ in Theorem 4 is the best possible (for $r \geq 3$).

References

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