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On generalized asymptotically equivalent double sequences through $(V, \lambda, \mu)-$summability

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Abstract

The purpose of this work is to introduce new generalizations of asymptotically equivalent double sequences which we call $S_{(\lambda,\mu)}-$ equivalent, $V_{(\lambda,\mu)}-$ equivalent, $C_{(1,1)}-$ equivalent, through $(V, \lambda, \mu)-$summability, and obtain some relevant connections between these notions.

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1 The first section

Marouf [1] introduced notion of asymptotically equivalent sequences in order to comparing the rate of growth of two sequences. Later, the idea is applied on many problems arising in the field of summability theory. In 2003, Patterson [2] presented statistical analogue of asymptotically equivalent sequences and studied some of their properties via statistical summability. Subsequently, many authors have shown their interest on asymptotically equivalent sequences in different directions (see [3], [4], [5], [6] and [7]). In present work we extend the idea of asymptotically equivalent double sequences through $(V, \lambda, \mu)-$ summability and obtain some results. We begin by recalling some definitions and results which form the base for present study.

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Definition 1. [1] The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent to a number L (denoted by $x \sim y$) provided that

$$\lim_{k\to\infty}\left(\frac{x_k}{y_k}\right) = L$$

In case, $L = 1$ we simply say x is equivalent to y.

Definition 2. [8] A number sequence $x = (x_k)$ is said to be statistically convergent to a number L provided that for every $\epsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{n}\ |\{k \le n : |x_k - L| \ge \epsilon\}| = 0;$$

where $|\{A\}|$ denotes the cardinality of a sets A, and $|a|$ denotes the absolute value of a number a. In this case, we write $S - \lim_{k\to\infty} x_k = L$ or $x_k \to L(S)$.
The next definition is a natural combination of Definition 1.1 and 1.2.

Definition 3. [2] The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistically equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : \left|\left(\frac{x_k}{y_k}\right) - L\right| \ge \epsilon\right\}\right| = 0,$$

(denoted by $x \sim^S y$) and simply asymptotically statistical equivalent if $L = 1$.

Let $\lambda = (\lambda_n)$ be a non decreasing sequence of positive real numbers tending to ∞ with $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1$. The generalized de la Vallée Pousin mean of $x = (x_k)$ is defined by

$$t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L (see [9]) provided that $t_n(x) \to L$ as $n \to \infty$. It is being noted that if we take $\lambda_n = n$, then (V, λ)-summability reduces to $(C, 1)$- summability.

Definition 4. [10] Let $\lambda = (\lambda_n)$ be a sequence as described above. A number sequence $x = (x_k)$ is said to be λ-statistically convergent to a number L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\ |\{k \in I_n : |x_k - L| \ge \epsilon\}| = 0.$$

In this case, we write $S_\lambda - \lim_{k\to\infty} x_k = L$ or $x_k \to L(S_\lambda)$.

The next definition is a natural combination of Definition 1.1 and 1.4.

Definition 5. [11] The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically λ-statistically equivalent of multiple L provided that for every$\epsilon > 0$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\left(\frac{x_k}{y_k}\right) - L\right| \ge \epsilon\right\}\right| = 0,$$

(denoted by $x \sim^{S_\lambda} y$) and simply asymptotically λ-statistical equivalent if $L = 1$.

Definition 6. [11] Let $\lambda = (\lambda_n)$ be a non decreasing sequence as described above. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically λ-equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |x_k - L| = 0,$$

(denoted by $x \sim^{(V,\lambda)} y$) and simply strongly asymptotically λ-equivalent if $L = 1$.

By convergence of a double sequence we mean convergence in the Pringsheim's sense [12] as given follow: A double sequence $x = (x_{ij})$ of numbers is said to be convergent to a number L (in the Pringsheim's sense) provided for each $\epsilon > 0$, there exists a positive integer m such that

$$|x_{ij} - L| < \epsilon \quad \text{whenever } i, j \geq m.$$

In this case, the number L is called the Pringsheim's limit of (x_{ij}) and we write $P - \lim_{i,j\to\infty} x_{ij} = L$. Further, a double sequence $x = (x_{ij})$ is said to be bounded if there exists a positive number M such that $|x_{ij}| \leq M$ for all i, j, i.e., if $\|x\|_{(2,\infty)} = \sup_{i,j} |x_{ij}| < \infty$. It is remarkable that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. Let l_2^∞ denotes the space of all bounded sequences of numbers. Mursaleen *et al.* [13] presented extension of Pringsheim's limit in term of statistical convergence for double sequences as follows:

Definition 7. [13] A double sequence $x = (x_{ij})$ of numbers is said to be statistically convergent to a number L provided for each $\epsilon > 0$

$$P - \lim_{n,m\to\infty} \frac{1}{nm} |\{i \leq n, j \leq m : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, the number L is called the statistical limit of x and we write $S(P) - \lim_{i,j\to\infty} x_{ij} = L$.

Definition 8. [6] The two non-negative double sequences $x = (x_{ij})$ and $y = (y_{ij})$ of numbers are said to be asymptotically statistically equivalent of multiple L provided that for every $\epsilon > 0$,

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \left| \left\{ i \leq n, j \leq m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| = 0,$$

(denoted by $x \sim^{S(P)} y$) and simply asymptotically statistical equivalent if $L = 1$.

Let $S(P)$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{S(P)} y$.

Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non decreasing sequences of positive real numbers tending to ∞ with

$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$ and

$\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$.

The generalized de la Vallée Pousin mean of $x = (x_{ij})$ is defined by

$$t_{mn}(x) = \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n\times I_m} x_{ij},$$

where $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$.

A double sequence $x = (x_{ij})$ is said to be $(V, \lambda, \mu)-$summable to a number L provided that $t_{mn}(x) \to L$ as $m, n \to \infty$. As in case of single sequences, if we choose $\lambda_n = n$ and $\mu_m = m$, then $(V, \lambda, \mu)-$summability reduces to $(C, 1, 1)-$summability. Let,

$$[C, 1, 1] = \left\{ x = (x_{ij}) : \exists\, L \in \mathbb{R}, P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1,j=1}^{m,n} |x_{ij} - L| = 0 \right\} \text{ and}$$

$$[V, \lambda, \mu] = \left\{ x = (x_{ij}) : \exists\, L \in \mathbb{R}, P - \lim_{m,n\to\infty} \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} |x_{ij} - L| = 0 \right\}.$$

Definition 9. [14] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ as described above. A double sequence $x = (x_{ij})$ of numbers is said to be $(\lambda, \mu)-$statistically convergent to a number L provided for every $\epsilon > 0$,

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} |\{(i, j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, the number L is called $(\lambda, \mu)-$statistical limit of the sequence $x = (x_{ij})$ and we write $S_{(\lambda,\mu)}(P) - \lim_{i,j\to\infty} x_{ij} = L$.

We now consider some new kind of asymptotically equivalent double sequences defined through $[V, \lambda, \mu]-$summability.

2 Main Results

Definition 10. The two double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be asymptotically Cesàro equivalent of multiple L (denoted by $x \sim^{C_{(1,1)}} y$) provided that

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \sum_{i=1,j=1}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right| = 0.$$

In case $L = 1$, we simply say x is asymptotically Cesàro equivalent to y.

Let $C_{(1,1)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{C_{(1,1)}} y$.

Definition 11. Let p be a positive real number. The two double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be strongly asymptotically $p-$Cesàro equivalent of multiple L (denoted by $x \sim^{C^p_{(1,1)}} y$) provided that

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \sum_{i=1,j=1}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p = 0.$$

In case $L = 1$, we simply say x is strongly asymptotically $p-$Cesàro equivalent to y.

Let $C^p_{(1,1)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{C^p_{(1,1)}} y$.

Remark 12. If $0 < p \leq q < \infty$, then $C^q_{(1,1)} \subseteq C^p_{(1,1)}$ and

$$C^p_{(1,1)} \cap l^\infty_2 = C^1_{(1,1)} \cap l^\infty_2 = C_{(1,1)} \cap l^\infty_2.$$

Theorem 13. *Let p be a positive real number such that $p \in (0, \infty)$, then $C^p_{(1,1)} \subseteq S(P)$.*

Proof. Let $p \in (0, \infty)$ and $x = (x_{ij})$ and $y = (y_{ij})$ be two double sequences such that $x \sim^{C^p_{(1,1)}} y$. For any $\epsilon > 0$, if we take

$$K_{mn} = \left\{ (i,j), i \leq n, j \leq m : \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \geq \epsilon \right\},$$

then we can write

$$\frac{1}{nm} \sum_{i=1,j=1}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p = \frac{1}{nm} \left\{ \sum_{(i,j) \in K_{mn}} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p + \sum_{(i,j) \notin K_{mn}} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \right\}$$

$$\geq \frac{1}{nm} \left\{ \sum_{(i,j) \in K_{mn}} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \right\}$$

$$\geq \frac{1}{nm} \left| \left\{ (i,j),\, i \leq n, j \leq m : \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \geq \epsilon \right\} \right|.$$

Since $x \sim^{C^p_{(1,1)}} y$, it follows that $x \sim^{S^{(P)}} y$. □

Theorem 14. *Let p be a positive real number such that $p \in (0, \infty)$, then $S(P) \cap l_2^\infty \subseteq C^p_{(1,1)}$.*

Proof. Let $p \in (0, \infty)$ and $x = (x_{ij})$, $y = (y_{ij}) \in l_2^\infty$ such that $x \sim^{S(P)} y$. Since $x = (x_{ij})$, $y = (y_{ij}) \in l_2^\infty$ so there is a real number M (say) such that for every i and j we have

$$\left| \frac{x_{ij}}{y_{ij}} - L \right| \leq M.$$

Since $x \sim^{S(P)} y$ so for given $\epsilon > 0$ and enough large m and n we can write

$$\begin{aligned} \frac{1}{nm} \sum_{i=1,j=1}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p &= \frac{1}{nm} \left\{ \sum_{\substack{i=1,j=1: \\ \left| \frac{x_{ij}}{y_{ij}} - L \right| \leq \epsilon}}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p + \sum_{\substack{i=1,j=1: \\ \left| \frac{x_{ij}}{y_{ij}} - L \right| > \epsilon}}^{m,n} \left| \frac{x_{ij}}{y_{ij}} - L \right|^p \right\} \\ &\leq (\epsilon)^p + \frac{M^p}{nm} \left| \left\{ i \leq n, j \leq m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| \\ &\leq 2\epsilon^p \end{aligned}$$

This shows that $x \sim^{C^p_{(1,1)}} y$. □

Definition 15. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non decreasing sequences of positive real numbers such that each tending to ∞ with

$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$ and

$\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1.$
The two non-negative double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be asymptotically $(\lambda, \mu)-$statistically equivalent of multiple L (denoted by $x \sim^{S_{(\lambda,\mu)}} y$) provided that for every $\epsilon > 0$,

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| = 0.$$

In case $L = 1$, we simply say x is asymptotically $(\lambda, \mu)-$statistically equivalent to y.

Let $S_{(\lambda,\mu)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{S_{(\lambda,\mu)}} y$. For the choose $\lambda_n = n$ and $\mu_m = m$, Definition 2.6 coincides with Definition 1.8.

Definition 16. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two sequences as in Definition 2.6. The two double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be strongly asymptotically $(\lambda, \mu)-$equivalent of multiple L (denoted by $x \sim^{V_{(\lambda,\mu)}} y$) provided that

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| = 0.$$

In case $L = 1$, we simply say x is strongly asymptotically $(\lambda, \mu)-$equivalent to y.

Let $V_{(\lambda,\mu)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{V_{(\lambda,\mu)}} y$.

Theorem 17. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two sequences as describe above, then we have following:*

(i) $x \sim^{V_{(\lambda,\mu)}} y$ implies $x \sim^{S_{(\lambda,\mu)}} y$ and the inclusion $V_{(\lambda,\mu)} \subset S_{(\lambda,\mu)}$ is proper.

(ii) If $x = (x_{ij})$, $y = (y_{ij}) \in l_2^\infty$ such that $x \sim^{S_{(\lambda,\mu)}} y$, then $x \sim^{V_{(\lambda,\mu)}} y$ and hence $x \sim^{C_{(1,1)}} y$ provided $x = (x_{ij})$ is not eventually constant.

(iii) $S_{(\lambda,\mu)}) \cap l_2^\infty = V_{(\lambda,\mu)} \cap l_2^\infty$.

Proof.

(i) Suppose $x = (x_{ij})$ and $y = (y_{ij})$ be two double sequences such that $x \sim^{V_{(\lambda,\mu)}} y$. For any $\epsilon > 0$, we can write

$$\begin{aligned} \sum_{(i,j)\in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| &\geq \sum_{\substack{(i,j)\in I_n \times I_m \\ |\frac{x_{ij}}{y_{ij}} - L| \geq \epsilon}} \left| \frac{x_{ij}}{y_{ij}} - L \right| \\ &\geq \epsilon \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right|. \end{aligned}$$

Since $x \sim^{V_{(\lambda,\mu)}} y$, so $P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| = 0$. This shows that $x \sim^{S_{(\lambda,\mu)}} y$.

We next give an example that shows the containment $V_{(\lambda,\mu)} \subset S_{(\lambda,\mu)}$ is proper. Define sequences $x = (x_{ij})$ and $y = (y_{ij})$ as follows:

$$x_{ij} = \begin{cases} ij, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq i \leq n \quad \text{and} \quad m - [\sqrt{\mu_m}] + 1 \leq j \leq m] \\ 0 & \text{otherwise} \end{cases}$$

and $y_{ij} = 1$ for all i and j.
Then $x = x_{ij} \notin l_2^{\infty}$ and for every $\epsilon (0 < \epsilon \leq 1)$ we have

$$\begin{aligned} &\frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - 0 \right| \geq \epsilon \right\} \right| \\ = &\frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : {}^{n-[\sqrt{\lambda_n}]+1 \leq i \leq n \text{ and}}_{m-[\sqrt{\mu_m}]+1 \leq j \leq m} \right\} \right| \\ &\leq \frac{[\sqrt{\lambda_n}\sqrt{\mu_m}]}{\lambda_n \mu_m}. \end{aligned}$$

It follows that

$$P - \lim_{n,m \to \infty} \frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - 0 \right| \geq \epsilon \right\} \right| = P - \lim_{n,m \to \infty} \frac{[\sqrt{\lambda_n}\sqrt{\mu_m}]}{\lambda_n \mu_m} = 0.$$

This shows that $x \sim^{S_{(\lambda,\mu)}} y$. Also note that

$$P - \lim_{n,m \to \infty} \frac{1}{\lambda_n \mu_m} \sum_{(i,j) \in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - 0 \right|$$

does not exists. Thus the inclusion $V_{(\lambda,\mu)} \subset S_{(\lambda,\mu)}$ is proper.
(ii) Let $x = (x_{ij})$, $y = (y_{ij}) \in l_2^{\infty}$ such that $x \sim^{S_{(\lambda,\mu)}} y$. Since $x = (x_{ij})$, $y = (y_{ij}) \in l_2^{\infty}$ so there is a real number M (say) such that for every i and j we have

$$\left| \frac{x_{ij}}{y_{ij}} - L \right| \leq M.$$

Also for given $\epsilon > 0$ and enough large m and n we can write

$$\begin{aligned} \frac{1}{\lambda_n \mu_m} \sum_{(i,j) \in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| &= \frac{1}{\lambda_n \mu_m} \sum_{\substack{(i,j) \in I_n \times I_m \\ \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon}} \left| \frac{x_{ij}}{y_{ij}} - L \right| + \frac{1}{\lambda_n \mu_m} \sum_{\substack{(i,j) \in I_n \times I_m \\ \left| \frac{x_{ij}}{y_{ij}} - L \right| < \epsilon}} \left| \frac{x_{ij}}{y_{ij}} - L \right| \\ &\leq \frac{M}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : \left| \frac{x_{ij}}{y_{ij}} - L \right| \geq \epsilon \right\} \right| + \epsilon. \end{aligned}$$

Since $x \sim^{S_{(\lambda,\mu)}} y$, it follows that the first part on right side of the above expression is zero, which immediately gives $x \sim^{V_{(\lambda,\mu)}} y$.
Furthermore, using the fact $(\frac{\lambda_n}{n}) \leq 1$ and $(\frac{\mu_m}{m}) \leq 1$, we have

$$\begin{aligned} \frac{1}{nm} \sum_{i=1,j=1}^{n,m} \left| \frac{x_{ij}}{y_{ij}} - L \right| &= \frac{1}{nm} \sum_{i=1,j=1}^{n-\lambda_n, m-\mu_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| + \frac{1}{nm} \sum_{i=n-\lambda_n+1, j=m-\mu_m+1}^{n,m} \left| \frac{x_{ij}}{y_{ij}} - L \right| \\ &\leq \frac{1}{\lambda_n \mu_m} \sum_{i=1,j=1}^{n-\lambda_n, m-\mu_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| + \frac{1}{\lambda_n \mu_m} \sum_{(i,j) \in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right| \\ &\leq \frac{2}{\lambda_n \mu_m} \sum_{(i,j) \in I_n \times I_m} \left| \frac{x_{ij}}{y_{ij}} - L \right|. \end{aligned}$$

Since $x \sim^{V_{(\lambda,\mu)}} y$, it follows that $x \sim^{C_{(1,1)}} y$.

(iii) This immediately follows from (i) and (ii). $\square$

Theorem 18. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two sequences as describe above. Then $S(P) \subset S_{(\lambda,\mu)}$ if and only if,* $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$ *and* $\liminf_{m\to\infty} \frac{\mu_m}{m} > 0$.

Proof. For given $\epsilon > 0$, we have

$$\left\{(i,j) \in I_n \times I_m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\} \subseteq \left\{i \leq n, j \leq m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\}.$$

Therefore,

$$\begin{aligned} \frac{1}{nm}\left|\left\{i \leq n, j \leq m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\}\right| &\geq \frac{1}{\lambda_n\mu_m}\left|\left\{(i,j) \in I_n \times I_m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\}\right| \\ &= \left(\frac{\lambda_n}{n}\right)\left(\frac{\mu_m}{m}\right)\frac{1}{\lambda_n\mu_m}\left|\left\{(i,j) \in I_n \times I_m : \left|\frac{x_{ij}}{y_{ij}} - L\right| \geq \epsilon\right\}\right|. \end{aligned}$$

Taking limit as $n, m \to \infty$ and using the assumption, we get $S(P) \subset S_{(\lambda,\mu)}$.

Conversely, suppose that $x = (x_{ij})$, $y = (y_{ij})$ be two double sequences such that $x \sim^{S(P)} y$. Assume, either $\liminf_{n\to\infty} \frac{\lambda_n}{n}$ or $\liminf_{m\to\infty} \frac{\mu_m}{m}$ or both are zero. Then we can choose two subsequences (n_p) and (m_q) such that $\frac{\lambda_{n_p}}{n_p} < \frac{1}{p}$and $\frac{\mu_{m_q}}{m_q} < \frac{1}{q}$. Define double sequences $x = (x_{ij})$ and $y = (y_{ij})$ as follows:

$$x_{ij} = \begin{cases} 1 & \text{if } i \in I_{n_p} \quad \text{and} \quad j \in I_{m_q} \qquad (p,q = 1,2,3,\ldots) \\ 0 & \text{otherwise.} \end{cases}$$

and $y_{ij} = 1$ for all i and j. Then clearly $x \sim^{C_{(1,1)}} y$ and therefore by Theorem 2.4, $x \sim^{S(P)} y$ which implies $x \sim^{S_{(\lambda,\mu)}} y$ as $S(P) \subset S_{(\lambda,\mu)}$. On the other hand the sequences $x = (x_{ij})$ and $y = (y_{ij})$ do not satisfy $x \sim^{V_{(\lambda,\mu)}} y$ which contradicts Theorem 2.8 (ii). Hence, we have $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$ and $\liminf_{m\to\infty} \frac{\mu_m}{m} > 0$. $\square$

Definition 19. Let p be a positive real number. The two double sequences $x = (x_{ij})$ and $y = (y_{ij})$ are said to be strongly asymptotically $V^p_{(\lambda,\mu)}-$equivalent of multiple L (denoted by $x \sim^{V^p_{(\lambda,\mu)}} y$) provided that

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n\mu_m} \sum_{(i,j)\in I_n\times I_m} \left|\frac{x_{ij}}{y_{ij}} - L\right|^p = 0.$$

In case $L = 1$, we simply say x is strongly asymptotically $V^p_{(\lambda,\mu)}-$equivalent to y.

Let $V^p_{(\lambda,\mu)}$ denotes the set of all sequences $x = (x_{ij})$ and $y = (y_{ij})$ such that $x \sim^{V^p_{(\lambda,\mu)}} y$.

Following Theorems are the analogue of Theorem 2.4 and 2.5, consequently their proofs can be obtained similarly.

Theorem 20. *Let p be a positive real number such that $p \in (0,\infty)$, then $V^p_{(\lambda,\mu)} \subseteq S^p_{(\lambda,\mu)}$.*

Theorem 21. *Let p be a positive real number such that $p \in (0,\infty)$, then $S^p_{(\lambda,\mu)} \cap l^\infty_2 \subseteq V^p_{(\lambda,\mu)}$.*

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